Bearing capacity of circular foundations reinforced with geogrid sheets

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Abstract

The ultimate bearing capacity of a circular footing, placed over a soil mass which is reinforced with horizontal layers of circular reinforcement sheets, has been determined by using the upper bound theorem of the limit analysis in conjunction with finite elements and linear optimization. For performing the analysis, three different soil media have been separately considered, namely, (i) fully granular, (ii) cohesive frictional, and (iii) fully cohesive with an additional provision to account for an increase of cohesion with depth. The reinforcement sheets are assumed to be structurally strong to resist axial tension but without having any resistance to bending; such an approximation usually holds good for geogrid sheets. The shear failure between the reinforcement sheet and adjoining soil mass has been considered. The increase in the magnitudes of the bearing capacity factors ($N_c$ and $N_{\gamma}$) with an inclusion of the reinforcement has been computed in terms of the efficiency factors $\eta_c$ and $\eta_{\gamma}$. The results have been obtained (i) for different values of $\phi$ in case of fully granular ($c=0$) and $c-\phi$ soils, and (ii) for different rates ($m$) at which the cohesion increases with depth for a purely cohesive soil ($\phi=0^\circ$). The critical positions and corresponding optimum diameter of the reinforcement sheets, for achieving the maximum bearing capacity, have also been established. The increase in the bearing capacity with an employment of the reinforcement increases continuously with an increase in $\phi$. The improvement in the bearing capacity becomes quite extensive for two layers of the reinforcements as compared to the single layer of the reinforcement. The results obtained from the study are found to compare well with the available theoretical and experimental data reported in literature.

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1. Introduction

Various forms of the reinforcement layers, such as geotextiles, geogrids and galvanized steel strips are often embedded in weak foundation soils primarily (i) to reduce footing settlements, and (ii) to increase the ultimate bearing capacity of foundations. The usage of the reinforcements in a soil medium became especially popular after the pioneering work of Vidal (1966). Subsequently, for assessing the effect of the reinforcements on load carrying capacity and settlement of the foundations, a number of researchers have performed extensive studies by using a series of models tests (Binquet and Lee, 1975; Fragaszy and Lawton, 1984; Guido et al., 1986; Khing et al., 1993; Omar et al., 1993; Das and Omar, 1994; Adams and Collin, 1997; Abu-Farsakh et al., 2013) and different computational approaches (Asaoka et al., 1994; Otani et al., 1998; Yu and Sloan, 1997; Huang and Hong, 2000; Blatz and Bathurst, 2003; Michalowski, 2004; Deb et al., 2007; Chakraborty and Kumar, 2012; Miyata and Bathurst, 2012;

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Asakereh et al., 2013; Kumar and Sahoo, 2013). These studies are generally meant for reinforced strip foundations. In the recent past, a few experimental and analytical studies have also been undertaken to examine the behavior of circular footings placed on reinforced soil media (Boushehrian and Hataf, 2003; Basudhar et al., 2007; Sireesh et al., 2009; Lovisa et al., 2010; Chakraborty and Kumar, 2012; Lavasan and Ghazavi, 2012; Ornek et al., 2012; Demir et al., 2013). Boushehrian and Hataf (2003) have performed a series of laboratory tests, along with a numerical analysis, to examine the effect of the depth of the first layer of the reinforcement, vertical spacing and number of reinforcement layers on the bearing capacity of foundations. Basudhar et al. (2007) have conducted an experimental study for circular footings resting on sand reinforced with geotextiles. Sireesh et al. (2009) have investigated the inclusion of geocell reinforced sand mattress in a clay medium for a circular foundation. Lovisa et al. (2010) have experimentally investigated the potential benefit of prestressing the geotextile layer in a reinforced soil circular foundation. Chakraborty and Kumar (2012), by using the lower bound theorem of the limit analysis in conjunction with finite elements and linear optimization, have determined the bearing capacity of a circular foundation placed over soils which are embedded with a single layer of horizontal circular reinforcement sheet. Demir et al. (2013) have carried out experimental and numerical investigations, by using Plaxis 3D, for determining the bearing capacity of a circular footing resting over granular fill reinforced with geogrid sheet overlying natural clay deposit. In the present research, the bearing capacity of a circular foundation embedded with single and two horizontal layers of the reinforcements in the form of circular geogrid sheets, has been determined by using the upper bound finite elements limit analysis in combination with linear optimization. The upper bound formulation, to incorporate the inclusion of the reinforcement sheets in the analysis, has been implemented from the work of Kumar and Sahoo (2013) for strip foundations on the basis of finite elements and linear optimization. Similar to the study of Kumar and Sahoo (2013), the circular reinforcement sheets in the present analysis are assumed to be structurally strong to resist axial tension but these reinforcements are assumed not to have any resistance to bending. The critical depths of the reinforcement layers, both for single and two layers of reinforcement sheets have been computed for different cases. Corresponding to the critical depths of the reinforcement sheets, the optimum diameter of the circular reinforcement sheet has also being evaluated. The results obtained from the analysis have been compared with that available from literature. The nodal velocity patterns have also been examined for a few typical cases.

