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On the Essential Spectrum*

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1. INTRODUCTION

For bounded self-adjoint operators A and B in a Hilbert space, Weyl's theorem [1] states that $\sigma_e(A + B) = \sigma_e(A)$ if B is compact and σ_e denotes the essential spectrum of an operator (in this case just the limit-points of the spectrum). On the other hand it is an immediate corollary of the spectral theorem that if $\sigma_e(A + B) = \sigma_e(A)$ for all bounded self-adjoint operators A, then (the self-adjoint) B is compact. Recently there has been much interest concerning extensions and applications of Weyl's theorem to unbounded operators in a Banach space.

The basic contention of this note is that one cannot extend Weyl's theorem significantly beyond a (rather strong) relative compactness requirement on the perturbation B, in general. To this end we shall: (a) exhibit limitations to the extendability of Weyl's theorem; and (b) give an analogue of the above-mentioned corollary of the spectral theorem for non-self-adjoint operators in Hilbert space.

In Section 2 we make precise the different definitions of the essential spectrum; in Section 3 we examine the possibility of extending Weyl's theorem to B which are not relatively compact and show that the approach of Schechter [2, 3] cannot be pushed further on to A^n-compactness ($n > 2$), in general. However, for self-adjoint A this can be done, since then A-boundedness and A^m-compactness for some $m > 1$ implies A^q-compactness for any $q > 1$, as will be shown in Section 3. In Section 4 we give the generalization of the above-mentioned corollary of the spectral theorem.

Throughout, all operators are assumed to be densely defined in a Banach space.

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2. Several Definitions of the Essential Spectrum

For self-adjoint operators in a Hilbert space there seems to be only one reasonable way of defining the essential spectrum σ_e: the set σ_ℓ of limit-points of the spectrum (where eigenvalues are counted according to their multiplicity and hence infinite dimensional eigenvalues are included); i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity. Weyl's theorem [1] (e.g., [4], p. 367) states that for A and B self-adjoint and B compact, $\sigma_\ell(A + B) = \sigma_\ell(A)$ holds. For self-adjoint operators it is easy to show that σ_ℓ is the largest subset of the spectrum with this property; this motivates another definition of the essential spectrum (for an arbitrary operator); the largest subset of the spectrum remaining invariant under arbitrary compact perturbations, sometimes denoted as

$$w(A) = \bigcap_{B \text{ compact}} \sigma(A + B).$$

A third way of defining the essential spectrum of arbitrary operators (and this actually is the most convenient one) is to define it as the complement of the set of λ such that $\lambda - A$ has certain Fredholm-properties, as follows. An operator A is called normally solvable if A is closed and $R(A)$ is closed; if in addition $\alpha(A)$ or $\beta(A)$ (where $\alpha(A) = \dim N(A)$ and $\beta(A) = \operatorname{codim} R(A)$; here $N(A)$ is the null-space, $R(A)$ the range of A) is finite A is called a semi-Fredholm operator; if both $\alpha(A) < \infty$ and $\beta(A) < \infty$ A is called a Fredholm operator; if $\alpha(A) = \beta(A) < \infty$ A is called a Fredholm operator with index zero (index $(A) = \kappa(A) = \alpha(A) - \beta(A)$). Let $\Delta_i(T) = \{\lambda \mid \lambda - T \in F_i\}$ for $i = 1, 2, 3, 4, 5$ where F_1 is the set of normally solvable operators, F_2 the set of semi-Fredholm operators, F_3 the set of Fredholm operators, F_4 the set of Fredholm operators with index zero, F_5 the set of Fredholm operators with index zero where a deleted neighborhood of 0 is in the resolvent set. We define $\sigma_e^i(T)$ to be the complement of $\Delta_i(T)$ in the complex plane. Then $\sigma_e^i(T)$ is the essential spectrum of T according to (1) Goldberg [5], (2) Kato [6], (3) Wolf [7], (4) Schechter [2, 3], (5) Browder [8].

