Note
Symmetric and resolvable $\lambda$-configurations constructed from block designs

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Received 1 April 2005; received in revised form 12 July 2007; accepted 14 August 2007
Available online 24 September 2007

Abstract

A $\lambda$-configuration $(v_r, b_k)$ is a finite incidence structure of $v$ points and $b$ blocks such that each block contains exactly $k$ points, each point lies on exactly $r$ blocks and two different points are connected by at most $\lambda$ blocks. If $v = b$ and hence $r = k$, then a $\lambda$-configuration is symmetric. From any block design we construct $\lambda$-configurations. Some block designs lead to symmetric $\lambda$-configurations, and some leads to resolvable $\lambda$-configurations.

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Keywords: $\lambda$-configuration; Block design; Resolvable $\lambda$-configuration

1. Introduction

Definition 1. A $\lambda$-configuration $(v_r, b_k)$ is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, such that

1. $|\mathcal{P}| = v$,
2. $|\mathcal{B}| = b$,
3. every element of $\mathcal{B}$ is incident with exactly $k$ elements of $\mathcal{P}$,
4. every element of $\mathcal{P}$ is incident with exactly $r$ elements of $\mathcal{B}$,
5. every pair of distinct elements of $\mathcal{P}$ is incident with at most $\lambda$ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points, and the elements of the set $\mathcal{B}$ are called blocks. If a point $P$ is incident with a block $x$, we write $PIx$.

A 2-configuration is called a spatial configuration. If $v = b$ and hence $r = k$, then a $\lambda$-configuration is symmetric.

Definition 2. A parallel class or resolution class in a $\lambda$-configuration is a set of blocks that partition the point set. A resolvable $\lambda$-configuration is a $\lambda$-configuration whose blocks can be partitioned into parallel classes.
Definition 3. Let $\mathcal{I}$ be an incidence structure with the set of points $\mathcal{P} = \{P_1, P_2, \ldots, P_v\}$ and the set of blocks $\mathcal{B} = \{x_1, x_2, \ldots, x_b\}$. The incidence matrix of $\mathcal{I}$ is a $b \times v$ matrix $M = (m_{ij})$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } P_j \text{ is incident with } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4. Let $M$ be the incidence matrix of an incidence structure $\mathcal{I}$. Denote by $M^t$ the transpose of $M$. The graph with adjacency matrix

$$\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}$$

is called the incidence graph of $\mathcal{I}$.

Definition 5. A $(v, k, \lambda)$ block design is a $\lambda$-configuration $(v_r, b_k)_\lambda$ such that every pair of points is incident with exactly $\lambda$ blocks. A $(v, 3, 1)$ block design is called a Steiner triple system.

For further basic definitions and properties of configurations and block designs we refer the reader to [1–3].

2. $\lambda$-configurations constructed from block designs

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a $(v, k, \lambda)$ block design. Let us define the incidence structure $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ as follows:

$\mathcal{P}_1 = \{(P, x) | P \in \mathcal{P}, \ x \in \mathcal{B}, \ PIx\}$,

$\mathcal{B}_1 = \{(P, x, Q) | P, Q \in \mathcal{P}, \ P \neq Q, \ x \in \mathcal{B}, \ PIx, \ QIx\}$,

$P_1 = (P, x_1), \ x_1 \in \mathcal{B}, \ P_1 I_1 x_1 \leftrightarrow P \in \{\bar{P}, \bar{Q}\}$.

Remark 1. Let $G$ be the incidence graph of a $(v, k, \lambda)$ block design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ and $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ be the incidence structure defined as above. Then we can describe the incidence structure $\mathcal{D}_1$ in the following way:

$\mathcal{P}_1$ is the set of all the edges of $G$,

$\mathcal{B}_1$ is the set of all paths of length 2 in $G$ with the first (and the last) vertex corresponding to a point of $\mathcal{D}$,

$P_1 \in \mathcal{P}_1, \ x_1 \in \mathcal{B}_1, \ P_1 I_1 x_1$ if and only if the union of the corresponding edge and path of length 2 is a path of length 2 or 3.

Theorem 1. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a $(v, k, \lambda)$ block design and $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ be the incidence structure defined as above. Then $\mathcal{D}_1$ is a $(v - 1)\lambda$-configuration $(v'_r, b'_k)_{(v-1)\lambda}$ with the following properties:

1. $v' = vr = bk$,
2. $b' = \left(\frac{v}{2}\right)\lambda$,
3. $k' = 2r$,
4. $r' = (v - 1)\lambda$,
5. every pair of points is incident with exactly $(v - 1)\lambda$ or $\lambda$ blocks,
6. every pair of blocks is incident with exactly $2r$, $r$ or $0$ points.

