Linear Maps between $C^*$-Algebras Whose Adjoints
Preserve Extreme Points of the Dual Ball

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We give a structural characterization of linear operators from one $C^*$-algebra into another whose adjoints map extreme points of the dual ball onto extreme points. We show that up to a $\mathcal{V}$-isomorphism, such a map admits of a decomposition into a degenerate and a non-degenerate part, the non-degenerate part of which appears as a Jordan $*$-morphism followed by a “rotation” and then a reduction. In the case of maps whose adjoints preserve pure states, the degenerate part does not appear, and the “rotation” is but the identity. In this context the results concerning such pure state preserving maps depend on and complement those of Størmer [1963, Acta Math. 110, 233–278, 5.6 and 5.7]. In conclusion we consider the action of maps with “extreme point preserving” adjoints on some specific $C^*$-algebras. © 1998 Academic Press

1. INTRODUCTION

It is clear from the remarks made in the abstract that the results concerning maps with “pure state preserving” adjoints, provide us with a valuable clue as to what objects we may regard as “non-commutative composition operators”. The value of these and the other results also lie in the fact that they indicate that results of this nature for $C(K)$ spaces are not merely isolated fragments, but rather indicative of a very deep $C^*$-algebraic structure reaching far beyond the simplicity of the commutative case.

The notation employed is fairly standard $C^*$-algebraic notation and for the most part amounts to a subtle interpolation of that of Bratteli and Robinson [BR], and Kadison and Ringrose [KR]. The main features are the following: $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ will be deemed to be typical $C^*$-algebras which for the sake of convenience we will assume to be unital. Given $\mathcal{A}$, the associated sets of
all states and all pure states of $\mathcal{A}$ will be denoted by $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{A}}$ respectively. If indeed $\mathcal{A}$ is concrete, $\mathcal{A}'$ denotes the commutant and $\mathcal{A}_w$ the set of all normal states. Functionals of a $C^*$-algebra will be denoted by $\rho$, $\omega$, with $\omega$ being reserved for the notation of states (usually pure). In this context, given a state $\omega$ of $\mathcal{A}$, $(\pi_\omega, h_\omega, \Omega_\omega)$ will denote the canonical cyclic representation of $\mathcal{A}$ engendered by $\omega$. Here $h_\omega$ is the relevant Hilbert space, $\Omega_\omega \in h_\omega$ the cyclic unit vector corresponding to $\omega$, and $\pi_\omega$ the canonical *-homomorphism from $\mathcal{A}$ into $B(h_\omega)$. Typical Hilbert spaces will be taken to be $h$ and $k$. Finally given any Banach space $X$, $(X)_1$ or $X_1$ if there is no danger of confusion, will denote the closed unit ball of $X$. In this context $\text{ext}(X_1)$ denotes the set of extreme points of $X_1$.

Regarding linear maps on $C^*$-algebras, a Jordan $(\ast)$-morphism is understood to be a mapping $\psi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\psi(AB + BA) = \psi(A)\psi(B) + \psi(B)\psi(A)$ and $\psi(A^\ast) = \psi(A)^\ast$ for all $A, B \in \mathcal{A}$. This concept is of course equivalent to that of a $C^*$-homomorphism which is defined to be a positive map preserving squares of self-adjoint elements. To see this one need only note that in general $(A + B)^2 - A^2 - B^2 = AB + BA$, and make use of the fact span($\mathcal{A}_w$) = $\mathcal{A}$.

Moreover given any Jordan $\ast$-morphism $\psi: \mathcal{A} \rightarrow \mathcal{B}$, $\psi(I) = E$ is easily seen to be an orthogonal projection with $\psi(A) = E\psi(A)E$ for all $A \in \mathcal{A}$. To see the latter fact one need only note that if indeed $\psi$ is a Jordan $\ast$-morphism, then $\psi(ABA) = \psi(A)\psi(B)\psi(A)$ for all $A, B \in \mathcal{A}$ [BR; p. 212]. Thus as a map into $\mathcal{B}_E$, $\psi$ then preserves the identity. In particular if $\mathcal{B}$ is concrete and $\psi$ a Jordan $\ast$-morphism with $\psi(\mathcal{A})^\ast = \mathcal{B}^\ast$, we must then have $\psi(I) = I$.

In this context we also observe that for our purposes we do not need to assume continuity of the operators we characterize, since the properties under consideration necessarily imply that these must even have norm one. In the case where $\psi: \mathcal{A} \rightarrow \mathcal{B}$ with $\omega \circ \psi \in \mathcal{P}_\mathcal{B}$ for every $\omega \in \mathcal{P}_\mathcal{A}$, this follows from [BR; 3.2.6] on noticing that by [KR; 4.3.8] we have $\psi \geq 0$ with $\psi(I) = I$.

In the case where $\rho \circ \psi \in \text{ext}(\mathcal{A}_w^\ast)$ for every $\rho \in \text{ext}(\mathcal{B}_w^\ast)$, we merely need to verify continuity and apply the Krein–Milman theorem to $\psi^\ast$. To see continuity in this case, given $A \in \mathcal{A}$, select $\omega_0, \omega_1 \in \mathcal{P}_\mathcal{A}$ so that $\omega_0\text{Re} \psi(A) = \|\text{Re} \psi(A)\|$ and $\omega_1(\text{Im} \psi(A)) = \|\text{Im} \psi(A)\|$ [KR; 4.3.8]. Then since $\omega_0 \circ \psi$, $\omega_1 \circ \psi \in \text{ext}(\mathcal{A}_w^T)$, they are both norm-one functionals and hence

$$
\|\psi(A)\| \leq \|\text{Re} \psi(A)\| + \|\text{Im} \psi(A)\|
$$

$$
= \omega_0 \left( \frac{1}{2} (\psi(A) + \psi(A)^\ast) \right) + \omega_1 \left( \frac{-i}{2} (\psi(A) - \psi(A)^\ast) \right)
$$

$$
= \frac{1}{2} [ \omega_0(\psi(A)) + \omega_0(\psi(A)^\ast) - i\omega_1(\psi(A)) + i\omega_1(\psi(A)^\ast) ]
$$

$$
\leq (\|\omega_0 \circ \psi\| + \|\omega_1 \circ \psi\|) |A| = 2 |A|
$$

as required.
2. MAPS WITH PURE STATE PRESERVING ADJOINTS: 
THE OVERTURE TO THE GENERAL CASE

Although the lemmas in this section may be deemed to be standard 
folklore and Theorem 5 judged to be a technical reworking of hard work 
done by Störm [Stö1], [Stö2]), its value lies in the fact that it does 
present a coherent framework within which to attack the more general case 
of maps whose adjoints preserve points of the unit ball. (To see that this 
case is indeed more general is none too trivial (cf. Corollary 20).) Like any 
good overture, this section and its lemmas presents in embryonic form the 
main ideas developed further later on. For this reason we have chosen to 
prove the lemmas in full.

**Lemmas 1.** Let \( \mathcal{A} \) be a \( \mathcal{C}^* \)-algebra and \( E \in \mathcal{A} \) a projection. Denote the 
reduction \( \mathcal{A} \to \mathcal{A}_E \) by \( \eta \). Then \( \omega \cdot \eta \) is a pure state of \( \mathcal{A} \) whenever \( \omega \) is a pure 
state of \( \mathcal{A}_E \). Conversely if \( \omega(E) = 1 \), then the restriction of \( \omega \) to \( \mathcal{A}_E \) is a pure 
state of \( \mathcal{A}_E \) whenever \( \omega \) is a pure state of \( \mathcal{A} \).

**Proof.** Suppose \( \omega \) is a pure state of \( \mathcal{A}_E \) and let \( \rho \) be a positive 
fractional on \( \mathcal{A} \) majorized by \( (\omega \cdot \eta) \). We show that then \( \rho(A) = \rho(EAE) \) for 
every \( A \in \mathcal{A} \). If this be true, then clearly \( \rho \) is of the form \( \rho_{E \cdot \eta} \) where \( \rho_E \) is the 
restriction of \( \rho \) to \( \mathcal{A}_E \). Since \( \omega \cdot \eta \geq \rho = \rho_E \cdot \eta \geq 0 \), it is clear that then 
\( \omega \geq \rho_E \geq 0 \). But then \( \rho_E \) will be a multiple of \( \omega \) on \( \mathcal{A}_E \) [KR; 3.4.6] and 
hence \( \rho = \rho_E \cdot \eta \) a multiple of \( \omega \cdot \eta \). By [KR; 3.4.6], \( \omega \cdot \eta \) must then be 
pure. In order to finally verify that \( \rho(A) = \rho(EAE) \) for every \( A \in \mathcal{A} \), it 
suffices to do this for the case \( A \in A^+ \) since span(\( A^+ \)) = \( \mathcal{A} \). Now if 
\( A \in A^+ \) and \( 0 \leq \rho \leq \omega \cdot \eta \), then surely \( 0 \leq (I - E) A(I - E) \), and hence 
\[
0 \leq \rho((E - E) A(I - E)) \leq \omega((E - E) A(I - E)) = \omega(0) = 0
\]
for every \( A \in A^+ \), that is
\[
\rho((I - E) A(I - E)) = 0.
\] (1)

Next appealing to (1) and applying [KR; 4.3.1], we get
\[
|\rho((I - E) A) E| \leq \rho(E^* E) \rho((I - E) A((I - E) A)^*)
\]
\[
= \rho(E) \rho((I - E) A^2(I - E)) = 0
\] (2)
for every \( A \in A^+ \), and hence also that
\[
|\rho(EA(I - E))| = |\rho((E^* E)(I - E))| = |\rho((E^* A)(I - E))|
\]
\[
= |\rho((I - E) A)| = 0.
\] (3)

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Combining (1), (2), and (3), we have $\rho(A) = \rho(EAE)$ for every $A \in \mathscr{A}^+$ as required.

Conversely if $\tilde{\omega} \in \mathscr{P}_\mathscr{A}$ with $\tilde{\omega}(E) = 1$, then it may be verified that $\tilde{\omega}(A) = \tilde{\omega}(EAE)$ for any $A \in \mathscr{A}$. As before it suffices to show this for the case $A \in \mathscr{A}^+$.

For $A \in \mathscr{A}^+$ it may easily be verified that

$$0 = \tilde{\omega}((I - E) A E) = \tilde{\omega}(E A(I - E)) = \tilde{\omega}((I - E) A(I - E)). \quad (4)$$

We show how to do this in one of the cases, the others being similar. Since $\tilde{\omega}(I - E) = \tilde{\omega}(I) - \tilde{\omega}(E) = 0$, we have by [KR; 4.3.1] that

$$|\tilde{\omega}((I - E) A E)|^2 \leq \tilde{\omega}((I - E)(I - E)^*) \tilde{\omega}((A E)^*(A E))
= \tilde{\omega}(I - E) \tilde{\omega}(E A^2 E) = 0$$

for every $A \in \mathscr{A}^+$. Thus $\tilde{\omega}(A) = \tilde{\omega}(EAE)$ for all $A \in \mathscr{A}^+$ by (4), as required. Clearly then $\tilde{\omega}$ is of the form $\omega_0 \cdot \eta$ where $\omega_0$ is the restriction of $\tilde{\omega}$ to $\mathscr{A}_E$.

Moreover $\omega_0$ is a state of $\mathscr{A}_E$ by [KR; 4.3.2] applied to the fact that $\tilde{\omega}(E) = 1$. Now finally if $\omega_0 \geq \rho \geq 0$ for some functional on $\mathscr{A}_E$, then since $\eta$ preserves order [KR; 4.2.7], we have $\tilde{\omega} = \omega_0 \cdot \eta \geq \rho \cdot \eta \geq 0$. Since $\tilde{\omega} \in \mathscr{P}_\mathscr{A}$, $\rho \cdot \eta$ must be a multiple of $\tilde{\omega}$ [KR; 3.4.6] and hence on restriction to $\mathscr{A}_E$, $\rho$ must then be a multiple of $\omega_0$. It follows that $\omega_0$ is a pure state of $\mathscr{A}_E$ [KR; 3.4.6].

**Lemma 2.** Let $\mathscr{A}$ be a von Neumann algebra, $E$ a projection in $\mathscr{A}$, and let $\eta$ be defined as before. Then $\rho \cdot \eta$ is a normal state on $\mathscr{A}$ whenever $\rho$ is a normal state on $\mathscr{A}_E$. Conversely if $\tilde{\rho}$ is a normal state of $\mathscr{A}$ with $\tilde{\rho}(E) = 1$, then the restriction of $\rho$ to $\mathscr{A}_E$ is a normal state of $\mathscr{A}_E$.

**Proof.** For the second part all we need to do is note that the restriction is a state by [KR; 4.2.3], and apply the definition [KR; 7.1.11]. To see the first part all we really need to do is to note that if $A_j$ is a monotone increasing net in $\mathscr{A}$ with least upper bound $A \in \mathscr{A}$, then $E A_j E$ is a monotone increasing net with l.u.b. $EAE$. Then, we use the fact that $\eta$ preserves order, and a combination of [BR; 2.4.1 and 2.4.19].

**Lemma 3.** Let $\mathscr{A}$ be a C*-algebra. If $\mathscr{A}$ is in its reduced atomic representation, then every pure state of $\mathscr{A}$ is normal (ultra-weakly continuous).

**Proof.** By [KR; 7.1.12], it suffices to show that all the pure states are vector states. First of all by the definition of the reduced atomic representation there is a maximal disjoint set of pure states, $\mathscr{M}$, in terms of which the representation is generated by the GNS construction. These pure states are then obviously vector states. Next given any $\omega \in \mathscr{P}_\mathscr{A}$, by the maximality
of $\mathcal{M}$, $\omega$ is unitarily equivalent to some $\omega_0 \in \mathcal{M}$ [KR; 10.2.6 and 10.3.7], say

$$\omega(A) = \omega_0(U^*AU) \quad A \in \mathcal{A}$$

where $U \in \mathcal{A}$ is unitary. But then if $\omega_0 \in \mathcal{M}$ corresponds to the vector state say $(A\Omega_0, \Omega_0)$, $A \in \mathcal{A}$, $\|\Omega_0\| = 1$, then surely $\omega$ corresponds to

$$\omega(A) = \langle U^*AU\Omega_0, \Omega_0 \rangle = \langle A(U\Omega_0), (U\Omega_0) \rangle \quad A \in \mathcal{A}$$

where $\|U\Omega_0\| = \|\Omega_0\| = 1$.

**Lemma 4.** If $\varphi$ is a pure normal state of a concrete $C^*$-algebra $\mathcal{A}$, the unique normal extension of $\varphi$ to $\mathcal{A}^\ast$, say $\varphi_\ast$, is a pure state of $\mathcal{A}^\ast$.

