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Rational quantum integrable systems of D_N type with polarized spin reversal operators

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Abstract

We study the spin Calogero model of D_N type with polarized spin reversal operators, as well as its associated spin chain of Haldane–Shastry type, both in the antiferromagnetic and ferromagnetic cases. We compute the spectrum and the partition function of the former model in closed form, from which we derive an exact formula for the chain's partition function in terms of products of partition functions of Polychronakos–Frahm spin chains of type A. Using a recursion relation for the latter partition functions that we derive in the paper, we are able to numerically evaluate the partition function, and thus the spectrum, of the D_N-type spin chain for relatively high values of the number of spins N. We analyze several global properties of the chain's spectrum, such as the asymptotic level density, the distribution of consecutive spacings of the unfolded spectrum, and the average degeneracy. In particular, our results suggest that this chain is invariant under a suitable Yangian group, and that its spectrum coincides with that of a Yangian-invariant vertex model with linear energy function and dispersion relation.

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1. Introduction

In a recent paper, a novel type of spin Calogero models and their associated spin chains of Haldane–Shastry type was introduced [1]. The distinguishing feature of these models is that they are constructed using a new representation of the Weyl group of the BC_N root system, obtained by replacing the standard spin reversal operators by an arbitrarily polarized version thereof. As shown in the latter reference, these models are exactly solvable for all such representations and, in particular, the partition function of the spin chains can be exactly computed using Polychronakos's freezing trick [2,3]. In this paper, we shall extend the above results to spin Calogero models and their corresponding spin chains of Haldane–Shastry type based on the D_N root system.

In order to present our work in an appropriate context, let us briefly recall the origin and significance of the latter models. The Haldane–Shastry (HS) spin chain, introduced independently by these authors in the late eighties [4,5], is perhaps the best known example of an exactly solvable one-dimensional lattice model with long-range interactions. More precisely, this model describes a circular array of equispaced spins with two-body interactions inversely proportional to the square of the (chord) distance between the spins. The motivation for introducing this chain was the construction of a simple model with an exact ground state given by the $U \to \infty$ limit of Gutzwiller's variational wavefunction for the ground state of the one-dimensional Hubbard model [6–8]. Over the years, the HS chain has appeared in many areas of interest both in Physics and Mathematics, such as fractional statistics and one-dimensional anyons [9–12], quantum entanglement [13], characterization of integrability vs. quantum chaos [14–17], quantum integrability via the asymptotic Bethe ansatz [18–20], Yangian quantum groups [12,21–23], and conformal field theory [18,24–26].

One of the key properties of the HS chain—already noted by Haldane and Shastry in their original papers—is its intimate connection with the scalar (trigonometric) Sutherland model [27,28]. This connection was subsequently elucidated by Polychronakos in Ref. [2], who showed how to derive the HS chain from the spin Sutherland model [29–31] by a technique that he called the "freezing trick". The main idea behind this technique is to note that when the coupling constant in the spin Sutherland model goes to infinity the particles tend to concentrate on the coordinates of the (essentially unique) minimum of the scalar part of the interaction potential, which are precisely the sites of the HS chain. Thus, in this limit the dynamical and the spin degrees of freedom decouple, and the latter are governed by the chain's Hamiltonian. Using this idea it is straightforward, for instance, to obtain the first integrals of the HS chain from their well-known counterparts for the spin Sutherland model. In fact, Polychronakos showed that applying the same procedure to the (rational) Calogero model [32] and its spin version [31] one obtains an integrable chain with non-equispaced sites and long-range interactions inversely proportional to the distance between the spins [2]. The spectrum of this chain—known in the literature as the Polychronakos–Frahm (PF) chain—was first studied numerically by Frahm [33] and then exactly computed by Polychronakos [3], who derived an exact formula for the partition function by means of the freezing trick. On the other hand, the partition function of the HS chain was only evaluated more than a decade later by some of the authors [14].

Both the Sutherland and Calogero models (and their corresponding HS and PF spin chains) are associated with the A_{N-1} root system, where N is the number of particles. Indeed, in these models the interactions only depend on the difference between the coordinates, and the spin operators appearing in the Hamiltonian are permutation operators, and thus generate a realization of the Weyl group of A_{N-1} type. In fact, there are versions of the Sutherland and Calogero

models associated to any (extended) classical root system [34]. Among these systems, those of BC_N, B_N, C_N and D_N type are by far the most studied in the literature, since they make it possible to construct integrable models with an arbitrary number of particles. By applying the freezing trick to the spin version of these models one obtains the corresponding generalizations of the HS and PF chains, that we shall collectively refer to as spin chains of HS type [35–41]. One of the fundamental features of the BC_N root system and its B_N and C_N degenerations is the fact that its Weyl algebra contains a family of reflection operators S_i ($i = 1, \ldots, N$) satisfying $S_i^2 = 1$. (In the case of the D_N root system, the Weyl group only contains products $S_i S_j$ with $i \neq j$.) In the spin chains studied in Refs. [35–41], the operators S_i are represented by spin reversal operators P_i (acting on the Hilbert space of the i-th particle), but this is by no means the only possible choice. As a matter of fact, in the novel version of the spin Calogero model of BC_N type and its corresponding (PF) chain introduced in Ref. [1], the operators S_i are represented instead by arbitrarily polarized spin reversal operators (PSRO) $P_i^{(m_1,m_2)}$, which act as the identity on the first m_1 elements of the spin basis and as minus the identity on the rest. These operators are equivalent under a similarity transformation to the usual spin reversal operators P_i only when $m_1 = m_2$ or $m_1 = m_2 \pm 1$, i.e., when there is minimal polarization. For the remaining values of the discrete parameters m_1 and m_2, the systems constructed in the latter reference differ from the standard Calogero and PF models of BC_N-type. In particular, when m_1 or m_2 are zero, the corresponding spin dynamical model reduces to the su(m)-invariant extension of the Calogero model studied by Simons and Altshuler [42]; see also [43].

In this paper we introduce the spin Calogero model of D_N-type with PSRO and its corresponding spin chain of HS type, i.e., the PF chain of D_N type with PSRO. As explained in Ref. [39], these models are singular limits of their corresponding BC_N counterparts, so that their spectrum cannot be obtained by setting to zero the parameter β in the latter models (cf. Eqs. (2.5) and (2.14)). This is also apparent at the level of the Hilbert space, which is the direct sum of the Hilbert spaces of two BC_N models with opposite chiralities. Thus, the models studied in this paper are not limiting cases of their BC_N versions in Ref. [1].

Our main result is the derivation of a closed-form expression for the partition function of the PF chain of D_N type with PSRO in terms of products of partition functions of type-A PF chains. Our approach is based on the computation of the spectrum and partition function of the corresponding spin Calogero model, from which the chain's partition function follows by a standard freezing trick argument. The structure of this partition function turns out to be more involved than that of its BC_N counterpart. In particular, it is not manifest that it is a polynomial in $q \equiv \mathrm{e}^{-1/(k_B T)}$, as follows from the freezing trick. Using the explicit expression for the partition function, we shall study several global properties of the chain's spectrum, such as the behavior of the level density and the average degeneracy when the number of spins tends to infinity. In particular, the fact that the number of distinct levels grows polynomially with the number of spins suggests that this model is isospectral to a Yangian-invariant vertex model of the kind studied in Ref. [44].

The paper is organized as follows. In Section 2 we recall the definition and main properties of the polarized spin reversal operators $P_i^{(m_1,m_2)}$, and construct the Hamiltonians of the D_N-type spin Calogero model with PSRO and its associated spin chain. Section 3 is devoted to the derivation of the closed-form expression of the chain's partition function, as explained in the previous paragraph. Using this expression, in Section 4 we analyze several global properties of the spectrum, providing strong numerical evidence of the Gaussian character of the level density when the number of spins is large enough. In Section 5 we extend the above results to the ferromagnetic version of the models under consideration. The paper ends with a brief section summarizing our

main results and presenting our conclusions, and a short technical Appendix establishing a useful recursion relation for the partition function of the PF chain of type A_{N-1}.

2. Construction of the models

For the purpose of describing the D_N-type Calogero model with polarized spin reversal operators, it is convenient to briefly summarize the construction of its BC_N counterpart [1]. To this end, let

$$\mathcal{S} = \big\langle |s_1, \ldots, s_N\rangle \big| s_i \in \{1, 2, \ldots, m\}\big\rangle \tag{2.1}$$

denote the internal spin space for N particles. As usual, the action of the spin exchange operator P_{ij} on $\mathcal{S}$ is defined as

$$P_{ij}|s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N\rangle = |s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N\rangle\,. \tag{2.2}$$

Let us denote the PSRO associated with the i-th particle as $P_i^{(m_1,m_2)}$, where m_1 and m_2 are two nonnegative integers satisfying the relation $m_1 + m_2 = m$. The action of $P_i^{(m_1,m_2)}$ on $\mathcal{S}$ is given by [1]

$$P_i^{(m_1,m_2)}|s_1, \ldots, s_i, \ldots, s_N\rangle = (-1)^{f(s_i)}|s_1, \ldots, s_i, \ldots, s_N\rangle\,, \tag{2.3}$$

where

$$f(s_i) = \begin{cases} 0, & 1 \leqslant s_i \leqslant m_1 \\ 1, & m_1 + 1 \leqslant s_i \leqslant m_1 + m_2. \end{cases} \tag{2.4}$$

In terms of these operators, the Hamiltonian of the BC_N-type Calogero model with PSRO is defined as [1]

$$\begin{aligned} H_{\mathrm{B},\epsilon}^{(m_1,m_2)} = &-\sum_i \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left(\frac{a + P_{ij}}{(x_{ij}^-)^2} + \frac{a + \widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2}\right) \\ &+ \beta a \sum_i \frac{\beta a - \epsilon P_i^{(m_1,m_2)}}{x_i^2} + \frac{a^2}{4} r^2\,, \end{aligned} \tag{2.5}$$

where the sums run from 1 to N, $a > \frac{1}{2}$, $\beta > 0$, $\epsilon = \pm 1$, $x_{ij}^{\pm} = x_i \pm x_j$, $r^2 = \sum_i x_i^2$, and

$$\widetilde{P}_{ij}^{(m_1,m_2)} = P_i^{(m_1,m_2)} P_j^{(m_1,m_2)} P_{ij}.$$

It can be shown that when m is even (resp. odd) and $m_1 = m_2$ (resp. $m_1 = m_2 \pm 1$), the PSRO in (2.3) is equivalent via a similarity transformation to the usual spin reversal operator P_i, which changes s_i into $m - s_i + 1$. As a result, for these special choices of m_1 and m_2, the Hamiltonian (2.5) reduces to that of the standard $su(m)$ spin Calogero model of BC_N type studied in Ref. [38]. As mentioned in the Introduction, another interesting special case is $m_1 = m$, $m_2 = 0$, for which the Hamiltonian (2.5) reduces to the Simons–Altshuler extension of the Calogero model.

