Some properties of matrices with signed null spaces

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Abstract

A matrix $A$ is said to have signed null space provided there exists a set $S$ of sign patterns such that the set of sign patterns of vectors in the null space of $\tilde{A}$ is $S$ for each $\tilde{A} \in Q(A)$. It is a generalization of a number of important qualitative matrix classes such as $L$-matrices, $S^*$-matrices, totally $L$-matrices, etc. In this paper, we obtain some new characterizations for matrices with signed null spaces. As applications, these results are used to obtain different proofs of some known properties and characterizations of matrices with signed null spaces, and are further used to study some special classes of matrices with signed null spaces.

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1. Introduction

The sign of a real number $a$, denoted by $\text{sgn}(a)$, is defined to be 1, $-1$ or 0, according to whether $a > 0$, $a < 0$ or $a = 0$. The sign pattern of a real matrix $A$, denoted by $\text{sgn}(A)$, is the $(0, 1, -1)$-matrix obtained from $A$ by replacing each entry by its sign. The set of real matrices with the same sign pattern as $A$ is called the qualitative class of $A$, and is denoted by $Q(A)$.

A real matrix $A$ is called an $L$-matrix provided every matrix in $Q(A)$ has linearly independent columns (note that it is defined in terms of rows in [3]). A square $L$-matrix is called a sign nonsingular matrix (SNS matrix).
The null space of a real matrix $A$, denoted by $\text{NS}(A)$, is the set of real column vectors $x$ such that $Ax = 0$. The qualitative null space of $A$, denoted by $\text{QNS}(A)$, is the set of sign patterns of the vectors in $\text{NS}(A)$. Namely,

$$\text{QNS}(A) = \{ \text{sgn} \, x \mid x \in \text{NS}(A) \}.$$ 

Let $A$ be an $m \times n$ matrix and $b$ an $m \times 1$ vector. The linear system $Ax = b$ is sign solvable provided each linear system $A'x = b'$ (where $A' \in \text{Q}(A)$ and $b' \in \text{Q}(b)$) has a solution and all such solutions have the same sign pattern.

In [4], Kim and Shader extend the concept of sign solvable linear system to a more general concept of linear system with signed solutions. The linear system $Ax = b$ is said to have signed solutions provided for each $A' \in \text{Q}(A)$ and $b' \in \text{Q}(b)$, the set of sign patterns of the solutions of $A'x = b'$ is equal to that of $Ax = b$.

It is not difficult to verify that the linear system $Ax = b$ is sign solvable if and only if it has signed solutions and $A'x = b'$ has a unique solution for each $A' \in \text{Q}(A)$ and $b' \in \text{Q}(b)$.

A real matrix $A$ is said to have signed null space, provided $Ax = 0$ has signed solutions. From this definition it is easy to see that $A$ has signed null space if and only if $\text{QNS}(A) = \text{QNS}(A')$ for each $A' \in \text{Q}(A)$.

It is shown in [4] that matrices with signed null spaces play an important role in the study of linear systems with signed solutions. Also some properties and characterizations of matrices with signed null spaces were given in [4,5] (e.g., see Theorems 2.A, 2.B and 2.C).

A submatrix $A_1$ of $A$ is called a column submatrix of $A$, if $A_1$ consists of a subset of columns of $A$. It is easy to see from the definition that if $A$ has signed null space, then so does each column submatrix of $A$.

The term rank of a matrix $A$, denoted by $\rho(A)$, is the maximal cardinality of a set of nonzero entries of $A$ no two of which lie on the same row or same column. The matrix $A$ is said to have “full row (or column) term rank” if $\rho(A)$ is equal to the number of rows (or columns) of $A$.

The following proposition follows easily from the definition and will be used several times later.

**Proposition 1.1.** Let $A$ be a matrix with signed null space and full column term rank. Then $A$ is an L-matrix.

**Proof.** Since $A$ has full column term rank, there exists $A' \in \text{Q}(A)$ such that $A'$ has linearly independent columns. Thus we have $\text{NS}(A') = \{0\}$ and so $\text{QNS}(A') = \{0\}$. Now $A$ has signed null space implies that $\text{QNS}(A) = \text{QNS}(A') = \{0\}$ for each $A \in \text{Q}(A)$. Thus $A$ has linearly independent columns and so $A$ is an L-matrix. \(\square\)

In this paper, we first obtain (in Section 2) some new characterizations of matrices with signed null spaces, then give some applications of these new characterizations in Sections 2 and 3. As the first application, we give different proofs of several properties and characterizations given in [4] for matrices with signed null spaces (the original proofs of these results are somewhat complicated and involve some results such as
separation theorem in convex analysis in some steps of the proofs). As the second application, we study (in Section 3) several special classes of matrices with signed null spaces, such as matrices with nearly signed null spaces, doubly signed null spaces and order preserving signed null spaces. We give complete characterizations of these special classes of matrices with signed null spaces.