**Proof.** It is an exercise to show that the ultra-weak continuity of $\varphi$ implies that the representation of $\mathcal{A}$ engendered by $\varphi$ is similarly continuous. (This can be seen by for example suitably adapting the first part of the proof of [BR; 2.4.24].) If $\pi_\omega$ is this representation, then by [KR; 10.1.10] it has a unique ultra-weakly continuous extension $\tilde{\pi}_\omega$ to all of $\mathcal{A}^\ast$. If now $\omega$ corresponds to the vector state $\omega_\Omega$ in the sense that $\varphi = \omega_\Omega \pi_\omega$, then surely $\omega_\Omega \tilde{\pi}_\omega$ is a normal ultra-weak extension of $\tilde{\varphi}$, and hence by the uniqueness of this extension we have $\tilde{\varphi} = \omega_\Omega \tilde{\pi}_\omega$. A combination of [KR; 10.2.3] and [KR; 10.2.5] applied to $\pi_\omega$ and $\tilde{\pi}_\omega$ respectively, reveal that $\omega$ is a pure state of $\mathcal{A}$.

**Theorem 5.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and $\pi$ the reduced atomic representation of $\mathcal{B}$. Then a linear mapping $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ has the property that $\omega \circ \varphi$ is a pure state whenever $\varphi$ is a pure state of $\mathcal{B}$ if and only if there exists a von Neumann algebra $\mathcal{R}$ acting on some Hilbert space $h$, a projection $E \in \mathcal{R}$, and a set $(F_i)\pi$ of mutually orthogonal central projections in $\mathcal{R}$ with $\sum_i F_i = 1_{\mathcal{R}}$, such that up to a (ultra-weakly continuous) $*$-isomorphic embedding $\Phi$ of $\pi(\mathcal{B})'$ in $\mathcal{R}$, $\pi(\mathcal{A})' \otimes \mathcal{R}$ appears as $\mathcal{R}E$ with $\Phi \circ \pi \circ \varphi$ of the form

$$(\Phi \circ \pi \circ \varphi)(A) = E\psi(A) E \quad \text{for all} \quad A \in \mathcal{A}.$$ 

Here $\psi$ is a Jordan $*$-morphism from $\mathcal{A}$ into $\mathcal{R}$ with the property that

$$(F_i \psi(\mathcal{A}) F_i)' = F_i \mathcal{R} F_i$$

(a slightly weaker condition than merely requiring $\psi(\mathcal{A})' = \mathcal{R}$).

**Proof.** First assume $\varphi$ to be of the form described in the hypothesis. Since $*$-isomorphisms clearly preserve pure states, $\Phi \circ \pi$ basically identifies
with $\Phi(x(\mathcal{A}))$ as far as we are concerned, and hence we may regard $\mathcal{A}$ as a subalgebra of $\mathcal{B}$ with the property that $\mathcal{B}'' = \mathcal{B}_E$. We proceed to show that $\varphi$ preserves pure states. Let $\omega \in \mathcal{P}_R$ be given. By Lemmas 3 and 4 there exists a unique extension $\tilde{\omega}$ of $\omega$ to all of $\mathcal{B}'' = \mathcal{B}_E$ which is pure and ultra-weakly continuous on $\mathcal{B}''$. By the uniqueness we may identify $\omega$ with $\tilde{\omega}$.

Considering Lemma 1, it follows that $\omega_E = \omega(E \cdot E)$ is a pure state on $\mathcal{A}$. Therefore by [KR, 4.3.14] $\omega_E(F_v) \in \{0, 1\}$ for every $v$. However since the $F_v$'s are mutually orthogonal central projections with $\bigvee F_v = 1$ and since $\omega_E$ is suitably continuous by Lemma 3, it follows that $1 = \omega_E(1) = \sum \omega_E(F_v)$ and hence that $\omega_E(F_v) = 1$ for precisely one of the $F_v$'s, say $\omega_E(F_{\theta}) = 1$.

Thus denoting $\omega_E(F_{\theta} \cdot F_{\theta})$ and $F_{\theta} \psi F_{\theta}$ by $\rho_\theta$ and $\psi_\theta$ respectively, it follows from [KR, 4.3.14] that

$$\omega_E \cdot \psi = \psi_\theta \cdot \rho_\theta.$$ 

Clearly it suffices to consider $\omega_E$ in terms of the von Neumann algebra $\mathcal{B}_E = (F_{\theta} \psi(\mathcal{A}) F_{\theta})''$ only. By Lemmas 1 and 2, $\rho_\theta$ does indeed define an ultra-weakly continuous pure state on $\mathcal{B}_E$. Thus we have reduced matters to the case where we have a von Neumann algebra $\mathcal{R}_0$, a Jordan-morphism $\psi_0 : \mathcal{A} \to \mathcal{R}_0$ with the property that the C*-algebra $\mathcal{C}$ generated by $\psi_0(\mathcal{A})$ has $\mathcal{R}_0$ as its double commutant, and an ultra-weakly continuous pure state $\rho_0$ on $\mathcal{R}_0$. Now assume that $\mathcal{C}$ is its own universal representation, and hence that $\mathcal{R}_0$ is the bidual of $\mathcal{C}$. If this was not the case we could have "lifted" the original description to this case by means of an application of [KR, 10.1.12] combined with Lemmas 1 and 2. Since $\rho_0$ is both pure and normal, it is an extreme point of the set of normal states. Finally by combining for example [KR, 7.4.2, 10.1.1 and 10.1.2], the set of normal states on $\mathcal{R}_0$ is isometrically isomorphic to the state space of $\mathcal{C}$ under restriction to $\mathcal{C}$. Hence the restriction of $\rho_0$ to $\mathcal{C}$ is a pure state of $\mathcal{C}$. On applying [Sto1, Corollary 5.8], we conclude that $(\omega_0 \cdot \psi_0)$ is a pure state of $\mathcal{C}$ as required.

For the converse we first show that for any pure state $\omega$ acting on $\mathcal{A}$, $\pi_\omega \cdot \varphi$ has the required form where $\pi_\omega$ corresponds to the canonical irreducible representation generated by $\omega$ [KR, 10.2.3], before deducing the result from this fact. This is basically a straightforward consequence of [Sto2, Thm 5.7].

Given $\omega \in \mathcal{P}_R$, we consider two cases:

If $\pi_\omega \cdot \varphi$ is a pure state on $\mathcal{A}$, then on denoting $\pi_\omega \cdot \varphi$ by $\rho$, let $\pi_{\rho}$ be the irreducible representation of $\mathcal{A}$ on a Hilbert space $h_{\rho}$ with cyclic unit vector $\Omega_{\rho}$, generated by $\rho$ by means of the GNS process. Since $\pi_{\rho}(\mathcal{A})$ is irreducible, $\pi_{\rho}(\mathcal{A})' = B(h_{\rho})$, and hence the orthogonal projection $E_{\rho}$ of $h_{\rho}$ onto the ray span{ $\Omega_{\rho}$, } belongs to $\pi_{\rho}(\mathcal{A})''$. Since now $E_{\rho}$ is of the form $E_{\rho} a = \langle a, \Omega_{\rho} \rangle \Omega_{\rho}$ for any $a \in h_{\rho}$, it follows that
\[ \langle E_\rho \pi_\rho(A) E_\rho a, b \rangle = \langle \pi_\rho(A) E_\rho a, E_\rho b \rangle \]
\[ = \langle \pi_\rho(A) \langle a, \Omega_\rho \rangle \Omega_\rho, \langle b, \Omega_\rho \rangle \Omega_\rho \rangle \]
\[ = \langle a, \Omega_\rho \rangle \Omega_\rho \langle b, \Omega_\rho \rangle \Omega_\rho \langle \pi_\rho(A) \Omega_\rho, \Omega_\rho \rangle \]
\[ = \langle a, \Omega_\rho \rangle \Omega_\rho \langle b, \Omega_\rho \rangle \Omega_\rho \rho(A) \cdot \| \Omega_\rho \|^2 \]
\[ = \rho(A) \langle a, \Omega_\rho \rangle \Omega_\rho, \langle b, \Omega_\rho \rangle \Omega_\rho \rangle \]
\[ = \langle \rho(A) E_\rho a, E_\rho b \rangle \]
\[ = \langle \rho(A) E_\rho a, b \rangle \]

for all \( A \in \mathcal{A} \) and all \( a, b \in \mathcal{H} \).

Hence \( \rho(A) E_\rho = E_\rho \pi_\rho(A) E_\rho \) for every \( A \in \mathcal{A} \). In the obvious way we may now identify \( \rho(\mathcal{A}) \) with \( \rho(\mathcal{A}) E_\rho = E_\rho \pi_\rho(\mathcal{A}) E_\rho \), from which it now follows that \( \pi_\omega \circ \varphi = \rho \) is of the required form.

Next suppose \( \omega \in \mathcal{P}_\mathcal{M} \), but that \( \pi_\omega \circ \varphi \) is not a pure state. In the notation of [Sto2, Thm 5.7] it now follows that

\[ \pi_\omega \circ \varphi = V^* \rho V \]

where \( \rho \) is a Jordan *-morphism with \( \rho(\mathcal{A})' = B(h) \) (i.e. the C*-algebra generated by \( \rho(\mathcal{A}) \) is irreducible), and \( V \) is a linear isometry from \( \mathcal{H}_\omega \) into \( \mathcal{H} \). Let \( E \in \mathcal{B}(\mathcal{H}) \) be the orthogonal projection onto \( V(\mathcal{H}_\omega) \). Since \( V|_E \) is the partial inverse of \( V \) on \( E(h) \) with \( V^* = V|_E h \cdot E \) and \( E V = V \), it follows that \( V \) generates a spatial *-isomorphism \( \Phi_\omega \) from \( B(E(h)) \) onto \( B(\mathcal{H}_\omega) \) such that \( \Phi_\omega(E(h)(h) E) = V^* B(h) V = B(h) = \pi_\omega(\mathcal{A})' \).

Clearly we may therefore assume \( \pi_\omega \circ \varphi \) to be of the form \( E_\rho E \) as required.

On applying Zorn's lemma, we may now select a maximal set of pure states \( \mathcal{M} \subset \mathcal{P}_\mathcal{M} \) such that the irreducible [KR; 10.2.3] GNS representations generated by any two elements of \( \mathcal{M} \) are mutually disjoint (pairwise inequivalent) [KR; 10.3.7]. Then surely \( \pi = \bigoplus_{\omega \in \mathcal{M}} \pi_\omega \) is faithful with

\[ \pi(\mathcal{A})' = \bigoplus_{\omega \in \mathcal{M}} B(h) = \bigoplus_{\omega \in \mathcal{M}} \pi_\omega(\mathcal{A})' \]

[KR; 10.3.10]. However we already know that for each \( \omega \in \mathcal{M} \), we may consider \( \mathcal{H}_\omega \) to be a subspace of a possibly larger Hilbert space \( \mathcal{K}_\omega \) such that \( \pi_\omega \circ \varphi \) is of the form \( E_\omega \psi_\omega E_\omega \) where \( E_\omega \) is the orthogonal projection of \( \mathcal{K}_\omega \) onto \( \mathcal{H}_\omega \), and \( \psi_\omega \) is a Jordan *-morphism from \( \mathcal{A} \) into \( B(\mathcal{K}_\omega) \) such that the C*-algebra generated by \( \psi_\omega(\mathcal{A}) \) is irreducible, that is \( \psi_\omega(\mathcal{A})' = B(\mathcal{K}_\omega) \) [BR; 2.3.8]. Now let \( \mathcal{H} \) be the von Neumann algebra \( \bigoplus_{\omega \in \mathcal{M}} B(\mathcal{K}_\omega) \), \( E \) the projection \( \bigoplus_{\omega \in \mathcal{M}} E_\omega \) and \( \psi \) the map \( \bigoplus_{\omega \in \mathcal{M}} \psi_\omega \). Since each \( \psi_\omega \) is a
Jordan *-morphism it can readily be verified that the same is true of $\psi$. Moreover

$$
\pi \circ \varphi = \bigoplus_{\omega \in \mathbb{A}} \pi_{\omega} \circ \varphi = \bigoplus_{\omega \in \mathbb{A}} E_{\omega} \psi_{\omega}(\cdot) E_{\omega} = E\psi(\cdot) E.
$$

Finally, for any $\omega_0 \in \mathbb{A}$, let $F_{\omega_0}$ be the orthogonal projection of $\bigoplus_{\omega \in \mathbb{A}} k_{\omega}$ onto the subspace corresponding to $k_{\omega_0}$. Clearly the $F_{\omega}$'s are orthogonal projections belonging to the centre of $\mathbb{A}$ such that $\bigvee_{\omega \in \mathbb{A}} F_{\omega} = I$ with $(F_{\omega_0}\psi(\mathcal{A}) F_{\omega})^\ast \equiv \psi_{\omega_0}(\mathcal{A})^\ast = B(k_{\omega_0}) \equiv F_{\omega_0} \mathbb{A} F_{\omega_0}$ for each $\omega \in \mathbb{A}$.

An analysis of the proof reveals a measure of dependence on some form of normality for the pure states. For this reason Theorem 5 was stated in terms of the reduced atomic representation. The following result puts the matter in context and enables us to restate Theorem 5 in terms of any faithful representation of $\mathbb{A}$ in which pure states are normal.

**Proposition 6.** Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a concrete $C^*$-algebra with the property that $\mathcal{A} \subset \mathcal{A}'$. Then there exists a projection $E \in \mathcal{A}' \cap \mathcal{A}''$ such that $\mathcal{A}_E$ affords an ultra-weakly continuous *-isomorphic copy of the reduced atomic representation of $\mathcal{A}$.