Since $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ contains the discrete parameters m_1, m_2 and ϵ, it is natural to inquire whether there exists any relation between models (2.5) with different sets of parameters. In fact, we shall now show that $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ is equivalent to $H_{\mathrm{B},-\epsilon}^{(m_2,m_1)}$ through a unitary transformation. To this end, consider the unitary operator T whose action on the spin space $\mathcal{S}$ is given by

$$T|s_1, \ldots, s_i, \ldots, s_N\rangle = |s_1', \ldots, s_i', \ldots, s_N'\rangle, \tag{2.6}$$

where

$$s_i' = \begin{cases} s_i + m_1, & 1 \leqslant s_i \leqslant m_2 \\ s_i - m_2, & m_2 + 1 \leqslant s_i \leqslant m_1 + m_2. \end{cases} \tag{2.7}$$

Using Eqs. (2.3) and (2.6) we easily obtain

$$T^\dagger P_i^{(m_1,m_2)} T |s_1, \ldots, s_i, \ldots, s_N\rangle = (-1)^{f(s_i')} |s_1, \ldots, s_i, \ldots, s_N\rangle . \tag{2.8}$$

From Eq. (2.7) it follows that $s_i' \in \{m_1 + 1, \ldots, m_1 + m_2\}$ for $1 \leqslant s_i \leqslant m_2$ and $s_i' \in \{1, \ldots, m_1\}$ for $m_2 + 1 \leqslant s_i \leqslant m_2 + m_1$, so that

$$f(s_i') = \begin{cases} 1, & 1 \leqslant s_i \leqslant m_2 \\ 0, & m_2 + 1 \leqslant s_i \leqslant m_2 + m_1. \end{cases} \tag{2.9}$$

Equations (2.8) and (2.9) clearly imply that

$$T^\dagger P_i^{(m_1,m_2)} T = -P_i^{(m_2,m_1)} . \tag{2.10}$$

It is also obvious from Eqs. (2.2) and (2.6) that

$$T^\dagger P_{ij} T = P_{ij} . \tag{2.11}$$

From Eqs. (2.5), (2.10) and (2.11) we readily obtain

$$T^\dagger H_{\mathrm{B},\epsilon}^{(m_1,m_2)} T = H_{\mathrm{B},-\epsilon}^{(m_2,m_1)}, \tag{2.12}$$

as claimed. In view of the above relation, it suffices to study the Hamiltonian (2.5) in the case $\epsilon = 1$. However, in the paper we shall intentionally keep the parameter ϵ in $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ in order to facilitate the comparison with its D_N counterpart that we shall introduce below.

Due to the nature of the singularities of the Hamiltonian $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$, its configuration space can be taken as one of the Weyl chambers of the BC_N root system, i.e., one of the maximal open subsets of $\mathbb{R}^N$ on which the functions $x_i \pm x_j$ and x_i have constants signs. We shall choose this configuration space as the principal Weyl chamber

$$C^{(\mathrm{B})} = \left\{ \mathbf{x} \in \mathbb{R}^N : 0 < x_1 < x_2 < \cdots < x_N \right\}, \tag{2.13}$$

where $\mathbf{x} \equiv (x_1, \ldots, x_N)$. The Hamiltonian $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ is thus defined on an appropriate dense subset of the Hilbert space $L^2(C^{(B)}) \otimes \mathcal{S}$. When $\epsilon = 1$, the spectrum of $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ was computed in Ref. [1] by constructing a suitable (non-orthogonal) basis of this Hilbert space in which this Hamiltonian acts triangularly.

As explained in the latter reference, from the spin dynamical model (2.5) one can construct a PF chain of BC_N type with PSRO by applying the freezing trick. The Hamiltonian of this chain is given by

$$\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)} = \sum_{i \neq j} \left[\frac{1 + P_{ij}}{(\zeta_i - \zeta_j)^2} + \frac{1 + \widetilde{P}_{ij}^{(m_1,m_2)}}{(\zeta_i + \zeta_j)^2} \right] + \beta \sum_i \frac{1 - \epsilon P_i^{(m_1,m_2)}}{\zeta_i^2} , \tag{2.14}$$

where the lattice sites ζ_i are related to the zeros y_i of the Laguerre polynomial $L_N^{\beta-1}$ by $y_i = \zeta_i^2/2$. The exact partition function of the chain (2.14) has also been computed in Ref. [1] by exploiting its connection with the spin dynamical model (2.5).

The Hamiltonian $H^{(m_1,m_2)}$ of the D_N-type spin Calogero model with PSRO is naturally defined by dropping the term related to the roots x_i in $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$, i.e., by setting $\beta = 0$ in Eq. (2.5). We thus obtain

$$H^{(m_1,m_2)} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left[\frac{a+P_{ij}}{(x_{ij}^-)^2} + \frac{a+\widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2}\right] + \frac{a^2}{4}r^2 . \tag{2.15}$$

It should be noted that, unlike its BC_N counterpart, the latter Hamiltonian does not depend on ϵ. Just as before, from Eqs. (2.10) and (2.11) it follows that $H^{(m_2,m_1)}$ is unitarily equivalent to $H^{(m_1,m_2)}$ under T:

$$T^\dagger H^{(m_1,m_2)} T = H^{(m_2,m_1)} . \tag{2.16}$$

Thus we can impose without loss of generality the restriction $m_1 \geqslant m_2$. Consequently, for any given m one can construct $\lfloor m/2+1 \rfloor$ inequivalent spin Calogero models of D_N type with PSRO, where $\lfloor\cdot\rfloor$ denotes the integer part. Among these models, only those with $m_1 = m_2$ (for even m) or $m_1 = m_2 + 1$ (for odd m) are unitarily equivalent to the su(m) spin Calogero model of D_N type with standard spin reversal operators introduced in Ref. [39].

As is the case with the latter model, the configuration space C of the Hamiltonian (2.15) can be taken as one of the maximal open subsets of $\mathbb{R}^N$ on which the linear functionals $x_i \pm x_j$ have constant signs. We shall again take C as the principal Weyl chamber of the D_N root system, namely

$$C = \left\{\mathbf{x}\in\mathbb{R}^N : |x_1| < x_2 < \cdots < x_N\right\} . \tag{2.17}$$

Note that this configuration space contains its BC_N counterpart (2.13) as a subset. As before, the Hamiltonian (2.15) is defined on a suitable dense subspace of the Hilbert space $L^2(C)\otimes\mathcal{S}$.

We shall next explain in detail how to construct the D_N-type PF chain with PSRO associated to the spin dynamical model (2.15) by means of Polychronakos's freezing trick. To begin with, note that the Hamiltonian $H^{(m_1,m_2)}$ can be decomposed as

$$H^{(m_1,m_2)} = H^{\mathrm{sc}} + a\hat{\mathcal{H}}^{(m_1,m_2)}(\mathbf{x}) , \tag{2.18}$$

where

$$H^{\mathrm{sc}} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a(a-1)\sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \frac{a^2}{4}r^2 \tag{2.19}$$

is the Hamiltonian of the scalar D_N Calogero model and

$$\hat{\mathcal{H}}^{(m_1,m_2)}(\mathbf{x}) = \sum_{i\neq j}\left[\frac{1+P_{ij}}{(x_{ij}^-)^2} + \frac{1+\widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2}\right] \tag{2.20}$$

is a spin-dependent multiplication operator. On the other hand, in the strong coupling limit $a\to\infty$ the coefficient of the dominant term (of order a^2) in the Hamiltonian (2.15) is given by

$$U(\mathbf{x}) = \sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \frac{r^2}{4} . \tag{2.21}$$

Hence as $a\to\infty$ the particles concentrate at the coordinates ξ_i of the unique minimum $\boldsymbol{\xi}$ of the potential $U(\mathbf{x})$ in the configuration space C [45], and the coordinate degrees of freedom of

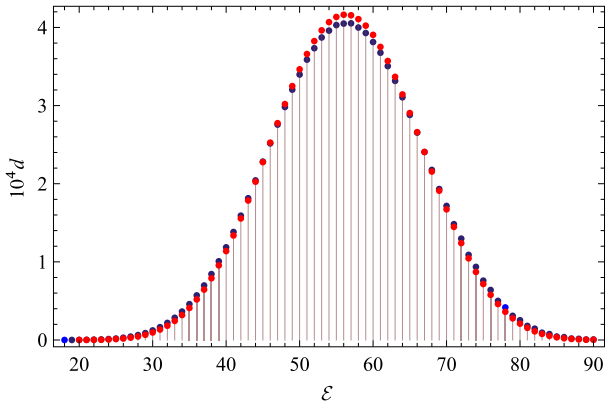

Fig. 1. Degeneracy d (in units of 10^4) versus energy $\mathcal{E}$ of the D_N chain (2.23) with $m_1 = 3$, $m_2 = 1$ (blue), compared to the su(4) D_N-type PF chain with standard spin reversal operators (red), for $N = 10$ spins. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$H^{(m_1,m_2)}$ decouple from the internal ones. By Eq. (2.18), in this limit the eigenvalues of $H^{(m_1,m_2)}$ are approximately given by

$$E_{ij} \simeq E_i^{\mathrm{sc}} + a\mathcal{E}_j, \tag{2.22}$$

where E_i^{sc} and $\mathcal{E}_j$ are two arbitrary eigenvalues of H^{sc} and

$$\mathcal{H}^{(m_1,m_2)} \equiv \hat{\mathcal{H}}^{(m_1,m_2)}(\boldsymbol{\xi}) = \sum_{i\neq j}\left[\frac{1+P_{ij}}{(\xi_i-\xi_j)^2} + \frac{1+\widetilde{P}_{ij}^{(m_1,m_2)}}{(\xi_i+\xi_j)^2}\right]. \tag{2.23}$$

We shall take Eq. (2.23) as the precise definition of the Hamiltonian of the D_N-type PF chain with PSRO. In fact, using Eqs. (2.10) and (2.11), it is easy to show that the Hamiltonians $\mathcal{H}^{(m_1,m_2)}$ and $\mathcal{H}^{(m_2,m_1)}$ are related by

$$T^{\dagger}\mathcal{H}^{(m_1,m_2)}T = \mathcal{H}^{(m_2,m_1)}. \tag{2.24}$$

Thus, we may assume without loss of generality that $m_1 \geqslant m_2$, so that there are again $\lfloor m/2+1 \rfloor$ inequivalent PF chains of D_N type with PSRO. Since the sites of these chains depend only on the scalar potential (2.21), the above models reduce to the su(m) PF chain of D_N type with standard spin reversal operators [39] when $m_1 = m_2$ (for even m) or $m_1 = m_2 + 1$ (for odd m). See, e.g., Fig. 1 for a comparison of the spectra of the D_N chain with PSRO (2.23) with $m_1 = 3, m_2 = 1$ and the su(4) D_N-type PF chain with standard spin reversal operators (corresponding to $m_1 = m_2 = 2$) for $N = 10$ spins.

