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On the Cohomology of Certain Topological Colimits of Pro-C-Groups

DION GILDENHUYS

Department of Mathematics, McGill University, Montreal, Canada

AND

LUIS RIBES

Department of Mathematics, Carleton University, Ottawa, Canada

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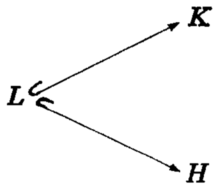
INTRODUCTION

Let **C** be a class of finite groups, closed under the formation of subgroups, group extensions and homomorphic images. In Section 1 we develop a cohomology theory for pairs (G, H) of pro-**C**-groups, (where H is a closed subgroup of G), on the same lines as Ribes' cohomology theory of pairs of abstract groups [15]. If G is the colimit (push-out) of the diagram



in the category **PC** of pro-**C**-groups, and the canonical map $K \to G$ is injective, then we have an *excision axiom* (Theorem 1.10)

$$H^n(G, H, -) \simeq H^n(K, L, -), \qquad n \geqslant 1.$$

If both canonical maps $K \to G$ and $H \to G$ are injective, then G is called *the amalgamated product of K and H over the common closed subgroup L* (see Ribes [16]), and we have

$$H^n(G, L, -) \simeq H^n(H, L, -) \oplus H^n(K, L, -), \qquad (n \geqslant 1),$$

(Proposition 1.11), as well as a Mayer–Vietoris sequence for the ordinary cohomology groups (Theorem 1.13). Section 2 is mainly formal in nature. We recall the definition of a category object $\mathscr{C}$ in a category $\mathscr{P}$ with pullbacks; and when $\mathscr{P}$ is the category *top* of topological spaces or the category *Ptop* of pointed topological spaces, we define the concepts of a *functor* from $\mathscr{C}$ into **PC**, and the *colimit* of such a functor. The first author learned about such things from A. Joyal, who dealt with similar concepts in the setting of the topos of Lawvere and Tierney. Just as a free discrete group is the colimit of a functor with domain a discrete category and values equal to the free group on one generator, the free pro-**C**-group generated by a pointed topological space is the topological colimit of a functor: $\mathscr{C} \to \mathbf{PC}$, where $\mathscr{C}$ is a category object in *Ptop* without nonidentity maps. More generally, the free pro-**C**-product defined in Gildenhuys and Ribes [7] of a family $\{G_x \mid x \in X\}$ of pro-**C**-groups, indexed by a pointed topological space $(X, *)$, with $G_* = (1)$ and $x \mapsto G_x$ locally constant outside $(*)$, is an example of a topological colimit of such a functor (Proposition 2.2). In Section 3 we study the cohomology of such free pro-**C**-products.

Given a discrete group $F^0(x_0, x_1, \ldots, x_{m+1})/(r)$ with one defining relator r and minimal set $\{x_0, x_1, \ldots, x_{m+1}\}$ of generators, assume that r belongs to the normal subgroup $N^0 = (x_0, x_1, \ldots, x_m)$, generated in the free group $F^0(x_0, x_1, \ldots, x_{m+1})$ by the elements $x_0, x_1, \ldots, x_m$. Very often r becomes more amenable when expressed in terms of the free generators $x_{i,j} = x_{m+1}^{-j} x_i x_{m+1}^{j}$ $(0 \leqslant i \leqslant m, j \in \mathbb{Z})$ of N^0. For one thing it becomes shorter. If r belongs to the subgroup $F_0{}^0$ of N^0 freely generated by the elements $\{x_{i,j} \mid h_i \leqslant j \leqslant h_i + n_i;\ 0 \leqslant i \leqslant m\}$, then $r_k = y^{-k} r y^k$ belongs to the subgroup $F_k{}^0$ of N^0 freely generated by $\{x_{i,j} \mid h_i + k \leqslant j \leqslant h_i + k + n_i;\ 0 \leqslant i \leqslant m\}$; and N^0/R^0, where $R^0 = (r)$, can be built up from the (simpler) one relator groups $F_k{}^0/(r_k)$ by a process of successive amalgamations followed by a passage to the direct limit (see Karass, Magnus, and Solitar [13, p. 252]). In the case of pro-p-groups with one defining relator, we can do something similar, but the direct limit (or colimit) in the category of groups has to be replaced by a *topological* colimit in the category of pro-p-groups (Proposition 2.3). Section 4 deals with the cohomology of pro-p-groups with single defining relator. As an illustration of our methods, we consider the defining relator

$$r = x^p((x, y), ((y, x), x))$$

(Example 4.5). If $p = 3$, Labute's method [10] does not apply. However, rewriting r in terms of the conjugates $x_j = y^{-j} x y^j$, it becomes more amenable

$$r = r_0 = x_0{}^3(x_0^{-1} x_1, (x_1^{-1}, x_0)^{x_0}).$$

Labute's method gives $\mathrm{cd}(F(x_0, x_1)/(r_0)) = 2$, and from our Theorem 4.4 we deduce that $\mathrm{cd}(F(x, y)/(r)) \leqslant 3$.

1. A Cohomology Theory for Pairs of Pro-C-Groups

Throughout this section, **C** will denote a nontrivial class of finite groups, closed under the formation of subgroups, extensions, and homomorphic images. Note that if the order of a group in **C** is divisible by a prime p, then **C** contains the Sylow p-groups of that group, and hence contains all finite p-groups. It follows that the free pro-**C**-group on one generator is of the form

$$\hat{\mathbb{Z}}_{\mathbf{C}} = \prod_{p \in S} \hat{\mathbb{Z}}_p,$$

where $\hat{\mathbb{Z}}_p$ denotes the ring of $p$adic integers, and S is the set of primes dividing the order of some group in **C**. So, $\hat{\mathbb{Z}}_{\mathbf{C}}$ is a pseudocompact ring, and for every pro-**C**-group G we can define a complete group algebra

$$\mathcal{O}\!\mathcal{U}G = \hat{\mathbb{Z}}_{\mathbf{C}}[[G]] = \varprojlim_{U} \hat{\mathbb{Z}}_{\mathbf{C}}[G/U]$$

(U runs through the open normal subgroups of G) which is again a pseudocompact ring (Brumer [2, Section 4]). Let $\mathscr{C}_{\mathbf{C}}{}^G$ be the category of discrete (topological) ($\mathcal{O}\!\mathcal{U}G$)-modules. Then $\mathscr{C}_{\mathbf{C}}{}^G$ is an abelian category with enough injectives (Brumer [2, Lemma 1.8]). Note that the discrete ($\mathcal{O}\!\mathcal{U}G$)-modules can also be characterized as discrete G-modules A that are **C**-torsion, in the sense that each element of A has finite order equal to a product of powers of primes in S (see Brumer [2, pp. 454, 455]). Given an abelian torsion group, i.e., a discrete $\hat{\mathbb{Z}}$-module, where

$$\hat{\mathbb{Z}} = \hat{\mathbb{Z}}_{\mathbf{F}} = \prod_{p} \hat{\mathbb{Z}}_p,$$

and **F** is the class of all finite groups, we denote by $T(A)$ (resp. $T'(A)$) the submodule of A consisting of all elements whose orders are products of powers of primes $p \in S$ (resp. $p \notin S$). One easily sees that $A = T(A) \oplus T'(A)$, and if $0 \to A' \to A \to A'' \to 0$ is an exact sequence of abelian torsion groups, then so is $0 \to T(A) \to T(A') \to T(A'') \to 0$.

Let H be a closed subgroup of G, let $A \in |\mathscr{C}_{\mathbf{C}}{}^G|$, and denote by $M_G{}^H(A)$ the induced module (Serre [18, Chapter I, 2.5] or Ribes [17, p. 143]). One has an exact sequence

$$0 \to A \xrightarrow{i} M_G{}^H(A) \to \Gamma(A) \to 0$$

in $\mathscr{C}_{\mathbf{C}}{}^{G}$ [18, I-13], $M_G{}^H: \mathscr{C}_{\mathbf{C}}{}^G \to \mathscr{C}_{\mathbf{C}}{}^G$ is an exact functor (see Ribes [17, Proposition 7.2]), and, by the 3×3 lemma [12, Lemma 5.1], $\Gamma: \mathscr{C}_{\mathbf{C}}{}^G \to \mathscr{C}_{\mathbf{C}}{}^G$ is also an exact functor.

Consider the abelian group

$$X(G, H, A) = \{f: G \to A \mid f(xy) = xf(y) + f(x), f \mid H = 0\}$$

of continuous crossed homomorphisms from G to A, vanishing on H.

Definition 1.1. The nth right derived functor of the left-exact functor $X(G, H, -)$ from $\mathscr{C}_{\mathbf{C}}{}^G$ into the category $\mathcal{Ab}$ of abelian groups, is denoted by $H^{n+1}(G, H, -)$, and $H^{n+1}(G, H, A)$ is called the $(n+1)$*st cohomology group of the pair* (G, H), *with coefficients in the discrete G-module* A $(n \geqslant 0)$.

We shall see that these cohomology groups are independent of $\mathbf{C}$, in the sense that we get the same groups if we view G and H as profinite groups, and A as an object of $\mathscr{C}_{\mathbf{F}}{}^G$.