**Proof.** Let $\omega \in \mathcal{P}_\mathcal{A}$ and let $\tilde{\omega}$ be the unique ultra-weakly continuous pure state extension of $\omega$ to all of $\mathcal{A}''$. We first show that regarding the GNS constructions corresponding to $\omega$ and $\tilde{\omega}$ respectively, we have $h_\omega = h_{\tilde{\omega}}$ with $\pi_{\omega} = \pi_{\tilde{\omega}}|_{\mathcal{A}}$. This effectively follows from the first part of the proof of Lemma 4 combined with the uniqueness in [BR; 2.3.16]. To see this directly note that $\|\pi_{\tilde{\omega}}(A) \Omega_{\omega}\|^2 = \omega(A^* A) = \tilde{\omega}(A^* A) = \|\pi_{\tilde{\omega}}(A) \Omega_{\omega}\|^2$ for all $A \in \mathcal{A}$ where $\Omega_{\omega}$, $\Omega_{\tilde{\omega}}$ are the relevant canonical cyclic vectors. It is obvious that the set $\{\pi_{\tilde{\omega}}(A) \Omega_{\omega} : A \in \mathcal{A}' \subseteq h_\omega\}$ affords an isometric copy of the dense subset $\{\pi_{\omega}(A) \Omega_{\omega} : A \in \mathcal{A}'\}$ of $h_\omega$. If we can show that the former is also dense in $h_{\tilde{\omega}}$, then surely $h_\omega = h_{\tilde{\omega}}$ in a canonical way, in which case we are done as regards the first part of the proof. To see this we first note that $\pi_{\tilde{\omega}}(\mathcal{A}''') = B(h_{\tilde{\omega}})$ by [KR; 7.1.7 and 10.2.3]. Then surely $\pi_{\tilde{\omega}}$ is ultra-weakly continuous [BR; 2.4.23]. Since $\mathcal{A}$ is ultra-weakly dense in $\mathcal{A}''$ [BR; 2.4.11], $\pi_{\tilde{\omega}}(\mathcal{A})$ must then be ultra-weakly dense in $\pi_{\tilde{\omega}}(\mathcal{A}'') = B(h_{\tilde{\omega}})$ by continuity, and hence even strongly dense by [BR, 2.4.11]. But then $\pi_{\tilde{\omega}}(\mathcal{A}) \Omega_{\omega} = \{\pi_{\tilde{\omega}}(A) \Omega_{\omega} : A \in \mathcal{A}'\}$ is dense in $h_\omega = B(h_{\tilde{\omega}}) \Omega_{\omega} = \{B \Omega_{\omega} : B \in \mathcal{A}'\}$ as required.

If we now apply [BR; 2.4.22 and 2.4.23] to the kernel of $\pi_{\omega}$, the existence of a projection $F_{\omega} \in \mathcal{A}' \cap \mathcal{A}''$ such that $F_{\omega} \mathcal{A} F_{\omega} = \pi_{\omega}(0)$ follows. Thus denoting $F_{\omega}$ by $E_{\omega}$, it follows that $E_{\omega}$ maps $E_{\omega} \mathcal{A} E_{\omega}$ (respectively $E_{\omega} \mathcal{A}'' E_{\omega}$) *-isomorphically onto $\pi_{\omega}(\mathcal{A})$ (respectively $\pi_{\omega}(\mathcal{A}'') = B(h_{\tilde{\omega}}) = \pi_{\omega}(\mathcal{A}''))$. If
now $\rho$ is another pure state on $A$ and in the same fashion we obtain $E_\rho \in 2A$ so that $E_{\omega_\rho} E_{\rho} \neq 0$, then $E_{\omega_\rho} E_{\rho} \leq (E_{\omega_\rho} E_{\rho} \cap E_{\rho} E_{\rho})$, and hence $E_{\omega_\rho} E_{\rho} E_{\rho}$ affords equivalent subrepresentations of $\pi_\omega(A) = \pi_\omega(A)$ and $\pi_\rho(A) = \pi_\rho(A)$. Thus by [KR; 10.3.4], $\pi_\omega$ and $\pi_\rho$ are then not disjoint. If now $A$ is a maximal family of pure states on $A$ such that the associated irreducible representations are pairwise inequivalent, then by [KR; 10.3.7] and the above, the projections $\{E_{\omega_\rho}; \omega_\rho \in A\} \subset A \cap A^\prime$ are pairwise orthogonal. Finally since each $E_{\omega_\rho} E_{\rho_\omega}$, $\omega_\rho \in A$, affords a copy of the irreducible representation $\pi_\omega(A)$ (with $E_{\omega_\rho} E_{\rho_\omega}$ corresponding to $\pi_\omega(A)$), it follows that with $E = \bigoplus_{\omega_\rho \in A} E_{\omega_\rho} E_{\omega_\rho}$, 

$$E(A) = \left( \bigoplus_{\omega_\rho \in A} E_{\omega_\rho} \right) A = \left( \bigoplus_{\omega_\rho \in A} E_{\omega_\rho} \right) E(A)$$

affords a copy of the reduced atomic representation $\bigoplus_{\omega_\rho \in A} \pi_\omega(A)$ with respect to the map $\bigoplus_{\omega_\rho \in A} \pi_\omega(A)$, such that $E(A) = \bigoplus_{\omega_\rho \in A} E_{\omega_\rho} A E_{\omega_\rho} E_{\omega_\rho}$ affords a copy of $\bigoplus_{\omega_\rho \in A} B(h_{\omega_\rho})$ with respect to the same map.

3. A STRUCTURAL CHARACTERIZATION OF MAPS WHOSE ADJOINTS PRESERVE EXTREME POINTS OF THE DUAL BALL

Having dealt with linear maps from one $C^*$-algebra into another which preserve pure states on composition, we now turn our attention to those maps which preserve the extreme points of the unit ball of the dual of the range space. We shall eventually see that the “pure state preserving” maps have this property. As might be expected this study does however require a number of not insubstantial lemmas, the first of which is based on [KR; 7.3.2]. The fundamental idea behind the first cycle of lemmas is to describe “extremal functional” in terms of pure states. In so doing we are then able to make use of the results of Section 1 to achieve the stated objective of this section.

**Lemma 7.** Let $\rho$ be a norm-one functional on a $C^*$-algebra $A$. If now we identify $A$ with its universal representation and extend $\rho$ to $A^* = A^{**}$, then by [KR; 7.3.2 and 10.1.2] there exists a partial isometry $V$ in $A^*$ and a normal state $\omega$ (that is $\omega \in A_{\omega}$) so that $\omega(A) = \rho(VA)$ and $\omega(V^*A) = \rho(A)$ for all $A \in A$. If indeed $\omega$ is a pure state of $A$, we may then assume $V$ to be a unitary element of $A$.

**Proof.** Let $\pi_\omega$ be the canonical irreducible representation of $A$ engendered by $\omega$. Since now

$$\rho(A) = \omega(V^*A) = \langle \pi_\omega(A) \Omega_\omega, \pi_\omega(V) \Omega_\omega \rangle \quad \text{for all } A \in A \quad (1)$$
with $\|\pi_\omega(V)\Omega_\omega\|^2 = \langle \pi_\omega(V^* V) \Omega_\omega, \Omega_\omega \rangle = \omega(V^* V) = \rho(V) = \omega(I) = 1$, there exists a unitary element $U \in \pi_\omega(A)^\prime = B(\mathcal{H}_\omega)$ with $U\Omega_\omega = \pi_\omega(V)\Omega_\omega$.

On applying [KR; 5.4.5] we conclude that there exists $H \in \mathcal{A}$ such that $\pi_\omega(H) = \pi_\omega(H^*)$, and that $\exp(i\pi_\omega(H))$ maps $\Omega_\omega$ onto $\pi_\omega(V)\Omega_\omega$. Replacing $H$ by $\frac{1}{2}(H + H^*)$ if necessary, we may assume $H$ to be self-adjoint. If now we select a sequence of polynomials $p_n$ such that $p_n \to \exp(i \cdot)$ uniformly on $[-\|H\|, \|H\|]$, then

$$
\pi_\omega(\exp(iH)) = \pi_\omega(\lim_n p_n(H)) = \lim_n \pi_\omega(p_n(H)) = \lim_n p_n(\pi_\omega(H)) = \exp(i\pi_\omega(H))
$$

where the convergence is in norm. Thus with $W = \exp(iH)$, $\pi_\omega(W)$ maps $\Omega_\omega$ onto $\pi_\omega(V)\Omega_\omega$. From (1) we then have that

$$
\rho(A) = \langle \pi_\omega(A) \Omega_\omega, \pi_\omega(W) \Omega_\omega \rangle = \omega(W^* A) \quad \text{for all} \quad A \in \mathcal{A}.
$$

But then $\omega(A) = \omega(IA) = \omega(W^* WA) = \rho(WA)$ for all $A \in \mathcal{A}$.  

Although the following is bound to be known, we are not aware of an explicit reference for it.

**Lemma 8.** For any C*-algebra $\mathcal{A}$, $\mathcal{S}_a$ is a face of $\mathcal{A}$.

**Proof.** Let $\omega$ be a state of $\mathcal{A}$ and $\rho_1$ and $\rho_2$ functionals in $\mathcal{A}$ with $\lambda \rho_1 + (1 - \lambda) \rho_2 = \omega$ for some $\lambda$ between 0 and 1. Since then

$$
\omega = \Re(\omega) = \frac{1}{2}(\lambda \rho_1 + (1 - \lambda) \rho_2) + (\lambda \rho_1 + (1 - \lambda) \rho_2)^* = \lambda (\frac{1}{2}(\rho_1^* + \rho_2^*)) + (1 - \lambda) (\frac{1}{2}(\rho_2 + \rho_2^*)) = \lambda \Re \rho_1 + (1 - \lambda) \Re \rho_2,
$$

it follows that

$$
1 = \omega(I) = \lambda \Re \rho_1(I) + (1 - \lambda) \Re \rho_2(I)
$$

$$
\leq \lambda \|\rho_1(I)\| + (1 - \lambda) \|\rho_2(I)\|
$$

$$
\leq \lambda \|\rho_1\| + (1 - \lambda) \|\rho_2\|
$$

$$
\leq \lambda + (1 - \lambda) = 1.
$$

But this can only be if

$$
1 = \Re(\rho_k(I)) = \|\rho_k(I)\| = \|\rho_k\| \quad k = 1, 2
$$
in which case

$$\rho_k(I) = 1 = \|\rho_k\| \quad k = 1, 2.$$  

Thus as required $\rho_1$ and $\rho_2$ are states by [KR; 4.3.2].

**Corollary 9.** If $\rho$ is a bounded functional on a $C^*$-algebra $\mathcal{A}$ related to a pure state $\omega$ in the manner described in the hypothesis of Lemma 7, then $\rho$ is an extreme point of $\mathcal{A}^\perp$.

**Proof.** First of all note that $\|\rho\| \leq \|\omega\| \|V\| = 1$. Now suppose $\rho_1$ and $\rho_2$ are elements of $\mathcal{A}^\perp$ such that $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$. Then surely

$$\omega(A) = \lambda \rho_1(V^*A) + (1 - \lambda) \rho_2(V^*A) \quad \text{for all } A \in \mathcal{A}.$$

Considering Lemma 8 alongside the fact that $\omega$ is pure, we have $\omega(A) = \rho(A(V^*A))$ for all $A \in \mathcal{A}$ and $k = 1, 2$. But then $\rho(A) = \rho_k(VV^*A)$ for all $A \in \mathcal{A}$.

Finally since by Lemma 7 we may assume $V$ to be unitary in $\mathcal{A}$, we therefore have that $\rho = \rho_1 = \rho_2$ as required.

We now see that in fact the converse of Corollary 9 also holds. This enables us to use the theory concerning pure states to treat the extreme points of the unit ball of the dual of a $C^*$-algebra.

**Lemma 10.** Let $\rho$, $\omega$, $\mathcal{A}$ and $V$ be as in Lemma 7. Then $\omega$ is pure whenever $\rho$ is an extreme point of $\mathcal{A}^\perp$.

**Proof.** Assume $\rho$ to be an extreme point of $\mathcal{A}^\perp$ and let $\omega_1$, $\omega_2$ be states on $\mathcal{A}$ with

$$\omega = \lambda \omega_1 + (1 - \lambda) \omega_2 \quad 0 < \lambda < 1.$$  

If now we identify $\rho$, $\omega$, $\omega_1$ and $\omega_2$ with their canonical ultra-weakly continuous extensions to $\mathcal{A}^* = \mathcal{A}^{**}$ [KR; 10.1.1], then surely

$$\rho(A) = \omega_1(V^*A) = \lambda \omega_1(V^*A) + (1 - \lambda) \omega_2(V^*A) \quad \text{for all } A \in \mathcal{A}^*.$$

Since $\rho$ is an extreme point of $\mathcal{A}^\perp$ with $\|\omega_1(V^*)\| \leq \|\omega_2\| \|V^*\| = 1$ for $k = 1, 2$, we conclude that $\omega_k(V^*A) = \rho(A)$ for all $A \in \mathcal{A}^*$ where $k = 1, 2$.

But then

$$\omega(A) = \rho(VA) = \omega_k(VV^*A) \quad \text{for all } A \in \mathcal{A}^*, k = 1, 2.$$  

Moreover since $0 \leq \omega_1 \leq \omega$ and $0 \leq (1 - \lambda) \omega_2 \leq \omega$ by (1), the fact that $I - V^*V \geq 0$ implies that

$$0 \leq \omega_1(I - V^*V) \leq \omega(I - V^*V) = 1 - \rho(V) = 1 - \omega(I) = 0.$$
Similarly $(1 - \lambda) \omega_2(I - V^*V) = 0$, and hence

$$\omega_k(I - V^*V) = 0 \quad \text{for} \quad k = 1, 2.$$  

But then for any $A \in \mathcal{A}''$, the fact that $I - V^*V$ is a projection considered alongside [KR; 4.3.1] implies that

$$0 \leq |\omega_k((I - V^*V)A)|^2 \leq \omega_k((I - V^*V)(I - V^*V^*)) \omega_k(A^*A) = \omega_k(I - V^*V) \omega_k(A^*A) = 0 \quad k = 1, 2.$$  

This fact combined with (2) now reveals that $\omega = \omega_1 = \omega_2$ as required.  

Having achieved the objective of describing extreme points of $\mathcal{A}''$ in terms of pure states, we are now able to duplicate the fundamental lemmas of Section 1 for the more general case.

**Lemma 11.** Let $\mathcal{A}$ be a concrete $C^*$-algebra with $P \subseteq N$ (e.g. the reduced atomic representation). Then every extreme point $\rho$ of $\mathcal{A}''$ has a unique ultra-weakly continuous norm-preserving extension $\tilde{\rho}$ to all of $\mathcal{A}''$. Moreover $\tilde{\rho}$ is an extreme point of $(\mathcal{A}'')'$.  