A brief remark on the relation between the D_N and BC_N spin chains with PSRO in Eqs. (2.23) and (2.14) is now in order. As shown in [39], the lattice sites of the former chain are given by $\xi_1 = 0$ and $\xi_i = \sqrt{2y_{i-1}}$ $(2 \leqslant i \leqslant N)$, where $y_k > 0$ denotes the k-th root of the generalized Laguerre polynomial L_{N-1}^1. From the well-known identity $NL_N^{-1}(y) = -yL_{N-1}^1(y)$ and the previous characterization of the sites ζ_i of the BC_N chain (2.14), it immediately follows that $\boldsymbol{\xi} = \lim_{\beta\to 0}\boldsymbol{\zeta}$. Although one may naively think that the Hamiltonian $\mathcal{H}^{(m_1,m_2)}$ is simply the

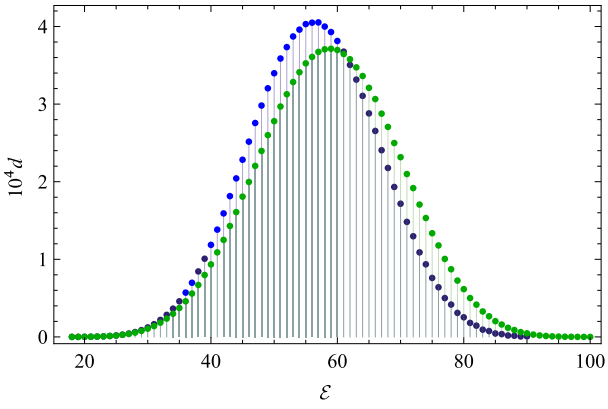

Fig. 2. Degeneracy d (in units of 10^4) versus energy $\mathcal{E}$ of the D_N chain (2.23) with $m_1 = 3$, $m_2 = 1$ and $N = 10$ spins (blue), compared to its BC_N counterpart in Eq. (2.14) (green; recall that the spectrum of the latter chain does not depend on β). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$\beta \to 0$ limit of its BC_N counterpart $\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)}$, this is certainly not the case. The point is that, although the roots ζ_i with $2 \leqslant i \leqslant N$ tend to finite nonzero limits when $\beta \to 0$, the first root ζ_1 tends to 0 in this limit. As a consequence, the $i = 1$ term of the last sum in Eq. (2.14) need not vanish as $\beta \to 0$, and in fact it can be shown [39] that

$$\lim_{\beta\to 0} \frac{\beta}{\zeta_1^2} = \frac{N}{2}. \tag{2.25}$$

Letting $\beta \to 0$ in Eq. (2.14) and using the latter identity we immediately obtain

$$\lim_{\beta\to 0} \mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)} = \mathcal{H}^{(m_1,m_2)} + \frac{N}{2}\big(1 - \epsilon P_1^{(m_1,m_2)}\big). \tag{2.26}$$

Thus, the $\beta \to 0$ limit of the Hamiltonian $\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ differs from its D_N counterpart $\mathcal{H}^{(m_1,m_2)}$ by the surface term or impurity interaction $N\big(1 - \epsilon P_1^{(m_1,m_2)}\big)/2$. It is easy to see that this term vanishes only for $\epsilon = 1$, $m_1 = m$, $m_2 = 0$ (or, equivalently, $\epsilon = -1$, $m_1 = 0$, $m_2 = m$). For any other choice of m_1 and m_2, this surface term is nonzero and does not commute with the Hamiltonian $\mathcal{H}^{(m_1,m_2)}$. Thus, except in the previously noted special cases, the spectrum of $\mathcal{H}^{(m_1,m_2)}$ cannot be obtained from that of $\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ by taking the $\beta \to 0$ limit. This fact is illustrated in Fig. 2, which shows that the spectra of these chains with $m_1 = 3$, $m_2 = 1$ and $N = 10$ spins are clearly different.

3. Spectrum and partition function

In this section, we shall compute in closed form the spectrum and partition function of the spin Calogero model of D_N type with PSRO in Eq. (2.15). This will enable us to compute the partition function $\mathcal{Z}^{(m_1,m_2)}$ of the D_N-type PF chain with PSRO (2.23) by a standard freezing trick argument. Indeed, from Eq. (2.22) it is straightforward to derive the following exact formula for

$\mathcal{Z}^{(m_1,m_2)}$ in terms of the partition functions $Z^{(m_1,m_2)}$ and Z of the spin dynamical model (2.15) and of its scalar counterpart (2.19):

$$\mathcal{Z}^{(m_1,m_2)}(T) = \lim_{a\to\infty} \frac{Z^{(m_1,m_2)}(aT)}{Z(aT)}. \tag{3.1}$$

Since Z has already been computed in Ref. [39], Eq. (3.1) provides an effective way of evaluating $\mathcal{Z}^{(m_1,m_2)}$ once $Z^{(m_1,m_2)}$ is known.

The key idea for deriving the spectrum of the spin Hamiltonian (2.15) is to observe that it can be obtained by applying a suitable projection to a simpler differential-difference operator H' acting on scalar functions. The spectrum of H' can be readily computed by constructing a (non-orthogonal) basis of its Hilbert space on which this operator acts triangularly. The spectrum of $H^{(m_1,m_2)}$ is then easily determined by projecting onto the Hilbert space of the latter operator.

More precisely, the auxiliary operator H' is given by [39]

$$H' = -\sum_i \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left[\frac{a}{(x_{ij}^-)^2}(a - K_{ij}) + \frac{a}{(x_{ij}^+)^2}(a - \tilde{K}_{ij})\right] + \frac{a^2}{4}r^2\,, \tag{3.2}$$

where K_{ij} and K_i are coordinate exchange and sign reversing operators, defined by

$$K_{ij}\,f(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = f(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N)\,,$$
$$K_i\,f(x_1,\ldots,x_i,\ldots,x_N) = f(x_1,\ldots,-x_i,\ldots,x_N)\,,$$

and $\widetilde{K}_{ij} \equiv K_i K_j K_{ij}$. The domain of the operator H' is of course a suitable dense subset of the Hilbert space $L^2(\mathbb{R}^N)$. The operator H' can be expressed in terms of the D_N-type rational Dunkl operators [46]

$$J_i^- = \frac{\partial}{\partial x_i} + a\sum_{j;\,j\neq i}\left[\frac{1}{x_{ij}^-}(1 - K_{ij}) + \frac{1}{x_{ij}^+}(1 - \tilde{K}_{ij})\right] \tag{3.3}$$

as [47]

$$H' = \rho(\mathbf{x})\left[-\sum_i (J_i^-)^2 + a\sum_i x_i\frac{\partial}{\partial x_i} + E_0\right]\rho(\mathbf{x})^{-1}\,, \tag{3.4}$$

where

$$\rho(\mathbf{x}) = \mathrm{e}^{-\frac{a}{4}r^2}\prod_{i<j}|x_i^2 - x_j^2|^a$$

is the ground state of the scalar Calogero model of D_N-type and

$$E_0 = Na\big(a(N-1) + \tfrac{1}{2}\big) \tag{3.5}$$

is its ground-state energy. A basis of this Hilbert space on which H' acts triangularly is provided by the functions

$$\phi_{\mathbf{n}}(\mathbf{x}) = \rho(\mathbf{x})\prod_i x_i^{n_i}\,, \qquad \mathbf{n} \equiv (n_1,\ldots,n_N), \tag{3.6}$$

where the n_i's are arbitrary non-negative integers. Indeed, since J_i^- lowers the degree $|\mathbf{n}| \equiv n_1 + \cdots + n_N$ of any monomial $\prod_i x_i^{n_i}$, from Eqs. (3.3) and (3.4) it immediately follows that

$$H'\,\phi_{\mathbf{n}}(\mathbf{x}) = E'_{\mathbf{n}}\,\phi_{\mathbf{n}}(\mathbf{x}) + \sum_{|\mathbf{m}|<|\mathbf{n}|} c_{\mathbf{mn}}\,\phi_{\mathbf{m}}(\mathbf{x})\,, \tag{3.7}$$

where the coefficients $c_{\mathbf{mn}}$ are real constants and

$$E'_{\mathbf{n}} = a|\mathbf{n}| + E_0\,. \tag{3.8}$$

As the diagonal elements of any upper triangular operator coincide with its eigenvalues, the spectrum of H' is given by Eq. (3.8).

The spectrum of the spin Hamiltonian $H^{(m_1,m_2)}$ can be derived from that of H' by noting that these Hamiltonians are formally related by

$$H^{(m_1,m_2)} = H'|_{K_{ij}\to -P_{ij},\, K_iK_j\to P_i^{(m_1,m_2)}P_j^{(m_1,m_2)}}. \tag{3.9}$$

In order to take advantage of this observation, we introduce the operator $\Lambda^{(m_1,m_2)}$ projecting the Hilbert space $L^2(\mathbb{R}^N)\otimes\mathcal{S}$ onto states that are antisymmetric under particle permutations and symmetric under the action of $K_iK_jP_i^{(m_1,m_2)}P_j^{(m_1,m_2)}$ for any $i\neq j$. In other words, the projector $\Lambda^{(m_1,m_2)}$ is determined by

$$\pi_{ij}\Lambda^{(m_1,m_2)} = -\Lambda^{(m_1,m_2)}, \qquad \pi_i^{(m_1,m_2)}\pi_j^{(m_1,m_2)}\Lambda^{(m_1,m_2)} = \Lambda^{(m_1,m_2)}, \tag{3.10}$$

where

$$\pi_{ij} \equiv K_{ij}P_{ij}\,, \qquad \pi_i^{(m_1,m_2)} \equiv K_iP_i^{(m_1,m_2)}\,, \tag{3.11}$$

so that

$$K_{ij}\Lambda^{(m_1,m_2)} = -P_{ij}\Lambda^{(m_1,m_2)}, \qquad K_iK_j\Lambda^{(m_1,m_2)} = P_i^{(m_1,m_2)}P_j^{(m_1,m_2)}\Lambda^{(m_1,m_2)}. \tag{3.12}$$

We shall now outline the construction of the projector $\Lambda^{(m_1,m_2)}$ in terms of the analogous projectors $\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}$ for the BC_N-type spin Calogero model with PSRO (2.14) with chirality $\varepsilon=\pm1$ (cf. [1]). To this end, recall that $\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}$ projects from the Hilbert space $L^2(\mathbb{R}^N)\otimes\mathcal{S}$ onto spin wavefunctions antisymmetric under particle permutations and with parity ±1 under $\pi_i^{(m_1,m_2)}$, i.e.,

$$\pi_{ij}\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)} = -\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}, \qquad \pi_i^{(m_1,m_2)}\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)} = \pm\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}. \tag{3.13}$$

