On the Essential Spectrum*

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1. Introduction

For bounded self-adjoint operators $A$ and $B$ in a Hilbert space, Weyl's theorem [1] states that $\sigma_+(A + B) = \sigma_+(A)$ if $B$ is compact and $\sigma_+$ denotes the essential spectrum of an operator (in this case just the limit-points of the spectrum). On the other hand it is an immediate corollary of the spectral theorem that if $\sigma_+(A + B) = \sigma_+(A)$ for all bounded self-adjoint operators $A$, then (the self-adjoint) $B$ is compact. Recently there has been much interest concerning extensions and applications of Weyl's theorem to unbounded operators in a Banach space.

The basic contention of this note is that one cannot extend Weyl's theorem significantly beyond a (rather strong) relative compactness requirement on the perturbation $B$, in general. To this end we shall: (a) exhibit limitations to the extendability of Weyl's theorem; and (b) give an analogue of the above-mentioned corollary of the spectral theorem for non-self-adjoint operators in Hilbert space.

In Section 2 we make precise the different definitions of the essential spectrum; in Section 3 we examine the possibility of extending Weyl's theorem to $B$ which are not relatively compact and show that the approach of Schechter [2, 3] cannot be pushed further on to $A^n$-compactness ($n > 2$), in general. However, for self-adjoint $A$ this can be done, since then $A$-boundedness and $A^m$-compactness for some $m > 1$ implies $A^n$-compactness for any $q > 1$, as will be shown in Section 3. In Section 4 we give the generalization of the above-mentioned corollary of the spectral theorem.

Throughout, all operators are assumed to be densely defined in a Banach space.

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2. Several Definitions of the Essential Spectrum

For self-adjoint operators in a Hilbert space there seems to be only one reasonable way of defining the essential spectrum \( \sigma_e \): the set \( \sigma_e \) of limit-points of the spectrum (where eigenvalues are counted according to their multiplicity and hence infinite dimensional eigenvalues are included); i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity. Weyl's theorem [1] (e.g., [4], p. 367) states that for \( A \) and \( B \) self-adjoint and \( B \) compact, \( \sigma_e(A + B) = \sigma_e(A) \) holds. For self-adjoint operators it is easy to show that \( \sigma_e \) is the largest subset of the spectrum with this property; this motivates another definition of the essential spectrum (for an arbitrary operator); the largest subset of the spectrum remaining invariant under arbitrary compact perturbations, sometimes denoted as

\[
\nu(A) = \bigcap_{B \text{ compact}} \sigma(A + B).
\]

A third way of defining the essential spectrum of arbitrary operators (and this actually is the most convenient one) is to define it as the complement of the set of \( \lambda \) such that \( \lambda - A \) has certain Fredholm-properties, as follows. An operator \( A \) is called normally solvable if \( A \) is closed and \( R(A) \) is closed; if in addition \( \alpha(A) \) or \( \beta(A) \) (where \( \alpha(A) = \dim N(A) \) and \( \beta(A) = \text{codim} R(A) \); here \( N(A) \) is the null-space, \( R(A) \) the range of \( A \)) is finite \( A \) is called a semi-Fredholm operator; if both \( \alpha(A) < \infty \) and \( \beta(A) < \infty \) \( A \) is called a Fredholm operator; if \( \alpha(A) = \beta(A) < \infty \) \( A \) is called a Fredholm operator with index zero (index \( \Delta(A) = \kappa(A) = \alpha(A) - \beta(A) \)). Let \( \Delta_i(T) = \{ \lambda \mid \lambda - T \in F_i \} \) for \( i = 1, 2, 3, 4, 5 \) where \( F_1 \) is the set of normally solvable operators, \( F_2 \) the set of semi-Fredholm operators, \( F_3 \) the set of Fredholm operators, \( F_4 \) the set of Fredholm operators with index zero, \( F_5 \) the set of Fredholm operators with index zero where a deleted neighborhood of 0 is in the resolvent set. We define \( \sigma_e(T) \) to be the complement of \( \Delta_i(T) \) in the complex plane. Then \( \sigma_e(T) \) is the essential spectrum of \( T \) according to (1) Goldberg [5], (2) Kato [6], (3) Wolf [7], (4) Schechter [2, 3], (5) Browder [8].