Lemma 1.2. *One has a natural isomorphism*

$$\mathrm{Hom}_G(\hat{\mathbb{Z}}_{\mathbf{C}}, \Gamma(A)) \cong X(G, H, A).$$

The proof proceeds almost exactly as in Ribes [15], Lemma 1.1, and is therefore omitted.

Lemma 1.3. *Let H be a closed subgroup of a pro-$\mathbf{C}$-group G, and let A be an injective object of $\mathscr{C}_{\mathbf{C}}{}^G$ or of $\mathscr{C}_{\mathbf{F}}{}^G$. Then the cohomology groups $H^n(H, A)$ are zero for $n \geqslant 1$.*

Proof. For every open normal subgroup U of G, let

$$A^U = \{a \in A \mid ua = a, u \in U\}$$

be the submodule of U-invariants. If A is an injective object of $\mathscr{C}_{\mathbf{C}}$, then A^U is easily seen to be an injective G/U module. By Ribes [17, Lemma 5.12], it is an injective (HU/U)-module, and from the isomorphism

$$H^n(H, A) \cong \varinjlim_{U} H^n(HU/U, A^U)$$

of Serre [18, I, Proposition 8], we deduce that $H^n(H, A) = 0$ for $n \geqslant 1$. The same argument applies if A is an injective object of $\mathscr{C}_{\mathbf{F}}{}^G$. ∎

PROPOSITION 1.4. *Let A be an object of $\mathscr{C}_{\mathbf{F}}^G$. Then $T(A)$ is an object of $\mathscr{C}_{\mathbf{C}}^G$, and one has natural isomorphisms of cohomology functors*

$$H^n(G, A) \cong H^n(G, T(A)),$$
$$H^{n+1}(G, H, A) \cong H^{n+1}(G, H, T(A)) \cong H^n(G, \Gamma(A)) \cong H^n(G, \Gamma(TA))$$

for all $n \geqslant 1$.

Proof. One easily sees that $T\colon \mathscr{C}_{\mathbf{F}}^G \to \mathscr{C}_{\mathbf{C}}^G$ is an exact functor mapping injectives to injectives, and

$$M_G^H \circ T = T \circ M_G^H, \qquad \Gamma \circ T = T \circ \Gamma.$$

Since both $H(G, -)$ and $H(G, T(-))$ are effaceable by injectives in $\mathscr{C}_{\mathbf{F}}$, and $\Gamma\colon \mathscr{C}_{\mathbf{C}}^G \to \mathscr{C}_{\mathbf{C}}^G$ is exact, the isomorphisms

$$H^n(G, -) \cong H^n(G, T(-)), \qquad H^n(G, \Gamma(-)) \cong H^n(G, \Gamma(T(-))) \qquad (n \geqslant 0)$$

follow from a standard comparison theorem (Ribes [17, Corollary 5.7]). Lemma 1.2 yields the natural isomorphism

$$X(G, H, A) \cong X(G, H, T(A)),$$

and we may as well assume that A is in $\mathscr{C}_{\mathbf{C}}^G$. Using the isomorphism

$$H^n(G, -) \cong \operatorname{Ext}^n_{\mathfrak{A}\iota G}(\hat{\mathbb{Z}}_{\mathbf{C}}, -), \qquad n \geqslant 0$$

(Brumer [2, Lemma 4.2(i)]), and applying $\operatorname{Hom}_{\mathfrak{A}\iota G}(\hat{\mathbb{Z}}_{\mathbf{C}}, -)$ to the exact sequence

$$0 \to A \to M_G^H(A) \to \Gamma(A) \to 0,$$

we obtain a long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_{\mathfrak{A}\iota G}(\hat{\mathbb{Z}}_{\mathbf{C}}, A) \to \operatorname{Hom}_{\mathfrak{A}\iota G}(\hat{\mathbb{Z}}_{\mathbf{C}}, M_G^H(A)) \to \operatorname{Hom}(\hat{\mathbb{Z}}_{\mathbf{C}}, \Gamma(A)) \\ &\to H^1(G, A) \to H^1(G, M_G^H(A)) \to H^1(G, \Gamma(A)) \to H^2(G, A) \\ &\to H^2(G, M_G^H(A)) \to \cdots. \end{aligned}$$

By Serre [18, I-12, Proposition 10], we have natural isomorphisms

$$H^n(G, M_G^H(A)) \cong H^n(H, A), \qquad n \geqslant 0,$$

and if A is an injective object of $\mathscr{C}_{\mathbf{C}}^G$, then

$$H^n(G, A) = 0 = H^n(H, A)$$

for all $n \geqslant 1$ (Lemma 1.3). From the above long exact sequence, we conclude that $H^n(G, \Gamma(A)) = 0$ for all $n \geqslant 1$, and by Lemma 1.2 and the standard comparison theorem (Ribes [17, Corollary 5.7]), we have

$$H^n(G, \Gamma(-)) \cong H^{n+1}(G, H, -), \qquad n \geqslant 1. \quad \blacksquare$$

PROPOSITION 1.5. *Let H be a closed subgroup of a pro-$\mathbf{C}$-group G, and let A be an object of $\mathscr{C}_{\mathbf{C}}{}^G$. There exists a long exact sequence*

$$0 \to A^G \xrightarrow{i} A^H \xrightarrow{\delta} H^1(G, H, A) \xrightarrow{j} H^1(G, A)$$
$$\xrightarrow{i} H^1(H, A) \xrightarrow{\delta} H^2(G, H, A) \xrightarrow{j} \cdots,$$

where the i's are restriction maps induced by the inclusion $H \subset G$.

Proof. Substitute A^G for $\operatorname{Hom}_{\mathfrak{A}\imath G}(\hat{\mathbb{Z}}_{\mathbf{C}}, A)$, A^H for $\operatorname{Hom}_{\mathfrak{A}\imath G}(\hat{\mathbb{Z}}_{\mathbf{C}}, M_G{}^H(A))$, $H^1(G, H, A)$ for $\operatorname{Hom}(\hat{\mathbb{Z}}_{\mathbf{C}}, \Gamma(A))$, $H^n(H, A)$ for $H^n(G, M_G{}^H(A))$ and $H^{n+1}(G, H, A)$ for $H^n(G, \Gamma(A))$ $(n \geqslant 1)$ in the long exact sequence of the proof of Proposition 1.4. $\blacksquare$

COROLLARY 1.6. *Let 1 denote the group with one element. Then*

$$H^n(G, 1, A) \cong H^n(G, A), \qquad n \geqslant 2, \quad A \in \mathscr{C}_{\mathbf{C}}{}^G.$$

LEMMA 1.7. *Let $H \subset K \subset G$ be embeddings of pro-$\mathbf{C}$-groups. Then $\{H^n(K, H, -) \mid n \geqslant 1\}$ is a universal sequence of connected functors in $\mathscr{C}_{\mathbf{F}}{}^G$ ("∂-foncteur universel" in the terminology of Grothendieck* [8]).

Proof. The sequence is certainly exact. One easily deduces from Lemma 1.3 and Proposition 1.5 that it is effaceable. $\blacksquare$

LEMMA 1.8. *Let $X(G, A)$ stand for the abelian group $X(G, 1, A)$ of continuous crossed homomorphisms from a pro-$\mathbf{C}$-group G into a discrete $\mathfrak{A}\imath G$-module A. Let F be a functor from a small category $\mathbf{I}$ into the category* **PC** *of pro-$\mathbf{C}$-groups. One has a natural isomorphism*

$$X(\varinjlim F, A) \cong \varprojlim X(F(-), A)$$

of abelian groups, where, for each $i \in |\mathbf{I}|$, the $F(i)$-module structure of A is induced by the canonical map: $F(i) \to \varinjlim F$.

Proof. Since A is a direct limit of finite $\mathfrak{A}\imath G$-modules, we may assume without loss in generality that A is a finite $\mathfrak{A}\imath G$-module and an abelian group in $\mathbf{C}$. For each $i \in |\mathbf{I}|$, let

$$\varphi_i \colon F(i) \to G = \varinjlim F, \qquad p_i \colon \varprojlim X(F(-), A) \to X(F(i), A)$$

be the canonical maps. There exists a unique homomorphism

$$\eta\colon X(G, A) \to \varprojlim X(F(-), A)$$

such that

$$p_i(\eta(e)) = e \circ \varphi_i\colon F(i) \to A$$

for all $i \in |\mathbf{I}|$ and $e \in X(G, A)$.