The projector $\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}$ can then be expressed as

$$\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)} = \frac{1}{2^N N!}\prod_{j=1}^{N}\left(1\pm\pi_j^{(m_1,m_2)}\right)\cdot\sum_{l=1}^{N!}\varepsilon_l\mathcal{P}_l\,, \tag{3.14}$$

where $\mathcal{P}_l$ denotes an element of the realization of the permutation group generated by the operators π_{ij} and ε_l is the signature of $\mathcal{P}_l$. From Eqs. (3.10) and (3.13) we conclude that

$$\Lambda^{(m_1,m_2)} = \Lambda_{\mathrm{B},+}^{(m_1,m_2)} + \Lambda_{\mathrm{B},-}^{(m_1,m_2)}\,. \tag{3.15}$$

Indeed, the right-hand side of the latter equation is clearly a projector, since

$$\Lambda_{\mathrm{B},+}^{(m_1,m_2)}\Lambda_{\mathrm{B},-}^{(m_1,m_2)} = \Lambda_{\mathrm{B},-}^{(m_1,m_2)}\Lambda_{\mathrm{B},+}^{(m_1,m_2)} = 0\,,$$

and it satisfies (3.10) on account of (3.13). Thus the space

$$V \equiv \Lambda^{(m_1,m_2)}(L^2(\mathbb{R}^N)\otimes\mathcal{S})$$

decomposes as the direct sum

$$V = V_{\mathrm{B},+} \oplus V_{\mathrm{B},-}, \qquad V_{\mathrm{B},\pm} \equiv \Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}(L^2(\mathbb{R}^N) \otimes \mathcal{S}). \tag{3.16}$$

We have already mentioned that, due to the impenetrable nature of the singularities of the Hamiltonian $H^{(m_1,m_2)}$, its Hilbert space can be taken as the space $L^2(C) \otimes \mathcal{S}$ of spin wavefunctions square integrable on the open set C in Eq. (2.17). On the other hand, any point in $\mathbb{R}^N$ not lying on the singular subset $x_i \pm x_j = 0$, $1 \leqslant i < j \leqslant N$, can be mapped in a unique way to a point in C by a suitable element of the D_N Weyl group, which is generated by coordinate permutations and sign reversals of an *even* number of coordinates [48]. Using this fact, it can be shown that $L^2(C) \otimes \mathcal{S}$ is actually isomorphic to the space V, and $H^{(m_1,m_2)}$ is equivalent to its natural extension to the latter space which (with a slight abuse of notation) we shall also denote by $H^{(m_1,m_2)}$. With this identification, in view of Eq. (3.12) we can write

$$H^{(m_1,m_2)} = H^{(m_1,m_2)}\Lambda^{(m_1,m_2)} = H'\Lambda^{(m_1,m_2)}, \tag{3.17}$$

where H' acts trivially (as the identity) on $\mathcal{S}$.

We shall now explain how the spectrum of $H^{(m_1,m_2)}$ can be derived from that of H' using the previous equation. To this end, note that by Eq. (3.16) the Hilbert space V is the closure of the linear subspace spanned by the spin wavefunctions

$$\psi_{\mathbf{n},\mathbf{s}}^{\epsilon}(\mathbf{x}) = \Lambda_{\mathrm{B},\epsilon}^{(m_1,m_2)}(\phi_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle), \qquad \epsilon = \pm, \tag{3.18}$$

where $|\mathbf{s}\rangle \equiv |s_1, \ldots, s_N\rangle$ is an arbitrary element of the canonical spin basis. In fact, the wavefunctions (3.18) with *fixed* ϵ span a subspace whose closure is the Hilbert space $V_{\mathrm{B},\epsilon}$. Clearly, the functions (3.18) are not linearly independent. Indeed, using Eq. (3.13) it is easy to show that these functions satisfy the relations

$$\psi_{\mathbf{n},\mathbf{s}}^{\epsilon}(\mathbf{x}) = -\psi_{\mathbf{n}',\mathbf{s}'}^{\epsilon}(\mathbf{x}), \qquad \psi_{\mathbf{n},\mathbf{s}}^{\epsilon}(\mathbf{x}) = \epsilon(-1)^{n_i+f(s_i)}\psi_{\mathbf{n},\mathbf{s}}^{\epsilon}(\mathbf{x}), \tag{3.19}$$

where $\mathbf{n}'$ and $\mathbf{s}'$ are respectively obtained from $\mathbf{n}$ and $\mathbf{s}$ by permuting any two of their components (the same for both). Due to these identities, the sets $\{\psi_{\mathbf{n},\mathbf{s}}^{+}(\mathbf{x})\}$ and $\{\psi_{\mathbf{n},\mathbf{s}}^{-}(\mathbf{x})\}$ are both linearly independent provided that the following three conditions are imposed on the quantum numbers $\mathbf{n}$ and $\mathbf{s}$:

i) To avoid overcounting, and for later convenience, we shall order the components of $\mathbf{n}$ as follows:

$$\mathbf{n} \equiv (\mathbf{n}_{\mathrm{e}}, \mathbf{n}_{\mathrm{o}}) = \big(\overbrace{2p_1, \ldots, 2p_1}^{k_1}, \ldots, \overbrace{2p_s, \ldots, 2p_s}^{k_s}, \overbrace{2q_1+1, \ldots, 2q_1+1}^{l_1}, \ldots, \overbrace{2q_t+1, \ldots, 2q_t+1}^{l_t}\big),$$

where $0 \leqslant s, t \leqslant N$, $p_1 > p_2 > \cdots > p_s \geqslant 0$ and $q_1 > q_2 > \cdots > q_t \geqslant 0$.

ii) By the second equation in (3.19), the allowed values of s_i corresponding to each n_i are given by

$$s_i \in \begin{cases} \{1, 2, \ldots, m_1\}, & \text{for even } n_i, \\ \{m_1+1, m_1+2, \ldots, m_1+m_2\}, & \text{for odd } n_i, \end{cases}$$

for the set $\{\psi_{\mathbf{n},\mathbf{s}}^{+}(\mathbf{x})\}$, and by

$$s_i \in \begin{cases} \{1, 2, \ldots, m_1\}, & \text{for odd } n_i, \\ \{m_1+1, m_1+2, \ldots, m_1+m_2\}, & \text{for even } n_i, \end{cases}$$

for the set $\{\psi^{-}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$.

iii) If $n_i = n_j$ and $i < j$ we shall take $s_i > s_j$, again to avoid overcounting.

If the above conditions are satisfied, each of the sets $\{\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$ ($\epsilon = \pm$) is a non-orthogonal basis of the corresponding subspace $V_{\mathrm{B},\epsilon}$, and the union of these sets provides a non-orthogonal basis of the whole Hilbert space V by Eq. (3.16). We shall next show that $H^{(m_1,m_2)}$ leaves invariant each of the subspaces $V_{\mathrm{B},\epsilon}$, and that it acts triangularly on the corresponding basis $\{\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$ provided that we (partially) order it by the total degree $|\mathbf{n}|$. Indeed, using Eqs. (3.15) and (3.17), and taking into account that $[H', \Lambda^{(m_1,m_2)}_{\mathrm{B},\epsilon}] = 0$ we obtain

$$H^{(m_1,m_2)}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = \Lambda^{(m_1,m_2)}_{\mathrm{B},\epsilon}\big((H'\phi_{\mathbf{n}}(\mathbf{x}))|\mathbf{s}\rangle\big). \tag{3.20}$$

From this equation and Eqs. (3.7) and (3.18) it readily follows that

$$H^{(m_1,m_2)}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = E'_{\mathbf{n}}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) + \sum_{|\mathbf{m}|<|\mathbf{n}|} C_{\mathbf{mn}}\psi^{\epsilon}_{\mathbf{m},\mathbf{s}'}(\mathbf{x}), \tag{3.21}$$

where the $C_{\mathbf{mn}}$'s are real constants and $\mathbf{s}'$ is a permutation of $\mathbf{s}$ such that $(\mathbf{m},\mathbf{s}')$ satisfies conditions i)–iii) above; see [39] for more details. By Eq. (3.21), the action of $H^{(m_1,m_2)}$ on the whole Hilbert space $V = V_{\mathrm{B},+} \oplus V_{\mathrm{B},-}$ is the direct sum of two upper triangular actions on each of the subspaces $V_{\mathrm{B},\pm}$. Consequently, the eigenvalues of this operator are given by

$$E^{\epsilon}_{\mathbf{n},\mathbf{s}} = E'_{\mathbf{n}} = a|\mathbf{n}| + E_0, \tag{3.22}$$

where $\epsilon = \pm$ and $(\mathbf{n},\mathbf{s})$ satisfies conditions i)–iii) above. Since the RHS of Eq. (3.22) does not depend on ϵ and $\mathbf{s}$, the eigenvalue associated with the quantum number $\mathbf{n}$ has an *intrinsic degeneracy* $d^{(m_1,m_2)}_{\mathbf{n}}$ coming from the two possible chiralities and the spin degrees of freedom. This intrinsic degeneracy is in fact the sum

$$d^{(m_1,m_2)}_{\mathbf{n}} = d^{(m_1,m_2)}_{\mathbf{n},+} + d^{(m_1,m_2)}_{\mathbf{n},-}, \tag{3.23}$$

where $d^{(m_1,m_2)}_{\mathbf{n},\epsilon}$ is the number of spin states satisfying conditions i)–iii) for the given $\mathbf{n}$ and ϵ. Using these conditions we readily obtain [1]

$$d^{(m_1,m_2)}_{\mathbf{n},+} = \prod_{i=1}^{s}\binom{m_1}{k_i}\prod_{j=1}^{t}\binom{m_2}{l_j}, \qquad d^{(m_1,m_2)}_{\mathbf{n},-} = d^{(m_2,m_1)}_{\mathbf{n},+}, \tag{3.24}$$

and therefore

$$d^{(m_1,m_2)}_{\mathbf{n}} = \prod_{i=1}^{s}\binom{m_1}{k_i}\prod_{j=1}^{t}\binom{m_2}{l_j} + \prod_{i=1}^{s}\binom{m_2}{k_i}\prod_{j=1}^{t}\binom{m_1}{l_j}. \tag{3.25}$$

Thus the spectrum of the D_N-type spin Calogero model with PSRO (2.15) is given by the RHS of Eq. (3.22), where each level possesses an intrinsic degeneracy given by Eq. (3.25). Of course, the actual degeneracy of an energy $a|\mathbf{n}| + E_0$ is the sum

$$\sum_{|\mathbf{n}'|=|\mathbf{n}|} d^{(m_1,m_2)}_{\mathbf{n}'},$$

where the sum is over all multiindices $\mathbf{n}'$ satisfying condition i) above.