**Proof.** By Lemmas 7 and 10, and Corollary 9 there exists a unitary $V$ so that $\rho(V\cdot) = \omega$ is a pure state of $\mathcal{A}$. By (Lemma 3 and) [KR; 10.1.11], $\omega$ has a unique norm preserving ultra-weakly continuous extension $\tilde{\omega}$ to all of $\mathcal{A}''$. Clearly $\tilde{\omega}(V^*A) = \tilde{\rho}(A)$ for all $A \in \mathcal{A}''$ then defines a norm preserving extension $\tilde{\rho}$ of $\rho$ to all of $\mathcal{A}''$, which is moreover ultra-weakly continuous by the ultra-weak continuity of $\tilde{\omega}$ combined with [BR; 2.4.2]. But then $\tilde{\rho}|_{\mathcal{A}} = \rho$ is ultra-weakly continuous on $\mathcal{A}$, and so the extension $\tilde{\rho}$ must be unique by [KR; 10.1.11]. Finally since $V$ is unitary and since $\tilde{\omega}$ is a pure state of $\mathcal{A}''$ by Lemma 4, it now follows from Corollary 9 that $\tilde{\rho}$ is an extreme point of $(\mathcal{A}'')'$.  

**Lemma 12.** Let $\mathcal{A} \subseteq B(h)$ be a concrete $C^*$-algebra, $E$ a projection in $\mathcal{A}$, and let $\eta$ be defined as in Lemma 1. Then for any ultra-weakly continuous functional $\rho$ on $\mathcal{A}_E$, $\rho \cdot \eta$ is an ultra-weakly continuous functional on $\mathcal{A}$ with $\|\rho \cdot \eta\| = \|\rho \cdot \eta\|$. Conversely if $\tilde{\rho}$ is ultra-weakly continuous on $\mathcal{A}$, then the restriction of $\tilde{\rho}$ to $\mathcal{A}_E$ is ultra-weakly continuous with respect to $\mathcal{A}_E$.  

**Proof.** This is a fairly obvious and easily verifiable consequence of Lemma 2 considered alongside the fact that each ultra-weakly continuous functional is a linear combination of normal states (see for example [KR; 7.4.7]). We therefore forgo the proof.
LEMMA 13. Let $\mathcal{A}$ be a $C^*$-algebra, $E$ a projection in $\mathcal{A}$, and $\rho$ an extreme point of $(\mathcal{A}_E^*)^\perp$. With $\eta$ defined as in Lemma 1, it then follows that $\rho \cdot \eta$ is an extreme point of $\mathcal{A}_E^\perp$. Conversely if $\tilde{\rho}$ is an extreme point of $\mathcal{A}_E^\perp$ with $\|\tilde{\rho}\|_E = 1$, then the restriction of $\tilde{\rho}$ to $\mathcal{A}_E$ is an extreme point of $(\mathcal{A}_E^*)_E$.

Proof. By Lemmas 7 and 10, and Corollary 9 there exists a unitary element $V$ of $\mathcal{A}_E$ such that $\omega = \rho(V \cdot)$ is a pure state of $\mathcal{A}_E$. By Lemma 1 $\omega \cdot \eta$ is a pure state of $\mathcal{A}$. But since $VE = EV = V$ with $V^*V = VV^* = E$, it is clear that $\tilde{V} = V + (I - E)$ is unitary in $\mathcal{A}$. Moreover since then

$$\omega \cdot \eta(A) = \rho(AV\omega E) = \rho(V\tilde{V}\omega E) = \rho(\tilde{\omega}(A))$$

for all $A \in \mathcal{A}$, it is fairly clear from Corollary 9 that $\rho \cdot \eta$ is then an extreme point of $\mathcal{A}_E^\perp$. Conversely suppose $\tilde{\rho}$ is an extreme point of $\mathcal{A}_E^\perp$ with $\|\tilde{\rho}\| = 1$ and assume that $\mathcal{A} \subset (B(h))$ is universally represented. By Lemma 11 we may identify $\tilde{\rho}$ with its unique ultra-weakly continuous extension to $\mathcal{A}^\ast$. But then Lemma 12 informs us that the restriction $\tilde{\rho}|_{\mathcal{A}_E}$, which we will henceforth denote by $\rho_0$, is an ultra-weakly continuous functional on $\mathcal{A}_E^\ast$, with $\|\rho_0\| = 1$ by assumption. Applying [KR; 7.3.2], we conclude that there exists a partial isometry $V \in \mathcal{A}_E^\ast$ and a normal state $\omega_0$ on $\mathcal{A}_E^\ast$ so that $\rho_0(VA) = \omega_0(A)$ and $\rho_0(A) = \omega_0(A^*)$ for all $A \in \mathcal{A}_E$. But then $EV = VE = V$ with $\tilde{\rho}(V) = \tilde{\rho}(V^*E) = \rho_0(V^*E) = \rho_0(E) = 1$ by [KR; 4.3.2]. Hence again by [KR; 4.3.2], $\tilde{\rho}(V^*A), A \in \mathcal{A},$ defines a state on $\mathcal{A}$. By Lemma 10 this state is necessarily a pure state of $\mathcal{A}$, and hence by Lemma 1, $\omega$ is a pure state of $\mathcal{A}_E$ on restriction to $\mathcal{A}_E$. Finally on considering [KR; 10.1.12 and 10.1.21] it is clear that we may assume $V \in \mathcal{A}_E \subset (B(h))^\ast$ (up to an isometric isomorphism) and hence by Corollary 9, on restriction to $\mathcal{A}_E^\ast$, $\rho_0$ is an extreme point of $(\mathcal{A}_E^*)_E$.

The final building block we need to achieve a general characterization of maps with “extreme point preserving” adjoints, is that of reducing this question to the case of maps from say $B(h)$ to $B(k)$, where $h$ and $k$ are Hilbert spaces. It seems that we need to take steps to ensure that all pure states are normal in order to achieve this.

LEMMA 14. Let $\mathcal{A}$ be a $C^*$-algebra and $\rho_1, \rho_2$ extreme points of $\mathcal{A}_E^\ast$ for which the associated (pure) states defined as in Lemma 7 are disjoint, then $\|\rho_1 - \rho_2\| = 2$.

Proof. Observe that Lemma 10 ensures that the associated states, say $\omega_1$ and $\omega_2$, are indeed pure. Moreover by Lemma 7 there exist unitaries $U_1, U_2$ in $\mathcal{A}$ so that $\rho_i(A) = \omega_i(U_i^*A)$ and $\rho_i(U_iA) = \omega_i(A)$ for all $A \in \mathcal{A}$, $i = 1, 2$. Now let $\pi_{\omega_i}$, $i = 1, 2$, be the representations engendered by $\omega_i$, $i = 1, 2$. If now we let $\pi = \pi_{\omega_1} \oplus \pi_{\omega_2}$ and if indeed $\omega_1$ and $\omega_2$ are disjoint,
then by [KR: 10.3.3(iii)] with $E'$ and $E''$ the projections of $h_1 \oplus h_2$ onto $\{0\} \oplus h_2$ and $h_1 \oplus \{0\}$ respectively, we surely have

$$\pi(\mathcal{A}) = \pi_{\omega_1}(\mathcal{A}) \oplus \pi_{\omega_2}(\mathcal{A}).$$

Hence we may select $V \in \mathcal{A}$ with $\pi_{\omega_1}(V) = \pi_{\omega_1}(U_1)$ and $\pi_{\omega_2}(V) = -\pi_{\omega_2}(U_2)$.

Moreover since then $\|\pi(V)\| = 1$, we may in fact select $V$ so that $\|V\| = 1$, since $\pi(\mathcal{A}) = \pi(\mathcal{A})_1$. Thus

$$2 \geq \|p_1 - p_2\| \geq \|p_1(V) - p_2(V)\| = \|\omega_1(V^* V) - \omega_2(V^* V)\|
= |\langle \pi_{\omega_1}(U_1^* V) \Omega_{\omega_1}, \Omega_{\omega_1} \rangle - \langle \pi_{\omega_1}(U_2^* V) \Omega_{\omega_1}, \Omega_{\omega_1} \rangle|
= |\langle \pi_{\omega_1}(U_1^* U_1) \Omega_{\omega_1}, \Omega_{\omega_1} \rangle + \langle \pi_{\omega_1}(U_2^* U_2) \Omega_{\omega_1}, \Omega_{\omega_1} \rangle|
= \langle \Omega_{\omega_1}, \Omega_{\omega_1} \rangle + \langle \Omega_{\omega_1}, \Omega_{\omega_1} \rangle = 2.$$

The next Lemma in this cycle is based on an adaptation of a technique of Størmer's [Sto2; 5.6].

**Lemma 15.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras.

(a) If $\psi: \mathcal{A} \to \mathcal{B}$ is a linear map for which $p \cdot \psi \in \text{ext}(\mathcal{A}^*)$ whenever $p \in \text{ext}(\mathcal{B}^*)$, then given any two unitarily equivalent pure states $\omega_1$, $\omega_2$ of $\mathcal{B}$, the pure states associated with $\omega_1$ and $\omega_1 \cdot \psi$ by means of the technique described in Lemma 7, are also unitarily equivalent.

(b) If $\mathcal{B} = B(h$) and if $\psi: \mathcal{A} \to \mathcal{B}$ is a linear map for which $p \cdot \psi \in \text{ext}(\mathcal{A}^*)$ whenever $p$ is an ultra-weakly continuous element of $\text{ext}(\mathcal{B}^*)$, then given any two unitarily equivalent ultra-weakly continuous pure states $\omega_1$ and $\omega_2$ of $\mathcal{B}$, the pure states associated with $\omega_1$ and $\omega_1 \cdot \psi$ are unitarily equivalent.

**Proof.** The proof of the two cases being very similar, we content ourselves with proving (a). In fact as regards the proof of (b) as compared to (a), the only additional piece of information we need for (b) is to note that the stated condition in (b) is sufficient to ensure that $\|\psi\| \leq 1$. To see this note that for any given $\epsilon > 0$ and $A \in \mathcal{A}$ we may select $x, y \in h$ with $\|x\| = \|y\| = 1$ so that $\|\psi(A)\| - \epsilon \leq |\langle \psi(A)x, y \rangle|$. From for example Corollary 9 and [KR; 4.6.8] we may easily deduce that the functional $\rho(T) = \langle Tx, y \rangle$, $T \in \mathcal{B}$, is an ultra-weakly continuous element of $\text{ext}(\mathcal{B}^*)$. But then $p \cdot \psi \in \text{ext}(\mathcal{A}^*)$, and hence $\|\psi(A)\| - \epsilon \leq |(p \cdot \psi)(A)| \leq \|p \cdot \psi\| \cdot \|A\| = \|A\|$. Now for (a) suppose there exists a unitary $U \in \mathcal{B}$ so that $\omega_{\mathcal{A}}(A) = \omega_{\mathcal{B}}(U^* AU)$ for all $A \in \mathcal{B}$ where $\omega_{\mathcal{A}}$ and $\omega_{\mathcal{B}}$ are pure states. If now $\pi_1$ is the canonical representation engendered by $\omega_{\mathcal{A}}$ on say $h_1$ with corresponding cyclic vector $\Omega$, then surely $\omega_{\mathcal{A}} = \omega_{\mathcal{A}} \cdot \pi_1$ and $\omega_{\mathcal{B}} = \omega_{\mathcal{A}} \cdot \pi_1$, where $\omega_{\mathcal{B}}$ and $\omega_{\mathcal{A}}$ are the vector states on $B(h_1)$ corresponding to $\Omega$ and $z = \pi_1(U) \Omega$ respectively. The rest of the proof is basically an adaptation of part of [Sto2; 5.6]. Now if $\pi_1(U) \Omega$ was merely a (modulus
one) scalar multiple of \( \Omega \), it trivially follows that \( \omega_{\Omega} = \omega_{x} \), and hence in this case we are done since then \( \omega_{1} = \omega_{x} = \pi_{1} = \omega_{2} \). Thus suppose \( \text{span}\{ \Omega, \pi_{1}(U) \Omega \} = k \) is a two-dimensional subspace of \( h_{1} \), and select \( x \in k \) so that \( x \perp \Omega \) with \( |x| = 1 \). Next select \( \lambda \in \mathbb{C}, |\lambda| = 1 \) so that \( \langle \pi_{1}(U) \Omega, \Omega \rangle = \langle \pi_{1}(U) \Omega, \Omega \rangle \). Since the vector state induced by \( \lambda \pi_{1}(U) \Omega \) is identical to \( \omega_{x} \), it follows that we may assume \( \langle \pi_{1}(U) \Omega, \Omega \rangle = \langle \pi_{1}(U) \Omega, \Omega \rangle \).

By similarly adjusting \( x \) if necessary, we may assume \( \langle \pi_{1}(U) \Omega, \Omega \rangle = \langle \pi_{1}(U) \Omega, \Omega \rangle \).

Now let \( w_{1} = \Omega, w_{2} = 2^{-1/2}(\Omega + x) \) and \( w_{3} = \pi_{1}(U) \Omega \). Since \( \{ \Omega, x \} \) is an ONB for \( k \), it is an easy exercise to show that \( \|w_{2}\| = 1 \) with \( \|w_{1} - w_{2}\|^{2} = 2 - \sqrt{2} < 1 \). Moreover this fact together with the foregoing implies that

\[
    w_{3} = \langle w_{3}, \Omega \rangle \Omega + \langle w_{3}, x \rangle x = |\langle w_{3}, \Omega \rangle| \Omega + |\langle w_{3}, x \rangle| x.
\]

But as \( |\langle w_{3}, \Omega \rangle| \leq 1 \) and \( |\langle w_{3}, x \rangle| \leq 1 \), we therefore have that

\[
    |\langle w_{3}, \Omega \rangle| + |\langle w_{3}, x \rangle| \geq |\langle w_{3}, \Omega \rangle|^{2} + |\langle w_{3}, x \rangle|^{2} = \|w_{3}\|^{2} = 1.
\]

Thus since \( \langle w_{3}, \Omega \rangle \geq 0 \) and \( \langle w_{3}, x \rangle \geq 0 \) by construction,

\[
    \|w_{2} - w_{3}\|^{2} = \|w_{2}\|^{2} - 2^{-1/2} \cdot 2(\|w_{3}\|^{2} + |\langle w_{3}, x \rangle|) + \|w_{3}\|^{2} = 2 - 2^{1/2}(\|w_{3}\|^{2} + |\langle w_{3}, x \rangle|) \leq 2 - 2^{1/2} < 1.
\]

Let \( v_{i} \) be the vector state on \( \mathcal{H}(h_{1}) \) engendered by \( w_{i}, i = 1, 2, 3 \). Clearly \( v_{1} \circ \pi_{1} \) is a pure state of \( \mathcal{H} \) for all \( i = 1, 2, 3 \) [KR; 10.2.3 and 10.2.5] with \( v_{1} \circ \pi_{1} = \omega_{1} \) and \( v_{3} \circ \pi_{1} = \omega_{2} \). Moreover if the pure states associated with \( v_{i} \circ \pi_{1} \circ \psi \) and \( v_{i+1} \circ \pi_{1} \circ \psi, i = 1, 2, \) are unitarily equivalent, the same is trivially true of \( \omega_{1} \circ \psi \) and \( \omega_{2} \circ \psi \). Finally observe that for any \( i = 1, 2, \) and any \( A \in \mathcal{A}, \) we have

\[
    |v_{i} \circ \pi_{1} \circ \psi(A) - v_{i+1} \circ \pi_{1} \circ \psi(A)|
    = \|\pi_{1} \circ \psi(A) w_{i}, w_{i} - v_{i} \circ \psi(A) w_{i+1}, w_{i+1}\|
    = \|\pi_{1} \circ \psi(A)(w_{i} - w_{i+1}, w_{i}) + \pi_{1} \circ \psi(A) w_{i+1}, w_{i} - w_{i+1}\|
    \leq 2 \|\pi_{1}\| \|\psi\| \|w_{i} - w_{i+1}\| \|A\|
    = (2 \|\psi\|)(2 - 2^{1/2}1^{1/2}) \|A\|.
\]

All that remains to be done is to note that since \( \psi^{*} \) preserves extreme points, we necessarily have \( \|\psi\| \leq 1 \), and then to apply Lemma 14.
THEOREM 16. Let \(h, k\) be Hilbert spaces.