2. Some characterizations of signed null spaces

First we introduce some notation and terminology.

Two \( m \times n \) real matrices \( A \) and \( B \) are said to be permutation equivalent, if \( A \) can be transformed to \( B \) by permuting its rows and columns.

A real matrix \( A \) is called an RSB matrix (also called row mixed matrix in [4]), if each row of \( A \) contains both positive and negative entries. The matrix \( A \) is a GRSB matrix (also called strictly row mixable matrix in [4]), if some matrix obtained from \( A \) by multiplying some of the columns of \( A \) by \(-1\) is an RSB matrix.

It is easy to see that if \( A \) is an \( m \times n \) RSB matrix, then for each \( m \times 1 \) real vector \( b \), there exists \( \tilde{A} \in \mathcal{Q}(A) \) such that the linear system \( \tilde{A}x = b \) has a solution. The same result holds for GRSB matrices.

The well-known characterization of \( L \)-matrices given in [3, Theorem 2.1.1] can be reformulated in terms of GRSB matrices in the following way (which will be used in the proof of Theorem 2.C later).

**Proposition 2.A** (Shao [6]). A real matrix \( A \) is not an \( L \)-matrix if and only if \( A \) is permutation equivalent to a matrix of the following block partitioned form:

\[
\begin{pmatrix}
A_1 & B \\
0 & A_2
\end{pmatrix},
\]

where \( A_1 \) is a GRSB matrix containing at least one column.

A matrix \( A_1 \) is called a GRSB kernel of a matrix \( A \), if \( A_1 \) is a GRSB matrix and \( A \) is permutation equivalent to a matrix of the form \( \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \).

It is easy to see that if \( A_1 \) is a GRSB kernel of some column submatrix of \( A \), then \( A_1 \) is also a GRSB kernel of \( A \).

Let \( [m] = \{1, \ldots, m\} \) and \( [n] = \{1, \ldots, n\} \). Let \( A \) be an \( m \times n \) matrix with \( m \leq n \). Let \( T_1, \ldots, T_{m(n)} \) be a fixed ordering of the subsets of \( [n] \) with cardinality \( m \). Then the row compound of \( A \), denoted by \( C(A) \), is the \( 1 \times \left( \begin{pmatrix} n \\ m \end{pmatrix} \right) \) row vector whose \( j \)th entry is the determinant \( \det A[\vdots | T_j] \) (where \( A[\vdots | T_j] \) denotes the column submatrix of \( A \) whose columns have indices in \( T_j \)). The matrix \( A \) has a signed row compound if \( \text{sgn}(C(A)) = \text{sgn}(C(\tilde{A})) \) for each \( \tilde{A} \in \mathcal{Q}(A) \). \( A \) is said to have a nonzero signed row compound if \( A \) has a signed row compound and \( C(A) \) is not a zero vector.

It is easy to see that \( A \) has a signed row compound if and only if each \( m \times m \) square submatrix of \( A \) with \( \rho(A) = m \) is an SNS matrix.
In [4], Kim and Shader show that there are close relationships between matrices with signed null spaces and matrices with signed row compound. They obtained a characterization of having signed null spaces for GRSB matrices in [4, Theorem 7]. The main part of this result can be written in the following way.

**Theorem 2.1.** Let $A$ be a GRSB matrix. Then the following conditions are equivalent:

(a) $A$ has signed null spaces.
(b) Each $p \times q$ GRSB kernel of $A$ has $p < q$.
(c) Each GRSB kernel of $A$ does not have full column term rank.
(d) $A$ has nonzero signed row compound.

(Note that condition (e) of [4, Theorem 7] is not quoted here, since our results and proofs will not involve this condition.)

It is also shown in [4, Corollary 8] that conditions (a)–(c) are equivalent even if we do not have the GRSB assumption.

In this section, we give several new characterizations of matrices with signed null spaces. We also use these characterizations to give different proofs of the above Theorem 2.1 [4, Corollary 8]. (Thus we will avoid using [4, Theorem 7, and Corollary 8] in the proofs of our results, especially in the proofs of the following Lemma 2.1 and Theorem 2.1).