It is worth mentioning at this point that the spectrum of the BC_N-type spin Calogero model with PSRO and chirality ϵ in Eq. (2.5) is also given by the RHS of Eq. (3.22), with E_0 replaced by [1]

$$E_{0,\mathrm{B}} = E_0 + N\beta a^2 .$$

Moreover, the intrinsic degeneracy of the energy $a|\mathbf{n}| + E_{0,\mathrm{B}}$ is given by $d_{\mathbf{n},\epsilon}^{(m_1,m_2)}$. It follows from Eq. (3.23) that the D_N spin Hamiltonian $H^{(m_1,m_2)}$ is (up to a constant) the direct sum of two BC_N-type spin Calogero models of opposite chiralities with PSRO. Using Eqs. (3.22) and (3.25), the canonical partition function of the D_N-type spin Calogero model with PSRO can be written as

$$Z^{(m_1,m_2)}(aT) = q^{E_0/a} \sum_{\mathbf{n}} d_{\mathbf{n}}^{(m_1,m_2)} q^{|\mathbf{n}|} , \qquad q \equiv \mathrm{e}^{-1/(k_{\mathrm{B}}T)} , \tag{3.26}$$

where the sum ranges over all multiindices $\mathbf{n}$ satisfying condition i) above. Similarly, the partition functions of the corresponding BC_N-type models (2.5) are given by

$$Z_{B,\pm}^{(m_1,m_2)}(aT) = q^{(E_{0,\mathrm{B}})/a} \sum_{\mathbf{n}} d_{\mathbf{n},\pm}^{(m_1,m_2)} q^{|\mathbf{n}|} \equiv Z_{B,\mp}^{(m_2,m_1)} . \tag{3.27}$$

From Eq. (3.23) it then follows that

$$\begin{aligned} q^{-E_0/a} Z^{(m_1,m_2)}(aT) &= q^{-(E_{0,\mathrm{B}})/a} \big[Z_{B,+}^{(m_1,m_2)}(aT) + Z_{B,-}^{(m_1,m_2)}(aT) \big] \\ &= q^{-(E_{0,\mathrm{B}})/a} \big[Z_{B,+}^{(m_1,m_2)}(aT) + Z_{B,+}^{(m_2,m_1)}(aT) \big] . \end{aligned} \tag{3.28}$$

In order to apply the freezing trick formula (3.1), we need only recall the expression for the partition function Z of the scalar Calogero model of D_N-type derived in Ref. [39], namely

$$q^{-E_0/a} Z(aT) = \big(1 + q^N\big) \prod_i (1 - q^{2i})^{-1} = q^{-(E_{0,\mathrm{B}})/a} \big(1 + q^N\big) Z_{\mathrm{B}}(aT) , \tag{3.29}$$

where Z_{B} denotes the partition function of the scalar Calogero model of BC_N type. Dividing Eq. (3.28) by Eq. (3.29) and applying the analog of the freezing trick formula (3.1) for the partition function $\mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}$ of the PF spin chain of BC_N type (2.14) we finally obtain

$$\mathcal{Z}^{(m_1,m_2)}(q) = \big(1 + q^N\big)^{-1} \Big[\mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}(q) + \mathcal{Z}_{\mathrm{B},+}^{(m_2,m_1)}(q) \Big] , \tag{3.30}$$

where from now on we shall use the variable $q = \mathrm{e}^{-1/(k_{\mathrm{B}}T)}$ in place of T. The partition function $\mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}$ can in turn be expressed in terms of the partition function $\mathcal{Z}_{\mathrm{A},k}^{(m)}(q)$ of the su(m) PF chain of type A with k spins with Hamiltonian

$$\mathcal{H}_A^{(m)} = \sum_{1 \leqslant i < j \leqslant k} \frac{1 + P_{ij}}{(\rho_i - \rho_j)^2} , \tag{3.31}$$

where ρ_i is the i-th zero of the Hermite polynomial of degree k. Indeed, it is shown in Ref. [1] that for $m_2 > 0$ we have

$$\mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}(q) = \sum_{k=0}^{N} q^{N-k} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} \mathcal{Z}_{\mathrm{A},k}^{(m_1)}(q^2)\, \mathcal{Z}_{\mathrm{A},N-k}^{(m_2)}(q^2) \qquad (m_2 > 0), \tag{3.32}$$

where the q-binomial coefficient $\left[\begin{matrix} N \\ k \end{matrix}\right]_{q^2}$ is defined as

$$\left[\begin{matrix} N \\ k \end{matrix}\right]_{q^2} = \frac{(q^2)_N}{(q^2)_k (q^2)_{N-k}}, \qquad (q^2)_j \equiv \prod_{i=1}^{j} (1 - q^{2i}). \tag{3.33}$$

Combining Eqs. (3.30) and (3.32) we finally arrive at the following expression for the partition function of the D_N-type PF chain with PSRO (2.23) in terms of its type A counterpart as

$$\mathcal{Z}^{(m_1,m_2)}(q) = \sum_{k=0}^{N} f_{N,k}(q)\, \mathcal{Z}^{(m_1)}_{\mathrm{A},k}(q^2)\, \mathcal{Z}^{(m_2)}_{\mathrm{A},N-k}(q^2) \qquad (m_2 > 0), \tag{3.34}$$

where $f_{N,k}(q)$ is given by

$$f_{N,k}(q) = \frac{q^{N-k} + q^k}{1 + q^N} \left[\begin{matrix} N \\ k \end{matrix}\right]_{q^2}. \tag{3.35}$$

The case $m_2 = 0$, for which $P_i^{(m_1,0)} = 1$ and the Hamiltonian (2.23) reduces to the rational version of the (trigonometric) Simons–Altshuler chain [42], deserves special attention. Indeed, in this case by Eq. (3.19) the components of the multiindex $\mathbf{n}$ are all even (resp. odd) for the eigenfunctions $\psi^{+}_{\mathbf{n},\mathbf{s}}$ (resp. $\psi^{-}_{\mathbf{n},\mathbf{s}}$). As shown in Ref. [1], this entails that for $m_2 = 0$ Eq. (3.32) should be replaced by

$$\mathcal{Z}^{(m_1,0)}_{\mathrm{B},+}(q) = \mathcal{Z}^{(m_1)}_{\mathrm{A},N}(q^2). \tag{3.36}$$

On the other hand, since $P_i^{(0,m_2)} = -1$ we have

$$\mathcal{H}^{(0,m_2)}_{\mathrm{B},+} = \mathcal{H}^{(m_1,0)}_{\mathrm{B},+} + \sum_i \frac{2\beta}{\xi_i^2} = \mathcal{H}^{(m_1,0)}_{\mathrm{B},+} + N$$

by Eqs. (A2)–(A5) of Ref. [38]. From (3.36) it then follows that

$$\mathcal{Z}^{(0,m_2)}_{\mathrm{B},+}(q) = q^N \mathcal{Z}^{(m_1)}_{\mathrm{A},N}(q^2), \tag{3.37}$$

and substituting into Eq. (3.30) we finally obtain

$$\mathcal{Z}^{(m_1,0)}(q) = \mathcal{Z}^{(m_1)}_{\mathrm{A},N}(q^2). \tag{3.38}$$

Note that, as shown in Ref. [1], the RHS of the latter equation also coincides with the partition function of the BC_N-type chain (2.14) with $\varepsilon = 1$ and $m_2 = 0$. This was to be expected, as the latter model reduces to its D_N counterpart (2.23) when $m_2 = \beta = 0$ and its spectrum does not depend on β.

As is well known, several equivalent closed-form expressions for the partition function of the A_{k-1}-type PF chain (3.31) exist in the in the literature [3,16,44,49]. For instance, Polychronakos [3] showed that this function is given by

$$\mathcal{Z}^{(m)}_{\mathrm{A},k}(q) = \sum_{k_1+\cdots+k_m=k} q^{\frac{1}{2}\sum_{i=1}^{m} k_i(k_i-1)} [k_1, \ldots, k_m]_q, \tag{3.39}$$

where the q-multinomial coefficient $[k_1, \ldots, k_m]_q$ is defined by

$$[k_1, \ldots, k_m]_q = \frac{(q)_{k_1+\cdots+k_m}}{\prod_{i=1}^{m} (q)_{k_i}} .$$

Another well-known expression for the partition function $\mathcal{Z}^{(m)}_{\mathrm{A},k}$ was derived in Ref. [16], namely

$$\mathcal{Z}^{(m)}_{\mathrm{A},k}(q) = \sum_{\mathbf{f}\in\mathcal{P}_k} d_m(\mathbf{f})\, q^{\sum_{j=1}^{r-1} \mathcal{F}_j} \prod_{j=1}^{k-r} (1 - q^{\mathcal{F}'_j}) . \tag{3.40}$$

Here $\mathcal{P}_k$ represents the set of all ordered partitions $\mathbf{f} \equiv \{f_1, f_2, \ldots, f_r\}$ of the integer k, $d_m(\mathbf{f}) = \prod_{i=1}^{r} \binom{m}{f_i}$, $\mathcal{F}_j = \sum_{i=1}^{j} f_i$ are the partial sums of $\mathbf{f}$, and the complementary partial sums are defined as $\{\mathcal{F}'_1, \mathcal{F}'_2, \ldots, \mathcal{F}'_{k-r}\} \equiv \{1, 2, \ldots, k\} \setminus \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r\}$. A related expression for the partition function of the chain (3.31) can be obtained by exploiting its connection with a one-dimensional classical vertex model consisting of $k+1$ vertices connected by k intermediate bonds [44]. Any possible state for this vertex model can be represented by a path configuration given by

$$\vec{s} \equiv \{s_1, s_2, \ldots, s_k\} , \tag{3.41}$$

where $s_i \in \{1, 2, \ldots, m\}$ denotes the spin state of the i-th bond. The energy function associated with this spin path configuration $\vec{s}$ is defined as

$$E^{(m)}(\vec{s}) = \sum_{j=1}^{k-1} j\, \theta(s_j - s_{j+1}) , \tag{3.42}$$

where θ is Heaviside's step function, defined as

$$\theta(x) = \begin{cases} 0 , & \text{if } x < 0 , \\ 1 , & \text{if } x \geqslant 0 . \end{cases} \tag{3.43}$$

Using the Yangian quantum group symmetry of the model (3.31), it can be shown that its partition function coincides with that of the one-dimensional vertex model with energy function (3.42) (cf. [44]). Thus $\mathcal{Z}^{(m)}_{\mathrm{A},k}(q)$ can be expressed as

$$\mathcal{Z}^{(m)}_{\mathrm{A},k}(q) = \sum_{\vec{s}} q^{E^{(m)}(\vec{s})} , \tag{3.44}$$

where the sum runs over all possible m^k spin path configurations. In particular, from Eq. (3.44) it follows that

$$\mathcal{Z}^{(1)}_{\mathrm{A},k}(q) = q^{\frac{1}{2}k(k-1)} .$$

Thus the partition function (3.34) with $m_2 = 1$ reduces to

$$\mathcal{Z}^{(m_1,1)}(q) = \sum_{k=0}^{N} q^{(N-k)(N-k-1)}\, f_{N,k}(q)\, \mathcal{Z}^{(m_1)}_{\mathrm{A},k}(q^2) . \tag{3.45}$$

It is obvious from any of the expressions (3.39), (3.40) or (3.44) that the partition function $\mathcal{Z}^{(m)}_{\mathrm{A},k}(q)$ is a polynomial in q. In particular, from Eq. (3.38) it follows that $\mathcal{Z}^{(m_1,0)}$ is an

even polynomial in q, and its energies are therefore even nonnegative integers. By Eq. (3.34), in order to show that the partition function of the D_N-type PF chain with PSRO is a polynomial in q when $m_2 > 0$ it suffices to prove that the coefficients $f_{N,k}(q)$ in Eq. (3.35) depend polynomially on q. Although it is well known that the q-binomial coefficient $\begin{bmatrix} N \\ k \end{bmatrix}_{q^2}$ in (3.33) is indeed an even polynomial in q of degree $2k(N-k)$ [50], it is not clear whether $f_{N,k}(q)$ is also a polynomial. In fact, we have verified that this is the case for a wide range of values of N and all $k \leqslant N$. We conjecture that this is true in general, so that when $m_2 > 0$ the energies of the spin chain (2.23) are also nonnegative integers. Note that the latter fact also follows from the freezing trick formula (2.22), Eq. (3.22) for the spectrum of the spin dynamical model (2.15) and the analogous formula for the scalar D_N-type Calogero model in Ref. [39].