(A) A continuous linear map \(\psi: \mathcal{K}(k) \rightarrow \mathcal{K}(h)\) has the property that 
\(\rho \cdot \psi \in \text{ext}(\mathcal{S}(k)^1)\) whenever \(\rho\) is an extreme point of \(\mathcal{S}(h)^1\), if and only if 
\(\psi\) is of precisely one of the following forms:

1. There exist injective partial isometries \(U: h \rightarrow k\) and \(V: h \rightarrow k\) such that either \(\psi(T) = U^*TV\) for all \(T \in \mathcal{K}(k)\) or \(\psi(T) = U^*c^*TV\) for all \(T \in \mathcal{K}(k)\). (Here \(c: k \rightarrow k\) is the anti-unitary operator induced by complex conjugation of the scalars.)

2. There exists a fixed unit vector \(w \in k\) and a surjective partial isometry \(V: k \rightarrow \mathcal{S}(h)\) such that either \(\psi(T) = JV(Tw)\) for all \(T \in \mathcal{K}(k)\) or \(\psi(T) = (JV(T^*)w)^*\) for all \(T \in \mathcal{K}(k)\), where \(J\) is the natural injection of \(\mathcal{S}(h)\) into \(k(h)\).

(B) An ultra-weakly continuous linear map \(\psi: \mathcal{B}(k) \rightarrow \mathcal{B}(h)\) has the property that 
\(\rho \cdot \psi \in \text{ext}(\mathcal{B}(k)^*)\) whenever \(\rho\) is an ultra-weakly continuous extreme point of \(\mathcal{B}(h)^*\), if and only if \(\psi\) is of precisely one of the following forms:

1. There exist injective partial isometries \(U: h \rightarrow k\) and \(V: h \rightarrow k\) such that either \(\psi(T) = V^*TU\) for all \(T \in \mathcal{B}(k)\) or \(\psi(T) = V^*c^*TU\) for all \(T \in \mathcal{B}(k)\). (Here \(c: k \rightarrow k\) is the anti-unitary operator induced by complex conjugation of the scalars.)

2. There exists a fixed unit vector \(w \in k\) and a surjective partial isometry \(V: k \rightarrow \mathcal{S}(h)\) such that either \(\psi(T) = JV(Tw)\) for all \(T \in \mathcal{B}(k)\) or \(\psi(T) = (JV(T^*)w)^*\) for all \(T \in \mathcal{B}(k)\), where \(J\) is the natural injection of \(\mathcal{S}(h)\) into \(k(h)\).

Proof. (A) Let \(\{e_j\}_J\) be a fixed orthonormal basis for \(h\). Since 
\[\text{ext}(\mathcal{K}(h)^*) = \text{ext} \mathcal{S}(h)^1 = \{u \otimes v: u, v\ \text{unit vectors in } h\},\]
we start the investigation by looking at the images \(\psi^*(e_{\lambda} \otimes e_{\mu})\), \(\lambda, \mu \in \Gamma\). Let us state a sublemma providing us with a criterion that will be repeatedly applied in what follows (its proof is a straightforward exercise):

Sublemma. Let \(\lambda_1, \lambda_2, \mu_1, \mu_2 \in \Gamma\) with \(\lambda_1 \neq \mu_1, \lambda_2 \neq \mu_2\), and \(u, v, w, z \in k\) be unit vectors. If \(\psi^*(e_{\lambda_1} \otimes e_{\mu_1}) = u \otimes v\) and \(\psi^*(e_{\lambda_2} \otimes e_{\mu_2}) = w \otimes z\), then we either have that \(u \parallel w\) and \(v \perp z\), or \(u \perp w\) and \(v \parallel z\). Similarly, if \(\psi^*(e_{\lambda_1} \otimes e_{\lambda_2}) = u \otimes v\) and \(\psi^*(e_{\mu_1} \otimes e_{\mu_2}) = w \otimes z\), then either \(u \parallel w\) and \(v \perp z\), or \(u \perp w\) and \(v \parallel z\).

On fixing \(\lambda_0 \in \Gamma\) and applying the sublemma to the sets \((\psi^*(e_{\lambda_0} \otimes e_{\lambda_1}))_{\lambda_1}\) and \((\psi^*(e_{\lambda_0} \otimes e_{\mu_0}))_{\mu_0}\), we deduce that there are four possibilities for their values:
where all \((u_z), (v_z), (w_z)\) are orthonormal systems in \(k\) (and \(w_z = u_z\) or \(v_z\), depending on the case).

In the following we will assume \((e_z)\) to be countable and let \(z_0 = 1\). The reason we may do this is that in each case it is enough to establish the action of \(\psi^*\) in terms of arbitrary countable subsets of \((e_z)\), containing \(e_{z_0}\), in order to establish the action of \(\psi^*\) in terms of all \((e_z)\).

Let us look at the case \((1, a)\) first. If we assume that \(\psi^*(e_2 \otimes e_3) = y \otimes v_1\) with \(y \leq u_2\) (one of the two possibilities allowed by the sublemma), then on comparing \(\psi^*(e_2 \otimes e_1)\) to \(\psi^*(e_2 \otimes e_1)\) \((i = 1, 2)\) and applying the sublemma, we have that \(\psi^*(e_2 \otimes e_1) = z \otimes v_1\) for some unit vector \(z\) such that \(z \leq y\). But then the sublemma applied to \(\psi^*(e_2 \otimes e_1)\) and \(\psi^*(e_2 \otimes e_3)\) would give \(u_1 \| y\) (since \(v_1 \leq v_2\) by assumption). Applying it to \(\psi^*(e_2 \otimes e_3)\) and \(\psi^*(e_2 \otimes e_1)\) gives in turn \(u_1 \| z\) (since \(v_1 \leq v_3\)). Hence, we would get \(y \| z\), a contradiction. Thus we must have \(\psi^*(e_2 \otimes e_3) = u_3 \otimes z\) where \(z\) is a unit vector with \(z \leq v_1\). Inductively applying the sublemma we have

\[
\psi^*(e_2 \otimes e_j) = u_2 \otimes z_j \quad \text{for all } j \in \mathbb{N}
\]

where \((z_j)\) is an ONS with \(z_1 = e_1\). More generally we may verify that for each \(m \in \mathbb{N}\),

\[
\psi^*(e_m \otimes e_j) = u_m \otimes z_j^{(m)} \quad \text{for all } j \in \mathbb{N}
\]

where \((z_j^{(m)})\) is an ONS with \(z_1^{(m)} = e_1\). Thus by symmetry it follows that there is only the following way to satisfy the sublemma in this case: we have \(\psi^*(e_i \otimes e_j) = e_i u_i \otimes e_j\) for all \(i, j\), where the \(e_i\) are complex numbers of modulus one (we set \(e_i := 1\) whenever \(i = j\)). Define \(U, V\) (injective) partial isometries \(h \to k\) by \(U e_i := u_i\) and \(V e_i := v_i\) for all \(i\). We then have

\[
V^* \psi^*(e_i \otimes e_j) U = e_i V^*(u_i \otimes v_j) U = e_i e_i \otimes e_j \quad \text{for all } j.
\]

Call \(\Psi: S_1(h) \to S_1(h), S \mapsto V^* \psi^*(S) U\). Then \(||\Psi|| = 1\) and \(\Psi\) acts on \(S_1(h)\) as a Schur multiplier with matrix \((e_{ij})\) (with respect to the basis \((e_i)\), of course). It is not difficult to prove that this contradicts \(||\Psi|| \leq 1\) (even in the case of two-dimensional \(k!\)) except when all the \(e_{ij}\) are the same unimodular number, i.e. equal to \(e_{11} = 1\). Accordingly, we can now write that \(\psi^*(e_i \otimes e_j) = V(e_i \otimes e_j) U^*\) for all \(i, j\) and so

\[
\psi^*(S) = VSU^* \quad \text{for all } S \in S_1(h).
\]
Finally, if $T \in K(h)$ and $S \in \mathcal{A}(k)$ are arbitrary, then

$$\text{tr}(S^*T) = \text{tr}((S^*S)^{1/2}T) = \text{tr}(SU^*TV)$$

and so

$$\psi(T) = U^*TV \quad \text{for all} \quad T \in K(k).$$

Let us now deal with the case (2a). Suppose that $\psi^*(e_2 \otimes e_3) = w_2 \otimes y$ where $y \in k$ is some unit vector orthogonal to $v_1$. On comparing $\psi^*(e_2 \otimes e_3)$ to $\psi^*(e_2 \otimes e_1)$ and $\psi^*(e_2 \otimes e_2)$ and applying the sublemma, we conclude that $\psi^*(e_2 \otimes e_3) = w_2 \otimes z$ with $z \in k$ a unit vector orthogonal to both $y$ and $v_1$.

Then the sublemma applied to $\psi^*(e_1 \otimes e_3)$ gives (since $y \perp v_1$) $w_2 \parallel w_2$. Keeping this in mind and applying the sublemma to $\psi^*(e_1 \otimes e_2)$ and $\psi^*(e_2 \otimes e_3)$ gives $u_1 \parallel w_2$. Hence, after looking at $\psi^*(e_1 \otimes e_3)$ and $\psi^*(e_2 \otimes e_3)$ we would get $z \parallel v_1$, a contradiction with the above. Thus we must have $\psi^*(e_2 \otimes e_3) = z \otimes v_1$ where $z \in k$ is a unit vector orthogonal to $w_2$.

Continuing inductively it follows that

$$\psi^*(e_2 \otimes e_j) = z_j \otimes v_1 \quad \text{for all} \quad j \in \mathbb{N},$$

and more generally that

$$\psi^*(e_m \otimes e_j) = z_j^{(m)} \otimes v_1 \quad \text{for all} \quad j \in \mathbb{N}$$

for each fixed $m \in \mathbb{N}$, where $(z_j^{(m)})_j$ is an ONS with $z_1^{(m)} = w_2$. By symmetry it follows that there is only the following way to satisfy the sublemma in this case: we must have $\psi^*(e_i \otimes e_j) = u_{ij} \otimes v_1$ for all $i, j$ (we renamed the $u_i$ to $u_{ij}$ and the $w_i$ to $u_{1i}$), where the vectors in the “matrix” $(u_{ij})$ form orthonormal systems along the rows and columns. Now, since $\|\psi^*\| \leq 1$, we have

$$\left\| \sum_{i=1}^n \alpha_i \left( \sum_{j=1}^n \beta_j u_{ij} \right) \right\|_{\mathcal{A}(k)} = \left\| \sum_{i=1}^n \alpha_i \beta_j (u_{ij} \otimes v_1) \right\|_{\mathcal{A}(k)} = \left\| \psi^* \left( \left( \sum_{i=1}^n \alpha_i e_i \right) \otimes \left( \sum_{j=1}^n \beta_j e_j \right) \right) \right\|$$

$$\leq \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{1/2} \left( \sum_{j=1}^n |\beta_j|^2 \right)^{1/2}$$

for all $(\alpha_i), (\beta_j) \in \mathbb{C}^n$ and all $n$. Fixing $(\beta_j)$, this implies that the sequence $(\sum_{j=1}^n \beta_j u_{ij})_{n\times1}$ must be orthogonal, which in turn—since $(\beta_j)$ is arbitrary—forces the family $(u_{ij})$ to be orthogonal (not only row- and columnwise!).
Let \( V: k \to S_2(h) \) be the surjective partial isometry such that \( V^* (e_i \otimes e_j) = u_{ij} \) for all \( i, j \). Then, if \( J \) is the natural injection \( S_2(h) \to K(h) \), we have
\[
\psi(T) = JV(Tv_1) \quad \text{for all } T \in K(k).
\]

It should now be clear how to proceed in the analysis of the remaining cases. To finish the proof of (A), one only needs to check that the conditions on \( \psi \) given in the statement are sufficient to ensure \( \psi^*(\text{ext } K(h))^* \subseteq \text{ext } K(k) \). Recall first that \( \text{ext}(K(h))^* = \{ u \otimes v : u, v \text{ unit vectors in } h \} \) (the same being true for \( h \) replaced by \( k \)). Suppose then that (1) holds and \( \psi \) is an operator \( K(k) \to K(h) \) such that \( \psi(T) = U^*TV \) for all \( T \in K(k) \) and for some fixed injective partial isometries \( U: h \to k \) and \( V: h \to k \). This implies that
\[
\psi^*(u \otimes v) = V(u \otimes v) U^* = Uu \otimes Vv \in \text{ext } K(k)^*
\]
for every pair of unit vectors \( u, v \in h \). If \( \psi \) were of the form \( \psi(T) = U^*T^*vV \) the argument would be entirely similar. On the other hand, if (2) is satisfied and \( \psi(T) = JV(Tw) \) for some fixed unit vector \( w \in k \), some fixed surjective partial isometry \( V: k \to S_2(h) \) and all \( T \in K(k) \) (and \( J \) the natural injection \( S_2(h) \to K(h) \), then
\[
\psi^*(u \otimes v) = V^*(u \otimes v) \otimes w \in \text{ext } K(k)^*
\]
for all unit vectors \( u, v \in h \). Again, if instead \( \psi \) were of the form \( \psi(T) = (JV((T^*)w))^* \) the argument would be similar.