Proof. It is obvious that $v' = vr = bk$. Two points of the design $\mathcal{D}$ determine $\lambda$ blocks in $\mathcal{D}_1$, since every pair of points in $(v, k, \lambda)$ block design is incident with exactly $\lambda$ blocks. Therefore $b' = \left(\frac{v}{2}\right)\lambda$.

If a block $x_1$ in $\mathcal{D}_1$ corresponds to an ordered triple $(P, x, Q)$, then $x_1$ is incident with points which correspond to $(P, y)$ or $(Q, z)$. Since the points $P$ and $Q$ in $\mathcal{D}$ are incident with $r$ blocks, $x_1$ is incident with $2r$ points.

A point $P_1 \in \mathcal{P}_1$ corresponding to an ordered pair $(P, x)$ is incident with blocks which correspond to $(P, y, Q)$, $Q \in \mathcal{P}_1$, $y \in \mathcal{B}_1$. We can choose $Q$ in $v - 1$ ways, and each pair $P, Q$ is incident with exactly $\lambda$ blocks.

Let $P_1$ be a point in $\mathcal{P}_1$ which corresponds to an ordered pair $(P, x)$. $R_1 \in \mathcal{P}_1$ corresponds to $(P, y)$ and $Q_1 \in \mathcal{P}_1$ corresponds to $(Q, z)$, $Q \neq P$. A block $x_1 \in \mathcal{B}_1$ is incident with $P_1$ if and only if it is incident with $R_1$. On the other
hand, pair of points \( P_1, Q_1 \) lies on \( \lambda \) blocks of the type \((P, w, Q)\), \( w \in B \). So, every pair of points in \( \mathcal{D}_1 \) is incident with exactly \((v - 1)\lambda\) or \( \lambda \) blocks.

Let \( P, Q, R, S \in \mathcal{P} \) be mutually different points in \( \mathcal{D} \) and \( x_1 = (P, x, Q), y_1 = (P, y, Q), z_1 = (Q, z, R) \) and \( w_1 = (R, w, S) \) be blocks in \( \mathcal{D}_1 \). The pair of blocks \( x_1, y_1 \) is incident with exactly \( 2r \) points, the pair \( x_1, z_1 \) with \( r \) points, while \( x_1 \) and \( w_1 \) do not have common points. Therefore, every pair of blocks in \( \mathcal{D}_1 \) is incident with exactly \( 2r\), \( r \) or 0 points. \( \square \)

**Corollary 1.** Let \( \mathcal{D} = (\mathcal{P}, \mathcal{B}, I) \) be a \((v, k, \lambda)\) block design. The \((v - 1)\lambda\)-configuration \( \mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1) \) is symmetric if and only if \( k = 3 \).

**Proof.** \( \mathcal{D}_1 \) is a symmetric \((v - 1)\lambda\)-configuration if and only if \( vr = \binom{v}{2} \lambda \). It is known that \( r(k - 1) = (v - 1)\lambda \) (see \([1, \text{Theorem 2.10, p. 10}]\)). Then \( r = (v - 1)\lambda/2 = r(k - 1)/2 \), so \( k = 3 \). \( \square \)

### 3. Resolvable \( \lambda \)-configurations constructed from block designs

Let \( \mathcal{D} = (\mathcal{P}, \mathcal{B}, I) \) be a \((v, k, \lambda)\) block design and \( P \in \mathcal{P} \) a point in \( \mathcal{D} \). Define an incidence structure \( \mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2) \) in the following way:

- \( \mathcal{P}_2 = \{(Q, x) \mid Q \in \mathcal{P}, Q \neq P, x \in \mathcal{B}, P I x, Q I x\}, \)
- \( \mathcal{B}_2 = \{(x, P, y) \mid x, y \in \mathcal{B}, P I x, P I y\}, \)
- \( P_2 = (Q, x), x_2 = (\tilde{x}, P, \tilde{y}), P_2I_2x_2 \Leftrightarrow x = \tilde{x} \) or \( x = \tilde{y} \).

**Theorem 2.** Let \( \mathcal{D} = (\mathcal{P}, \mathcal{B}, I) \) be a \((v, k, \lambda)\) block design, \( P \in \mathcal{P} \) and \( \mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2) \) be the incidence structure defined as above. Then \( \mathcal{D}_2 \) is a \((r - 1)\)-configuration \((v'_r, b'_k)_{r-1}\) with the following properties:

1. \( v' = r(k - 1) \),
2. \( b' = \binom{r}{2} \),
3. \( k' = 2(k - 1) \),
4. \( r' = r - 1 \),
5. every pair of points is incident with exactly \((r - 1)\) or 0 blocks,
6. every pair of blocks is incident with exactly \((k - 1)\) or 0 points.