Two $m \times n$ real matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be conformal, if $a_{ij}b_{ij} \geq 0$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Let

$$A = \begin{pmatrix} A' & u & v \\ 0 & 1 & -1 \end{pmatrix},$$

(2.1)

where the columns $u$ and $v$ are conformal. Then matrix $B = (A' u + v)$ is called a conformal contraction matrix of $A$.

The concept of conformal contraction was first introduced in [3].

It is not difficult to see (as also pointed out in [4]) that if $B$ is a conformal contraction matrix of $A$, then $A$ has signed row compound if and only if $B$ has.

The following proposition will be used in Lemma 2.1 and Theorem 2.1.

**Proposition 2.1.** Let $A$ be an $m \times n$ matrix with nonzero signed row compound and $X$ be a submatrix of $A$ obtained by deleting one column of $A$. Then one of the following two conditions hold:

1. $X$ also has nonzero signed row compound.
2. $A$ is permutation equivalent to a matrix of the form $\begin{pmatrix} Y_1 & 0 \\ Y_2 & Y_3 \end{pmatrix}$, where $Y_1$ is an SNS matrix and $Y_2$ has nonzero signed row compound.

**Proof.** If (1) does not hold, then clearly $X$ does not have full row term rank. So by the well-known König’s theorem [2] $X$ contains (thus $A$ contains) a (nonvacuous) $p \times q$
zero submatrix with \( p + q = n \). Thus, \( A \) is permutation equivalent to a matrix of the form \( \begin{pmatrix} Y_1 & 0 \\ Z & Y_2 \end{pmatrix} \) where \( Y_1 \) is a square matrix. Now \( A \) has nonzero signed row compound implies that \( Y_1 \) is an SNS matrix and \( Y_2 \) has nonzero signed row compound.

Note that if in Proposition 2.1 we further assume that \( A \) is a GRSB matrix, then condition (2) does not hold since \( Y_1 \) is not a GRSB matrix (as an SNS matrix). So in this case condition (1) holds.

**Lemma 2.1.** Let \( A \) be an \( m \times n \) RSB matrix with nonzero signed row compound. Then there exists some positive column vector \( x \in \mathbb{R}^n \) in the null space \( \text{NS}(A) \).

**Proof.** It is obvious from the hypothesis that \( n \geq m + 1 \). Let \( A_i \) be the matrix obtained from \( A \) by deleting the \( i \)th column of \( A \). We divide the proof into the following two cases.

**Case 1:** There are two indices \( i \) and \( j \) in \([n]\) (where \( i \neq j \)) such that both \( A_i \) and \( A_j \) are still RSB matrices, say \( i = 1 \) and \( j = 2 \).

By Proposition 2.1, \( A_1 \) and \( A_2 \) also have nonzero signed row compound. Using induction on \( n \), there exist \( y_1 \) and \( y_2 \) in \( \mathbb{R}^{n-1} \) with \( y_i \neq 0 \) such that \( A_i y_i = 0 \) (\( i = 1, 2 \)). Write

\[
y_1 = (a_1, \ldots, a_{n-1})^T \quad \text{and} \quad y_2 = (b_1, \ldots, b_{n-1})^T
\]

and take

\[
x_1 = (0, a_1, \ldots, a_{n-1})^T \quad \text{and} \quad x_2 = (b_1, 0, b_2, \ldots, b_{n-1})^T.
\]

Then \( A_i y_i = 0 \) implies that \( A x_i = 0 \) (\( i = 1, 2 \)). Now take \( x = x_1 + x_2 \in \mathbb{R}^n \), then clearly \( x \neq 0 \) and \( A x = 0 \).

**Case 2:** There exists at most one index \( i \) in \([n]\) such that \( A_i \) is still an RSB matrix.

Note that if some \( A_i \) is still an RSB matrix, then we must have \( n \geq m + 2 \) by Proposition 2.1. So in any case there exists a subset \( T \) of \([n]\) with \( |T| = m + 1 \) such that \( A_j \) is not an RSB matrix for each \( j \in T \).

Since \( |T| > m \), by the pigeonhole principle, there exists two different indices \( i \) and \( j \) in \( T \) and some row index \( r \in [m] \) such that the \( r \)th row of both matrices \( A_i \) and \( A_j \) is not row sign balanced. It follows that the \( r \)th row of \( A \) contains exactly two nonzero entries.