(B) All we need to note is that on \( B(h) \) and \( B(k) \) the ultra-weak and weak* topologies coincide. Hence since \( K(h)^* = B(h) \) and \( K(k)^* = B(k) \) and since by hypothesis \( \psi \) is weak*-continuous, it follows that on restriction to \( K(h)^* \), \( \psi^* \) is a well defined map from \( K(h)^* \) into \( K(k)^* \). Moreover this restriction maps \( \text{ext}(K(h)^*) \) into \( \text{ext}(K(k)^*) \) if and only if \( \psi^* \) maps the weak*-continuous elements of \( \text{ext}(B(h)^*) \) into (the weak*-continuous elements of) \( \text{ext}(B(k)^*) \). The result now follows by applying a similar argument as was used in proving (A).

**Lemma 17.** Let \( \mathcal{A} \) be a C*-algebra and \( \psi: \mathcal{A} \to B(h) \) a linear map. The, \( \psi^* \) maps the ultra-weakly continuous extreme points of \( B(h)^* \) into \( \text{ext}(\mathcal{A})^* \) if and only if there exists a Hilbert space \( k \) such that \( \psi = \pi \cdot \psi \), where \( \pi(\mathcal{A}) \subseteq B(k) \) is an irreducible representation of \( \mathcal{A} \) on \( k \) and \( \psi := B(k) \to B(h) \) is an ultra-weakly continuous linear map with the property that \( \rho \cdot \psi \in \text{ext}(B(k)^*) \) whenever \( \rho \) is an ultra-weakly continuous element of \( \text{ext}(B(h)^*) \).

**Proof.** First assume \( \psi \) to be for the form \( \psi = \psi \cdot \pi \). By the hypothesis all we then really need to check is that \( \rho \cdot \pi \in \text{ext}(\mathcal{A})^* \) whenever \( \rho \) is an ultra-weakly continuous extreme point of \( B(k) \). By [BR, 2.4.6] and the
extremality of \( \rho \), \( \rho \) is of the form
\[ \rho(A) = \langle Ax, y \rangle \]
for some \( x, y \in k \) with \( \|x\| = \|y\| = 1 \). As in the proof of Lemma 7 we may now select a unitary
\( U \in \mathcal{A} \) with \( \pi(U)^* y = x \). Then
\[
\rho \cdot \pi(UA) = \langle \pi(A) x, \pi(U)^* y \rangle = \langle \pi(A) x, x \rangle \quad A \in \mathcal{A},
\]
defines a pure state on \( \mathcal{A} \) by [KR; 10.2.5], and hence \( \rho \cdot \pi \) is an extreme point of \( \mathcal{A}^\ast \) by Corollary 9.

Conversely assume that \( \psi^* \) maps the ultra-weakly continuous elements of \( \text{ext}(B(k)^\ast) \) into \( \text{ext}(\mathcal{A}^\ast) \). Now let \( \omega_0 \) be a fixed vector state of \( B(h) \).
Since \( \omega_0 \) is pure [KR; 4.6.68], \( \omega_0 \cdot \psi \in \text{ext}(\mathcal{A}^\ast) \) by hypothesis. Thus by Lemmas 7 and 10, and Corollary 9 there exists a unitary \( U \in \mathcal{A} \) so that
\[
v(A) = \omega_0 \cdot \psi(UA), \quad A \in \mathcal{A},
\]
defines a pure state on \( \mathcal{A} \). Now let \( \pi \) be the irreducible representation of \( \mathcal{A} \) on some Hilbert space \( h_v = h \), engendered by the GNS process applied to \( v \) [KR; 10.2.3]. If \( \Omega \) is the canonical cyclic vector in \( k \) corresponding to \( v \), then surely
\[
\langle \omega_0 \cdot \psi(A) \rangle = \langle \pi(A) \Omega, \pi(U) \Omega \rangle \quad \text{for every } A \in \mathcal{A}. \tag{1}
\]
Now let \( \omega \) be any other ultra-weakly continuous pure state of \( B(h) \). By [KR; 7.1.12] and the extremality of \( \omega \), \( \omega \) is precisely a vector state of \( B(h) \) and hence unitarily equivalent to \( \omega_0 \). But then by Lemma 15(b) the pure state associated with \( \omega \cdot \psi \) is unitarily equivalent to \( v \). Considering Lemma 7, this effectively means that there exist unitaries \( V \) and \( W \) in \( \mathcal{A} \) so that
\[ \omega \cdot \psi(WW^*A^*W) = v(A) \quad \text{for all } A \in \mathcal{A}. \]
Consequently
\[
\omega \cdot \psi(A) = v(W^*V^*AW) = \langle \pi(A) \pi(W) \Omega, \pi(VW) \Omega \rangle \tag{2}
\]
for all \( A \in \mathcal{A} \). If now \( \pi(A) = 0 \), it is clear from the above that then \( \omega \cdot \psi(A) = 0 \) for all vector states of \( B(k) \) (ultra-weakly continuous elements of \( \text{ext}(B(k)^\ast) \)). Since by the polarization identity the vector states of \( B(h) \) separate the points of \( B(h) \), it follows that \( \psi(A) = 0 \) whenever \( \pi(A) = 0 \), i.e. \( \psi^{-1}(0) = \pi^{-1}(0) \).
Thus \( \psi \) induces a well defined linear map \( \tilde{\psi} \) from \( \mathcal{A}/\pi^{-1}(0) \) into \( B(h) \). Since \( \pi \) effectively identifies \( \mathcal{A}/\pi^{-1}(0) \) with \( \pi(\mathcal{A}) \), we may assume \( \tilde{\psi} \) to be acting from \( \pi(\mathcal{A}) \) into \( B(h) \), in which case \( \tilde{\psi} = \psi \cdot \pi \) by construction. Moreover as was seen in for example (2) above, for any vector state (ultra-weakly continuous pure state) \( \omega \) of \( B(h) \), \( \omega \cdot \tilde{\psi} \) is ultra-weakly continuous on \( \pi(\mathcal{A}) \). Hence by [KR; 7.1.12], \( \omega \cdot \tilde{\psi} \) is ultra-weakly continuous for every \( \omega \in \mathcal{A} \). (B(h)).
Now if \( \rho \) is ultra-weakly continuous, then so is \( \rho^* \) [BR; 2.4.2], and hence by this fact and [KR; 7.4.7], each ultra-weakly continuous functional may be written as a linear combination of at most four normal states. Clearly then \( \tilde{\psi}^* \) maps ultra-weakly continuous functionals onto ultra-weakly continuous functionals. We conclude that \( \tilde{\psi} \) must be ultra-weakly continuous.
Since $\pi(\mathcal{A})$ is irreducible ($\pi(\mathcal{A})^* = B(k)$) it follows that $\pi(\mathcal{A})$ is ultra-weakly dense in $B(k)$ [BR; 2.4.15]. Thus the result follows on noting that $\psi$ has a unique ultra-weakly continuous extension to all of $B(k)$ [KR; 10.1.10], which we may identify with $\tilde{\psi}$ itself.

**Lemma 18.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and $\psi: \mathcal{A} \to \mathcal{B}$ a linear map. Then $\psi^*$ maps $\text{ext}(\mathcal{B}^*_1)$ into $\text{ext}(\mathcal{A}^*_1)$ if and only if for every irreducible representation $\pi(\mathcal{B}) \in B(h)$ of $\mathcal{B}$, $(\pi \cdot \psi)^*$ maps the ultra-weakly continuous elements of $\text{ext}(B(h))$ into $\text{ext}(\mathcal{A}^*_1)$.

**Proof.** Suppose that for every irreducible representation $\pi$ of $\mathcal{B}$, $\pi \cdot \psi$ satisfies the relevant condition stated above. Given any extreme point $\rho$ of $\mathcal{B}^*_1$, Lemmas 10 and 7 imply that for some unitary $V \in \mathcal{B}$, $\omega(A) = \rho(VA)(A \in \mathcal{A})$ defines a pure state of $\mathcal{B}$. If now $(\pi_\omega, h_\omega, \Omega_\omega)$ is the canonical irreducible [KR; 10.2.3] GNS representation engendered by $\omega$, then surely

$$\rho(A) = \omega(V^*A) = \langle \pi_\omega(A) \Omega_\omega, \pi_\omega(V) \Omega_\omega \rangle \quad A \in \mathcal{A}.$$  

Thus $\rho \cdot \psi$ is of the form $\rho_0 \cdot \pi_\omega \cdot \psi$ where $\rho_0$ is defined by $A \mapsto \langle A \Omega_\omega, \pi_\omega(V) \Omega_\omega \rangle$, $A \in B(h_\omega)$. Since now $\rho_0$ is clearly an ultra-weakly continuous element of $\text{ext}(B(h_\alpha))$ (see for example Corollary 9 and [KR; 4.6.68]), the hypothesis ensures that $\rho \cdot \psi = \rho_0 \cdot \pi_\omega \cdot \psi$ belongs to $\text{ext}(\mathcal{B}^*_1)$.

Conversely suppose that for some irreducible representation $\pi(\mathcal{B})$ of $\mathcal{B}$ on say $h$ there exists an ultra-weakly continuous extreme point $\rho$ of $B(h)^*$ such that $\rho \cdot \pi \cdot \psi$ does not belong to $\text{ext}(\mathcal{A}^*_1)$. The lemma then follows on verifying that $\rho : \pi \in \text{ext}(\mathcal{A}^*_1)$. To see this note that the extremality of $\rho$ alongside [BR; 2.4.6] ensures that $\rho := (\rho_0 \cdot \pi_\omega \cdot \psi)$ belongs to $\text{ext}(\mathcal{B}^*_1)$. Thus $\rho \cdot \pi \in \text{ext}(\mathcal{A}^*_1)$ by Corollary 9.

Finally, with all the groundwork done, we are now ready to verify the desired characterization. A slight drawback regarding this characterization is the atomistic description given to the so-called “degenerate” part of such maps. A more global description of such maps would have been desirable, but may however not be possible. What is immediately obvious is the recognizable
similarity between the commutative case and the non-degenerate part in the general case.

**Theorem 19.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras, and \( \psi: \mathcal{A} \to \mathcal{B} \) a linear operator. Then the following are equivalent:

(a) \( \rho \circ \psi \) is an extreme point of \( \mathcal{A}^*_1 \) whenever \( \rho \) is an extreme point of \( \mathcal{B}^*_1 \).

(b) For any irreducible representation \( \pi \) on say \( \mathcal{H} \), there exists a Hilbert space \( k \) such that \( \pi \circ \psi \) is of precisely one of the following forms:

1. There exist injective isometries \( U: \mathcal{H} \to k \) and \( V: \mathcal{H} \to k \) and a \( * \)-(anti) morphism \( \pi \) from \( \mathcal{A} \) into \( \mathcal{B} \) with irreducible range such that
   \[
   \pi \circ \psi(A) = V^* \pi(A) U \quad \text{for all} \quad A \in \mathcal{A}.
   \]

2. There exists an irreducible representation \( \pi \) of \( \mathcal{A} \) on \( k \), a fixed unit vector \( w \), and a surjective partial isometry \( V: k \to \mathcal{B}(h) \) such that either
   
   (i) \( \pi \circ \psi(A) = (J \pi(A)^* w) \) for all \( A \in \mathcal{A} \)
   
   or
   
   (ii) \( \pi \circ \psi(A) = (J \pi(A) w)^* \) for all \( A \in \mathcal{A} \).

   Here \( J \) is the natural injection of \( \mathcal{B}(h) \) into \( \mathcal{B}(k) \).

(c) With \( \mathcal{B} \) in its reduced atomic representation there exists a projection \( E \in \mathcal{B} \cap \mathcal{B}' \) such that \( \psi \) decomposes into a degenerate part

\[
\psi_{I-E}: \mathcal{A} \to \mathcal{B}_{I-E}: A \to (I-E) \pi(A)(I-E)
\]

and a non-degenerate part

\[
\psi_E: \mathcal{A} \to \mathcal{B}_E: A \to E \pi(A) E
\]

each with the following structure:

1. For every irreducible representation \( \pi_0(\mathcal{B}_{I-E}) \subset \mathcal{B}(h_0) \) of \( \mathcal{B}_{I-E} \) there exists an irreducible representation \( \pi_0(\mathcal{A}) \subset \mathcal{B}(k_0) \) of \( \mathcal{A} \), a unit vector \( w_0 \in k_0 \), and an embedding \( V_0 \) of \( k_0 \) into \( \mathcal{B}(h_0) \) such that \( V_0((k_0)_1) \) is ultra-

   weakly dense in \( \mathcal{B}(h_0)_1 \) with either

   \[
   \pi_0 \circ \psi_{I-E}(A) = V_0(\pi_0(A) w_0) \quad \text{for all} \quad A \in \mathcal{A}
   \]

   or

   \[
   \pi_0 \circ \psi_{I-E}(A) = (V_0(\pi_0(A)^* w_0))^* \quad \text{for all} \quad A \in \mathcal{A}.
   \]

2. There exists a von Neumann algebra \( \mathcal{B} \) acting on some Hilbert space \( \mathcal{H} \), a partial isometry \( W \in \mathcal{B} \) with initial projection \( E_1 \) and final projection \( E_2 \), and a Jordan \( * \)-morphism \( \psi: \mathcal{A} \to \mathcal{B} \) such that up to \( * \)-isomorphic
equivalence, $\mathbb{B}$ appears as $\mathbb{B}_E$ (or alternatively $\mathbb{B}_E'$) with $\psi_E^*$-isomorphically equivalent to the mapping

$$\mathcal{A} \to \mathbb{B}_E: A \to E_1 \varphi(A) WE_1$$

(or alternatively

$$A \to \mathbb{B}_E': A \to E_2 W\varphi(A) E_2.$$  

In addition $\varphi(\mathcal{A})$ has the density property that for some set $(F_v)$ of mutually orthogonal projections in $\mathbb{B} \cap \mathbb{B}'$ with $\sum F_v = I$, we have $(F_v \varphi(\mathcal{A}) F_v)' = \mathbb{B}_E'$ for each $v$.

**Proof.** The equivalence of (a) and (b) is an immediate consequence of Theorem 16(B) and Lemmas 17 and 18. We therefore need only verify that (a) follows from (c), and that (c) follows from (b).