2. Problem statement

A rigid rough circular footing of diameter, \( d \), is placed over a soil mass reinforced with either a single or a group of two layers of horizontal circular reinforcement sheets. The point of the application of the resultant load (\( Q \)) coincides with its axis of the footing. The ground surface is horizontal without any external surcharge pressure. The soil mass is assumed to be homogenous, isotropic, perfectly plastic, and it follows an associated flow rule and the Mohr–Coulomb’s failure criterion; it is known that the Mohr–Coulomb criterion is generally accepted as a good approximation for modeling the failure of soils (Abbo and Sloan, 1995; Davis and Selvadurai, 2002). Fig. 1(a) depicts the positions of the reinforcements sheets in a soil medium. The single layer reinforcement sheet is assumed to be placed at a depth \( h \) from the ground surface. In the case of two layers of reinforcements, the upper sheet is placed at a depth \( h_1 \) from ground surface, and the vertical spacing between the two layers of the reinforcement is equal to \( h_2 \). It is required to compute the ultimate bearing capacity of the foundation due to an inclusion of the reinforcement sheet(s). It is also intended to determine the critical positions of reinforcements so that the increase in the bearing capacity, with the usage of the reinforcements, becomes always the maximum. Corresponding to the critical depths, it is also aimed to find the optimum diameter (\( D_{opt} \)) of the circular reinforcements.

Three different types of soil media have been separately considered for doing the analysis, these are: (i) fully granular soil (\( c=0 \)), (ii) cohesive frictional soil, and (ii) fully cohesive soil (\( \phi=0^\circ \)) with an additional provision to account for an increase in the value of cohesion with depth. Bishop (1966) found that for saturated normally consolidated and lightly over consolidated clays, the cohesion of soil mass under undrained condition increases almost linearly with depth. Therefore, in the case of a purely cohesive soil (\( \phi=0 \)), the cohesion of soil mass at any depth (\( z \)) below the ground surface is defined by means of the following expression:

\[
c = c_0 + \frac{m c_0 z}{d}
\]

(1)

where (i) \( c \) and \( c_0 \) are the values of soil cohesion at a depth \( z \) and along ground surface, respectively, and (ii) \( m \) is a non-dimensional factor which indirectly defines the rate at which the cohesion increases with depth.

3. Assumptions and modeling of soil-reinforcement interference

The reinforcement sheets are assumed to have sufficient resistance to axial tension without any structural failure (breakage), but these reinforcements are assumed not to offer any resistance to bending. It needs to be mentioned that the reinforcement sheets in the form of geogrids and geotextiles do not have substantial resistance against bending (Boushehrian and Hataf, 2003). The tensile resistance of the geogrid sheet is usually much greater than that of geotextile (Lawson and Kempton, 1995), the present analysis will, therefore, remain generally applicable to soils that are reinforced with geogrid sheets; it is noted from the available literature that the axial tensile resistance of the reinforcement sheets varies approximately between (i) \( 26 \text{ kN/m} – 105 \text{ kN/m} \) for geogrid sheets (Koerner, 1994; Demir et al., 2013; Chehab et al., 2007,
(ii) 20 kN/m–40 kN/m for geotextiles (Koerner, 1994; Lovisa et al., 2010; Fourie and Fabian 1987), and (iii) 930 kN/m–1710 kN/m for galvanized steel sheets with thickness 3 mm (ASTM, 2011). The reinforcement sheets with relatively higher rigidity, such as galvanized steel sheets/strips, will also offer some resistance against bending apart from axial tension. Hence, the present analysis, if extended to foundations with reinforcements in the form of galvanized sheets/strips, would lead to a rather conservative estimate in finding the improvement of the bearing capacity. Note that, irrespective of the reinforcement type, a shear failure can always take place between the reinforcement sheet and adjoining soil mass. On the other hand, if the reinforcement sheet is weak in axial tensile, for instance a geotextile sheet, it will lead to structural failure of the reinforcements before the commencement of the shear failure between the reinforcement layer and adjoining soil mass. In that case, the improvement in the bearing capacity due to an inclusion of the reinforcements will be smaller than that predicted from the present analysis.

In the present analysis, the thickness of the reinforcement sheets is assumed to be negligible. To simulate the reinforcement sheet, a number of additional nodes are chosen in the radial direction along the position of the reinforcement layer. For the reinforcement nodes: (i) the velocity (u) along the radial direction has been specified equal to zero, and (ii) no constraint has been imposed for the velocity (v) in the vertical direction. Note that the stretching of the reinforcement, if any, has been neglected while specifying u = 0 along the reinforcement layer. The velocity jump along the reinforcement sheet–soil interface is governed by an associated flow rule, that is, \( \Delta v = |\Delta u| \tan \phi \); where (i) \( \Delta v \) is the velocity jump in the vertical (normal) direction and (ii) \( \Delta u \) denotes the velocity jump in the radial (tangential) direction. In the present analysis it has been assumed that (i) the adhesion between the soil mass and the reinforcement sheet is equal to the magnitude of the soil cohesion (c) at that level, and (ii) the interface friction angle between the soil mass and reinforcement sheet is equal to \( \phi \).