**Proof.** \( P \) is incident with \( r \) blocks and each block contains \( k \) points, so \( b' = \binom{r}{2} \) and \( v' = r(k - 1) \).

Let \( x_2 = (x, P, y) \) be a block in \( \mathcal{D}_2 \). Since the blocks \( x \) and \( y \) both are incident with \( k - 1 \) points other then \( P \), \( x_2 \) is incident with \( 2(k - 1) \) points form \( \mathcal{B}_2 \).

Let \( P_2 = (Q, x) \) be a point in \( \mathcal{D}_2 \). Blocks incident with \( P_2 \) are of the form \((x, P, y)\). \( P \) is incident with \( r \) blocks from \( \mathcal{B} \), so we have \( r - 1 \) possibilities for choosing the block \( y \).

Points \( P_2 = (Q, x) \) and \( Q_2 = (R, x) \) are incident with \( r - 1 \) common blocks. \( P_2 \) and a point \( R_2 = (S, y), y \neq x \), are incident with exactly one common block, the block \((x, P, y)\).

A pair of blocks \( x_2 = (x, P, y) \) and \( y_2 = (y, P, z) \) is incident with \( k - 1 \) points which correspond to \((Q, y), Q \in \mathcal{P}, Q \neq P, \) while blocks \( x_2 = (x, P, y) \) and \( z_2 = (z, P, w) \) do not have common points. \( \square \)

Let \( S \) be a set, \( r \) an even number and \(|S| = r \). Then there exist \( r - 1 \) partitions \( R_1, \ldots, R_{r-1} \) of \( S \) into 2-subsets, such that for every 2-subset \(|x, y| \subset S \) there exists exactly one partition \( R_i \) such that \(|x, y| \in R_i \).

**Corollary 2.** Let \( \mathcal{D} = (\mathcal{P}, \mathcal{B}, I) \) be a \((v, k, \lambda)\) block design and \( P \in \mathcal{P} \). If \( r \) is even, then \( \mathcal{D}_2 \) is a resolvable \((r - 1)\)-configuration. Blocks of \( \mathcal{D}_2 \) are partitioned into \( r - 1 \) parallel classes, each class consists of \( r/2 \) blocks.

**Proof.** Let \( \{x_1, x_2, \ldots, x_r\} \) be blocks of the design \( \mathcal{D} \) which are incident with the point \( P \). Further, let \( R_1, R_2, \ldots, R_{r-1} \) be pairwise disjoint partitions of the set \( \{x_1, x_2, \ldots, x_r\} \) into 2-subsets. Each partition \( R_i, i = 1, \ldots, r - 1 \), determines \( r/2 \) parallel blocks of \((r - 1)\)-configuration \( \mathcal{D}_2 \), i.e., one parallel class. Since \( R_1, R_2, \ldots, R_{r-1} \) are mutually disjoint partitions, they induce a partition of the set \( \mathcal{B}_2 \). \( \square \)
Corollary 3. Let $D = (\mathcal{P}, \mathcal{B}, I)$ be a $(v, k, \lambda)$ block design, $P \in \mathcal{P}$ and $D_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$ be the incidence structure defined as above. Then the dual structure $D_2$ of $D$ is a $(v', b', k')$-configuration with the following properties:

1. $v' = \left(\frac{r}{2}\right)$,
2. $b' = r(k - 1)$,
3. $k' = r - 1$,
4. $r' = 2(k - 1)$,
5. every pair of points is incident with exactly $(k - 1)$ or 0 blocks,
6. every pair of blocks is incident with exactly $(r - 1)$ or 1 points.

Remark 2. If $D = (\mathcal{P}, \mathcal{B}, I)$ is a $(v, 3, \lambda)$ block design, then $\overline{D}_2$ is a spatial configuration. Especially, if $D$ is a Steiner triple system then $\overline{D}_2$ is a spatial configuration.

Corollary 4. Let $D = (\mathcal{P}, \mathcal{B}, I)$ be a $(v, k, \lambda)$ block design. Then $D_2$ and $\overline{D}_2$ are symmetric if and only if $k = (r + 1)/2$.

Corollary 5. Let $D = (\mathcal{P}, \mathcal{B}, I)$ be a residual design of a Hadamard $(m - 1, \frac{1}{2}m - 1, \frac{1}{4}m - 1)$ design. Then $D$ is a $(\frac{1}{2}m, \frac{1}{4}m - 1)$ block design, $D_2$ is a symmetric $(\frac{1}{2}m - 2)$-configuration and $\overline{D}_2$ is a symmetric $(\frac{1}{4}m - 1)$-configuration.

Proof. $D$ is a $(\frac{1}{2}m, \frac{1}{4}m - 1)$ block design and each point of $D$ is incident with exactly $\frac{1}{2}m - 1$ blocks. □

References