Remarks. We have \( \sigma_e \subseteq \sigma_e \) for \( i < j \); for \( T \) self-adjoint \( \sigma_e \cap \sigma_e^2 = \sigma_e^5 \) (\( \sigma_e^1 \) does not contain isolated eigenvalues of infinite multiplicity). Coburn [9] showed that for \( T \) bounded and either hypernormal or Toeplitz, \( \sigma_e \cap \sigma_e^4 = \sigma_e^5 \); Kato [6, Theorem IV 5.33] observes that \( T \) bounded and \( \sigma_e \cap \sigma_e^3 \) countable implies \( \sigma_e \cap \sigma_e^2 = \sigma_e^5 \). For non-closed \( T \) (which we will not consider in this paper) \( \sigma_e^1 \) is equal to the whole plane (hence this holds for \( \sigma_e \) for all \( i \)). Closed \( T \) with empty \( \sigma_e \) are characterized by Kaashoek and Lay [10]. The definitions of \( \sigma_e \) are motivated by observing that \( \sigma_e = \nu, \sigma_e \cap \sigma_e \cap \sigma_e^2 = \nu \) as defined above, \( \sigma_e \) and \( \sigma_e \) are preserved under compact perturbations, \( \sigma_e \) consists of those \( \lambda \)
for which $\gamma(\lambda - T) = 0$ where $\gamma$ denotes the minimum modulus (see [6]);
$s_\varepsilon^1$, as used in [5], is actually $s_\varepsilon^2$ since the operators there involve
ordinary derivatives.

One might wish to define the following two subclasses of semi-Fredholm operators, namely $F_\alpha$, the class of semi-Fredholm operators with $\alpha(T) < \infty$ and $F_\beta$ the class of semi-Fredholm operators with $\beta(T) < \infty$, since certain
results can be obtained only for these two subclasses considered separately (e.g. see Goldberg [5], Kato [6]). Thus one could define $s_\varepsilon^\alpha(T)$ and $s_\varepsilon^\beta(T)$ as
the complements of

$$
\Delta_\alpha(T) = \{ \lambda \mid \lambda - T \text{ is in class } F_\alpha \} \quad \text{or} \quad \Delta_\beta(T) = \{ \lambda \mid \lambda - T \text{ is of class } F_\beta \},
$$

respectively. However, these sets do not satisfy the “symmetry condition”
$s_\varepsilon(T') = s_\varepsilon(T)$ (in Banach spaces) or $s_\varepsilon(T^*) = \overline{s_\varepsilon(T)}$ (in Hilbert spaces)
which seems to be natural and quite convenient. It can be verified that
$s_\varepsilon^i (i = 1, \ldots, 5)$ do satisfy this condition.

If $T$ is not closed, both $T$ and (if $B$ is any “reasonable” perturbation) $T + B$ will have $s_\varepsilon^i = C$ for $i = 1, \ldots, 5$. For this reason we will restrict ourselves
to closed $T$, and $B$ with $D(B) \supset D(T)$ and such that $T + B$ is closed (this
for example is guaranteed if $B$ is $T$-bounded with relative bound less than
one).

3. Weyl’s Theorem

Weyl’s original theorem states that for bounded self-adjoint operators $A$
and $B$ in a Hilbert space, $s_\varepsilon(A + B) = s_\varepsilon(A)$ for every $A$ if $B$ is compact.
As is well known, this result has been extended to arbitrary closed $A$ in a
Banach space and $B$ which are $A$-compact. More precisely, for operators of
this type, Weyl’s theorem is valid for $s_\varepsilon^i, i = 2, 3, 4, (i = 2, 3$ see Kato [6],
for $i = 4$ see Schechter [3]). Under additional assumptions (e.g., if the com-
plement of $s_\varepsilon^4$ is connected and neither $\rho(A)$ nor $\rho(A + B)$ is empty, where $\rho$
is the resolvent set) Weyl’s theorem holds for $i = 5$ (see Schechter [3],
Browder [8]). Recently Schechter [2, 3] examined whether Weyl’s theorem
can be extended beyond $A$-compact perturbations and he found the following
result.

**Theorem S** [2, 3]. Let $B$ be $A^2$-pseudo-compact and suppose
$\Delta_\alpha(A) \cap \Delta_\beta(A + B)$ is nonempty. Then $s_\varepsilon^\beta(A) = s_\varepsilon^\beta(A + B)$.