Consider the commutative diagram in **PC** with split exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \hat{G} & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & G & \longrightarrow & 1 \\
 & & \Vert & & \uparrow{\scriptstyle \hat{\varphi}_i} & & \uparrow{\scriptstyle \varphi_i} & & \\
0 & \longrightarrow & A & \longrightarrow & \widehat{F(i)} & \underset{\sigma_i}{\overset{\pi_i}{\rightleftarrows}} & F(i) & \longrightarrow & 1
\end{array}$$

where $\hat{G}$ is the space $A \times G$ with the product topology and with multiplication defined by

$$(a, g)(a', g') = (a + ga', gg');$$

$F(i)$ is the product space $A \times F(i)$, with multiplication defined by

$$(a, h)(a', h') = (a + \varphi_i(h)a', hh'),$$

and

$$\pi(a, g) = g, \qquad \sigma(g) = (0, g), \qquad \pi_i(a, h) = h, \qquad \sigma_i(h) = (0, h),$$
$$\hat{\varphi}_i(a, h) = (a, \varphi_i(h)).$$

For each $f \in \varprojlim X(F(-), A)$ and $i \in |\mathbf{I}|$, we define a map

$$t_i\colon F(i) \to \widehat{F(i)}, \qquad t_i(h) = (p_i(f)(h), h), \qquad h \in F(i).$$

It is immediately verified that each t_i is a continuous homomorphism, and the maps $\hat{\varphi}_i \circ t_i\colon F(i) \to \hat{G}$ induce a unique morphism $s\colon G \to \hat{G}$ such that $s \circ \varphi_i = \hat{\varphi}_i \circ t_i$ for all $i \in |\mathbf{I}|$. Now,

$$\pi(s(\varphi_i(h))) = \pi(\hat{\varphi}_i(t_i(h))) = \varphi_i(h)$$

for all $i \in |\mathbf{I}|$, and hence $\pi \circ s = id_G$. So we can write $s(g) = (e(g), g)$, where $e\colon G \to A$ is easily seen to be a continuous crossed homomorphism. We define

$$\theta\colon \varprojlim X(F(-), A) \to X(G, A)$$

by $\theta(f) = e$. One verifies with no difficulty that θ and η are inverse isomorphisms, and are natural in A. ∎

PROPOSITION 1.9. *Let G be the colimit (or pushout) in the category* **PC** *of a diagram consisting of maps $\alpha_i\colon H \to G_i$, $(i \in I)$, and let $\varphi_i\colon G_i \to G$ be the canonical maps $(i \in I)$. Let $\alpha = \varphi_i \circ \alpha_i\colon H \to G$. One has a natural isomorphism*

$$X(G, \alpha(H), A) \cong \prod_{i \in I} X(G_i, \alpha_i(H), A),$$

where A is a discrete $\mathcal{O}llG$-module, and hence also a discrete G_i-module, by the maps φ_i.

Proof. As in the preceding lemma, we may suppose, without loss in generality, that A is a finite abelian group in **C**. Let

$$p_i\colon \prod_{i \in I} X(G_i, \alpha_i(H), A) \to X(G_i, \alpha_i(H), A)$$

be the canonical projection, and define a homomorphism

$$\eta\colon X(G, \alpha(H), A) \to \prod_{i \in I} X(G_i, \alpha_i(H), A),$$

by writing $p_i(\eta(e)) = e \circ \varphi_i$ for all $e \in X(G, \alpha(H), A)$, $i \in I$.

As in the previous proof, one has a commutative diagram in **PC**, with split exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \hat{G} & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & G & \longrightarrow & 1 \\
 & & \| & & \uparrow{\scriptstyle \hat{\varphi}_i} & & \uparrow{\scriptstyle \varphi_i} & & \\
0 & \longrightarrow & A & \longrightarrow & \hat{G}_i & \underset{\sigma_i}{\overset{\pi_i}{\rightleftarrows}} & G_i & \longrightarrow & 1
\end{array}$$

where $\hat{G}$ is the product space $A \times G$, with multiplication defined by

$$(a, g)(a', g') = (a + ga', gg'), \qquad a, a' \in A, \quad gg' \in G$$

and $\hat{G}_i$ is the product space $A \times G_i$, with multiplication defined by

$$(a, g_i)(a', g_i') = (a + \varphi_i(g_i)a', g_i g_i'), \qquad a, a' \in A, \quad h, h' \in F(i).$$

For each $f \in \prod_{i \in I} X(G_i, \alpha_i(H), A)$, define a map $t_i\colon G_i \to \hat{G}_i$ by $t_i(g_i) = (p_i(f)(g_i), g_i)$. It is easily verified that each t_i is a continuous homomorphism, and the maps $\hat{\varphi}_i \circ t_i\colon G_i \to \hat{G}$ induce a unique morphism $s\colon G \to \hat{G}$ such that $s \circ \varphi_i = \hat{\varphi}_i \circ t_i$ for all $i \in I$. Now

$$\pi(s(\varphi_i(g_i))) = \pi(\hat{\varphi}_i(t_i(g_i))) = \varphi_i(g_i)$$

for all $i \in I$ and $g_i \in G_i$, and hence $\pi \circ s = id_G$. So, we can write $s(g) = (e(g), g)$, where $e: G \to A$ is easily seen to be a continuous crossed homomorphism that is trivial on $\alpha(H)$. We define

$$\theta: \prod_{i \in I} X(G_i, \alpha_i(H), A) \to X(G, \alpha(H), A)$$

by $\theta(f) = e$. One easily verifies that θ and η are inverse isomorphisms, and are natural in A. ▌

Let $H \subset G$, $L \subset K$ be pro-**C**-groups. Let $\varphi: K \to G$ be a continuous group homomorphism with $\varphi L \subset H$. If A is a discrete $(\mathcal{O}lG)$-module then it possesses a natural $(\mathcal{O}lK)$-module structure induced by φ. Then φ induces a natural homomorphism

$$\varphi^1: X(G, H, A) \to X(K, L, A)$$

given by

$$(\varphi^1 f)x = f(\varphi x),$$

which in turn induces mappings

$$\varphi^n: H^n(G, H, A) \to H^n(K, L, A).$$

THEOREM 1.10 (The Excision Axiom). *Let L be a common closed subgroup of two pro-**C**-groups H and K, and suppose that the pushout G in **PC** of $L \subset H$ and $L \subset K$ has the property that the canonical map: $K \to G$ is injective. Then the morphisms*

$$\varphi^n: H^n(G, H, -) \to H^n(K, L, -) \qquad (n \geqslant 1)$$

of functors: $\mathscr{C}_{\mathbf{C}}{}^G \to \mathcal{O}l$, induced by the inclusion

$$\varphi: (K, L) \to (G, H),$$

are isomorphisms.

Proof. It follows from Lemma 1.7 and the standard comparison theorem [17, Corollary 5.7], that it suffices to show that

$$\varphi^1: X(G, H, -) \to X(K, L, -)$$

is an isomorphism. So, let M be an object of $\mathscr{C}_{\mathbf{C}}{}^G$, and suppose that $f: K \to M$ is a continuous crossed homomorphism that annihilates L. Then, by Proposition 1.9, f and the trivial map $0: H \to M$ induce a continuous crossed homomorphism $\eta(f): G \to M$. Clearly, the map

$$\eta: X(K, L, M) \to X(G, H, M)$$

and the restriction $\varphi_M{}^1$ are inverse isomorphisms. ▌

PROPOSITION 1.11. *Suppose that* $\mathbf{C}$ *is contained in another class* $\mathbf{C}'$ *of finite groups, closed under the formation of subgroups, homomorphic images and extensions. Given pro-*$\mathbf{C}$*-groups* H_i, $i \in I$, *with a common closed subgroup* L, *we may view* H_i *as pro-*$\mathbf{C}'$*-groups, and we now assume the existence of their amalgamated product*

$$G = \coprod_{i \in I\, L} H_i$$

in the category $\mathbf{PC}'$. *Then*

$$H^n(G, L, A) \cong \prod_{i \in I} H^n(H_i, L, A)$$

for $n \geqslant 1$ *and* $A \in \mathscr{C}_{\mathbf{C}'}^G$, *where the canonical projections are induced by the inclusions*: $(H_i, L) \to (G, L)$.

Proof. By Proposition 1.4, we may without loss in generality take $\mathbf{C} = \mathbf{C}'$, and, by Lemma 1.7 and the standard comparison theorem [17, Corollary 5.7], it suffices to refer to Proposition 1.9, which gives the result for dimension 1. ▮

COROLLARY 1.12 (Neukirch [14]). *Let* $G = H \coprod K$ *be the coproduct in the category* $\mathbf{PC}$ *of two pro-*$\mathbf{C}$*-groups* H *and* K, *and let* A *be an object of* $\mathscr{C}_{\mathbf{C}}$. *Then*

$$H^n(G, A) = H^n(H, A) \oplus H^n(K, A)$$

for $n \geqslant 2$.

Proof. Put $L = 1$ in Proposition 1.11 and apply Corollary 1.6. ▮

THEOREM 1.13 (A Mayer–Vietoris sequence). *Assume the existence of the amalgamated product* $G = H \coprod_L K$ *in* $\mathbf{PC}$ *of two pro-*$\mathbf{C}$*-groups* H *and* K *over a common closed subgroup* L, *and let* A *be an object of* $\mathscr{C}_{\mathbf{C}}{}^G$. *Then the following sequence is exact*:

$$\begin{aligned} 0 &\to X(G, A) \to X(H, A) \oplus X(K, A) \to X(L, A) \to H^2(G, A) \to \cdots \\ &\to H^n(L, A) \xrightarrow{\Delta} H^{n+1}(G, A) \xrightarrow{\Phi} H^{n+1}(H, A) \oplus H^{n+1}(K, A) \\ &\xrightarrow{\Psi} H^{n+1}(L, A) \to \cdots, \end{aligned}$$

where

$$\Delta\colon H^n(L, A) \xrightarrow{\delta} H^{n+1}(K, L, A) \xrightarrow{(\varphi^{n+1})^{-1}} H^{n+1}(G, H, A) \xrightarrow{j} H^{n+1}(G, A),$$

with δ *and* j *as in Proposition* 1.5, φ^{n+1} *as in Theorem* 1.10; Φ *is the direct sum of the maps induced in cohomology by the inclusions* $H \hookrightarrow G$ *and* $K \hookrightarrow G$; $\Psi(\nu_1, \nu_2) = h_1^{n+1}(\nu_1) - h_2^{n+1}(\nu_2)$, *where* h_1^{n+1} *and* h_2^{n+1} *are maps induced in* *cohomology by the inclusions* $h_1: L \hookrightarrow H$ *and* $h_2: L \hookrightarrow K$ *respectively,* $\nu_1 \in H^{n+1}(H, A)$, $\nu_2 \in H^{n+1}(K, A)$.