Now by suitably permuting the rows and columns of \( A \) we may assume that

\[
A = \begin{pmatrix} A' & u & v \\ 0 & a & -b \end{pmatrix} \quad \text{(where} \ a > 0 \ \text{and} \ b > 0)\).
\]

Since \( A \) has nonzero signed row compound, \( u \) and \( v \) must be conformal.

Let

\[
\tilde{A} = \begin{pmatrix} A' & \frac{1}{a}u & \frac{1}{b}v \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = (A' - \frac{1}{a}u + \frac{1}{b}v).
\]

Then \( B \) is a conformal contraction matrix of \( \tilde{A} \in Q(A) \). Thus (as mentioned above) \( B \) is also an RSB matrix with nonzero signed row compound. By induction there exists some
\[ y = (y_1, \ldots, y_{n-1})^T > 0 \] such that \( By = 0 \). Now take \( x = (y_1, \ldots, y_{n-2}, y_{n-1}/a, y_{n-1}/b)^T > 0 \), then it is easy to verify that \( Ax = 0 \). \( \square \)

The following theorem is in some sense similar to part (a) \( \Leftrightarrow \) (d) of Theorem 2.A (or [4, Theorem 7]), except that the hypothesis “\( A \) is a GRSB matrix” is replaced by the condition \( \rho(A) = m \) which might be easier to verify and to use in some circumstances.

**Theorem 2.1.** Let \( A \) be an \( m \times n \) real matrix with \( \rho(A) = m \). Then the following three conditions are equivalent:

1. \( A \) has signed null space.
2. \( A \) has signed row compound.
3. \( A \) has nonzero signed row compound.

**Proof.** (1) \( \Rightarrow \) (2): Let \( A_1 \) be an \( m \times m \) submatrix of \( A \) with \( \rho(A_1) = m \). Then (1) implies that the column submatrix \( A_1 \) also has signed null space. So \( \rho(A_1) = m \) implies that \( A_1 \) is an \( L \)-matrix by Proposition 1.1. Thus \( A_1 \) is an SNS matrix and so (2) follows.

(2) \( \Rightarrow \) (3): Since \( \rho(A) = m \), we can take \( \tilde{A} \in Q(A) \) with rank \( \tilde{A} = m \). Then the row compound \( C(\tilde{A}) \neq 0 \). So (2) implies that \( C(A) \neq 0 \) and thus (3) follows from (2).

(3) \( \Rightarrow \) (1): Take \( A_i \in Q(A) \), \( (i = 1, 2) \) and take \( x \in \mathbb{R}^n \) with \( A_1 x = 0 \).

Case 1: Assume \( x \) contains no zero entries.

Let \( D \) be a diagonal matrix with diagonal entries in \( \{1, -1\} \) such that \( Dx > 0 \). Then \( (A_1 D)(Dx) = A_1 x = 0 \). So \( A_1 D \) is an RSB matrix and thus \( A_2 D \) also is. By Lemma 2.1, there exists \( u > 0 \) such that \( A_2 D u = 0 \). Let \( \tilde{x} = D u \), then we have \( A_2 \tilde{x} = 0 \) and \( \tilde{x} \in Q(x) \).

Case 2: Assume \( x \) contains some zero entry, say \( x = (0, y) \), where \( y \in \mathbb{R}^{n-1} \).

Let \( B_i \) be the submatrix of \( A_i \) obtained by deleting the first column of \( A_i \), \( (i = 1, 2) \). Then \( A_1 x = 0 \) implies that \( B_1 y = 0 \). Now we use Proposition 2.1 for the matrices \( A_1 \) and \( B_1 \). If condition (2) of Proposition 2.1 holds, then in the form \( \begin{pmatrix} Y_1 & 0 \\ Z & Y_2 \end{pmatrix} \), \( Y_2 \) has signed null space by induction. So \( A_1 \) (and thus \( A \)) also has signed null space. If condition (1) of Proposition 2.1 holds, then \( B_1 \) also has nonzero signed row compound. By induction \( B_1 \) has signed null space and thus there exists \( z \in Q(y) \) with \( B_2 z = 0 \). Take \( \tilde{x} = \begin{pmatrix} 0 \\ z \end{pmatrix} \), then we have \( \tilde{x} \in Q(x) \) and \( A_2 \tilde{x} = B_2 z = 0 \).