Here $B$ is called $A^2$-pseudo-compact if any sequence $\{x_n\} \subset D(A^2)$ with

$$
\|x_n\| + \|Ax_n\| + \|Bx_n\| + \|BAx_n\| + \|A^2x_n\| \leq M
$$

contains a subsequence $\{x_{n_k}\}$ such that $\{Bx_{n_k}\}$ converges.
Remarks. (1) Although stated for the larger class of operators $B$ which are only $A^2$-pseudo-compact, the hypotheses of Theorem S require that $B$ be $A^2$-compact, i.e., any sequence $\{x_n\} \subset D(A^2)$ with $\|x_n\| + \|A^2x_n\| \leq M$ contains a subsequence $\{x_{n_k}\}$ such that $\{Bx_{n_k}\}$ converges. This follows from the fact that $B$ is $A$-bounded and $A$ is $A^2$-bounded (since $D(B) \supset D(A) \supset D(A^2)$ and all three operators $A$, $B$ and $A^2$ are closed). Therefore it seems that the class of $A^2$-pseudo-compact operators (and similarly the class of $A$-pseudo-compact operators of [2, 3]) is superfluous.

(2) For self-adjoint operators in Hilbert space Theorem S implies: If $A$ is self-adjoint, $B$ symmetric, $A$-bounded with relative bound less than one and $A^2$-compact, then $\sigma_{e}^3(A) = \sigma_{e}^3(A + B)$.

(3) Under additional assumptions, similar result for $\sigma_e^4$ and $\sigma_e^5$ are obtained in [2, 3]; e.g., replace 3 by 4 in Theorem S.

(4) Theorem S suggests a "theorem" of the following form: "Let $\Delta_{3}(A) \cap \Delta_{3}(A + B)$ be nonempty, $B$ $A^m$-compact for some $m = 1, 2, \ldots$. Then $\sigma_{e}^3(A) = \sigma_{e}^3(A + B)$". The following example shows that this cannot be true (in general) for $m > 2$: Let $A$ and $B$ be the following (matrix-) operators in $\ell_2$,

$$A = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 2 & 1 \\
0 & 0 & 3 \\
0 & 1 & 0
\end{pmatrix} = \left(\begin{bmatrix} 0 & n \\ 1 & 0 \end{bmatrix}\right), \quad B = \left(\begin{bmatrix} 0 & -n \\ -1 & n \end{bmatrix}\right),$$

with

$$D(B) = D(A) = \left\{ x = \{\xi_n\} \mid \sum_{k=1}^{\infty} |\xi_{2k-1}|^2 + \sum_{k=1}^{\infty} |k\xi_{2k}|^2 < \infty \right\}.$$ 

Since $A^3 = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$, it is quite easy to see that $B$ is $A^2$-compact; the following may be calculated: $\Delta_{3}(A) = C$, $\Delta_{3}(A + B) = C - \{0\}$. Hence $\Delta_{3}(A) \cap \Delta_{3}(A + B)$ is not empty but $\Delta_{3}(A) \neq \Delta_{3}(A + B)$, which implies $\sigma_{e}^3(A) \neq \sigma_{e}^3(A + B)$. Since $A^3$ is $A^m$-bounded for $m > 3$, this example serves to negate the above "theorem" for all $m > 2$.

(5) For self-adjoint operators Theorem S does extend. If $A$ is self-adjoint, $D(B) \supset D(A)$ and $\Delta_{3}(A) \cap \Delta_{3}(A + B)$ is nonempty (hence $B$ is $A$-bounded), then if $B$ is $A^p$-compact for some real $p \geq 1$ it follows that $\sigma_{e}^3(A) = \sigma_{e}^3(A + B)$. The proof consists of proving that $B$ actually is $A^2$-compact. This is a consequence of the following.
Lemma 3.1. Let \( A \) be a self-adjoint operator in the Hilbert space \( H \), \( E_\lambda \) the corresponding spectral resolution: let \( B \) be \( A \)-bounded and such that the restriction \( B_N \) of \( B \) to \( H_N = (E_N - E_{-N}) H \) is a compact operator for every positive \( N \). Then \( B \) is \( f(A) \)-compact whenever \( f(\lambda) \) is a (locally bounded) complex-valued function such that \( |f(\lambda)|/\lambda \to \infty \) as \( |\lambda| \to \infty \).

Proof. For bounded \( A \) and sufficiently large \( N \) we have \( H_N = H \); hence if \( A \) is bounded, \( B \) is compact. In the following we may assume that \( A \) is unbounded. Equipped with the inner product

\[
\langle x, y \rangle_f = \langle f(A)x, f(A)y \rangle + \langle x, y \rangle
\]

and the corresponding norm, \( D(f(A)) \) is a Hilbert space and \( H_N \) are closed subspaces. The restriction \( B_N \) of \( B \) to \( H_N \) is compact in both the original norm and in the \( f \)-norm \( \| \cdot \|_f \). Since \( B \) is \( A \)-bounded we have for \( x \in D(A) \) that

\[
\| B - B_N \|_f = \sup_{x \in H_N} \frac{\| Bx \|}{\| x \|_f} \leq \sup_{x \in H_N} \frac{C\| Ax \| + \| x \|}{\| f(A)x \| + \| x \|_f} \to 0
\]

as \( N \to \infty \) (here \( \| B - B_N \|_f \) is the operator norm of \( B - B_N \) as an operator from \( D(f(A)) \) into \( H \)). This implies that \( B \) is compact as an operator from \( D(f(A)) \) into \( H \), i.e., \( B \) is \( f(A) \)-compact.