4. Statistical properties of the spectrum

A characteristic property of *all* spin chains of Haldane–Shastry type is the fact that their level density approaches a Gaussian distribution as the number of spins tends to infinity. This property has been rigorously proved for the chains of A_{N-1} type and their related one-dimensional vertex models [51,52], and has been numerically checked for the B_N, BC_N and D_N type chains with standard spin reversal operators [37–39,41]. More recently, it has been established that the level density of the BC_N-type PF chain with PSRO shows a similar behavior [1]. It is therefore of interest to ascertain whether the level density of the D_N-type spin chain with PSRO in Eq. (2.23) becomes normally distributed as the number of spins tends to infinity. In fact, Figs. 1 and 2 clearly suggest that this is actually the case. We shall restrict ourselves in the rest of this section to the case $m_2 > 0$, since for $m_2 = 0$ the spectrum of the chain (2.23) is twice that of an su(m_1) PF chain of A_{N-1} type (with the same degeneracies) on account of Eq. (3.38).

The spectrum of the spin chain (2.23) can be determined for any fixed N by evaluating its partition function (3.34) with the help of, e.g., MATHEMATICA. It turns out that the most efficient way to compute the partition function $\mathcal{Z}^{(m)}_{\mathrm{A},k}$ appearing in the latter equation is using the recursion relation

$$\mathcal{Z}^{(m)}_{\mathrm{A},k}(q) = \sum_{l=1}^{\min(m,k)} \binom{m}{l} q^{k-l} \prod_{i=1}^{l-1} (1-q^{k-i}) \cdot \mathcal{Z}^{(m)}_{\mathrm{A},k-l}(q) \tag{4.1}$$

with the initial condition $\mathcal{Z}^{(m)}_{\mathrm{A},0}(q) = 1$ (see Appendix A). In this way it is possible to evaluate the partition function $\mathcal{Z}^{(m_1,m_2)}(q)$ on a standard desktop computer for relatively high values of N (of the order of 50) and, say, $m_1 + m_2 \leqslant 4$. Our computations show that the energy levels of the D_N-type spin chain with PSRO are always a set of consecutive integers. This result is consistent with the fact that the spectrum of all previously studied rational spin chains of HS type is a set of consecutive integers [3,38,39], including the rational spin chain of BC_N type with PSRO introduced in Ref. [1]. For this reason, in order to test the Gaussian character of the level density of the chain (2.23) as $N \to \infty$ one can compare directly its normalized level density

$$f(\mathcal{E}) = m^{-N} \sum_{i=1}^{L} d_i \, \delta(\mathcal{E} - \mathcal{E}_i) \,, \qquad m \equiv m_1 + m_2 \,, \tag{4.2}$$

where $\mathcal{E}_1 < \cdots < \mathcal{E}_L$ are the distinct energy levels and d_i is the degeneracy of $\mathcal{E}_i$, with the Gaussian distribution

$$g(\mathcal{E}) = \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{(\mathcal{E}-\mu)^2}{2\sigma^2}} \tag{4.3}$$

with parameters μ and σ given by the mean and standard deviation of the spectrum, respectively. More precisely, the level density of the chain (2.23) is asymptotically Gaussian provided that

$$\frac{d_i}{m^N} \simeq g(\mathcal{E}_i)\,, \qquad N \gg 1.$$

In order to check the validity of the latter equation for any given m_1, m_2 and N we need to compute the corresponding values of μ and σ. We shall next show that, as is the case with other spin chains of HS type, these parameters can be easily evaluated in closed form from their definition

$$\mu = m^{-N}\, \mathrm{tr}\, \mathcal{H}^{(m_1,m_2)}, \qquad \sigma^2 = m^{-N}\, \mathrm{tr}\Big[\big(\mathcal{H}^{(m_1,m_2)}\big)^2\Big] - \mu^2\,. \tag{4.4}$$

The traces appearing in (4.4) can be computed in essentially the same way as for the BC_N-type PF chain with PSRO (2.14), using the traces of the spin operators P_{ij}, $P_i^{(m_1,m_2)}$ and $\widetilde{P}_{ij}^{(m_1,m_2)}$ given in Ref. [1]. Proceeding in this way we obtain

$$\mu = \left(1 + \frac{1}{m}\right) \sum_{i\neq j} (h_{ij} + \widetilde{h}_{ij}), \tag{4.5}$$

$$\sigma^2 = 2\left(1 - \frac{1}{m^2}\right) \sum_{i\neq j} (h_{ij}^2 + \widetilde{h}_{ij}^2) + \frac{4}{m^2}(t^2-1) \sum_{i\neq j} h_{ij}\widetilde{h}_{ij}\,, \tag{4.6}$$

where $t \equiv m_1 - m_2$ and

$$h_{ij} = (\xi_i - \xi_j)^{-2}\,, \qquad \widetilde{h}_{ij} = (\xi_i + \xi_j)^{-2}\,.$$

The sums in Eqs. (4.5)–(4.6) can be evaluated by taking the $\beta \to 0$ limit of the corresponding formulas in Appendix A of Ref. [38]. We thus obtain

$$\mu = \frac{1}{2}\left(1 + \frac{1}{m}\right) N(N-1), \tag{4.7}$$

$$\sigma^2 = \frac{1}{36}\left(1 - \frac{1}{m^2}\right) N(N-1)(4N+1) + \frac{1}{4m^2} N(N-1)(t^2-1)\,. \tag{4.8}$$

We have checked that the normalized level density of the spin chain (2.23) is indeed in excellent agreement with the Gaussian distribution (4.3) for different values of m_1, m_2, and even moderately large values of $N \gtrsim 15$. As an example, in Fig. 3 we compare the normalized level density of the chain (2.23) with $m_1 = 3$, $m_2 = 1$, for $N = 10$ and $N = 20$ spins, respectively, with the corresponding Gaussian distribution (4.3). It is apparent from these plots that the fit, already quite good for $N = 10$, improves significantly for $N = 20$. This is confirmed by computing the RMSE errors for both fits, which are respectively equal to 3.66×10^{-2} and 2.18×10^{-2}. For comparison purposes, we note that this error decreases to 1.11×10^{-2} for $N = 50$ spins.

Another interesting property of the spectrum of the chain (2.23) is connected to the distribution of the spacings between consecutive levels of the unfolded spectrum [53], which in this case is given by

$$s_i = (\eta_{i+1} - \eta_i)/\Delta, \qquad i = 1, \ldots, L-1,$$

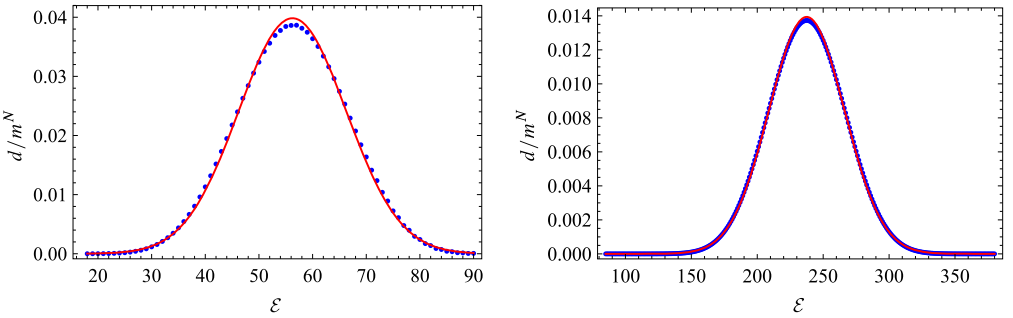

Fig. 3. Left: level density of the chain (2.23) with $m_1 = 3$, $m_2 = 1$ and $N = 10$ (blue dots) compared to the Gaussian distribution (4.3) (continuous red line). Right: analogous plot for $N = 20$ spins. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $\Delta = (\eta_L - \eta_1)/(L-1)$, $\eta_i = \eta(\mathcal{E}_i)$, and

$$\eta(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} g(\mathcal{E}')\mathrm{d}\mathcal{E}' = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\mathcal{E}-\mu}{\sqrt{2}\sigma}\right)\right].$$

According to a celebrated conjecture due to Berry and Tabor [54], the distribution of these spacings for a "generic" quantum integrable system should be Poissonian, i.e., $p(s) = \mathrm{e}^{-s}$. On the other hand, a fundamental conjecture in quantum chaos due to Bohigas, Giannoni and Schmit [55] posits that the spacings distribution for a fully chaotic quantum system invariant under time reversal should follow Wigner's law

$$p(s) = (\pi s/2)\exp(-\pi s^2/4)\,,$$

characteristic of the Gaussian orthogonal ensemble in random matrix theory [56]. In fact, it has been shown that the spacings distribution of a large class of integrable spin chains of Haldane–Shastry type follows neither Poisson's nor Wigner's law [1,14,16,38,57]. More precisely, it is shown in Refs. [16,38,58] that the cumulative spacings density $P(s) \equiv \int_0^s p(s')\mathrm{d}s'$ of a quantum system with equispaced energy levels and asymptotically Gaussian level density follows the "square root of a logarithm law"

$$P(s) \simeq 1 - \frac{2}{\sqrt{\pi}\, s_{\max}}\sqrt{\log\left(\frac{s_{\max}}{s}\right)}\,, \qquad s_{\max} \equiv \frac{\mathcal{E}_L - \mathcal{E}_1}{\sqrt{2\pi}\,\sigma}\,, \tag{4.9}$$

provided that a few mild technical conditions are satisfied. We have just shown that the energy levels of the rational D_N chain with PSRO (2.23) are equispaced and its level density is asymptotically Gaussian, and it can be easily checked using the formulas for $\mathcal{E}_1$ and $\mathcal{E}_L$ below that the technical assumptions in Ref. [58] are satisfied. Thus the spacings distribution of this chain is again approximately given by Eq. (4.9). It should be noted that for a more precise test of the validity of the Berry–Tabor conjecture one should restrict oneself to eigenspaces with well-defined quantum numbers corresponding to the main symmetries of the model. On the other hand, the fact that the spacings distribution of the whole spectrum is not Poissonian suggests that the Berry–Tabor conjecture does not hold in these eigenspaces, since the superposition of even a small number of Poissonian distributions is also Poissonian [59].