4. Problem domain and boundary conditions

Owing to the fact that the velocity distribution remains symmetric about the vertical axis passing through the center of the footing, only one half of the total domain in an \( r-z \) plane has been employed for carrying out the analysis. The chosen planar domain and associated velocity conditions along the boundaries of the domain are shown in Fig. 1(a). The depth (H) of the domain below the footing and the horizontal extent (Lc) of the domain from the edge of the footing are chosen in a way such that (i) the regions of significant velocities are contained well within the chosen domain and (ii) further extension in the size of the domain hardly brings any change in the magnitude of the collapse load. By performing a number of trials, it was noted that (i) the depth (H) of the domain lies between 1.8d and 3.8d, and (ii) the horizontal extent (Lc) of the domain ranges between 3d and 7d. The higher values of \( L_c \) and \( H \) need to be chosen for greater values of \( \phi \); for \( \phi = 40^\circ \), \( H = 3.8d \) and \( L_c = 7d \). These chosen values of \( H \) and \( L_c \) were found to be generally adequate since the magnitudes of the nodal velocities become almost equal to zero much before the chosen domain boundaries. No velocity boundary constraints were imposed on ground surface. The nodes lying along the center line of the footing (MO) are restrained to move in the radial direction (\( u = 0 \)). As \( H \) and \( L_c \) are chosen to be extremely large, the vertical and the radial velocities along the boundaries OR and RP were specified to be zero.

5. Finite element meshes

The chosen domain has been discretized into a number of three-noded constant strain triangular elements and each node remains unique to a particular element. Velocity discontinuities are permitted along the interfaces of all the adjacent elements. At each node, there remain two basic unknowns, namely, horizontal velocity (u) and vertical velocity (v). The velocities are assumed to vary linearly within each element by using linear shape functions. Fig. 1(b) indicates the reinforcement nodes and the kinematic velocity boundary conditions along the interface of the reinforcement and the adjoining soil mass. No explicit elements are chosen to model the rigid footing. The nodes along the footing are restrained to move in the radial direction (\( u = 0 \)), and along these nodes, the resultant velocities are assumed to be vertical and uniform (\( v = V_0 \)), where \( V_0 \) is the velocity of footing at collapse. A typical finite element mesh for \( \phi = 30^\circ \) is depicted in Fig. 1(c); where the parameters \( E, N \) and \( D_c \) refer to the total number of (i) elements, (ii) nodes, and (iii) discontinuities, respectively.

6. Analysis

An upper bound axisymmetric finite elements limit analysis has been used by incorporating the plastic strains both within elements and along the velocity discontinuities. The axisymmetric formulation is simply a modification over the plane strain methodology presented earlier by Sloan (1989) and Sloan and Kleeman (1995). The present formulation is based on the application of Haar and von Karman (1909) hypothesis, that is, the hoop stress (\( \sigma_\theta \)) is kept closer to the minor principal stress (\( \sigma_3 \)) in an \( r-z \) plane. To obtain an upper bound solution it is required to construct a velocity field which satisfies (i) the compatibility condition of plastic strains within each element; (ii) kinematically admissible velocity discontinuity conditions; and (iii) the prescribed velocity boundary conditions. After constructing such a velocity field, the upper bound magnitude of the collapse load is obtained by equating the rate of the total work done by the external loads to the total internal power dissipation. The constraints which arise from the usage of the velocity discontinuities and the velocity boundary conditions become exactly the same as that given by Sloan and Kleeman (1995) and Kumar and Kouzer (2007), and these constraints are, therefore, not repeated herein. Following Bottero et al. (1980), the original Mohr–Coulomb yield function is linearized by using a regular yield polygon of p sides. Due to prior unknown magnitude of \( \sigma_3 \), the imposition of the von-Karman
condition ($\sigma_0 = \sigma_3$) is not directly possible. Three additional inequality constraints, that is, (i) $\sigma_0 \geq \sigma_1$, (ii) $\sigma_0 \geq \sigma_2$, and (iii) $\sigma_3 \leq \sigma_0$ were imposed; where $\sigma_3$ represents the minor (minimum) compressive normal stress at failure, that is, $\sigma_3 = 0.5 (1 - \sin \phi) \sigma_0 + 0.5 (1 - \sin \phi) \sigma_2 + c \cos \phi$. These constraints restrict the magnitude of $\sigma_0$ within a range of $\sigma_3$ and the maximum of $(\sigma_1, \sigma_2)$. This condition generates three additional yield functions in addition to the Mohr–Coulomb yield criterion. Each of this yield function is associated with one additional plastic multiplier rate as compared to an equivalent plane strain problem. Hence, for each element, the total number of plastic multiplier rates becomes equal to $p + 3$. The accuracy of the solution improves with an increase in the number of plastic multiplier rates becomes equal to $p + 3$. As the linearized form of the Mohr–Coulomb yield function is used in the present formulation, the first order derivative of the yield function with respect to stress variables, which are required while applying the associated flow rule, become eventually independent of the stresses. The equality constraints (Eqs. 2–4) which are generated by imposing the associated flow rule become a little different from that presented earlier by Sloan and Kleeman (1995) and Kumar and Kouzer (2007) for a typical plane strain problem. For the axisymmetric formulation, these constraints take the following form:

$$a_{11} x_1 - a_{12} x_2 = 0$$

where

$$a_{11} = \frac{1}{2A_e} \begin{bmatrix} z_{11} & 0 & z_{12} & 0 \\ 0 & r_{12} & 0 & r_{13} \\ r_{23} & z_{23} & 0 & z_{21} \\ \frac{1}{r} & 0 & \frac{1}{r} & 0 \end{bmatrix} \quad ; \quad z_{ij} = z_i - z_j ;$$

$$r_{ji} = r_j - r_i$$

and

$$A_e = \text{area of the triangular element and } r \text{ is radial distance between the axis of symmetry and centroid of the element.}$$

$$a_{12} = \begin{bmatrix} A_1 & \ldots & A_{12} & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 & \ldots & b_{13} \\ c_1 & \ldots & c_{12} & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{r} \\ \frac{1}{r} \end{bmatrix} \quad ; \quad (4x6)$$

$$x_1 = \left\{ u_1, v_1, u_2, v_2, u_3, v_3 \right\}^T \quad ; \quad -\infty \leq x_1 \leq \infty$$

$$x_2 = \left\{ \lambda_1, \ldots, \lambda_6, \lambda_{p+1}, \lambda_{p+2}, \lambda_{p+3} \right\}^T \quad ; \quad x_2 \geq 0$$

where $x_1$ and $x_2$ are the vectors containing nodal velocities and plastic multipliers for an element $e$. Following Kumar and Kouzer (2007), the objective function $(Q)$ is expressed as:

$$Q = \sum_{e=1}^{n_e} P_{1e} + \sum_{e=1}^{n_e} P_{2e} + \sum_{e=1}^{n_e} P_{3d} \quad ; \quad \text{where } P_{1e} \text{ and } P_{2e} \text{ represent the power dissipation within an element due to body forces and plastic straining of elements, respectively and } P_{3d} \text{ defines the power dissipation due to plastic shearing along the velocity discontinuity lines. The expressions for } P_{1e}, P_{2e} \text{ and } P_{3d} \text{ for the present axisymmetric formulation are given herein}$$

$$P_{1e} = c_1^T x_1; \quad \text{where } c_1^T = \left( \gamma A_e / 3 \pi \right) \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad (1x6)$$

$$P_{2e} = c_2^T x_2; \quad \text{where } c_2^T = 2c_2 \cos \phi (2 \pi r) A_e \begin{bmatrix} 1 & \ldots & 1 & 0 & 0 & 0.5 \end{bmatrix} \quad (1x(p+3))$$

$$P_{3d} = c_3^T x_3; \quad \text{where } c_3^T = (\pi r d L d c / 2) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad (1x4)$$

The upper bound limit analysis is finally formulated as a linear programming problem in which the magnitude of collapse load $(Q_a)$ needs to be minimized subject to a number of linear equality constraints. The optimization problem finally takes the following form:

Minimize: $Q$ \quad (6)

Subjected to:

$$A_{11} x_1 - A_{12} x_2 = B_1$$

$$A_{21} x_1 - A_{22} x_3 = B_2$$

$$A_{31} x_1 = B_3 \quad ; \quad x_2, x_3 \geq 0$$

where $(i)$ $x_1, x_2$ and $x_3$ are the global vectors corresponding to the local vectors $x_1, x_2$ and $x_3$, respectively, and (iii) $A_{11}$ and $A_{12}$ are the global vectors associated with the local coefficient matrices $a_{11}$ and $a_{12}$; the matrices $A_{21}, A_{22} and A_{31}$ become exactly the same as provided in Sloan and Kleeman (1995) and Kumar and Kouzer (2007) for the plane strain problem. For solving the problem, the computer code is written in MATLAB. In the present analysis, a fully rough interface is assumed to exist between (i) circular reinforcement sheet and adjoining soil mass, and (ii) footing surface and underlying soil medium.

7. Definition of the efficiency factor

The magnitude of the collapse load $(Q_a)$ per unit area of the circular footing, without any surcharge pressure, in the presence of the reinforcement has been expressed by using the following standard bearing capacity expression:

$$q_u = \frac{Q_a}{\pi(d/2)^2} = c N_c \eta_c + 0.5 \gamma d N_p \eta_p$$

where $N_c$ and $N_p$ are the bearing capacity factors, in the absence of any reinforcement, due to the components of soil cohesion and unit weight, respectively; for a soil with $\phi = 0^\circ$ and $m > 0$, $c$ in Eq. (7) refers to the cohesion value at ground surface. The factors $\eta_c$ and $\eta_p$ are termed as the efficiency factors due to the components of soil cohesion and unit weight, respectively; the values of these factors become equal to unity in the absence of any reinforcement. It needs to be mentioned that the bearing capacity factors themselves have been computed by considering the circular shape of the foundation.
and, therefore, no shape factors have been introduced in Eq. 7. Note that the bearing capacity factors and corresponding efficiency factors have been obtained by considering that the principle of superposition remains applicable. Since the principle of superposition results in a conservative estimate, the failure load thus obtained will always be safe (Davis and Booker, 1971; Bolton and Lau, 1993; Kumar and Sahoo, 2013). For computing $\eta_c$, the value of $\gamma$ has been kept equal to

\begin{align*}
\eta_c &= \frac{E}{N}
\end{align*}
zero, on the other hand, for determining \( \eta_\gamma \), the value of \( c \) is kept equal to zero.