Theorem 3.2. Let \( A \) be self-adjoint, \( D(B) \supset D(A) \) and \( \Delta_d(A) \cap \Delta_d(A + B) \) not empty, and let \( B \) be \( g(A) \)-compact where \( g \) is some locally bounded (measurable) function. Then \( \sigma_e^d(A) = \sigma_e^d(A + B) \).

Proof. \( B \) is clearly \( A \)-bounded (from \( D(B) \supset D(A) \) together with \( \Delta_d(A) \cap \Delta_d(A + B) \) not empty), and that \( B \) is \( A^2 \)-compact then follows from Lemma 3.1 with \( f(\lambda) = \lambda^2 \), provided that \( B_N \) is compact for all positive \( N \). The latter may be seen as follows: let \( \{x_n\} \) be a sequence in \( H_N \) with \( \| x_n \| \leq 1 \); then by the spectral theorem:

\[
\| g(A)x_n \|^2 + \| x_n \|^2 \leq \sup_{\lambda \in [-N,N]} |g(\lambda)|^2 + 1 \leq C_N,
\]

so that a subsequence \( \{Bx_{n_k}\} \) converges due to the \( g(A) \)-compactness of \( B \). Thus \( \sigma_e^d(A) = \sigma_e^d(A + B) \) by Theorem 5.

Extensions of this result to \( \sigma_e^4 \) and \( \sigma_e^5 \) are available under the hypotheses of [2, 3]. We notice that neither \( B \) nor \( A + B \) are required to be self-adjoint.

4. A Converse of Weyl’s Theorem

The theorem of v. Neumann [13] (e.g. Riesz-Sz.-Nagy [4], p. 367) is a converse of Weyl’s theorem and asserts that if two bounded self-adjoint
operators \( A \) and \( A' \) have the same essential spectrum, then there is an operator \( A'' \) unitarily equivalent to \( A' \) such that \( A'' - A - A'' \) is a compact operator. As an immediate consequence of this theorem one has the corollary: If \( B \) is self-adjoint and \( \sigma_e(A) = \sigma_e(A + B) \) for all bounded self-adjoint operators \( A \), then \( B \) is compact. This corollary is also an immediate consequence of the spectral theorem for self-adjoint operators; for \( A = 0 \) one has \( \sigma_e(B) = \sigma_e(0) = \{0\} \) and every self-adjoint operator with essential spectrum \( \{0\} \) is compact. (From this corollary we see that for preservation of the essential spectrum there is no (general) weaker requirement than compactness; all weaker requirements must depend on the unperturbed operator, as for example relative compactness in the case of unbounded operators).

A strict generalization of v. Neumann's theorem to non-self-adjoint operators is not possible, since this would imply that from \( \sigma_e(B) = 0 \) it would follow that \( B \) is compact; in \( \ell_2, B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) is certainly not compact even though \( \sigma_e^1 = \cdots = \sigma_e^5 = \{0\} \). However the above-mentioned corollary can be generalized as follows: this possibility was suggested to one of the authors by P. Rejto.

**Theorem 4.1.** Let \( B \) be a bounded operator in a Hilbert space, \( \sigma_e^i(A) = \sigma_e^i(A + B) \) for all bounded operators \( A \). Then \( B \) is compact. Here \( i = 2, 3, 4, \) or \( 5 \).

**Proof.** We write \( \sigma_e \) for \( \sigma_e^i \). Applying the assumption to \( A = 0, A = B^* \) and \( A = -B^* \) we get:

\[
\begin{align*}
\{0\} &= \sigma_e(0) = \sigma_e(B) = \overline{\sigma_e(B^*)} = \sigma_e(B + B^*), \\
\{0\} &= \sigma_e(B) = \sigma_e(-B) = \sigma_e(-B^*) = \sigma_e(B - B^*).
\end{align*}
\]

Hence the bounded self-adjoint operators \( B + B^* \) and \( i(B - B^*) \) have essential spectrum \( \{0\} \). This implies that \( B + B^* \) and \( i(B - B^*) \) are compact, hence \( B \) is compact.

**References**

ON THE ESSENTIAL SPECTRUM