Combining Cases 1 and 2, we conclude that \( A \) has signed null space. \( \square \)

Next, we consider the characterizations of matrices with signed null spaces in the cases \( \rho(A) \neq m \). For this purpose, we first introduce the following notion of term rank decomposed form.

**Definition 2.1.** A matrix of the following lower triangular block form

\[
\begin{pmatrix}
B & 0 \\
C & D
\end{pmatrix}
\]  

is called a term rank decomposed form, if \( B \) has full column term rank and \( D \) has full row term rank.
Note that in Definition 2.1, we allow the special case where $B$ (or $D$) is vacuous (then $C$ must also be vacuous).

The well-known theorem of König [2] asserts that every matrix $A$ is permutation equivalent to a term rank decomposed form (2.2), where $A = B$ for the special case where $A$ has full column term rank and $A = D$ for the special case where $A$ has full row term rank.

Now we consider when a matrix of term rank decomposed form has a signed null space. For convenience, we first give the following definition, where a linear system $Ax = b$ is sign inconsistent if $\tilde{A}x = \tilde{b}$ is inconsistent for each $\tilde{A} \in Q(A)$ and $\tilde{b} \in Q(b)$ (see [6]).

**Definition 2.2.** An $m \times n$ matrix $A$ is said to be sign consistentable if for each $m \times 1$ column vector $b$, the linear system $Ax = b$ is not sign inconsistent.

(Not that if $A'x = b'$ has a solution for some $A' \in Q(A)$ and $b' \in Q(b)$, then there exists some $A'' \in Q(A)$ such that $A''x = b$ has a solution, since there exists a diagonal matrix $D$ with all the diagonal entries positive such that $b' = Db$.)

For example, if $A$ has full row term rank, then $A$ is sign consistentable; Also if $A$ is a GRSB matrix, then $A$ is sign consistentable by the comments on GRSB matrices at the beginning of this section.

**Proposition 2.2.** Let $A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$. Then we have:

1. If $D$ is sign consistentable and $A$ has signed null space, then $B$ also has signed null space.
2. If $B$ is an $L$-matrix and $D$ has signed null space, then $A$ also has signed null space.

**Proof.** (1) Take $B_i \in Q(B)$, $(i = 1, 2)$ and take $y_1$ with $B_1y_1 = 0$. Since $D$ is sign consistentable, the linear system $Dx = -Cy_1$ is not sign inconsistent. So there exists $D_1 \in Q(D)$ such that $D_1x = -Cy_1$ has a solution, say $x = x_1$. Now take $A_i \in Q(A)$ with $A_i = \begin{pmatrix} B_i & 0 \\ C_i & D_i \end{pmatrix}$, $(i = 1, 2)$ (where $D_2$ can be taken arbitrarily in $Q(D)$). Clearly we have $\tilde{A}_1 (\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}) = 0$. Since $A$ has signed null space, there exists $\begin{pmatrix} y_2 \\ x_2 \end{pmatrix} \in Q(\begin{pmatrix} y_1 \\ x_1 \end{pmatrix})$ such that $A_2 (\begin{pmatrix} y_2 \\ x_2 \end{pmatrix}) = 0$. It follows that we have $y_2 \in Q(y_1)$ such that $B_2y_2 = 0$. So $B$ has signed null space.

(2) Take $A_i = \begin{pmatrix} B_i & 0 \\ C_i & D_i \end{pmatrix} \in Q(A)$, $(i = 1, 2)$ and take $x_1 = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}$ with $A_1 (\begin{pmatrix} y_1 \\ z_1 \end{pmatrix}) = 0$. Then we have $B_1y_1 = 0$, and thus $y_1 = 0$ (since $B$ is an $L$-matrix) and $D_1z_1 = 0$. Since $D$ has signed null space, there exists $z_2 \in Q(z_1)$ such that $D_2z_2 = 0$. Now take $x_2 = \begin{pmatrix} 0 \\ z_2 \end{pmatrix}$. Then it is easy to verify that $x_2 \in Q(x_1)$ and $A_2x_2 = 0$. This implies that $A$ also has signed null space. □

The next theorem give a characterization of a general matrix $A$ to have a signed null space in terms of the term rank decomposed form.
Theorem 2.2. Let $A$ be a real matrix. Then the following three conditions are equivalent:

1. $A$ has signed null space.
2. If \( \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \) is a term rank decomposed form which is permutation equivalent to $A$, then $B$ is an $L$-matrix (possibly vacuous) and $D$ has nonzero signed row compound (possibly vacuous).
3. $A$ is permutation equivalent to a matrix of the form \( \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \), where $B$ is an $L$-matrix (possibly vacuous) and $D$ has nonzero signed row compound (possibly vacuous).