(c) $\Rightarrow$ (a). Suppose that $\psi$ is of the form described in (c). Given any $\rho \in \text{exi}(\mathbb{B}^*_E)$, apply Lemmas 7 and 10 to obtain a unitary $V \in \mathbb{B}$ such that $A \to \rho(VA), A \in \mathbb{B}$, defines a pure state of $\mathbb{B}$. First of all notice that by Lemma 11 we may identify $\rho$ and $\rho(V \cdot)$ with its unique extension to $\mathbb{B}''$. Thus with $E$ as in the hypothesis, an application of [KR; 4.3.14] reveals that $\rho(V E) \in \{0, 1\}$ (since $E^2 = E$). Hence again by [KR; 4.3.14] either

$$\rho(A) = \rho(V V^* A) \rho(V E) = \rho(V (V^* A) E) = \rho(E A)$$

for all $A \in \mathbb{B}$ when $\rho(VE) = 1$, or similarly

$$\rho(A) = \rho((I - E) A) \quad \text{for all} \quad A \in \mathbb{B}$$

if indeed $\rho(VE) = 0$ (that is $\rho(V(I - E)) = 1$). Now if $\rho(V(I - E)) = 1$, then $\rho$ effectively annihilates $\mathbb{B}_E$, and by Lemma 13 defines an extreme point of $(\mathbb{B}_E')^*$. Now given any extreme point $\tilde{\rho}$ of $(\mathbb{B}_E')^*$, related to some pure state $\tilde{\omega}$ in the manner described in Lemmas 7 and 10, and Corollary 9, it follows that $\tilde{\rho}$ is of the form

$$\tilde{\rho}(A) = \langle \pi_\varphi(A) x, y \rangle \quad \text{for all} \quad A \in \mathcal{A}$$

where $\pi_\varphi(\mathbb{B}_E \cap \mathbb{B}) \in \mathcal{B}(h_0)$ is the canonical irreducible representation engendered by $\tilde{\omega}$, and $x$ and $y$ suitable unit vectors. Since the functional $\omega_{x,y}(A) = \langle Ax, y \rangle$, $A \in \mathcal{B}(h_0)$, is ultra-weakly continuous on $\mathcal{B}(h_0)$, it follows from the hypothesis that $\omega_{x,y}$ assumes its norm on $V_0((k_0)_{1})$, and hence that $\omega_{x,y} \circ V_0$ is a norm one functional of $k_0$. Similarly the ultra-weak continuity of $\omega_{x,y}^*$ ensures that $\omega_{x,y}^* \circ V_0$ is also a norm-one functional of $k_0$. The two cases being similar we now assume that

$$\pi_0 \circ \psi_{I-E}(A) = (V_0(A^* w_0))^* \quad \text{for all} \quad A \in \mathcal{A}.$$
Now since $\omega_{x,y}^* \circ V_0$ has norm one, there exists a unit vector $z \in k_0$ so that
$$(\omega_{x,y}^* \circ V_0)(p) = \langle p, z \rangle \quad \text{for all } p \in k_0.$$ But then

$$\tilde{\rho} \circ \psi_{I,E}(A) = (\omega_{x,y} \circ \pi_0 \circ \psi_{I,E})(A)$$
$$= \omega_{x,y}((V_0(\pi_0(A^*) w_0))^*)$$
$$= \omega_{x,y}^* (V_0(\pi_0(A^*) w_0))$$
$$= \langle \pi_0(A)^* \omega_0, z \rangle$$
$$= \langle \pi_0(A) z, w_0 \rangle \quad \text{for all } A \in \mathcal{A}.$$ If now as in the proof of Lemma 7 we select a unitary $U \in \mathcal{A}$ so that $\pi_0(U^*) = w_0 = z$, then $\tilde{\rho} \circ \psi_{I,E}(U)A = \langle \pi_0(U) \pi_0(A) z, w_0 \rangle = \langle \pi_0(A) z, z \rangle$, $A \in \mathcal{A}$, defines a pure state of $\mathcal{A}$ [KR; 10.2.5] and so an application of Corollary 9 reveals that $\tilde{\rho} \circ \psi_{I,E} \in \text{ext}(\mathcal{A}^*)$ as required.

If on the other hand $\rho(VE) = 1$, then on arguing as before, $\rho$ defines an extreme point of $(\mathcal{B}E)^*$ and annihilates $\mathcal{B}_{I,E}$. We may therefore replace $\mathcal{B}$ by $\mathcal{B}_E$ and assume that $I = E$. Since in addition *-isomorphisms trivially preserve extreme points of $\mathcal{B}^*$, we assume for the moment that $\mathcal{B}^* = \mathcal{B}_{E_1}$ and that $\psi$ is of the form

$$\psi(A) = E_2 W \psi(A) E_2 \quad \text{for all } A \in \mathcal{A}. \tag{1}$$

We show that it is sufficient to consider only this case. To see this note that by hypothesis $W$ defines a unitary mapping from $E_1 h$ onto $E_2 h$, and hence $\mathcal{B}_{E_1} \to \mathcal{B}_{E_2}; A \to E_1 W^* A W E_1$ defines a spatial *-isomorphism from $\mathcal{B}_{E_1}$ onto $\mathcal{B}_{E_2}$. Since in addition for all $A \in \mathcal{B}$ we have

$$E_1 \psi(A) W E_1 = W^* W \psi(A) W E_1 = E_1 W^* (E_2 W \psi(A) E_2) W E_1,$$

it is clear that the one case is *-isomorphic to the other, and hence we may restrict attention to the case outlined in (1) above. Now apply Lemmas 7 and 10 to find a unitary $V \in \mathcal{B}_{E_2}$ such that

$$\omega(A) = \rho(VA) \quad \text{for all } A \in \mathcal{B}_{E_2}$$
defines a pure state of $\mathcal{B}_{E_2}$. Denoting the map $\mathcal{B} \to \mathcal{B}_{E_2}; A \to E_2 AE_2$ by $\eta_2$, Lemma 1 reveals that $\omega \circ \eta_2$ is a pure state of $\mathcal{B}^*$ with

$$\rho \circ \eta_2(E_2 A) = \rho \circ \eta_2(A) \quad \text{for all } A \in \mathcal{B}.$$
Now since $V^*V = VV^* = E_2$, it can easily be verified that $W^*V$ is a partial isometry with initial projection $E_2$ and final projection $E_1$. Since then

$$
\rho \cdot \eta_\delta(W(W^*V)) = \rho \cdot \eta_\delta(E_2V)
$$

$$
= \rho(VE_2AE_2)
$$

$$
= \omega \cdot \eta_\delta(A) \quad \text{for all} \quad A \in \mathcal{R},
$$

Corollary 9 reveals that $A \rightarrow \rho \cdot \eta_\delta(WA)$ defines an extreme point of $\mathcal{R}_\delta^*$ which is moreover ultra-weakly continuous by Lemma 12 and [BR; 2.4.2]. The problem thus reduces to showing that the adjoint of some Jordan-morphism $\psi: \mathcal{A} \rightarrow \mathcal{R}$ for which $\psi(\mathcal{A})$ has the stated density condition, maps ultra-weakly continuous elements of $\text{ext}(\mathcal{R}_\delta^*)$ into $\text{ext}(\mathcal{A}_\delta^*)$. Hence assume $\psi$ to be such a mapping and let $\rho$ be an ultra-weakly continuous element of $\text{ext}(\mathcal{R}_\delta^*)$. By Lemmas 7 and 10 there exists a unitary $V \in \mathcal{R}$ and an ultra-weakly continuous pure state $\omega$ on $\mathcal{R}$ with $\rho(A) = \omega(V^*A)$ and $\rho(VA) = \omega(A)$ for every $A \in \mathcal{R}$.

Now let $(F_v)$ be the family of mutually orthogonal central projections described in the hypothesis. As in the proof of Theorem 5 we may apply [KR; 4.3.14] to conclude that $\omega(F_v) \in [0, 1]$ for every $v$, and then make use of the fact that $\omega(VF_v) = \omega(I) = 1$ to conclude that $\omega(F_v) = 1$ for precisely one $v$, say $\omega(F_{v_0}) = 1$. Then surely by [KR; 4.3.14],

$$
\rho(A) = \omega(V^*A) = \omega(V^*F_{v_0}A) = \rho(F_{v_0}A)
$$

(2)

for every $A \in \mathcal{R}$. Since moreover $\rho(F_{v_0}V) = \omega(F_{v_0}) = 1$, it now follows from Lemmas 12 and 13 that the restriction of $\rho$ to $\mathcal{R}_{F_{v_0}}$ is an ultra-weakly continuous extreme point of $(\mathcal{R}_{F_{v_0}})^*$. From this fact and (2) it is clear that we may replace $\mathcal{R}$ by $\mathcal{R}_{F_{v_0}}$ and hence effectively assume that $\psi(\mathcal{A})' = \mathcal{R}$ since $(F_{v_0}\psi(\mathcal{A}) F_{v_0})' = \mathcal{R}_{F_{v_0}}$ by hypothesis. Next apply [BR; 3.2.2] to obtain a projection $E \in \mathcal{R} \cap \mathcal{R}$ such that $E \psi(A)$, $A \in \mathcal{A}$, defines a $*$-homomorphism and $\psi(I - E) \psi(A)$, $A \in \mathcal{A}$, a $*$-antimorphism. Now by [KR; 4.3.14] we have $\omega(E) \in [0, 1]$ and hence again by [KR; 4.3.14], for any $A \in \mathcal{R}$ we either have

$$
\rho(A) = \omega(V^*A) \omega(E) = \omega(V^*EA) = \rho(EA) \quad \text{if} \quad \omega(E) = 1
$$

(3)

or similarly

$$
\rho(A) = \rho(I - E) A \quad \text{if} \quad \omega(I - E) = 1.
$$

(4)

We show that we may assume $\psi$ to be either a $*$-morphism, or a $*$-antimorphism by considering the two cases separately.
Case 1 $(\alpha(E) = 1)$. Denote the $C^*$-algebra generated by $\psi(A)$ by $\mathcal{C}$. Then since $\mathcal{C}^* = \mathcal{R}$, we surely have $(\mathcal{E} \mathcal{C} \mathcal{E})^* = \mathcal{R}$. As $E\psi(\cdot)$ is a $*$-morphism, $E\psi(\mathcal{A}) \mathcal{E}$ is already a $C^*$-algebra and hence without too much ado we have $E\psi(\mathcal{A}) \mathcal{E} = \mathcal{E} \psi(\mathcal{A}) \mathcal{E}$. Moreover as $1 = \alpha(E) = \rho(E V) \leq ||\rho||_{\mathcal{A} \mathcal{E} \mathcal{E}} \leq 1$, it is clear from Lemmas 12 and 13 that the restriction of $\rho \cdot \eta_{\mathcal{A}}$ to $\mathcal{R} \mathcal{E}$ is a ultra-weakly continuous extreme point of $(\mathcal{R} \mathcal{E})^*$. Since by (2) $\rho$ vanishes on $\mathcal{R} \mathcal{E}$, the assertion follows.

Case 2 $(\alpha(E) = 0$, i.e. $\alpha(I - E) = 1)$. The proof of this case is virtually identical to Case 1, and is therefore omitted.

Recapitulating, we have thus effectively reduced the situation in question to

$$\psi : \mathcal{A} \rightarrow \psi(\mathcal{A})^* = \mathcal{R}$$

where $\mathcal{A}$ is a $C^*$-algebra, $\mathcal{R}$ a von Neumann algebra, and $\psi$ a $*$-(anti) morphism with the immediate task at hand being to show that $\rho \cdot \psi \in \text{ext}(\mathcal{A}^*)$ for very ultra-weakly continuous extreme point $\rho$ of $\mathcal{R}^*$. If now we apply [KR; 10.1.12], then in the notation of [KR; 10.1.12], it follows from for example [BR; 2.4.23] that $\mathcal{R}$ may be identified with $P\Phi(\mathcal{C})^* P$ as far as we are concerned. Here $\mathcal{C}$ is of course the $C^*$-algebra $\psi(\mathcal{A})$. If next we apply Lemmas 12 and 13, it is clear that $\rho$ extends to an ultra-weakly continuous extreme point of $(\mathcal{R} \mathcal{E})^*$. Since by (2) $\rho$ vanishes on $\mathcal{R} \mathcal{E}$, the assertion follows.

However as $\psi^*$ is a $*$-(anti) morphism, we have that $\|\psi^*\| = 1$ and $\psi^* ((\mathcal{A}^*)^*) = \psi^* (\mathcal{A}^*)^*$. Hence we may select a sequence $(A_n) \subset (\mathcal{A}^*)^*$ with $\rho \cdot \psi^* (A_n) = ||A_n|| = 1$. Since $\|\rho \cdot \psi^* (A_n)\| = ||A_n\| \|\psi^*\| = 1$, it is clear that $\rho \cdot \psi^*$ is then a norm-one ultra-weakly continuous functional on $\mathcal{A}$. Thus by [KR; 7.3.2] there exists a normal state $\omega_{\mathcal{A}}$ on $\mathcal{A}^*$ and a partial isometry $W \in \mathcal{A}^*$ with

$$\omega_{\mathcal{A}} (A) = \rho \cdot \psi^* (WA) \quad \text{and} \quad \omega_{\mathcal{A}} (W^* A) = \rho \cdot \psi^* (A)$$

for all $A \in \mathcal{A}^*$. Since $\psi^*$ is a $*$-(anti)morphism, $\psi^* (W)$ is a partial isometry in $\mathcal{R} = \psi(\mathcal{A})^*$S. We conclude by considering two cases.

Case 1 ($\psi^*$ a $*$-homomorphism). If we define $\omega_{\mathcal{A}} : \psi(\mathcal{A}) \rightarrow \mathcal{C}$ by

$$\omega_{\mathcal{A}} (\psi (A)) = \rho (\psi^* (W) \psi (A)) = \rho (\psi^* (WA)) = \omega_{\mathcal{A}} (A)$$
for every $A \in \mathcal{A}$ and uniquely extend $\omega_1$ to $\psi(\mathcal{A})^*\mathcal{B} = \mathcal{B}$, then since $\omega_1(I) = \omega_B(I) = \omega_B(I) = 1$, $\omega_1$ is a state of $\mathcal{B}$ [KR; 43.2], which is moreover ultra-weakly continuous [KR; 10.1.1]. Lemma 10 and the fact that $\omega_1$ is “related” to $\rho$ by means of $\psi^*(W)$ by (5) above, now reveals that $\omega_1$ is in fact pure. But then $\omega_1 \cdot \psi$ must be a pure state of $\mathcal{A}$ by Theorem 5, and hence since $\omega_1 \cdot \psi$ is just the restriction of $\omega_B$ to $\mathcal{A}$, $\omega_B$ is a (normal) pure state by Lemma 4. Thus by Corollary 9 and (5), the restriction of $\rho \cdot \psi^*$ to $\mathcal{A}$, $\rho \cdot \psi$, is an extreme point of $\mathcal{A}^*$.  