The proof is formally the same as in Eilenberg and Steenrod [3, Theorem 15.3(c), p. 43].

Remark 1.14. Barr and Beck have proved (see [1, Section 7, p. 297 and Section 9, p. 310]) that the analogue of Theorem 1.13 is valid in a very general setting in the presence of Proposition 1.11, namely for a class of categories tripleable over *sets*, and triple cohomology. The category **PC** is tripleable over *sets* (see Kennison and Gildenhuys [9]); however, we did not try to ascertain whether the usual cohomology groups of pro-**C**-groups are obtainable from this triple.

2. Topological Colimits of Pro-C-Groups

Let $\mathscr{P}$ be a category with pullbacks. A *category object* in $\mathscr{P}$ is a 6-tuple $\mathscr{C} = (F, X, \alpha, \beta, \mu, m)$, where $\mu: X \hookrightarrow F$ is a monomorphism in $\mathscr{P}$, α and β are maps $F \to X$, called the *domain map* and *codomain map*, respectively, such that $\alpha\mu = \beta\mu = id_F$;

$$\begin{array}{ccc} M & \longrightarrow & F \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ F & \xrightarrow[\beta]{} & X \end{array}$$

is a pullback in $\mathscr{P}$, and $m: M \to F$, called *composition*, is a map satisfying certain more or less obvious conditions. We are only interested in the case where $\mathscr{P}$ is the category *top* of topological spaces, or the category *Ptop* of pointed topological spaces, in which case these conditions can be expressed by requiring that $\mathbf{U}\mathscr{C}$ be a (small) category, where the objects of $\mathbf{U}\mathscr{C}$ are the elements of X, the maps are the elements of F, the identity map on $x \in X$ is $\mu(x)$, the domain (resp. codomain) of $f \in F$ is the object $\alpha(f)$ (resp. $\beta(f)$), and composition $\circ$ is defined as follows. Suppose that $f, f' \in F$ and $\alpha(f') = \beta(f)$. Let $1 = \{1\}$ be the terminal object in $\mathscr{P}$, and define $g: 1 \to F$, $g': 1 \to F$ be $g(1) = f$, $g'(1) = f'$; then g and g' determine a unique map $h: 1 \to M$, and we let $f' \circ f = m(h(1))$. In order for $\mathbf{U}\mathscr{C}$ to be a category, we need among other things that $\alpha(f' \circ f) = \alpha(f)$ and $\beta(\rho' \circ \rho) = \beta(\rho)$. We will call $\mathbf{U}\mathscr{C}$ the *underlying category* of $\mathscr{C}$.

Let **C** be a class of finite groups, closed under the formation of subgroups, finite products and homomorphic images.

A *functor*: $\mathscr{C} \to \mathbf{PC}$ from a category object $\mathscr{C} = (F, X, \alpha, \beta, \mu, m)$ of the category $\mathscr{P} = top$ (resp. $\mathscr{P} = Ptop$) into the category **PC** of pro-**C**-groups is a pair $T = (\pi, \tau)$, where $\pi\colon E \to (X, *)$ is a map in $\mathscr{P}$ such that for every $x \in X$, the *fiber* $G_x = \pi^{-1}(\{x\})$ (resp. $G_x = \pi^{-1}(\{x, *\})$) is a pro-**C**-group and $G_x \hookrightarrow E$ is a morphism in $\mathscr{P}$;

$$\begin{array}{ccc} \alpha^*(E) & \longrightarrow & E \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \pi} \\ F & \xrightarrow{\ \alpha\ } & X \end{array} \qquad\qquad \begin{array}{ccc} \beta^*(E) & \xrightarrow{\ \beta''\ } & E \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \pi} \\ F & \xrightarrow{\ \beta\ } & X \end{array}$$

are pullback diagrams in $\mathscr{P}$ and $\tau\colon \alpha^*(E) \to \beta^*(E)$ is a map in $\mathscr{P}$, with $\beta'\tau = \alpha'$ and the property that $UT\colon \mathbf{U}\mathscr{C} \to \mathbf{PC}$, defined as follows, is a functor in the usual sense. For every object $x \in X$ of $\mathbf{U}\mathscr{C}$, write $(UF)(x) = G_x$. For each $t \in G_{x_1}$ and $f \in F$ with $\alpha(f) = x_1$, $\beta(f) = x_2$, let $\hat{t}\colon 1 \to \alpha^*(E)$ be the map induced by $1 \to E$, $1 \mapsto t$ and $1 \to F$, $1 \mapsto f$. Then $\beta(\beta'(\tau(\hat{t}(1)))) = \beta(f) = x_2$, so that $\beta''(\tau(\hat{t}(1))) \in \pi^{-1}(\{x_2, *\}) = G_{x_2}$. So, $(UT)(f)\colon G_{x_1} \to G_{x_2}$ is well defined by writing $(UT)(f)(t) = \beta''(\tau(\hat{t}(1)))$.

A *morphism* $\varphi\colon T = (\pi, \tau) \to T' = (\pi', \tau')$ *of functors from a category object* $\mathscr{C}$ *of* $\mathscr{P}$ *into* **PC** is a map $\varphi\colon E \to E'$ in $\mathscr{P}$, where $\pi\colon E \to X$ and $\pi'\colon E' \to X$, such that $\pi'\varphi = \pi$, the following diagram commutes

$$\begin{array}{ccc} \alpha^*(E) & \xrightarrow{\ \tau\ } & \beta^*(E) \\ {\scriptstyle \alpha^*(\varphi)}\downarrow & & \downarrow{\scriptstyle \beta^*(\varphi)} \\ \alpha^*(E') & \xrightarrow{\ \tau'\ } & \beta^*(E') \end{array}$$

and the restriction of φ to the fiber G_x above $x \in X$ defines a morphism: $G_x \to G_x'$ in **PC**. Here $G_x = \pi^{-1}(\{x\})$, $G_x' = (\pi')^{-1}(\{x\})$ (resp. $G_x = \pi^{-1}(\{x, *\})$, $G_x' = (\pi')^{-1}(\{x\})$ if $\mathscr{P} = top$ (resp. *Ptop*).

One easily sees that the functors from the category object $\mathscr{C}$ of $\mathscr{P}$ into **PC**, and morphisms of these functors, form a category $\mathbf{P}\mathscr{C}^{\mathscr{C}}$.

To every pro-**C**-group G, there corresponds a *constant functor* $K(G) = (\pi, \tau)\colon \mathscr{C} \to \mathbf{PC}$, where π is the projection from the product $G \circ X$ of G and X in $\mathscr{P}$, onto X, and

$$\tau\colon \alpha^*(G \circ X) = G \circ F \to G \circ F = \beta^*(G \circ X)$$

is the identity map.

A pair (G, η) consisting of a pro-**C**-group G and a morphism $\eta\colon T \to K(G)$ in $\mathscr{P}^{\mathscr{C}}$ is said to be a *topological colimit* of a functor $T\colon \mathscr{C} \to \mathscr{P}$ if for every other pro-**C**-group G' and morphism $\varphi\colon T \to K(G')$, there exists a unique morphism $\psi\colon G \to G'$ in **PC**, such that $K(\psi)\eta = \varphi$.

PROPOSITION 2.1. *Let $T: \mathscr{C} \to \mathbf{PC}$ be a functor from a category object $\mathscr{C}$ of $\mathscr{P}$ into the category* **PC** *of pro-**C**-groups, where $\mathscr{P}$ is the category top or the category Ptop. Then the topological colimit of T exists and is unique up to isomorphism.*

Proof. Let $UT: \mathbf{U}\mathscr{C} \to \mathbf{PC}$ be the corresponding underlying functor, and L its colimit in **PC**. Let $\mathscr{C} = (F, X, \alpha, \beta, \mu, m)$, $T = (\pi, \tau)$, $\pi: E \to X$. For each $x \in X$, one has a canonical morphism

$$\eta_x: G_x \to L$$

of pro-**C**-groups, where $G_x = \pi^{-1}(\{x\})$ if $\mathscr{P} = top$ and $G_x = \pi^{-1}(\{x, *\})$ if $\mathscr{P} = Ptop$ and $*$ is the distinguished point of X.