8. Results and comparison

The computations were carried out for a single as well as for two layers of the reinforcement sheet. The following sections present the bearing capacity components of the circular footing resting over unreinforced soil as well as soil medium reinforced with an inclusion of single and double layer of geogrid sheets.

8.1. For an unreinforced soil medium

Table 1 presents the computed values of \( N_c \) and \( N_\gamma \) for a rough unreinforced circular foundation. It is observed that the bearing capacity factors increase continuously with an increase in the values of both \( \phi \) and \( m \). The values of the factors \( N_c \) and \( N_\gamma \) for different values of \( \phi \), obtained from the present analysis, were compared with (i) the solution of Erickson and Drescher (2002) based on FLAC, (ii) the solutions on the basis of the method of the stress characteristics given by De Simone (1985), Cassidy and Houlsby (2002) and Martin (2004, 2005), (iii) the lower and upper bound solutions given by Lyamin et al. (2007) based on the three-dimensional finite element limit analysis, and (iv) the lower bound limit analysis with finite elements and linear programming obtained by Kumar and Khatri (2011). Table 1 also provides the comparison of the obtained results with that reported in literature. The obtained results appear to be quite convincing as compared to different results from literature. As it was expected, the present values of the bearing capacity factors are found to be a little higher than the solutions obtained by using the method of the

![Fig. 2. The variation of the efficiency factor with \( h/d \) for a single layer of reinforcement in (a) sand, (b) \( c=\phi \) soil and (c) clay with \( \phi=0 \).](image)
stress characteristics. Note that the present $N_f$ solutions are found to be slightly lower (better) than the upper bound solution given by Lyamin et al. (2007) based on the three-dimensional finite element limit analysis. In Table 1, the obtained magnitudes of $N_c$ for $\phi=0^\circ$ but with different values of $m$, were also compared with (i) the solutions of Khatri and Kumar (2008), (ii) the solutions of Houlsby and Martin (2003) based on the method of the stress characteristics, and (iii) the upper bound solution of Kusakabe et al. (1986) based on rigid blocks mechanism. The present upper bound values of $N_c$ are found to be slightly lower than the upper bound solution given by Kusakabe et al. (1986). Overall, the values of the bearing capacity factors, $N_c$ and $N_p$, computed from the present analysis compare quite well with the different solutions reported from literature.

8.2. For a reinforced soil medium

For both single and double layers of the reinforcements sheets, the computations were carried out for (i) three different values of $\phi$, namely, $30^\circ$, $35^\circ$ and $40^\circ$ for computing $\eta_f$, (ii) three different values of $\phi$, namely, $10^\circ$, $20^\circ$ and $30^\circ$ for determining $\eta_f$; and (iii) three different values of $m$, namely, 0, 0.5 and 1 for computing $\eta_c$ with $\phi=0^\circ$.

8.2.1. Single layer of reinforcement

Fig. 2(a)–(c) presents the variation of $\eta_f$ and $\eta_c$ with respect to changes in $h/d$ with an inclusion of the single layer of reinforcement sheet. Fig. 2(a) and (b) provides the variation of $\eta_f$ and $\eta_c$ for different values of $\phi$, and Fig. 2(c) presents the variation of $\eta_c$ for different values of $m$ with $\phi=0^\circ$. These three figures give a clear impression that for each case there always exists a certain critical depth ($h_{cr}$) of the reinforcement layer corresponding to which the values of $\eta_f$ and $\eta_c$ become always the maximum. Beyond this $h_{cr}$, the confining effect of the reinforcement sheet does not impart any additional benefit on the increment of the magnitude of collapse load. The values of $h_{cr}$ along with corresponding maximum efficiency factors are exclusively provided in Table 2. It is observed that the value of $h_{cr}$ increases with an increase in $\phi$ but reduces with an increase in $m$. The peak values of the efficiency factors, associated with $h_{cr}$, increase continuously with increases in both $\phi$ and $m$. The critical depth ($h_{cr}$) of the reinforcement varies between (i) 0.25$d$ and 0.40$d$ in case of $\eta_f$, (ii) 0.20$d$ and 0.40$d$ in case of $\eta_c$ for $c-\phi$ soil mass, and (iii) 0.10$d$–0.175$d$ in case of $\eta_c$ for $\phi=0$. Corresponding to $h_{cr}$, the maximum values of $\eta_f$ and $\eta_c$ have been found to vary between (i) 1.58 and 1.89 in the case of $\eta_f$, (ii) 1.15 and 1.38 in the case of $\eta_c$ for $c-\phi$ soil mass, and (iii) 1.11 and 1.15 in the case of $\eta_c$ for undrained clay. The magnitude of $\eta_c$ remains greater than $\eta_f$; this can be attributed due to an additional effect of the confining (over-burden) stresses in increasing the shear resistance along the reinforcement-soil interface. Even in the case of cohesive-frictional soil, since the unit weight of soil mass is not equal to zero, while computing the total bearing capacity in the presence of reinforcements, one needs to use the terms containing $\eta_f$ as well as $\eta_c$. Since the factor $\eta_c$ remains smaller than $\eta_f$, the ratio of the total bearing capacity with the reinforcements to that without reinforcements will become obviously lower for a cohesive-frictional soil than that for a pure granular soil. It implies the relative advantage of employing