Remarks. We have $\sigma_e^i \subseteq \sigma_e^j$ for $i \leqslant j$; for T self-adjoint $\sigma_e^2 = \sigma_e^5$ (σ_e^1 does not contain isolated eigenvalues of infinite multiplicity). Coburn [9] showed that for T bounded and either hypernormal or Toeplitz, $\sigma_e^4 = \sigma_e^5$; Kato [6, Theorem IV 5.33] observes that T bounded and σ_e^2 countable implies $\sigma_e^2 = \sigma_e^5$. For non-closed T (which we will not consider in this paper) σ_e^1 is equal to the whole plane (hence this holds for σ_e^i for all i). Closed T with empty σ_e^3 are characterized by Kaashoek and Lay [10]. The definitions of σ_e^i are motivated by observing that $\sigma_e^5 = \sigma_\ell$, $\sigma_e^4 = w$ as defined above, σ_e^3 and σ_e^2 are preserved under compact perturbations, σ_e^1 consists of those λ

for which $\gamma(\lambda - T) = 0$ where γ denotes the minimum modulus (see [6]); σ_e^1, as used in [5], is actually σ_e^2 since the operators there involve ordinary derivatives.

One might wish to define the following two subclasses of semi-Fredholm operators, namely F_α the class of semi-Fredholm operators with $\alpha(T) < \infty$ and F_β the class of semi-Fredholm operators with $\beta(T) < \infty$, since certain results can be obtained only for these two subclasses considered separately (e.g. see Goldberg [5], Kato [6]). Thus one could define $\sigma_e^\alpha(T)$ and $\sigma_e^\beta(T)$ as the complements of

$$\Delta_\alpha(T) = \{\lambda \mid \lambda - T \text{ is in class } F_\alpha\} \quad \text{or} \quad \Delta_\beta(T) = \{\lambda \mid \lambda - T \text{ is of class } F_\beta\},$$

respectively. However, these sets do not satisfy the "symmetry condition" $\sigma_e(T') = \sigma_e(T)$ (in Banach spaces) or $\sigma_e(T^*) = \overline{\sigma_e(T)}$ (in Hilbert spaces) which seems to be natural and quite convenient. It can be verified that σ_e^i $(i = 1,\ldots, 5)$ do satisfy this condition.

If T is not closed, both T and (if B is any "reasonable" perturbation) $T + B$ will have $\sigma_e^i = C$ for $i = 1,\ldots, 5$. For this reason we will restrict ourselves to closed T, and B with $D(B) \supset D(T)$ and such that $T + B$ is closed (this for example is guaranteed if B is T-bounded with relative bound less than one).

3. Weyl's Theorem

Weyl's original theorem states that for bounded self-adjoint operators A and B in a Hilbert space, $\sigma_e(A + B) = \sigma_e(A)$ for every A if B is compact. As is well known, this result has been extended to arbitrary closed A in a Banach space and B which are A-compact. More precisely, for operators of this type, Weyl's theorem is valid for σ_e^i, $i = 2, 3, 4$, (for $i = 2, 3$ see Kato [6], for $i = 4$ see Schechter [3]). Under additional assumptions (e.g., if the complement of σ_e^4 is connected and neither $\rho(A)$ nor $\rho(A + B)$ is empty, where ρ is the resolvent set) Weyl's theorem holds for $i = 5$ (see Schechter [3], Browder [8]). Recently Schechter [2, 3] examined whether Weyl's theorem can be extended beyond A-compact perturbations and he found the following result.

Theorem S [2, 3]. *Let B be A^2-pseudo-compact and suppose $\Delta_3(A) \cap \Delta_3(A + B)$ is nonempty. Then $\sigma_e^3(A) = \sigma_e^3(A + B)$.*

Here B is called A^2-pseudo-compact if any sequence $\{x_n\} \subset D(A^2)$ with

$$\| x_n \| + \| Ax_n \| + \| Bx_n \| + \| BAx_n \| + \| A^2x_n \| \leqslant M$$

contains a subsequence $\{x_{n_k}\}$ such that $\{Bx_{n_k}\}$ converges.