We define $\nu: E \to L$ by $\nu(e) = \eta_{\pi(e)}(e)$. Let Φ be the family of open normal subgroups N of L, such that $\nu^{-1}(gN)$ is open in E, for every coset gN of N in L. Let $G = \varprojlim_{N\in\Phi} L/N$ (with $G = (1)$ if $\Phi = \varnothing$). For each $N \in \Phi$, let $p_N: G \to L/N$ be the canonical projection of G onto the discrete group L/N. Then the maps $p_N \circ \nu: E \to L/N$ are continuous and induce a morphism $\eta': E \to G$ in $\mathscr{P}$. The maps η' and π induce a map η from E into the product $G \circ X$ of G and X in $\mathscr{P}$, and η defines a morphism $\eta: T \to K(G)$ in $\mathscr{P}^{\mathscr{C}}$. One easily verifies that the pair (G, η) is a topological colimit of T. Uniqueness is clear. ▌

PROPOSITION 2.2. *Let $(X, *)$ be a pointed compact Hausdorff totally disconnected space, and let $\{G_x \mid x \in X\}$ be a family of pro-**C**-groups with $G_* = (1)$, and such that the map $x \mapsto G_x$ is locally constant on $X\backslash\{*\}$. Then the free pro-**C**-product (see Gildenhuys and Ribes [7]) of these pro-**C**-groups is a topological colimit of a functor from a category object of Ptop into* **PC**.

Proof. We recall the definition of the étale space $E = \bigvee_{x\in X} G_x$. As a pointed set, E is the coproduct of the pointed sets $(G_x, 1)$, $x \in X$. For all $x \in X\backslash\{*\}$, there exists a so-called constant open neighborhood U of x in $X\backslash\{*\}$, with $G_x = G_y$ for all $x, y \in U$, and for such a set U we define

$$p_U: U \times G_x \to E, \qquad (u, t) \mapsto t \in A_u, \quad (u, t) \in U \times G_x.$$

A subset W of E is open iff

(i) for every constant open subset U of X, the set $p_U^{-1}(W)$ is open with respect to the product topology on $U \times G_x$, $(x \in U)$;

(ii) if W contains the distinguished point 1 of E, there is a neighborhood V of $*$ in X, such that $G_y \subset W$ whenever $y \in V$.

The map $\pi: E \to X$ is defined by $\pi(1) = *$ and $\pi(e) = x$ if $e \in G_x\backslash\{1\}$.

Let $\mathscr{C}$ be the category object $(X, X, id_X, id_X, id_X, id_X)$ of *Ptop*. Clearly $T = (\pi, id_X): \mathscr{C} \to \mathbf{PC}$ is a functor. There is a bijective correspondance between maps $\eta: T \to K(G)$, $(\eta: E \to G \circ X)$ of functors in $\mathbf{PC}^{\mathscr{C}}$ and maps $\eta': E \to G$ in *Ptop* whose restrictions to the fibers $\pi^{-1}(\{x, *\})$, $x \in X$, are morphisms of pro-**C**-groups. The pair (G, η') is a free pro-**C**-product of $\{G_x \mid x \in X\}$ iff for each morphism φ from E into the underlying pointed space of a pro-**C**-group H such that $\varphi \mid \pi^{-1}(\{x, *\})$ is a morphism in **PC**, there exists a unique morphism $\psi: G \to G'$ of pro-**C**-groups such that $\psi \circ \eta' = \varphi$. Clearly this condition is equivalent to (G, η) being a colimit of T. ▮

We will now look at pro-p-groups $G = F(x_0, x_1, ..., x_{m+1})/(r)$ $(m \geqslant 0)$ with one defining relator r, which belongs to the Frattini subgroup F^* of $F = F(x_0, x_1, ..., x_{m+1})$. (If $r \notin F^*$, then G is free.) Changing the basis of F, if necessary, we may assume without loss in generality that r belongs to the closed normal subgroup $N = (x_0, x_1, ..., x_m)$ of F, generated by $x_0, x_1, ..., x_m$. We write $R = (r)$ and $x_{i,j} = x_{m+1}^{-j} x_i x_{m+1}^{j}$ $(i \in \{0, 1, ..., m\}, j \in \hat{\mathbb{Z}}_p)$. We know that N is the free pro-**C**-group generated by the homeomorphic image

$$\{x_{i,j} \in N \mid i \in \{0, 1, ..., m\}, j \in \hat{\mathbb{Z}}_p\}$$

of the product $\{0, 1, ..., m\} \times \hat{\mathbb{Z}}_p$ of the discrete space $\{0, 1, ..., m\}$ and the underlying space of the ring of $p$adic integers, under the map

$$\omega: X \to N, \qquad (i, j) \mapsto x_{i,j}.$$

(See Gildenhuys and Lim [6, Corollary 2.2].) It follows that N is also freely generated by $\omega(X)$, where $X = \{0, 1, ..., m\} \times \mathbb{Z}$, and $\mathbb{Z}$ has the p-adic topology. We now suppose that r belongs to the closed subgroup C of N generated by $x_{i,j}$, $j = h_i, h_i + 1, ..., h_i + n_i$, $n_i \geqslant 0$, $i = 0, 1, ..., m$. (If r is a (finite) word in the generators $x_0, x_1, ..., x_{m+1}$, this assumption is always justified.) Since we can replace the basis $x_0, x_1, ..., x_{m+1}$ by the basis

$$\{x_{m+1}^{-h_i} x_i x_{m+1}^{h_i} \mid i = 0, 1, ..., m\} \cup \{x_{m+1}\}$$

if necessary [18, I-Proposition 2.5], we may assume without loss in generality that $h_i = 0$ for all $i = 0, 1, ..., m$. Let $r_j = x_{m+1}^{-j} r x_{m+1}^{j}$ $(j \in \hat{\mathbb{Z}}_p)$ and identify the free pro-p-group

$$F_0 = F(x_{0,0}, x_{0,1}, ..., x_{0,n_0}; x_{0,0}, ..., x_{0,n_1}; ...; x_{m,0}, ..., x_{m,n_m})$$

with its obvious image C in N. For every $j \in \hat{\mathbb{Z}}_p$, the free pro-p-group F_j generated by the finite set

$$\{x_{i,h+j} \mid h = 0, 1, ..., n_i; i = 0, 1, ..., m\},$$

can also be identified in an obvious way with a closed subgroup of N, containing r_j. For every $j \in \hat{\mathbb{Z}}_p$ one has a natural map

$$\gamma_j\colon G_j = F_j/(r_j) \to N/R$$

(in general not injective, see Gildenhuys [5, Remark (i)]). For each $j \in \mathbb{Z}_p$, let H_j be the free pro-p-group generated by the set

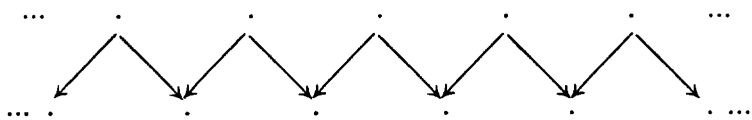
$$\{x_{i,h+j} \mid 0 \leqslant h \leqslant n_i - 1;\ i = 0, 1,\dots, m\}.$$

For each $j \in \hat{\mathbb{Z}}_p$, there are two maps

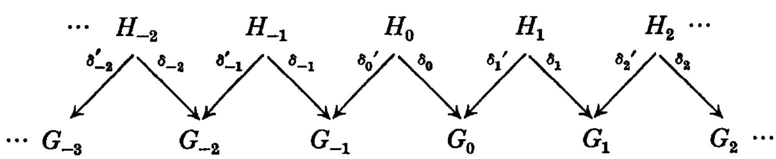
$$\delta_j\colon H_j \to G_j, \qquad \delta_j'\colon H_j \to G_{j-1}$$

that send each $x_{i,k}$ to its natural image in G_j and G_{j-1}, respectively.

PROPOSITION 2.3. *The closed normal subgroup N/R of G is a topological colimit of a functor $T\colon \mathscr{C} \to \mathbf{P}_p$ from a category object $\mathscr{C}$ of top into the category $\mathbf{P}_p$ of pro-p-groups, where the underlying category $\mathbf{U}\mathscr{C}$ of $\mathscr{C}$ is represented by the infinite diagram*



and the underlying functor $UT\colon \mathbf{U}\mathscr{C} \to \mathbf{P}_p$ maps this diagram onto the diagram



Proof. Let

$$\mathscr{C} = (\mathbb{Z} \cup \mathbb{Z} \cup \mathbb{Z} \cup \mathbb{Z}, \mathbb{Z} \cup \mathbb{Z}, \alpha, \beta, \mu, m),$$

where $\mathbb{Z}$ has the $p$adic topology and the symbol $\cup$ denotes the coproduct in *top*. Let E_1 (resp. E_2) be the product $G_0 \times \mathbb{Z}$ (resp. $H_0 \times \mathbb{Z}$) in *top*. The maps $\pi_1\colon E_1 \to \mathbb{Z}$ and $\pi_2\colon E_2 \to \mathbb{Z}$ are projections and $\pi\colon E = E_1 \cup E_2 \to \mathbb{Z} \cup \mathbb{Z}$ is their coproduct. The functor $T\colon \mathscr{C} \to \mathbf{P}_p$ is of the form $T = (\pi, \tau)$. Note that for each $j \in \mathbb{Z}$ one has an isomorphism $\sigma_j\colon G_0 \to G_j$ and an isomorphism $\tau_j\colon H_0 \to H_j$. The pro-p-group G_j (resp. H_j) is identified with the fiber $\pi_1^{-1}(\{j\})$ (resp. $\pi_2^{-1}(\{j\})$). It is now clear how τ, α, β, μ, and m are to be defined, in order for the conditions of Proposition 2.3 to be satisfied.