<table>
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<th>Type of soils</th>
<th>Layer of reinforcement sheets</th>
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<tbody>
<tr>
<td></td>
<td>Single</td>
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<td></td>
<td>Present analysis</td>
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<td></td>
<td>Present analysis</td>
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<tr>
<td></td>
<td>$h_{cr}/d$</td>
</tr>
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<td>Sand $\phi$</td>
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<tr>
<td>c–$\phi$ soil $\phi$</td>
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<tr>
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<td>40°</td>
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<tr>
<td>$\phi=0^\circ$ soil $m$</td>
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<td>0.5</td>
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*Lower bound limit analysis in conjunction with finite elements and linear programming.
the reinforcement in a granular soil as compared to that in a cohesive soil.

Fig. 2(a) and (b) illustrate the comparison of the present results with that obtained by Chakraborty and Kumar (2012) by using the lower bound limit analysis. Note that the present solution provides a little higher magnitudes of $\eta_\gamma$ and $\eta_c$ as compared to the corresponding lower bound solution, however, the difference between the two solutions in all the cases has been found to be only marginal.

Fig. 3(a)–(c) provides the variation of the efficiency factors with changes in the diameter ($D$) of the reinforcement sheets; these figures have been generated correspond to the critical position ($h_{cr}$) of the reinforcement. For different values of $\phi$, Fig. 3(a) provides the variation of $\eta_\gamma$ with $D/d$ for a cohesionless soil. Fig. 3(b) and (c) presents the variation of $\eta_c$ with $D/d$ (i) for different values of $\phi$ for a cohesive-frictional soil mass, and (ii) for different values of $m$ with $\phi=0$, respectively. It is observed that the values of $\eta_\gamma$ and $\eta_c$ increase continuously with an increase in $D/d$ up to a certain diameter of the reinforcement sheets, and beyond that there seems to be hardly any improvement in the bearing capacity with an increase in $D/d$; note that the improvement in the bearing capacity reduces very significantly when the diameter of the reinforcement sheet becomes smaller than that of the footing diameter. The diameter of the reinforcement sheet beyond which the increment in the efficiency factor almost ceases is termed as the optimum extent of the reinforcement sheet; this diameter of the sheet is referred to as $D_{opt}$. It is noted that the value of $D_{opt}$ increases continuously with an increase in $\phi$ but reduces on the other hand with an increase in $m$. It needs to be mentioned here that in all the cases the rate of the increment of the efficiency factors with an increase in $D/d$ becomes very small for $D/d > 1.5$; however, for soils with greater values of $\phi$, the improvement in the bearing capacity still continues but at a much lower rate. The value of $D_{opt}$ has been found to vary generally between (i) $2d$ and $3.55d$ for fully granular soils, (ii) $1.85d$ and $2.55d$ for $c-\phi$ soils and (iii) $1.05d$ and $1.35d$ for $\phi=0$ soil.
8.2.2. Two layers of reinforcement