Proof. (2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): By Theorem 2.1, $D$ has signed null space. Thus, by Proposition 2.2, $A$ also has signed null space.

(1) $\Rightarrow$ (2): If $A$ has signed null space, then the column submatrix \( \begin{pmatrix} 0 \\ D \end{pmatrix} \) also has and thus $D$ has signed null space. Since in the term rank decomposed form, $D$ has full row term rank. So $D$ has nonzero signed row compound by Theorem 2.1.

Now $D$ has full row term rank also implies that $D$ is sign consistentable. So (1) of Proposition 2.2 implies that $B$ has signed null space. But $B$ has full column term rank (in the term rank decomposed form), so $B$ must be an $L$-matrix by Proposition 1.1. □

We now use the characterizations given in Theorem 2.2 to give a different proof of part (a) $\iff$ (d) of Theorem 2.A [4, Theorem 7] in the following Theorem 2.B.

Theorem 2.B. Let $A$ be an $m \times n$ GRSB matrix. Then $A$ has signed null space if and only if $A$ has nonzero signed row compound.

Proof. The sufficiency part obviously follows from Theorem 2.1. We now prove the necessity part. Let \( \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \) be a term rank decomposed form which is permutation equivalent to $A$. Then $B$ is an $L$-matrix by Theorem 2.2. If $B$ is not vacuous, then $A$ is an GRSB matrix implies that $B$ also is. Thus $B$ is not an $L$-matrix, a contradiction. So we have $A = D$ and hence $A$ has full row term rank. The result now follows directly from Theorem 2.1. □

As another application of Theorem 2.2, we consider the relationships between matrices with signed null spaces and matrices whose rank is uniquely determined by whose sign pattern.

Definition 2.3. A real matrix $A$ is called a rank preserving matrix, if $\rho(A) = \rho(B)$ for each matrix $B \in Q(A)$.

Example 2.1. If $A$ is a rank preserving matrix with full column term rank, then $A$ is an $L$-matrix.
Proposition 2.3. If $A$ is an $m \times n$ matrix having signed null space, then $A$ is rank preserving.

Proof. Take any $B \in Q(A)$. Then $\text{QNS}(A) = \text{QNS}(B)$ by hypothesis. From this it follows easily that each pair of corresponding column subsets (with the same index subsets of $[n]$) of $A$ and $B$ have the same linear dependence relation. Thus, we have $\text{rank}(A) = \text{rank}(B)$ and the result follows. □

The next theorem is another characterization of matrices with signed null spaces which will be used to give a different proof of [4, Corollary 8] in Theorem 2.C.

Theorem 2.3. Let $A$ be an $m \times n$ real matrix. Then the following three conditions are equivalent:

1. $A$ has signed null space.
2. Each column submatrix $A_1$ of $A$ is rank preserving.
3. Each column submatrix $A_1$ of $A$ with full column term rank is an L-matrix.

Proof. (1) $\Rightarrow$ (2): $A$ has signed null space implies that $A_1$ has signed null space. Thus $A_1$ is rank preserving by Proposition 2.3.

(2) $\Rightarrow$ (3): This follows directly from Example 2.1.

(3) $\Rightarrow$ (1): Let \(\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}\) be a term rank decomposed form which is permutation equivalent to $A$. Without loss of generality, we may assume that $A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$. Since $D$ has full row term rank, there exists some square column submatrix of $D$ with full row term rank. Let $D_1$ be any such square column submatrix of $D$ and let $A_1 = \begin{pmatrix} B & 0 \\ C & D_1 \end{pmatrix}$. Then $A_1$ is a column submatrix of $A$ with full column term rank. By (3), $A_1$ is an L-matrix. So $D_1$ is an SNS matrix (and thus a barely L-matrix) and $B$ is also an L-matrix [3, Lemma 2.2.8]. It follows that $D$ has nonzero signed row compound and $B$ is an L-matrix. So $A$ has signed null space by Theorem 2.2. □

Now we give a different proof of [4, Corollary 8] in the following Theorem 2.C.

Theorem 2.C. Let $A$ be an $m \times n$ matrix. Then the following three conditions are equivalent:

1. $A$ has signed null space.
2. Each $p \times q$ GRSB-kernel $A_1$ of $A$ has $p < q$.
3. Each $p \times q$ GRSB-kernel $A_1$ of $A$ has $p(A_1) < q$.