One of the characteristic properties of both the original Haldane–Shastry and the Polychronakos–Frahm spin chains of A_{N-1} type is their invariance under the quantum group $Y(\mathrm{sl}(m))$.

From the existence of such a large symmetry group one should expect that the spectrum of these chain exhibits a high degree of degeneracy. In fact, it is shown in Ref. [60] that the spectrum of these models is far more degenerate than that of a generic Yangian-invariant system, due to their equivalence to a vertex model of the form (3.42) with a very simple dispersion relation. Indeed, as shown in the latter reference, the number $\nu^{(m)}$ of distinct levels of a generic $Y(\mathrm{sl}(m))$-invariant spin system with a large number of sites N behaves as λ_m^N, where $1 < \lambda_m < 2$ is the highest real root of the polynomial $\lambda^m - \lambda^{m-1} - \cdots - 1$. In contrast, $\nu^{(m)}$ grows as a polynomial in N for all spin chains of HS type associated with the A_{N-1} root system. For instance, in the case of the type A_{N-1} PF chain this polynomial is simply given by $\mathcal{E}_L - \mathcal{E}_1 + 1$, since its spectrum is a set of consecutive integers. From the explicit expressions for the maximum and minimum energies of this model in Ref. [16] we easily obtain

$$\nu^{(m)} = \frac{1}{2}\left(1 - \frac{1}{m}\right)N^2 + \frac{l(m-l)}{2m} + 1 \qquad \text{(PF chain)}\,, \tag{4.10}$$

where $l = N \bmod m$. The situation is far less clear for spin chains of HS type associated to other root systems, with either standard or polarized spin reversal operators. On the one hand, the presence of these spin reversal operators breaks $\mathrm{su}(m)$ invariance, so that it is not obvious whether these models are invariant under a suitable quantum group, let alone $Y(\mathrm{sl}(m))$. On the other hand, it has been observed that the spectrum of some of these chains is also highly degenerate, which seems to indicate the presence of a large symmetry group.

In the particular case of the D_N-type chain with PSRO in Eq. (2.23), the number of distinct energy levels can again be exactly computed under the assumption (which we have numerically checked) that the spectrum consists of consecutive integers. Indeed, it suffices to evaluate the maximum and minimum energies $\mathcal{E}_{\max}^{(m_1,m_2)}$ and $\mathcal{E}_{\min}^{(m_1,m_2)}$, in terms of which the number $\nu^{(m_1,m_2)}$ of distinct energy levels is given by

$$\nu^{(m_1,m_2)} = \mathcal{E}_{\max}^{(m_1,m_2)} - \mathcal{E}_{\min}^{(m_1,m_2)} + 1.$$

In the first place, the maximum energy can be easily computed by taking into account that P_{ij} and $\tilde{P}_{ij}^{(m_1,m_2)}$ are self-adjoint operators whose square is the identity, so that their eigenvalues are ± 1. Moreover, it is clear that a state of the form $|s, s, \ldots, s\rangle$ is a simultaneous eigenvector of all the operators P_{ij} and $\tilde{P}_{ij}^{(m_1,m_2)}$ with eigenvalue 1. Hence the maximum energy of the chain (2.23) is given by

$$\mathcal{E}_{\max}^{(m_1,m_2)} = 2\sum_{i\neq j}\left[(\xi_i - \xi_j)^{-2} + (\xi_i + \xi_j)^{-2}\right] = N(N-1), \tag{4.11}$$

where the sum was evaluated in [39]. On the other hand, by Eq. (3.30) the minimum energy is given by

$$\mathcal{E}_{\min}^{(m_1,m_2)} = \min\left(\mathcal{E}_{\mathrm{B},+}^{(m_1,m_2)}, \mathcal{E}_{\mathrm{B},+}^{(m_2,m_1)}\right), \tag{4.12}$$

where $\mathcal{E}_{\mathrm{B},+}^{(m_1,m_2)}$ is the minimum energy of the BC_N-type chain (2.14) with $\epsilon = +1$. The latter energy was computed in Ref. [1], with the result

$$\mathcal{E}_{\mathrm{B},+}^{(m_1,m_2)} = (N-l)(N+l-m_1)/m + (l-m_1)\theta(l-m_1) \tag{4.13}$$

where $l \equiv N \bmod m$ and θ is Heaviside's function (cf. Eq. (3.43)). Using the above relation it is straightforward to check that if $m_1 \geqslant m_2$ we have $\mathcal{E}_{\mathrm{B},+}^{(m_1,m_2)} \leqslant \mathcal{E}_{\mathrm{B},+}^{(m_2,m_1)}$, and therefore

$$\mathcal{E}_{\min}^{(m_1,m_2)} = \mathcal{E}_{\mathrm{B},+}^{(m_1,m_2)}, \qquad m_1 \geqslant m_2. \tag{4.14}$$

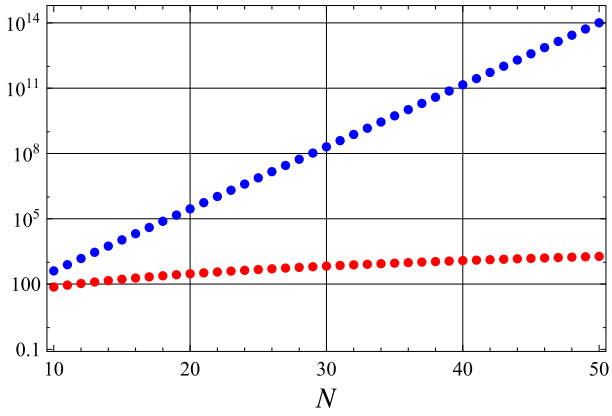

Fig. 4. Logarithmic plot of the number of distinct energy levels of the rational D_N-type chain with PSRO (2.23) with $m_1 = 3$, $m_2 = 1$ (red dots) and of a generic su(4) Yangian spin model (red dots) for $10 \leqslant N \leqslant 50$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

From Eqs. (4.11) and (4.14), and the assumption that the energy levels are equispaced, we finally obtain the following closed formula for the number of distinct energy levels of the D_N chain (2.23):

$$\nu^{(m_1,m_2)} = \left(1 - \frac{1}{m}\right)N^2 - \frac{m_2}{m}\,N + \frac{l(l-m_1)}{m} - (l-m_1)\theta(l-m_1) + 1\,. \tag{4.15}$$

Thus, it is apparent that $\nu^{(m_1,m_2)}$ is a quadratic polynomial in N, as is the case with the PF chain of A_{N-1} type (cf. Eq. (4.10)). In particular, the spectrum of the chain (2.23) exhibits a very high degeneracy, much larger than that of a generic Yangian-invariant su(m) spin model; see, e.g., Fig. 4.

The low number $\nu^{(m_1,m_2)}$ of distinct energy levels of the model (2.23) entails an extremely high average degeneracy $d^{(m_1,m_2)} \equiv m^N/\nu^{(m_1,m_2)}$, which in turn suggests the existence of a large symmetry group. More precisely, it was shown in Ref. [60] that the polynomial growth of the number of distinct energy levels of the spin chains of HS type associated to the A_{N-1} root system is ultimately due to the equivalence of these chains to a Yangian-invariant vertex model of the form (3.42) with a suitable dispersion relation. This observation makes it reasonable to conjecture that the D_N-type spin chain with PSRO (2.23) is also invariant under a suitable Yangian group, and that its spectrum coincides with that of a vertex model analogous to (3.42) with an appropriate energy function.

5. The ferromagnetic models

We shall consider in this section the ferromagnetic counterparts of the D_N-type spin Calogero model with PSRO (2.15) and its corresponding spin chain (2.23), with Hamiltonians respectively given by

$$H_{\mathrm{F}}^{(m_1,m_2)} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left[\frac{a - P_{ij}}{(x_{ij}^-)^2} + \frac{a - \widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2}\right] + \frac{a^2}{4}r^2 \tag{5.1}$$

and

$$\mathcal{H}_{\mathrm{F}}^{(m_1,m_2)} = \sum_{i\neq j}\left[\frac{1-P_{ij}}{(\xi_i-\xi_j)^2} + \frac{1-\widetilde{P}_{ij}^{(m_1,m_2)}}{(\xi_i+\xi_j)^2}\right]. \tag{5.2}$$

The spectrum of the ferromagnetic spin Calogero model (5.1) can be studied in a similar way as its antiferromagnetic counterpart, following the procedure described in Section 3. To begin with, we note that the Hamiltonian (5.1) and the auxiliary operator (3.2) are related by

$$H_{\mathrm{F}}^{(m_1,m_2)} = H'\big|_{K_{ij}\to P_{ij},\, K_iK_j\to P_i^{(m_1,m_2)}P_j^{(m_1,m_2)}}. \tag{5.3}$$

Hence, the operator $\Lambda^{(m_1,m_2)}$ in Section 3 should be replaced by the projector $\Lambda_{\mathrm{s}}^{(m_1,m_2)}$ onto states *symmetric* under simultaneous exchange of the particles' spatial and spin coordinates, and with parity $+1$ under the product of an *even* number of operators $\pi_i^{(m_1,m_2)}$ (cf. (3.11)). The new projection operator is the sum

$$\Lambda_{\mathrm{s}}^{(m_1,m_2)} = \Lambda_{\mathrm{B,s},+}^{(m_1,m_2)} + \Lambda_{\mathrm{B,s},-}^{(m_1,m_2)}$$

of the symmetric analogs of the BC_N-type projectors in Section 3, determined by

$$\pi_{ij}\Lambda_{\mathrm{B,s},\pm}^{(m_1,m_2)} = \Lambda_{\mathrm{B,s},\pm}^{(m_1,m_2)}, \qquad \pi_i^{(m_1,m_2)}\Lambda_{\mathrm{B,s},\pm}^{(m_1,m_2)} = \pm\Lambda_{\mathrm{B,s},\pm}^{(m_1,m_2)}. \tag{5.4}$$

As explained in Section 3 for the antiferromagnetic case, the operator $H_{\mathrm{F}}^{(m_1,m_2)}$ is equivalent to its natural extension to the Hilbert space

$$V = V_{\mathrm{B,s},+}\oplus V_{\mathrm{B,s},-}, \qquad V_{\mathrm{B,s},\pm} \equiv \Lambda_{\mathrm{B,s},\pm}^{(m_1,m_2)}(L^2(\mathbb{R}^N)\otimes\mathcal{S}). \tag{5.5}$$

A set of (non-orthogonal) vectors whose linear span is dense in each of the Hilbert spaces $V_{\mathrm{B,s},\pm}$ can be constructed in much the same way as in the antiferromagnetic case, replacing $\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}$ by $\Lambda_{\mathrm{B,s},\pm}^{(m_1,m_2)}$ in (3.18). Due to the symmetry of $\Lambda_{\mathrm{B,s},\pm}^{(m_1,m_2)}$ under permutations, in order to obtain a basis of these Hilbert spaces we must replace condition iii) in Section 3 by

iii′) $s_i \geqslant s_j$ if $n_i = n_j$ and $i < j$.