There exists a unique map η_1: $E_1 \to N/R$ that sends $e \in G_j \subset E$ to $\gamma_j(e)$ (i.e., the image $\bar{x}_{i,h}$ of $x_{i,h}$, $0 \leqslant i \leqslant m, j \leqslant h \leqslant n_i + j$, is sent to its natural image in N/R), and has the property that $\eta_1 \mid G_j$: $G_j \to N/R$ is a morphism in $\mathbf{P}_p$ for each $j \in \mathbb{Z}$. For every open normal subgroup W of N/R, there exists a natural number k, such that the images of $x_{i,h}$ and $x_{i,t}$ are congruent mod W, whenever $h \equiv t \bmod p^k\mathbb{Z}$, $0 \leqslant i \leqslant m$. Hence,

$$\gamma_j(\sigma_j(e)) \equiv \gamma_t(\sigma_t(e)) \bmod W$$

whenever $j \equiv t \bmod p^k\mathbb{Z}$, $e \in G_0$, and η_1 is continuous. Moreover, it has the property that $\eta_1 \mid G_j$: $G_j \to N/R$ is a morphism in $\mathbf{P}_p$ for each $j \in \mathbb{Z}$. Similarly, one has a map η_2: $E_2 \to N/R$ that sends $x_{i,h} \in H_j \subset E_2$, $0 \leqslant i \leqslant m$, $j \leqslant h \leqslant n_i + j - 1$ to its natural image in N/R, and has the property that $\eta_2 \mid H_j$ is a morphism in $\mathbf{P}_p$ for each $j \in \mathbb{Z}$. The maps η_1 and η_2 now induce a map η': $E \to N/R$ in *top*, and the maps η' and π induce a map η: $E \to (N/R) \times (\mathbb{Z} \cup \mathbb{Z})$, which can be viewed as a morphism: $T \to K(N/R)$ in $\mathbf{P}_p^{\mathscr{C}}$. We proceed to verify that η: $T \to K(N/R)$ satisfies the universal property of a topological colimit. So, let φ: $T \to K(G')$ be a morphism in $\mathbf{P}_p^{\mathscr{C}}$; then the composition of φ: $E \to G' \times (\mathbb{Z} \cup \mathbb{Z})$ and the projection $G' \times (\mathbb{Z} \cup \mathbb{Z}) \to G'$ gives a morphism φ': $E \to G'$ in *top*. For every open normal subgroup V of G', there exists a natural number k such that if $h \equiv j \bmod p^k\mathbb{Z}$, then $\varphi'(\bar{x}_{i,h}) \equiv \varphi'(\bar{x}_{i,j})$ and $\varphi'(x_{i,h}) \equiv \varphi'(x_{i,j}) \bmod V$, where $\bar{x}_{i,h}$ denotes the image of $x_{i,h}$ in some $G_j \subset E_1 \subset E$ $(j \leqslant h \leqslant j + n_i$, $0 \leqslant i \leqslant m)$, and $x_{i,h}$ has been identified with its image in $H_j \subset E_2 \subset E$ $(j \leqslant h \leqslant j + n_i - 1$, $0 \leqslant i \leqslant m)$. Since N is freely generated by the topological space $\{x_{i,h} \mid 0 \leqslant i \leqslant m,\ h \in \mathbb{Z}\}$, there exists a unique map θ_V: $N \to G'/V$ that sends $x_{i,h}$ to the image of $\varphi'(x_{i,h})$ in G/V. Moreover, the restriction of φ' to each fiber G_j is a continuous homomorphism; hence $\theta_V(r_j) = 1$ for all $j \in \hat{\mathbb{Z}}_p$, and θ_V induces a map θ_V': $N/R \to G'/V$. The maps θ_V' now induce the desired map ψ: $N/R \to G' = \varprojlim G'/V$, for which $\psi \circ \eta' = \varphi'$, and hence $K(\psi) \circ \eta = \varphi$: $T \to K(G')$ in $\mathbf{P}_p^{\mathscr{C}}$. The uniqueness of ψ is easily verified. ∎

3. On the Cohomology of Free Pro-C-Products of Pro-C-Groups

Let $\mathbf{C}$ be a nontrivial class of finite groups, closed under the formation of subgroups, extensions and homomorphic images, and let $(X, *)$ be a pointed compact Hausdorff totally disconnected topological space. Let $\{G_x \mid x \in X\}$ be a family of pro-$\mathbf{C}$-groups, such that $G_* = (1)$ and $x \mapsto G_x$ is locally constant outside $\{*\}$. There exists a family $\mathbf{R}$ of open equivalent relations R on X such that $G_x = G_y$ whenever xRy and not $xR*$. Writing

$G_{xR} = G_x$ and G^R for the coproduct of the finite set $\{G_{xR} \mid xR \in X/R\}$ of pro-**C**-groups (here xR denotes the equivalence class of x), we have an isomorphism

$$\coprod_{x \in X} G_x = \varprojlim_{R \in \mathbf{R}} G^R,$$

where the left side denotes the free pro-**C**-product of the family $\{G_x \mid x \in X\}$ (see Gildenhuys and Ribes [7, Proposition 2.1]).

PROPOSITION 3.1. *For every discrete $\mathfrak{A}G$-module A, where $\mathfrak{A}G = \mathbb{Z}_{\mathbf{C}}[[G]]$ (see Section 1), one has a natural isomorphism*

$$H^n\left(\coprod_{x \in X} G_x, A\right) = \varinjlim_{R \in \mathbf{R}} \bigoplus_{xR \in X/R} H^n(G_{xR}, A^R), \qquad n \geqslant 2,$$

where

$$A^R = \{a \in A \mid ka = a, k \in K_R\}$$

is the submodule of invariants under the kernel K_R of the canonical projection:

$$G = \coprod_{x \in X} G_x \to G^R = \coprod_{x \in X/R} G_{xR},$$

and the G_{xR} module structure on A^R is induced by the canonical inclusion $G_{xR} \hookrightarrow G^R$.

Proof. The natural isomorphism

$$\bigoplus_{xR \in X/R} H^n(G_{xR}, A^R) \cong H^n(G^R, A^R)$$

is an immediate consequence of Corollary 1.12, and the result now follows from Serre [18, I-Proposition 8]. ∎

COROLLARY 3.2 (Neukirch [14]). *If G is the restricted free pro-**C**-product of a family $\{G_x\}_{x \in X}$ of pro-**C**-groups, then one has a natural isomorphism:*

$$H^n(G, A) \cong \bigoplus_{x \in X} H^n(G_x, A), \qquad n \geqslant 2,$$

where A is a discrete $\mathfrak{A}G$-module.

Proof. Let $\bar{X} = X \cup \{*\}$ be the one point compactification of the discrete space X, and $G_* = (1)$, then

$$G \cong \coprod_{x \in \bar{X}} G_x \cong \varprojlim_{R \in \mathbf{R}} G^{(R)},$$

and $\mathbf{R}$ admits a cofinal subset $\mathbf{R}'$ of equivalence relations whose equivalence classes either contain $*$ or consist of a single element of X. Clearly

$$H^n(G, A) \cong \varinjlim_{R \in \mathbf{R}'} \bigoplus_{xR \in X/R} H^n(G_{xR}, A) \cong \bigoplus_{x \in X} H^n(G_x, A). \quad \blacksquare$$

4. Cohomology of Pro-p-Groups with Single Defining Relator

We keep the notation of Section 2. For every natural number k, let F^k be the free pro-p-group generated by

$$\{x_{i,j} \mid 0 \leqslant j \leqslant p^k + n_i - 1, 0 \leqslant i \leqslant m\}.$$

We identify F^k in an obvious way with a closed subgroup of N. Let $(r_0, r_1, ..., r_{p^k-1})$ be the closed normal subgroup of F^k generated by $r_0, r_1, ..., r_{p^k-1}$, and let

$$G^k = F^k/(r_0, r_1, ..., r_{p^k-1}).$$

Let k_0 be a fixed natural number such that $p^{k_0} \geqslant n_i$ for all $i = 0, 1, ..., m$. For $k \geqslant k_0$, write

$$\mathbb{Z}/p^k\mathbb{Z} = \{0, 1, ..., p^k - 1\}$$

and for $i \in \mathbb{Z}/p^k\mathbb{Z}$, write

$$r_i' = \pi(r_i) \in E_k = F(x_{i,j} \mid 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant p^k - 1),$$

and $D_k = E_k/(r_0', r_1', ..., r_{p^k-1})$, where

$$\pi: N = F(x_{i,j} \mid 0 \leqslant i \leqslant m, j \in \hat{\mathbb{Z}}_p) \to E_k$$

is induced by the canonical projection $\pi': \hat{\mathbb{Z}}_p \to \mathbb{Z}/p^k\mathbb{Z}$, i.e., r_i' is obtained from r_i by writing r_i as a limit of sequence of words w_n in the letters

$$x_{0,i}, x_{0,i+1}, ..., x_{0,i+n_0}; x_{1,i}, ..., x_{1,i+n_1}; ...; x_{m,i}, ..., x_{m,i+n_m}$$

and replacing $x_{i,j}$ by $x_{i,h}$ where h is the image in $\mathbb{Z}/p^k\mathbb{Z}$ of $j \in \hat{\mathbb{Z}}_p$. Let K_k be the closed subgroup of G^k generated by the images of the elements

$$x_{i,j+p^k}^{-1} x_{i,j}, \qquad 0 \leqslant j \leqslant n_i - 1, \quad 0 \leqslant i \leqslant m.$$

Clearly $N/R = \varprojlim D_k$ (see also Gildenhuys [5]).