Case 2 ($\psi^*$ an anti-morphism). As a start we observe that since $\rho$ is an ultra-weakly continuous extreme point of $\mathcal{B}_1^*$, it is an easy exercise to show that the same is true of $\rho^*$. Now define $\omega_B : \psi(\mathcal{A}) \rightarrow \mathcal{B}$ by

$$\omega_B(\psi(A)) = \rho^*(\psi^*(W^* \psi(A)) \quad \text{for all} \quad A \in \mathcal{A}.$$ 

Observe that for any self-adjoint $A \in \mathcal{A}$ we have

$$\omega_B(\psi(A)) = \rho(\psi^*(W^* \psi(A))) = \rho((\psi^*W^* \psi(A))^*)$$

$$= \rho(\psi(A) \psi^*(W)) = \rho(\psi^*W)A)$$

$$= \omega_B(A) = \omega_B(A). \quad (6)$$

By linearity the above holds for span$(\mathcal{A}_0) = \mathcal{A}$. Since $\omega_B(I) = \omega_B(\psi(I)) = \omega_B(I) = 1$ with $\|A\| = \|\rho^*\| = \|\psi^*\| = 1$, it is a state of $\psi(\mathcal{A})$. By Lemma 10 applied to the fact that $\omega_1$ is “related” to $\rho^*$ by means of $\psi^*(W^*)$, this state is pure, and by Theorem 5 $\omega_B \cdot \psi$ is then a pure state of $\mathcal{A}$. But from (6) above, $\omega_B \cdot \psi = \omega_B \cdot \psi$. Thus by Lemma 4, $\omega_B$ is the unique ultra-weakly continuous pure extension of $\omega_B \cdot \psi$ to all of $\mathcal{A}^*$. As in Case 1, Corollary 9 now reveals that $\rho \cdot \psi$ is an extreme point of $\mathcal{A}^*_1$.

It remains to verify that (c) follows from (b).

(b) $\Rightarrow$ (c). To see this assume (b) to be true, and apply Zorn’s lemma to obtain a maximal set $\mathcal{M}$ of pure states of $\mathcal{A}$ for which the associated (irreducible) GNS representations of $\mathcal{A}$ are pairwise inequivalent. Then surely $\pi = \bigoplus_{\omega \in \mathcal{M}} \pi_\omega$ corresponds to the reduced atomic representation of $\mathcal{A}$ (here $(\pi_\omega, h_\omega, \Theta_\omega)$ is the canonical representation engendered by $\omega \in \mathcal{A}$). Now partition $\mathcal{M}$ into the two disjoint classes $\mathcal{M}_1$ and $\mathcal{M}_2$ where for $i = 1, 2$, $\omega \in \mathcal{M}_i$ (respectively $\omega \in \mathcal{M}_2$) if and only if $\omega \in \mathcal{M}$ and $\pi_\omega \cdot \psi$ is of the form (1) (respectively (2)) as described in (b). Now let $E \in \mathcal{B}(\bigoplus_{\omega \in \mathcal{M}_1} h_\omega)$ be the canonical projection of $\bigoplus_{\omega \in \mathcal{M}_1} h_\omega$ onto the subspace corresponding to $\bigoplus_{\omega \in \mathcal{M}_1} h_\omega$.

By construction it is now clear that $E \in \bigoplus_{\omega \in \mathcal{M}_1} \pi_\omega(\mathcal{B})$. Moreover since on restriction to the subspace corresponding to $h_\omega$ (for some given $\omega \in \mathcal{M}$)
E corresponds to either the identity or the zero-operator on \( h_w \), it is clear from [KR; 10.3.10] that \( E \in \pi(\mathcal{B}) \) since \( \mathcal{B} \) is unital. Now given any irreducible representation \( \pi \) of \( \pi(\mathcal{B})_{I-E} \), it follows from the maximality of \( \mathcal{M} \) that \( \pi(\mathcal{B})_{I-E} \) is equivalent to \( \pi_{\omega_0}(\mathcal{B}) \) for some \( \omega_0 \in \mathcal{M} \). Hence \( A \mapsto \pi(I-E \pi(A)) \), \( A \in \mathcal{B} \), has the same kernel as \( \pi_{\omega_0} \). However the presence of \( I-E \) leads us to conclude that for any \( \omega \in \mathcal{M}_1 \) there is some \( A \in \mathcal{B} \) with \( \pi_{\omega}(A) \neq 0 \) and \( \pi(I-E \pi(A)) = 0 \). Hence \( \omega_0 \in \mathcal{M}_2 \). From this fact and the equivalence of \( \pi(\pi(\mathcal{B})_{I-E}) \) and \( \pi_{\omega_1}(\mathcal{B}) \), we conclude that \( \pi \circ (I-E \pi(\mathcal{B})) \) is of the form described in (b(2)) as required. To conclude the proof we consider the mapping \( \mathcal{A} \mapsto \pi(\mathcal{B})_E ; A \mapsto E(\pi \circ \psi(\mathcal{A}))E \) and show that it is of the requisite form described in (c(2)). Recall that each \( \pi_{\omega} \circ \psi \), \( \omega \in \mathcal{M}_1 \), is of the form

\[
\pi_{\omega} \circ \psi(A) = V_{\omega}^* \varpi_{\omega}(A) U_{\omega} \quad \text{for all} \quad A \in \mathcal{A}
\]

for some irreducible \( \ast \)-(anti)morphism \( \varpi_{\omega} \) from \( \mathcal{A} \) into \( B(k_w) \). Now let \( h = \bigoplus_{\mathcal{A}_1} k_w \) and \( \mathcal{B} = \bigoplus_{\mathcal{A}_1} B(k_w) \). If for each \( \omega \in \mathcal{M}_1 \) we let \( F_{\omega} \) be the canonical projection of \( h \) onto the subspace corresponding to \( k_w \), and if we let \( \varphi = \bigoplus_{\mathcal{A}_1} \varpi_{\omega} \), then it may easily be verified that \( \varphi \) is a Jordan \( \ast \)-morphism (since each \( \varpi_{\omega} \) is), that \( (F_{\omega})_{\mathcal{A}_1} \subseteq \mathcal{B} \cap \mathcal{B}^* \) is mutually orthogonal with \( \bigoplus_{\mathcal{A}_1} F_{\omega} = I \), and that \( \varphi(\mathcal{A}) = \bigoplus_{\mathcal{A}_1} \varpi_{\omega}(\mathcal{A}) \) has the required density property in terms of \( (F_{\omega})_{\mathcal{A}_1} \). Finally note that since by hypothesis the mappings \( V_{\omega} = h_{\omega} \mapsto k_w \) and \( U_{\omega} : h_{\omega} \mapsto k_w \), \( \omega \in \mathcal{M}_1 \), referred to in (7) are injective partial isometries, it follows that the same is true of

\[
V : \bigoplus_{\mathcal{A}_1} h_{\omega} \mapsto h, \quad U : \bigoplus_{\mathcal{A}_1} h_{\omega} \mapsto h
\]

and \( U = \bigoplus_{\mathcal{A}_1} U_{\omega} \). Since effectively \( \bigoplus_{\mathcal{A}_1} h_{\omega} \) appears as the image of \( E \), we may suppose \( V \) and \( U \) to be acting from \( E(\bigoplus_{\mathcal{A}_1} h_{\omega}) \). Thus by construction

\[
\pi \circ \psi_E(A) = V^* \varphi(A) U \quad \text{for all} \quad A \in \mathcal{A}.
\]

The injectivity of \( U \) and \( V \) now imply that \( U^*U = I_E = V^*V \), with in addition \( W = UV^* = \bigoplus_{\mathcal{A}_1} U_{\omega} V_{\omega}^* \in \bigoplus_{\mathcal{A}_1} B(k_w) = \mathcal{B} \). Notice further that now

\[
W^*W = V(U^*U) V^* = R_V \quad \text{and} \quad WW^* = U(V^*V) U^* = R_U.
\]

Hence \( W \) is a partial isometry with initial projection \( R_V \), the range projection of \( V \), and final projection \( R_U \), the range projection of \( U \). Let \( E_1 = R_V \) and \( E_2 = R_U \). Then

\[
\pi \circ \psi_E(A) = V^* \varphi(A) U = U^* W \varphi(A) U
\]
for all $A \in \mathcal{A}$. However note that since $U$ is an injective partial isometry from $E(\oplus_{1} h_{\omega})$ into $E_{2}(h)$, the mapping $\pi(b)_{E_{2}} \to \mathcal{A}_{E_{2}}: A \to UAU^{*}$ now turns out to be a spatial $*$-isomorphism. Hence up to $*$-isomorphism $(\pi(b)_{E_{2}})^{*}$ appears as $\mathcal{A}_{E_{2}}$ with $\pi \cdot \psi_{E}$ corresponding to the map

$$U(U^{*}W_{\varphi}(A)U^{*}) = E_{2}W_{\varphi}(A)E_{2}, \quad A \in \mathcal{A}.$$ 

In a similar fashion one can show that up to a spatial $*$-isomorphism induced by $V$, $(\pi(b)_{E_{2}})^{*}$ appears as $\mathcal{A}_{E_{2}}$, with $\pi \cdot \psi_{E}$ now corresponding to the map

$$E_{1}W_{\varphi}(A)W_{E_{1}}, \quad A \in \mathcal{A}.$$ 

With Theorems 5 and 19 now at our disposal, the conclusion regarding maps with pure state preserving adjoints is now immediately obvious.

Corollary 20. Let $\mathcal{A}$, $\mathcal{B}$ be $C^{*}$-algebras and $\psi: \mathcal{A} \to \mathcal{B}$ a linear map with the property that $\omega \cdot \psi \in \mathcal{P}_{\mathcal{A}}$ whenever $\omega \in \mathcal{P}_{\mathcal{A}}$. Then $\rho \cdot \psi \in \text{ext}(\mathcal{A})^{*}$ whenever $\rho \in \text{ext}(\mathcal{B}^{*})^{*}$.

Proof. Consider Theorem 5 alongside Theorem 19.

4. MAPS ON SOME SPECIFIC SPACES

In conclusion, to provide information about what may happen in the commutative case, we consider the action of maps with “extreme point preserving” adjoints on some uniform algebras, hereby considering the commutative $C^{*}$-algebras in particular, where the maps in question reduce to ordinary compositions of a multiplication and of a composition operator.

If $A \subset C(X)$ is a uniform algebra over $X$, let $\partial_{A}$ be its Šilov boundary. Denote by $p_{A}$ the so-called Choquet (or strong) boundary of $A$ (this is the set of all $p$-points in $X$, or generalized peak points—see [Gam]). Also, if $x \in X$ and $\delta_{x}$ is the functional corresponding to the evaluation at $x$, denote by $\eta_{x}$ its restriction to $A$. The following Lemma is very probably well-known, but we have not been able to find a reference for it:

Lemma 21. Let $A \subset C(X)$ be a uniform algebra. Then

$$\text{ext}(A) = \partial D \cdot p_{A} \quad \text{and} \quad \text{ext}(A^{*}) = \partial D \cdot \delta_{A}.$$ 

Proof. “⇒”: Let $x \in X$ be a $p$-point, and suppose we have $0 \leq t \leq 1$ and $\varphi, \psi \in B_{A}$, such that $\eta_{x} = t\varphi + (1-t)\psi$. Clearly, if $\psi$ (resp. $\psi$) are Hahn–Banach extensions of $\varphi$ (resp. $\varphi$), then $\xi := t\varphi + (1-t)\psi$ is an extension...
of $\eta_x$. But since for $p$-points such an extension is unique [G], we must have $\xi = \delta_x$. Now, $\delta_x \in \text{ext}(C(X)^*_p)$, and so $\delta_x = \varphi = \psi$ and, in particular, $\eta_x = \varphi = \psi$. Consequently, $\eta_x \in \text{ext}(A^*_p)$.

"⊂": We first prove that $\text{ext}(A^*_p) \supset \partial D \cdot \partial A$. Let $\delta \in \text{ext}(A^*_p)$, and consider the set $S := \{\mu \in C(X)^*_p : |\mu| = 1, \int f \, d\mu = \delta(f), \forall f \in A \text{ and } \supp \mu \subset \partial A\}$. $S$ is $\text{w}^*$-compact in $C(X)^*_p$ and can be partially ordered by setting $\mu \leq v$ iff $\supp \mu \subset \supp v$. By the $\text{w}^*$-compactness of $S$ Zorn's lemma applies to give $\delta$ with minimal support $\partial A$.

Suppose now $x \in \partial A \setminus \partial p_A$. Then, by the Bishop-deLeeuw theorem [Gam, 12.9] there exists a probability measure $\mu$ on the $\sigma$-algebra generated by $p_A$ and the Borel sets in $X$, such that $\mu(p_A) = 1$ and $\delta_x(f) = f(x) = \int f \, d\mu$ for all $f \in A$. Let $y \in p_A \cap \supp \mu$, and let $f \in A$ be a function with $f(y) = 1$ and $|f(x)| < 1/2$. Choosing a small open neighborhood $E$ of $y$ we can ensure that $|1/\mu(E) \times \int_E f \, d\mu - 1| < 1/2$ and $\mu(p_A \setminus E) \neq 0$ (clearly, $\mu$ cannot be concentrated on $y$). So, we write

$$
\delta_x(g) = \mu(E) \left( \frac{1}{\mu(E)} \int_{E} g \, d\mu \right) + (1 - \mu(E)) \left( \frac{1}{1 - \mu(E)} \int_{E^c} g \, d\mu \right)
$$

for all $g \in A$. If $\delta_x$ were an extreme point of $B_{A^*}$, then we would have for the above $f$ the contradiction $\delta_x(f) = 1/\mu(E) \int_{E} f \, d\mu$.

Finally, the statement about $\text{ext}(A^*_p)$ follows immediately from the fact $\partial A = \partial A^*$.

**Corollary 22.** If $A$ is logmodular then $\text{ext}(A^*_p) = \partial D \cdot \partial A$. 
This follows from the fact that multiplicative functionals on logmodular algebras have unique representing measures [Gam, II.4.2], and from [Gam, II.11.3] which states that a point $x \in \partial A$ is a p-point if and only if the point mass at $x$ is the only representing measure for the point evaluation at $x$.

Using the above corollary (and classical results such as: every point in $\partial D$ is a peak point for the disc algebra, $H_\infty$ is logmodular--see [Gam, Gar]) it is now more or less immediate to deduce the following (where for the sake of simplicity we call extremal an operator $\Phi$ such that $\Phi^*$ sends extreme points of the domain ball to extreme points of the range ball):

**Theorem 23.** (a) An operator $\Phi$ on the disc algebra $A$ is extremal if and only if $\Phi = M_\psi C_\varphi$, where $\psi$ and $\varphi$ are finite Blaschke products. (b) An operator $\Phi$ on $H_\infty$ is extremal if and only if $\Phi = M_\psi C_\varphi$, where $\psi$ and $\varphi$ are inner functions.

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