As a result, the spectrum of the ferromagnetic model (5.1) is still given by Eq. (3.22), but the corresponding degeneracy factor $d_{\mathbf{n}}^{(m_1,m_2)}$ in (3.25) should be replaced by

$$\begin{aligned} d_{\mathrm{F},\mathbf{n}}^{(m_1,m_2)} = &\prod_{i=1}^{s}\binom{m_1+k_i-1}{k_i}\prod_{j=1}^{t}\binom{m_2+l_j-1}{l_j} \\ &+\prod_{i=1}^{s}\binom{m_2+k_i-1}{k_i}\prod_{j=1}^{t}\binom{m_1+l_j-1}{l_j}. \end{aligned} \tag{5.6}$$

Using this formula for the degeneracy factor and proceeding as in Section 3, we find that the partition function of the D_N-type ferromagnetic spin chain (5.2) is given by the following analog of Eq. (3.30):

$$\mathcal{Z}_{\mathrm{F}}^{(m_1,m_2)}(q) = \left(1+q^N\right)^{-1}\left[\mathcal{Z}_{\mathrm{B,F},+}^{(m_1,m_2)}(q) + \mathcal{Z}_{\mathrm{B,F},+}^{(m_2,m_1)}(q)\right], \tag{5.7}$$

where $\mathcal{Z}^{(m_1,m_2)}_{\mathrm{B,F,+}}$ denotes the partition function of the ferromagnetic counterpart of the rational BC_N-type chain (2.14). Proceeding as in Ref. [1] one can readily prove the ferromagnetic version of Eq. (3.32), namely

$$\mathcal{Z}^{(m_1,m_2)}_{\mathrm{B,F,+}}(q) = \sum_{k=0}^{N} q^{N-k} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} \mathcal{Z}^{(m_1)}_{\mathrm{A,F},k}(q^2)\, \mathcal{Z}^{(m_2)}_{\mathrm{A,F},N-k}(q^2) \qquad (m_2 > 0)\,. \tag{5.8}$$

Here $\mathcal{Z}^{(m)}_{\mathrm{A,F},k}$ denotes the partition function of the ferromagnetic version of the su(m) PF chain of type A_{N-1} (3.31) with k spins, obtained replacing P_{ij} by $-P_{ij}$ in the latter equation. Finally, from Eqs. (5.7) and (5.8) we immediately obtain the following explicit formula for the partition function of the ferromagnetic chain (5.2):

$$\mathcal{Z}^{(m_1,m_2)}_{\mathrm{F}}(q) = \sum_{k=0}^{N} f_{N,k}(q)\, \mathcal{Z}^{(m_1)}_{\mathrm{A,F},k}(q^2)\, \mathcal{Z}^{(m_2)}_{\mathrm{A,F},N-k}(q^2) \qquad (m_2 > 0)\,, \tag{5.9}$$

where $f_{N,k}(q)$ is again given by (3.35). For $m_2 = 0$, proceeding exactly as in Section 3 we obtain

$$\mathcal{Z}^{(m_1,0)}_{\mathrm{F}}(q) = \mathcal{Z}^{(m_1)}_{\mathrm{A,F},N}(q^2)\,. \tag{5.10}$$

Several explicit expressions for the partition function $\mathcal{Z}^{(m)}_{\mathrm{A,F},k}$ of the ferromagnetic PF chain of A_{k-1} type appearing in the previous formulas are known in the literature. The first of these expressions is the analog of Eq. (3.39), namely

$$\mathcal{Z}^{(m)}_{\mathrm{A,F},k}(q) = \sum_{k_1+\cdots+k_m=k} [k_1,\ldots,k_m]_q\,.$$

Alternatively, $\mathcal{Z}^{(m)}_{\mathrm{A,F},k}$ may be obtained from Eq. (3.40) replacing $d_m(\mathbf{f})$ by its ferromagnetic version $d_{\mathrm{F},m}(\mathbf{f}) \equiv \prod_{i=1}^{r} \binom{m+f_i-1}{f_i}$. Finally, $\mathcal{Z}^{(m)}_{\mathrm{A,F},k}$ is also given by the RHS of Eq. (3.44) with $\theta(x)$ replaced by $1-\theta(x)$ in the definition (3.42) of $E^{(m)}(\vec{s})$. From any of these explicit formulas for $\mathcal{Z}^{(m)}_{\mathrm{A,F},k}(q)$, it follows that this function is a polynomial in q. By Eqs. (5.9)–(5.10) the same is true for the partition function of the chain (2.23), provided that the coefficient $f_{N,k}(q)$ is a polynomial in q.

As is well known, the partition functions of the A_{N-1}-type ferromagnetic and antiferromagnetic PF spin chains satisfy a certain duality relation [3,61,62]. In fact, a similar relation also holds for PF chains associated with other root systems [1,38,39]. In order to establish a duality relation between the partition functions of the ferromagnetic and antiferromagnetic spin chains of D_N type with PSRO, it suffices to observe that their Hamiltonians (5.2) and (2.23) are related by

$$\mathcal{H}^{(m_1,m_2)}_{\mathrm{F}} + \mathcal{H}^{(m_1,m_2)} = 2\sum_{i\neq j} \left[(\xi_i-\xi_j)^{-2} + (\xi_i+\xi_j)^{-2}\right] = N(N-1) \tag{5.11}$$

(cf. Eq. (4.11)). This obviously implies that the eigenvalues of $\mathcal{H}^{(m_1,m_2)}_{\mathrm{F}}$ and $\mathcal{H}^{(m_1,m_2)}$ are also related by (5.11), so that their partition functions satisfy the duality relation

$$\mathcal{Z}^{(m_1,m_2)}_{\mathrm{F}}(q) = q^{N(N-1)}\, \mathcal{Z}^{(m_1,m_2)}(q^{-1}). \tag{5.12}$$

6. Conclusions and outlook

We introduce the D_N spin Calogero model with PSRO and its associated spin chain of HS type, namely the D_N PF chain with PSRO. We solve the former model by finding a suitable (non-orthonormal) basis of its Hilbert space on which its Hamiltonian acts triangularly. From the spectrum of this model we are able to compute its partition function in closed form, which yields the partition function of the spin chain via Polychronakos's freezing trick. More precisely, we show that the latter partition function can be expressed in terms of the partition function of the type-A PF chain. Since the type-A partition function can be efficiently evaluated using a simple recursion formula that we also derive in this paper, we are able to exactly compute the spectrum of the D_N-type chain for relatively high values of N. In this way, we are able to study several global properties of the spectrum of the latter chain. In particular, we provide strong numerical evidence showing that its energy levels are a sequence of consecutive integers, and that its level density becomes normally distributed when the number of spins tends to infinity. From these facts we conclude that the spacings between consecutive levels of the unfolded spectrum follows a "square-root-of-a-logarithm" distribution, characteristic of most spin chains of HS type. We also determine the number of distinct energy levels of the spin chain, showing that it is a second-degree polynomial in N, as is the case with the PF chain of A_{N-1} type. For spin chains of HS type related to the A_{N-1} root system, it is known [60] that the polynomial growth of the number of distinct levels is a consequence of the fact that these models are equivalent to a Yangian-invariant vertex model with linear energy function and polynomial dispersion relation. Our results strongly suggest that this is also the case for the present model, a conjecture which certainly deserves further study. In particular, the validity of this conjecture would also point out at the existence of a suitable Yangian symmetry for both the D_N-type spin chain and the spin Calogero model with PSRO, as is the case with the rational and trigonometric Calogero–Sutherland models of A_{N-1}-type and their associated spin chains.

The present work suggests some possible future developments. Among them, the most natural one would be to address the extension of our results to the Sutherland (both trigonometric and hyperbolic) models of BC_N, B_N and D_N type and their related spin chains. From a more mathematical standpoint, the fact that the chain's spectrum consists of integers leads us to conjecture that the function $f_{N,k}(q)$ in Eq. (3.35) is a polynomial in q. Although this conjecture can be easily checked numerically, we have not been able to find an analytic proof thereof using the properties of q-binomial coefficients.

Acknowledgements

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Appendix A. Recursion relation for the partition function of the PF chain of A_{k-1} type

We shall provide in this appendix a short derivation of the recursion relation (4.1) satisfied by the partition function $\mathcal{Z}^{(m)}_{\mathrm{A},k}$ of the su(m) PF chain of A_{k-1} type. The main idea behind the proof is to decompose the multiindex $\mathbf{f} \in \mathcal{P}_k$ in Eq. (3.40) as

$$\mathbf{f} = (f_1, \ldots, f_{r-1}, l) \equiv (\tilde{\mathbf{f}}, l)\,,$$

with $1 \leqslant l \leqslant \min(m,k)$ and $\tilde{\mathbf{f}} \in \mathcal{P}_{k-l}$. Setting $s = r-1$ we have

$$\mathcal{F}_1 + \cdots + \mathcal{F}_{r-1} = \tilde{\mathcal{F}}_1 + \cdots + \tilde{\mathcal{F}}_{s-1} + \mathcal{F}_{r-1} = \tilde{\mathcal{F}}_1 + \cdots + \tilde{\mathcal{F}}_{s-1} + k - l\,, \tag{A.1}$$

and therefore

$$\{\mathcal{F}'_1, \ldots, \mathcal{F}'_{k-r}\} = \{\tilde{\mathcal{F}}'_1, \ldots, \tilde{\mathcal{F}}'_{k-l-s}\} \cup \{k-l+1, \ldots, k-1\}\,. \tag{A.2}$$

Substituting (A.1) and (A.2) into Eq. (3.40) we obtain

$$\begin{aligned}\mathcal{Z}^{(m)}_{\mathrm{A},k}(q) &= \sum_{l=1}^{\min(m,k)} \binom{m}{l} q^{k-l} \prod_{i=1}^{l-1}(1-q^{k-i}) \cdot \sum_{\tilde{\mathbf{f}} \in \mathcal{P}_{k-l}} \prod_{i=1}^{s} \binom{m}{\tilde{f}_i} q^{\tilde{\mathcal{F}}_1 + \cdots + \tilde{\mathcal{F}}_{s-1}} \prod_{i=1}^{k-l-s} \left(1 - q^{\tilde{\mathcal{F}}'_i}\right) \\ &\equiv \sum_{l=1}^{\min(m,k)} \binom{m}{l} q^{k-l} \prod_{i=1}^{l-1}(1-q^{k-i}) \cdot \mathcal{Z}^{(m)}_{\mathrm{A},k-l}(q)\,,\end{aligned}$$

as claimed. As to the initial condition, from Eq. (3.40) with $k=1$ it easily follows that $\mathcal{Z}^{(m)}_{\mathrm{A},1}(q) = m$. From the recursion relation (4.1) with $k=1$ we easily obtain $\mathcal{Z}^{(m)}_{\mathrm{A},0}(q) = 1$.

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