Proof. (1) $\Rightarrow$ (2): Let $A_1$ be a $p \times q$ GRSB-kernel of $A$. Thus $A$ is permutation equivalent to some $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, and so $A_1$ has signed null space since $A$ has. Thus by Theorem 2.A (since $A_1$ is a GRSB matrix) $A_1$ has nonzero signed row compound, and so $p \leq q$. If $p = q$, then $A_1$ is an SNS matrix, contradicting the fact that $A_1$ is a GRSB matrix. So we have $p < q$. 

(2) ⇒ (3): This is obvious since $\rho(A_1) \leq p$.

(3) ⇒ (1): Suppose to the contrary that (1) is not true. Then by Theorem 2.3 $A$ contains some column submatrix $X$ with full column term rank which is not an $L$-matrix. So by Proposition 2.4 $X$ is permutation equivalent to a matrix of the form

$$\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$$

where $A_1$ is a GRSB matrix. So $A_1$ is a GRSB-kernel of $X$, and hence of $A$. On the other hand, $X$ has full column term rank implies that $A_1$ also has full column term rank, contradicting (3). ◻

3. Some further applications

In this section, we use the characterizations given in Section 2 to further study some special classes of matrices with signed null spaces.

A matrix $A$ is called a nearly $L$-matrix [1,9], if $A$ is not an $L$-matrix, but each matrix obtained from $A$ by deleting one column of $A$ is an $L$-matrix. Similar to nearly $L$-matrix, we have the following definition.

**Definition 3.1.** A matrix $A$ is said to have nearly signed null space, if $A$ does not have signed null space, but each matrix obtained from $A$ by deleting one column of $A$ has signed null space.

It is obvious that each matrix which does not have signed null space contains a column submatrix which has nearly signed null space.

Now we use the results obtained in Section 2 to give the following characterization of matrices with nearly signed null space.

**Theorem 3.1.** An $m \times n$ matrix $A$ has nearly signed null space if and only if $A$ is a nearly $L$-matrix with full column term rank.

**Proof.** *Sufficiency:* If $A$ is a nearly $L$-matrix, then $A$ is not an $L$-matrix. So $A$ has full column term rank implies that $A$ does not have signed null space by Proposition 1.1. Also, each matrix obtained from $A$ by deleting one column of $A$ has signed null space since it is an $L$-matrix. So $A$ has nearly signed null space.

*Necessity:* We divide the proof into the following three cases.

**Case 1:** $A$ has full column term rank. Then the result obviously follows from Proposition 1.1.

**Case 2:** $A$ has full row term rank. Then $A$ has nearly signed null space $\Rightarrow$ $A$ does not have signed null space $\Rightarrow$ $A$ does not have signed row compound (by Theorem 2.1) $\Rightarrow$ $A$ contains a (square) column submatrix $A_1$ of order $m$ with $\rho(A_1) = m$ which is not an SNS matrix (thus $A_1$ does not have signed null space by Proposition 1.1).

Now $A$ has nearly signed null space, so we must have $A = A_1$ and thus we are back to Case 1.

**Case 3:** $A$ has neither full column term rank nor full row term rank.

Then by König’s Theorem $A$ is permutation equivalent to a term rank decomposed form

$$\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix},$$

where $D$ is a $p \times q$ matrix and $p < q$ since $A$ does not have full
column term rank. Now $A$ has nearly signed null space implies that each proper column submatrix of $D$ has signed null space, thus each square submatrix $X$ of order $p$ of $D$ with $\rho(X) = p$ is an SNS matrix. So we conclude that $D$ has nonzero signed row compound.

Now let $D_1$ be an SNS submatrix of $D$ of order $n$ and let $A_1 = \begin{pmatrix} B & 0 \\ C & D_1 \end{pmatrix}$. Then $A_1$ has signed null space by hypothesis. Thus $B$ is an $L$-matrix by Theorem 2.2. Using Theorem 2.2 again to $A$ we conclude that $A$ also has signed null space, a contradiction. So Case 3 cannot happen and the result is proven.

As a comment on Theorem 3.1, we notice that a nearly $L$-matrix does not necessarily have full column term rank. For example, every $S$*-matrix is a nearly $L$-matrix which does not have full column term rank.

**Definition 3.2** (Shao [7]). Let $a, b$ be two real numbers, $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ real matrices.