LEMMA 4.1. *Let A be a discrete $\mathfrak{A}(N/R)$-module, where $\mathfrak{A}(N/R) = \hat{\mathbb{Z}}_p[[N/R]]$. Then A can be viewed as a discrete $\mathfrak{A}G^k$-module by the obvious map $G^k \to N/R$, and one has a natural isomorphism*

$$X(N/R, A) \simeq \varinjlim_{k \geqslant k_0} X(G^k, K_k, A),$$

where the direct limit is taken with respect to the maps defined in the proof below.

Proof. If $j \geqslant k \geqslant k_0$, the map $X(G^k, K_k, A) \to X(G^j, K_j, A)$ is induced by a map $q_{j,k}$: $G^j \to G^k$, which in turn is induced by $q'_{j,k}$: $F^j \to F^k$, defined as follows. If $h \geqslant p^j$, let $q'_{j,k}(x_{i,h}) = x_{i,h-p^j+p^k}$, and if

$$h \in \{0, 1,..., p^j - 1\} = \mathbb{Z}/p^j\mathbb{Z},$$

let t be the image of h in $\mathbb{Z}/p^k\mathbb{Z} = \{0, 1,..., p^k - 1\}$, and define $q'_{j,k}(x_{i,h}) = x_{i,t}$. Since each $r_h \in F^j$ $(0 \leqslant h \leqslant p^j - 1)$ involves sequences of letters

$$x_{i,0}, x_{i,1}, ..., x_{i,n_i} \qquad (0 \leqslant i \leqslant m)$$

of length $\leqslant n_i \leqslant p^{k_0} \leqslant p^k$, one has $q'_{j,k}(r_h) = r_t$, and the induced map $q_{j,k}$: $G^j \to G^k$ is therefore well defined.

Suppose now that $\gamma \in X(N/R, A)$. Since γ is continuous, there exists a natural number $k \geqslant k_0$ such that $\gamma(\bar{x}_{i,h}^{-1} \cdot \bar{x}_{i,j}) = 0$ whenever $j \equiv h \bmod p^k\hat{\mathbb{Z}}_p$, where $\bar{x}_{i,h}$ denotes the image of $x_{i,h}$ in N/R. So, the composite

$$\delta_k\colon G_k \to N/R \xrightarrow{\gamma} A$$

is a continuous crossed homomorphism that annihilates K_k. If

$$\alpha_k\colon X(G^k, K_k, A) \to \varinjlim_{k \geqslant k_0} X(G^k, K_k, A)$$

denotes the canonical map, then it is immediately verified that

$$\theta\colon X(N/R, A) \to \varinjlim_{k \geqslant k_0} X(G^k, K_k, A)$$

is well defined by $\theta(\gamma) = \alpha_k(\delta_k)$, (where k depends on γ), and θ is a homomorphism of abelian groups.

To define its inverse, suppose that $\epsilon \in X(G^k, K_k, A)$, $(k \geqslant k_0)$. Its image generates a finite abelian subgroup A' of A and, since the action of N/R on A is continuous, one can find a natural number $j \geqslant k$ such that $\bar{x}_{i,t}a' = \bar{x}_{i,s}a'$ whenever $t \equiv s$. mod $p^j\hat{\mathbb{Z}}_p$ and $a' \in A'$, where $\bar{x}_{i,t}$ and $\bar{x}_{i,s}$ denote the images of $x_{i,t}$ and $x_{i,s}$ in N/R. It follows that A' is a D_j-module.

Moreover, the image of $\psi = \epsilon \circ q_{j,k}\colon G^j \to A$ is contained in A', and ψ annihilates the elements

$$e_{i,h} = \bar{x}_{i,h}^{-1} \cdot \bar{x}_{i,h+p^j}, \qquad 0 \leqslant h \leqslant n_i - 1, \quad 0 \leqslant i \leqslant m,$$

where $\bar{x}_{i,h}$ now denotes the image of $x_{i,h}$ in G^j. If $g \in G^j$, then

$$\psi(g^{-1}e_{i,h}g) = (1 - g^{-1}e_{i,h}g)\,\psi(g) + g^{-1}\psi(e_{i,h}) = 0.$$

Thus ψ annihilates conjugates of $e_{i,h}$, products of conjugates of $e_{i,h}$ and their inverses, and limits of sequences of such products. It follows that ψ induces a continuous crossed homomorphism

$$\eta_k(\epsilon)\colon N/R \to D_j \cong G^j/M \to A' \subset A,$$

where M is the closed normal subgroup of G^j generated by the elements $e_{i,h}$ $(0 \leqslant h \leqslant n_i - 1, 0 \leqslant i \leqslant m)$, and $N/R \to D_j \cong G^j/M$ are the obvious maps. One easily verifies that the maps

$$\eta_k\colon X(G^k, K_k, A) \to X(N/R, A)$$

induce a homomorphism

$$\eta\colon \varinjlim X(G^k, K_k, A) \to X(N/R, A),$$

and that η and θ are inverse isomorphisms, natural in A. ∎

PROPOSITION 4.2. *Suppose that*

(i) *G_0 has cohomological dimension $\leqslant n$, where $n \geqslant 2$;*

(ii) *for all $k \geqslant k_0$, K_k is freely generated by the images of $x_{i,j}^{-1}x_{i,p^k+j}$ in G^k, where $0 \leqslant j \leqslant n_i - 1$, $0 \leqslant i \leqslant m$;*

(iii) *for every $k \geqslant k_0$ and discrete $\mathcal{Cl}G^k$-module M, the restriction map* Res: *$H^1(G^k, M) \to H^1(K_k, M)$ is injective;*

(iv) *the map $\gamma_0\colon G_0 \to N/R$ is injective;*

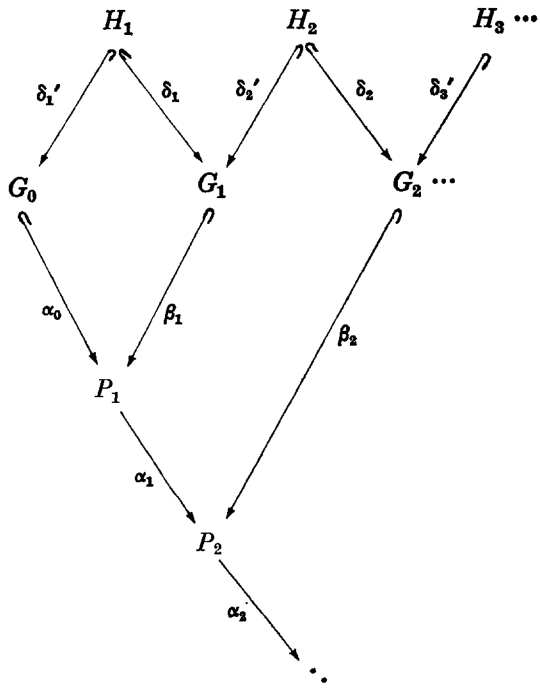
(v) *For every $k \geqslant k_0$, the obvious maps: $G_i \to G^k$ are injective, for $i = 0, 1, \ldots, p^k - 1$;*

(vi) *the maps $\delta_0\colon H_0 \to G_0$ and $\delta_1'\colon H_1 \to G_0$ are injective. One then has a natural isomorphism:*

$$H^{q+1}(N/R, A) \cong \varinjlim_{k \geqslant k_0} H^q(G^k, K_k, A),$$

where A denotes a discrete $\mathcal{Cl}(N/R)$-module, and $q \geqslant 2$. Furthermore, N/R and each G^k, $k \geqslant k_0$, has cohomological dimension $\leqslant n$.

Proof. Clearly all the maps γ_j, δ_j, δ_j' are injective ($j \in \mathbb{Z}$). The pro-p-group G^k can be obtained from the pro-p-groups $G_0, G_1, \ldots, G_{p^k-1}$ by a process of necessive push-outs, as indicated in the diagram below.



where $P_{p^k-1} = G^k$, and $\gamma_0\colon G_0 \to P_1 \to P_2 \to \cdots \to P_{p^k-1} = G^k$. Note that the maps β_i are injective, by (v), and we may consider them as inclusions. By the excision axiom (Theorem 1.10),

$$H^{n+1}(P_2, G_2, A) \cong H^{n+1}(P_1, H_2, A),$$
$$H^{n+1}(P_3, G_3, A) \cong H^{n+1}(P_2, H_3, A), \text{ etc.,}$$
$$H^{n+1}(G^k, G_{p^k-1}, A) \cong H^{n+1}(P_{p^k-2}, H_{p^k-1}, A)$$

for every discrete $\mathcal{O}l(N/R)$-module A. Since α_0 and β_1 are injective, we can apply Theorem 1.13 to the first push-out, to obtain an exact sequence:

$$\cdots \to H^n(H_1, A) \xrightarrow{\Delta} H^{n+1}(P_1, A) \xrightarrow{\Phi} H^{n+1}(G_0, A) \oplus H^{n+1}(G_1, A) \to \cdots.$$