In order to determine the maximum value of $\eta_\gamma$ and $\eta_c$, for two layers of reinforcement sheets, a number of independent computations were performed for various combinations of the values of $h_1/d$ and $h_2/d$. The variation of $\eta_\gamma$ and $\eta_c$ with $h_1/d$ and $h_2/d$ for different cases are being presented in Figs. 4–6; (i) Fig. 4 shows the variation of $\eta_\gamma$ with $h_1/d$ and $h_2/d$ for three different values of $\phi$, namely, 30°, 35° and 40°, (ii) Fig. 5 illustrates the variation of $\eta_c$ with $h_1/d$ and $h_2/d$ for three different values of $\phi$, namely, 10°, 20° and 30°, and (iii) Fig. 6 presents the variation of $\eta_c$ with $h_1/d$ and $h_2/d$ for three different values of $m$, namely, 0, 0.5 and 1.0. These figures were generated by performing extensive computation runs for several values of $h_2/d$ for a certain chosen value of $h_1/d$ constant. In all the cases, it is observed that there exists a critical value of $h_1$ and $h_2$, namely, $h_{1,cr}$ and $h_{2,cr}$, corresponding to which the magnitudes of the efficiency factor become always the maximum. For different types of soil, the maximum values of $\eta_\gamma$ and $\eta_c$ and corresponding values of $h_{1,cr}$ and $h_{2,cr}$ have been included in Table 2. It can be noted that in all the cases, as compared to a single layer of the reinforcement sheet, the improvement in the values of the bearing capacity factors becomes significantly higher in the case of two layers of reinforcement sheets; the difference between the two cases becomes especially quite predominant when the reinforcements are embedded in frictional soils with relatively higher values of $\phi$. For most of the cases, the values of $h_{1,cr}$ for two layers of the reinforcement are found to be slightly smaller than the corresponding values of $h_{cr}$ with a single layer of reinforcement. Note that in all the cases, both the layers of the reinforcement have been found to cause a significant effect on the values of $\eta_\gamma$ and $\eta_c$. In all the cases, beyond $h_{1,cr}$ and $h_{2,cr}$, the values of $\eta_\gamma$ and $\eta_c$ reduce continuously with a further increase in the values of $h_1$ and $h_2$. Note that for $h_1 > h_{1,cr}$, the values of $\eta_\gamma$ and $\eta_c$ reduce quite suddenly, and for which case, the effect of $h_2$ on the improvement in the bearing capacity has been found to be only very marginal.

Table 3 shows the comparison of the present results with that the experimental data reported in literature for circular model footings placed on sand which is reinforced with geogrid sheets. The experimental results are due to (i) Boushehrian and Hataf (2003) with $\phi=38°$ for a single as
well as two layers of the reinforcement sheets, and (ii) Lovisa et al. (2010) with \( \phi = 31^{\circ} \) for a single layer of the reinforcement. It can be seen that the results from the present analysis match reasonably well with the experimental data. Since it is known that the upper bound solution always provides higher magnitude of the collapse load than the true value (Sloan and Kleeman, 1995), in all the cases, the values of \( \eta_r \) from the present upper bound analysis have been found to be marginally greater than the corresponding experimental data; the difference between the two becomes especially more predominant for foundations embedded with two layers of reinforcement sheets. Due to lack of availability of the experimental data, the comparison of the obtained results with model tests’ results could not be made at present for reinforced earth foundations either in fully cohesive or in cohesive frictional soils.

Corresponding to the critical positions of the two reinforcements, Fig. 7(a)–(c) presents the variation of the efficiency factor with \( D/d \) for three different cases. Note that in the present analysis, the diameters of the two layers of the reinforcements are kept exactly the same. It is observed that similar to a single layer of reinforcement, the values of the efficiency factors increase with an increase in the diameter of the sheet up to a certain extent and thereafter no further increment in the efficiency factor occurs. The values of \( D_{\text{opt}}/d \) corresponding to the optimum values of \( h_{1_{\text{cr}}} \) and \( h_{2_{\text{cr}}} \) are exclusively provided in Table 2. The value of \( D_{\text{opt}}/d \) varies generally between (i) 3.15\( D \) and 3.80\( D \) for cohesionless soil, (ii) 2.10\( D \) and 4.10\( D \) for \( c/c_0 \) soils and (iii) 1.40\( D \) and 1.80\( D \) for \( \phi = 0 \) with varying cohesion. It is to be noted that the values of \( D_{\text{opt}}/d \) increase continuously with an increase in \( \phi \) but reduces with an increase in \( m \). It can also be seen that as compared to the single layer of the reinforcement, the values of \( D_{\text{opt}}/d \) are found to be greater for two layers of the reinforcements.

### 8.3. Nodal velocities patterns

The nodal velocity diagrams illustrate the magnitudes and the directions of soil movement at various points within soil mass. Figs. 8–10 depict the nodal velocities without and with the employment of single and two layers of reinforcement corresponding to optimum position and optimum diameter of the reinforcement sheet (s) for different types of soils; (i) Fig. 8 corresponds to a cohesionless soil with \( \phi = 30^{\circ} \), (ii) Fig. 9 illustrates the nodal velocities for a cohesive weightless medium with \( \phi = 30^{\circ} \), and (iii) Fig. 10 illustrates the nodal velocities for purely cohesive soil (\( \phi = 0^{\circ} \)) with \( m = 0 \). It can be noted that as compared to the velocities of the soil particles just beneath the footing edge, the magnitudes of the velocities along the ground surface near to the footing edge are found to be significantly higher. The velocity discontinuities are found to be prominent near the footing edge and these discontinuities

<table>
<thead>
<tr>
<th>Single layer of reinforcement</th>
<th>Double layers of reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi (^{\circ}) )</td>
<td>( h_{1d} )</td>
</tr>
<tr>
<td>Present work</td>
<td>31</td>
</tr>
</tbody>
</table>
| Lovisa et al. (2010) | 31 | 0.30 | 1.50 | \( ^a \) By using biaxial geogrid. | \( ^b \) By using woven geotextile.

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**Table 3**

A comparison of the obtained values of \( \eta_r \) with the experimental work of Boushehrian and Hataf (2003) and Lovisa et al. (2010).