1. We say that $b$ is sign majorized by $a$, denoted by $b \preccurlyeq a$, if $b = 0$ or $\text{sgn}(b) = \text{sgn}(a)$.
2. We say that $B$ is sign majorized by $A$, denoted by $B \preccurlyeq A$, if $b_{ij} \preccurlyeq a_{ij}$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

It is easy to see that $B \preccurlyeq A$ if and only if $B$ can be obtained from some $\tilde{A} \in Q(A)$ by replacing some nonzero entries of $\tilde{A}$ by zero.

Quite a number of qualitative properties of matrices are preserved under the above defined “signed majorized” order (if the term rank is also preserved). For example, if $A$ is an SNS matrix or $S^2$NS matrix (or even more generally, a matrix with signed generalised inverse, see [7,8]) and if $A_1 \preccurlyeq A$ with $\rho(A_1) = \rho(A)$, then (it can be proven by using the graph theoretical methods that) $A_1$ also is. From this it follows that if $A$ has nonzero signed row compound and $A_1 \preccurlyeq A$ with $\rho(A_1) = \rho(A)$, then $A_1$ also has nonzero signed row compound.

But for the property of having signed null space we are now considering, the situation is slightly different. For example, let

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

(3.1)

and let $A_1$ be the matrix obtained from $A$ by replacing all the entries in the last row by zero. Then it is easy to see that $A_1 \preccurlyeq A$ and $\rho(A_1) = \rho(A) = 3$. But $A$ has signed null space (since $A$ is an $L$-matrix, see [3, p. 6]) and $A_1$ does not have signed null space by Proposition 1.1 (since $A_1$ is not an $L$-matrix). In view of this example, we make the following definition.
Definition 3.3. A matrix $A$ is said to have order preserving signed null space, if each matrix $A_1$ with $A_1 \preceq A$ and $\rho(A_1) = \rho(A)$ has signed null space.

We also note that if a matrix $A$ has signed null space, then $A^T$ does not necessarily have signed null space (for example, see the matrix in (3.1)). So we make the following definition (which has close relationship with Definition 3.3 as will be shown in Theorem 3.2).

Definition 3.4. A matrix $A$ is said to have doubly signed null space, if both $A$ and $A^T$ have signed null space.

The following theorem gives characterizations for matrices having order preserving signed null spaces and matrices having doubly signed null spaces.

Theorem 3.2. The following three conditions are equivalent.

1. $A$ has doubly signed null space.
2. $A$ is permutation equivalent to a matrix of the form \( \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \) where both $D$ and $B^T$ (might be vacuous) have nonzero signed row compound.
3. $A$ has order preserving signed null space.

Proof. (1) $\iff$ (2) follows directly from Theorem 2.2.

(2) $\Rightarrow$ (3): Let $A_1$ be a matrix with $A_1 \preceq A$ and $\rho(A_1) = \rho(A)$. Then $A_1$ is permutation equivalent to a matrix of the form \( \begin{pmatrix} B_1 & 0 \\ C_1 & D_1 \end{pmatrix} \) where $B_1 \preceq B$, $C_1 \preceq C$ and $D_1 \preceq D$. Since both $D$ and $B^T$ have full row term rank and $\rho(A_1) = \rho(A)$, we conclude that $\rho(B_1) = \rho(B)$ and $\rho(D_1) = \rho(D)$. It follows from the comments after Definition 3.2 that both $D_1$ and $B_1^T$ have nonzero signed row compound, thus $A_1$ also has signed null space by Theorem 2.2 and so (3) holds.

(3) $\Rightarrow$ (2): Let \( \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \) be a matrix of term rank decomposed form which is permutation equivalent to $A$. Without loss of generality we may assume that $A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$. Then (3) $\Rightarrow$ $A$ has signed null space $\Rightarrow$ $D$ has nonzero signed row compound (by Theorem 2.2). Now suppose to the contrary that $B^T$ does not have nonzero signed row compound, then we may write (for the sake of simplicity) $B = \begin{pmatrix} B' \\ 0 \end{pmatrix}$ where $B'$ is a square “non-SNS” matrix with full (row or column) term rank. Now take $A_1 = \begin{pmatrix} B' & 0 \\ 0 & D \end{pmatrix}$. Then it is easy to see that $A_1 \preceq A$ and $\rho(A_1) = \rho(A)$. But $A_1$ does not have signed null space by Theorem 2.2 since $B'$ is not an $L$-matrix, contradicting (3).

References