REMARKS. (1) Although stated for the larger class of operators B which are only A^2-pseudo-compact, the hypotheses of Theorem S require that B be A^2-compact, i.e., any sequence $\{x_n\} \subset D(A^2)$ with $\| x_n \| + \| A^2 x_n \| \leqslant M$ contains a subsequence $\{x_{n_k}\}$ such that $\{Bx_{n_k}\}$ converges. This follows from the fact that B is A-bounded and A is A^2-bounded (since $D(B) \supset D(A) \supset D(A^2)$) and all three operators A, B and A^2 are closed). Therefore it seems that the class of A^2-pseudo-compact operators (and similarly the class of A-pseudo-compact operators of [2, 3]) is superfluous.

(2) For self-adjoint operators in Hilbert space Theorem S implies: If A is self-adjoint, B symmetric, A-bounded with relative bound less than one and A^2-compact, then $\sigma_e^3(A) = \sigma_e^3(A + B)$.

(3) Under additional assumptions, similar result for σ_e^4 and σ_e^5 are obtained in [2, 3]; e.g., replace 3 by 4 in Theorem S.

(4) Theorem S suggests a "theorem" of the following form: "Let $\Delta_3(A) \cap \Delta_3(A + B)$ be nonempty, B A^m-compact for some $m = 1, 2,\ldots$. Then $\sigma_e^3(A) = \sigma_e^3(A + B)$". The following example shows that this cannot be true (in general) for $m > 2$: Let A and B be the following (matrix-) operators in ℓ_2,

$$A = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & 0 & \\ & & 0 & 2 & & & \\ & & 1 & 0 & & & \\ & 0 & & & 0 & 3 & \\ & & & & 1 & 0 & \\ & & & & & & \ddots \end{bmatrix} = \left(\begin{bmatrix} 0 & n \\ 1 & 0 \end{bmatrix}\right), \qquad B = \left(\begin{bmatrix} 0 & -n \\ -1 & n \end{bmatrix}\right),$$

with

$$D(B) = D(A) = \left\{ x = \{\xi_n\} \,\middle|\, \sum_{k=1}^{\infty} | \xi_{2k-1} |^2 + \sum_{k=1}^{\infty} | k\xi_{2k} |^2 < \infty \right\}.$$

Since $A^3 = (\begin{bmatrix} 0 & n^2 \\ n & 0 \end{bmatrix})$, it is quite easy to see that B is A^3-compact; the following may be calculated: $\Delta_3(A) = C$, $\Delta_3(A + B) = C - \{0\}$. Hence $\Delta_3(A) \cap \Delta_3(A + B)$ is not empty but $\Delta_3(A) \neq \Delta_3(A + B)$, which implies $\sigma_e^3(A) \neq \sigma_e^3(A + B)$. Since A^3 is A^m-bounded for $m > 3$, this example serves to negate the above "theorem" for all $m > 2$.

(5) For self-adjoint operators Theorem S does extend. If A is self-adjoint, $D(B) \supset D(A)$ and $\Delta_3(A) \cap \Delta_3(A + B)$ is nonempty (hence B is A-bounded), then if B is A^p-compact for some real $p \geqslant 1$ it follows that $\sigma_e^3(A) = \sigma_e^3(A + B)$. The proof consists of proving that B actually is A^2-compact. This is a consequence of the following.

LEMMA 3.1. *Let A be a self-adjoint operator in the Hilbert space H, E_λ the corresponding spectral resolution: let B be A-bounded and such that the restriction B_N of B to $H_N = (E_N - E_{-N}) H$ is a compact operator for every positive N. Then B is $f(A)$-compact whenever $f(\lambda)$ is a (locally bounded) complex-valued function such that $| f(\lambda)/\lambda | \to \infty$ as $| \lambda | \to \infty$.*

PROOF. For bounded A and sufficiently large N we have $H_N = H$; hence if A is bounded, B is compact. In the following we may assume that A is unbounded. Equipped with the inner product

$$\langle x, y \rangle_f = \langle f(A)\, x, f(A)\, y \rangle + \langle x, y \rangle$$

and the corresponding norm, $D(f(A))$ is a Hilbert space and H_N are closed subspaces. The restriction B_N of B to H_N is compact in both the original norm and in the f-norm $\| \cdot \|_f$. Since B is A-bounded we have for $x \in D(A)$ that

$$\| B - B_N \|_f = \sup_{x \in H_N^\perp} \frac{\| Bx \|}{\| x \|_f} \leqslant \sup_{x \in H_N^\perp} \frac{C(\| Ax \| + \| x \|)}{(\| f(A)\, x \|^2 + \| x \|^2)^{1/2}} \to 0$$

as $N \to \infty$ (here $\| B - B_N \|_f$ is the operator norm of $B - B_N$ as an operator from $D(f(A)$ with the norm $\| \cdot \|_f$ into H). This implies that B is compact as an operator from $D(f(A))$ into H, i.e., B is $f(A)$-compact.