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\[
\begin{array}{cccccc}
\phi (^{\circ}) & h_{1d} & \eta_r & \phi (^{\circ}) & h_{2d} & \eta_r \\
\hline
38 & 0.33 & 1.78 & 38 & 0.33 & 0.46 \\
31 & 0.30 & 1.62 & Boushehrian and Hataf (2003) & 38 & 0.33 & 0.46 \\
Lovisa et al. (2010) & 31 & 0.30 & 1.50 & Boushehrian and Hataf (2003) & 38 & 0.33 & 0.46 \\
\end{array}
\]

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**Fig. 7.** The variation of the efficiency factors with \( D/d \) for two reinforcement layers with optimum values of \( h_1 \) and \( h_2 \) in (a) sand, (b) \( c - \phi \) soil and (c) clay with \( \phi = 0 \).

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**Fig. 8.** The nodal velocity diagrams illustrate the magnitudes and the directions of soil movement at various points within soil mass. Figs. 8–10 depict the nodal velocities without and with the employment of single and two layers of reinforcement corresponding to optimum position and optimum diameter of the reinforcement sheet (s) for different types of soils; (i) Fig. 8 corresponds to a cohesionless soil with \( \phi = 30^{\circ} \), (ii) Fig. 9 illustrates the nodal velocities for a cohesive weightless medium with \( \phi = 30^{\circ} \), and (iii) Fig. 10 illustrates the nodal velocities for purely cohesive soil (\( \phi = 0^{\circ} \)) with \( m = 0 \). It can be noted that as compared to the velocities of the soil particles just beneath the footing edge, the magnitudes of the velocities along the ground surface near to the footing edge are found to be significantly higher. The velocity discontinuities are found to be prominent near the footing edge and these discontinuities...
gradually diminish continuously away from the footing edge. The region in which the magnitudes of the velocities becomes significant than the rest part of the soil domain is termed as the zone of influence. It is observed from the figures that for all the cases the extent of the zone of influence, especially in the vertical downward direction, becomes significantly greater for a reinforced soil mass than that compared to unreinforced soil. It is also observed that the presence of cohesion in $c - \phi$ soil enhances the size of the zone of influence as compared to cohesionless soil for the same value of $\phi$. The magnitudes of the velocities surrounding the footing edge are found to be generally greater for the reinforced soil bed in comparison to the unreinforced soil mass. It can be noted from Fig. 10 that the extent of the zone of influence for a purely cohesive soil is much lower than that in the case of cohesive-frictional and pure granular soils. Note that a significant amount of soil movement is being hindered by the inclusion of the reinforcement sheet (s); it is indicated with the formation of white zones underneath the reinforcement layer. It is observed that the extent of this white zone, below the reinforcement sheet, becomes greater below the lower reinforcement sheet as compared to the upper reinforcement sheet.

9. Remarks

The present study deals only with the ultimate limit state. The analysis presented in the current research cannot take into account the serviceability limit state. The stiffness parameters (in terms of Young’s modulus and Poisson ratio) of either the reinforcement sheet or soil mass cannot be incorporated in the analysis. The present analysis is, therefore, based on only the shear strength parameters of (i) soils, and (ii) reinforcement-soil interface. The settlements of the footings for a given applied load can be predicted by using the displacement based elasto-plastic finite element method in which case the complete constitutional model of the material needs to be incorporated before arriving at the load-settlement response of the footings till the ultimate failure. Note that in the elasto-plastic finite
element method, one may employ a certain settlement criterion for computing the bearing capacity. However, in the finite element limit analysis, no such criterion is needed.

10. Conclusions

The bearing capacity of a rigid circular rough footing placed over a soil mass reinforced with a single and a group of two layers of horizontal circular reinforcement sheets has been computed by using the upper bound finite element limit analysis in combination with linear optimization. The analysis is based on the assumption that the reinforcement sheets can resist axial tension but not the bending moment. The improvement in ultimate bearing capacity is presented in terms of the efficiency factors $\eta_\gamma$ and $\eta_\phi$, which need to be multiplied with the respective bearing capacity factors $N_\gamma$ and $N_\phi$. The magnitudes of $\eta_\gamma$ and $\eta_\phi$ increase continuously with increases in the friction angle ($\phi$) of the soil mass. The analysis clearly reveals that the inclusion of the reinforcement causes a significant increase in the bearing capacity especially when the soil medium is reinforced with two layers of the reinforcement sheets. For all the cases there exists a certain critical depth of the reinforcement sheet (s) corresponding to which the magnitudes of $\eta_\gamma$ and $\eta_\phi$ become always the maximum. The critical depth and the critical spacing between the upper and lower layer of reinforcement sheets have been established for three different types of soil media. The required diameter of the reinforcement sheet has been found to increase with an increase in the value of friction angle. It is found that for two layers of reinforcements embedded in sand, with the value of $\phi$ varying between 30° and 45°, corresponding to the critical position of the reinforcement, the optimum diameter of the circular reinforcement sheets lies within the range of 3.15$d$–3.80$d$. The results available from the analysis have been found to compare well with different theoretical and experimental data available from literature.

References