THEOREM 3.2. *Let A be self-adjoint, $D(B) \supset D(A)$ and $\Delta_3(A) \cap \Delta_3(A + B)$ not empty, and let B be $g(A)$-compact where g is some locally bounded (measurable) function. Then $\sigma_e{}^3(A) = \sigma_e{}^3(A + B)$.*

PROOF. B is clearly A-bounded (from $D(B) \supset D(A)$ together with $\Delta_3(A) \cap \Delta_3(A + B)$ not empty), and that B is A^2-compact then follows from Lemma 3.1 with $f(\lambda) = \lambda^2$, provided that B_N is compact for all positive N. The latter may be seen as follows: let $\{x_n\}$ be a sequence in H_N with $\| x_n \| \leqslant 1$; then by the spectral theorem:

$$\| g(A)\, x_n \|^2 + \| x_n \|^2 \leqslant \sup_{\lambda \in [-N, N]} | g(\lambda) |^2 + 1 \leqslant C_N ,$$

so that a subsequence $\{Bx_{n_k}\}$ converges due to the $g(A)$-compactness of B. Thus $\sigma_e{}^3(A) = \sigma_e{}^3(A + B)$ by Theorem S.

Extensions of this result to $\sigma_e{}^4$ and $\sigma_e{}^5$ are available under the hypotheses of [2, 3]. We notice that neither B nor $A + B$ are required to be self-adjoint.

4. A CONVERSE OF WEYL'S THEOREM

The theorem of v. Neumann [13] (e.g. Riesz-Sz.-Nagy [4], p. 367) is a converse of Weyl's theorem and asserts that if two bounded self-adjoint operators A and A' have the same essential spectrum, then there is an operator A'' unitarily equivalent to A' such that $A''' = A - A''$ is a compact operator. As an immediate consequence of this theorem one has the corollary: If B is self-adjoint and $\sigma_e(A) = \sigma_e(A + B)$ for all bounded self-adjoint operators A, then B is compact. This corollary is also an immediate consequence of the spectral theorem for self-adjoint operators; for $A = 0$ one has $\sigma_e(B) = \sigma_e(0) = \{0\}$ and every self-adjoint operator with essential spectrum $\{0\}$ is compact. (From this corollary we see that for preservation of the essential spectrum there is no (general) weaker requirement than compactness; all weaker requirements must depend on the unperturbed operator, as for example relative compactness in the case of unbounded operators).

A strict generalization of v. Neumann's theorem to non-self-adjoint operators is not possible, since this would imply that from $\sigma_e(B) = 0$ it would follow that B is compact; in ℓ_2, $B = (\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right])$ is certainly not compact even though $\sigma_e^1 = \cdots = \sigma_e^5 = \{0\}$. However the above-mentioned corollary can be generalized as follows: this possibility was suggested to one of the authors by P. Rejto.

THEOREM 4.1. *Let B be a bounded operator in a Hilbert space, $\sigma_e^i(A) = \sigma_e^i(A + B)$ for all bounded operators A. Then B is compact. Here $i = 2, 3, 4,$ or 5.*

PROOF. We write σ_e for σ_e^i. Applying the assumption to $A = 0$, $A = B^*$ and $A = -B^*$ we get:

$$\{0\} = \sigma_e(0) = \sigma_e(B) = \overline{\sigma_e(B)} = \sigma_e(B^*) = \sigma_e(B + B^*),$$

$$\{0\} = \sigma_e(B) = \sigma_e(-B) = \sigma_e(-B^*) = \sigma_e(B - B^*).$$

Hence the bounded self-adjoint operators $B + B^*$ and $i(B - B^*)$ have essential spectrum $\{0\}$. This implies that $B + B^*$ and $i(B - B^*)$ are compact, hence B is compact.

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