Since H_1 is free and $n \geqslant 2$, Φ is injective, and since $\operatorname{cd}(G_0) \leqslant n$, and hence $\operatorname{cd}(G_j) \leqslant n$ for all $j \in \mathbb{Z}$, we have $H^{n+1}(P_1, A) = 0$. By the exact sequence,

$$\cdots \to H^n(H_{j+1}, A) \xrightarrow{\delta} H^{n+1}(P_j, H_{j+1}, A) \xrightarrow{j} H^{n+1}(P_j, A)$$
$$\xrightarrow{i} H^{n+1}(H_{j+1}, A) \to \cdots$$

of Proposition 1.5, we have

$$H^{n+1}(P_j, H_{j+1}, A) \cong H^{n+1}(P_j, A), \qquad j = 1, 2, \ldots, p^k - 2.$$

Applying the same exact sequence to the pair (P_j, G_j), we obtain

$$H^{n+1}(P_j, A) \cong H^{n+1}(P_j, G_j, A)/\delta(H^n(G_j, A)), \qquad 1 \leqslant j \leqslant p^k - 1,$$

and it follows from the above isomorphisms that

$$H^{n+1}(G^k, A) = 0, \qquad \text{i.e.,} \qquad \operatorname{cd} G^k \leqslant n.$$

Suppose now that A is an injective $\mathcal{O}l(N/R)$-module. Then, by Proposition 1.3, $H^n(G_j, A) = 0 = H^n(H_j, A)$ for $n \geqslant 1$ and $j \in \mathbb{Z}$. From the above exact sequence of Theorem 1.13, we deduce that $H^{q+1}(P_1, A) = 0$ for $q \geqslant 1$, and from the exact sequence of Proposition 1.5, we deduce that

$$H^{q+1}(P_j, H_{j+1}, A) \cong H^{q+1}(P_j, A) \cong H^{q+1}(P_j, G_j, A)/\delta(H^q(G_j, A))$$

for $q \geqslant 1$. The isomorphisms

$$H^{q+1}(P_j, G_j, A) \cong H^{q+1}(P_{j-1}, H_j, A), \qquad 2 \leqslant j \leqslant p^k - 1,$$

of the excision theorem (Theorem 1.10) now imply that $H^{q+1}(G^k, A) = 0$ for $q \geqslant 1$. The exact sequence

$$\cdots \to H^1(G^k, A) \xrightarrow{\text{Res}} H^1(K_k, A) \xrightarrow{\delta} H^2(G^k, K_k, A) \xrightarrow{j} H^2(G^k, A)$$
$$\to H^2(K_k, A) \to \cdots$$

and the hypotheses (ii) and (iii) imply that

$$H^2(G^k, K_k, A) \cong H^2(G^k, A) = 0.$$

Also, if $q \geqslant 2$, then $H^q(K_k, A) = 0 = H^{q+1}(K_k, A)$ and

$$H^{q+1}(G^k, K_k, A) \cong H^{q+1}(G^k, A) = 0.$$

So, the cohomology functor $\{H^{q+1}(G^k, K_k, -)\}_{q \geqslant 0}$ is effaceable by injective discrete $\mathcal{O}l(N/R)$-modules, and it follows from Lemma 4.1 and the standard comparison theorem [17, Corollary 5.7] that

$$H^{q+1}(N/R, A) \cong \varinjlim_{k \geqslant k_0} H^q(G^k, K_k, A)$$

for *every* discrete $\mathcal{O}l(N/R)$-module A. If $q \geqslant n$, then the exact sequence

$$\cdots \to H^q(K_k, A) \xrightarrow{\delta} H^{q+1}(G^k, K_k, A) \xrightarrow{j} H^{q+1}(G^k, A) \xrightarrow{i} H^{q+1}(K_k, A) \to \cdots$$

shows that

$$H^{q+1}(G^k, K_k, A) \cong H^{q+1}(G^k, A) = 0,$$

which completes the proof. ∎

Suppose from now on that $m = 0$ and write

$$x = x_0, \quad y = x_1, \quad z_j = y^{-j}xy^j, \quad j \in \hat{\mathbb{Z}}_p, \quad n = n_0.$$

PROPOSITION 4.3. *Suppose that r_0 belongs to the closed normal subgroup of $F(z_0, z_1, ..., z_n)$ generated by z_0. Then K_k is free for all $k \geqslant k_0$, and* Res: $H^1(G^k, A) \to H^1(K_k, A)$ *is onto for every discrete $\mathcal{U}G^k$-module A.*

Proof. Define an automorphism α on

$$F^k(z_0, z_1, ..., z_{p^k+n-1})$$

by

$$\alpha(z_i) = \begin{cases} z_i & \text{for } 0 \leqslant i < p^k \\ z_{i-p^k}^{-1} \cdot z_i & \text{for } p^k \leqslant i \leqslant p^k + n - 1 \end{cases}$$

(see Serre [18, I-Proposition 25]). Clearly, the elements r_i $(0 \leqslant i \leqslant p^k - 1)$ are contained in the closed normal subgroup of F^k generated by $\alpha(z_i) = z_i$ $(0 \leqslant i \leqslant p^k - 1)$. Since $\{\alpha(z_i) \mid i = 0, 1, ..., p^k + n - 1\}$ is a basis for F^k, we can define a map

$$\beta: F^k \to F(z_0, z_1, ..., z_{n-1})$$

by

$$\beta(\alpha(z_i)) = \begin{cases} 1 & \text{if } 0 \leqslant i < p^k, \\ z_{i-p^k} & \text{if } p^k \leqslant i \leqslant p^k + n - 1. \end{cases}$$

Then $\beta(r_i) = 1$ for all $i = 0, 1, ..., p^k - 1$, and β induces a map $\gamma: G^k \to F(z_0, z_1, ..., z_{n-1})$. This map has a right inverse $\delta: F(z_0, z_1, ..., z_{n-1}) \to G^k$, defined by sending z_i to the image of $\alpha(z_{i+p^k})$, $(i = 0, 1, ..., n - 1)$; and δ defines an isomorphism φ of $F(z_0, z_1, ..., z_{n-1})$ onto K_k. Given a continuous crossed homomorphism $\epsilon: K_k \to A$, the map

$$\lambda: G^k \xrightarrow{\gamma} F(z_0, z_1, ..., z_{n-1}) \xrightarrow[\cong]{\varphi} K_k \xrightarrow{\epsilon} A$$

is a continuous crossed homomorphism, whose restriction to K_k is ϵ. It follows that

$$\text{Res}: H^1(G^k, A) \to H^1(K_k, A)$$

is onto. ∎

THEOREM 4.4. *Suppose that*

(i) $n \geqslant 1$, *and* r_0 *belongs to the closed normal subgroup of* $F(z_0, z_1, \dots, z_n)$ *generated by* z_0;

(ii) $G_0 = F(z_0, z_1, \dots, z_n)/(r_0)$ *has cohomological dimension* $\leqslant q$, *where* $q \geqslant 2$;

(iii) *there exists a* (t, p)*-filtration* $\bar{\omega}$ *on* G_0 *such that*:

(a) *the elements* $\operatorname{gr} \bar{z}_0, \operatorname{gr} \bar{z}_1, \dots, \operatorname{gr} \bar{z}_n$ *of the corresponding mixed Lie algebra* $\operatorname{gr} G_0$ *are distinct, where* $\bar{z}_i$ *denotes the image of* z_i *in* G_0;

(b) *the sets* $S = \{\operatorname{gr} \bar{z}_0, \operatorname{gr} \bar{z}_1, \dots, \operatorname{gr} \bar{z}_{n-1}\}$ *and* $T = \{\operatorname{gr} \bar{z}_1, \operatorname{gr} \bar{z}_2, \dots, \operatorname{gr} \bar{z}_n\}$ *freely generate free mixed Lie algebras* L_S *and* L_T *in* $\operatorname{gr} G_0$;

(c) $L_S \cap L_T$ *is freely generated by* $S \cap T$ (*with* $L_S \cap L_T = (0)$ *if* $S \cap T = \varnothing$).

Then

$$\operatorname{cd}(N/R) \leqslant q, \qquad \operatorname{cd}(G^k) \leqslant q, \qquad (k \geqslant k_0), \qquad \operatorname{cd} F(x, y)/R \leqslant q + 1,$$

and

$$H^j(N/R, A) \simeq \varinjlim_{k \geqslant k_0} H^j(G^k, K_k, A)$$

for every $j \geqslant 2$ *and discrete* $\mathcal{Cl}(N/R)$*-module* A.

Proof. The second statement follows from the first, by virtue of the exact sequence

$$1 \to N/R \to F(x, y)/R \to F(y) \to 1$$

and Serre [18, I-Proposition 15].

The element r_0 belongs to the Frattini subgroup of $F(z_0, z_1, \dots, z_n)$, because of (iii). Note that (iii) is precisely the hypothesis of Gildenhuys [5, Theorem 1], and, hence, the hypotheses (iv), (v), and (vi) of Proposition 4.2 are satisfied. The remaining hypotheses of this Proposition are satisfied by virtue of Proposition 4.3, and the result follows. ∎

EXAMPLE 4.5. Let $r = x^p((x, y), ((y, x), x))$.

Case 1: $p > 3$.

Let $\tau_1 = 1/4$, $\tau_2 = 1/10$, and let ω be the (x, τ, p)-filtration on $F(x, y)$. Then

$$\omega(((x, y), ((y, x), x))) = 3/4 + 1/5 < 5/4 = \omega(x^p);$$

so that

$$\operatorname{gr} r = \operatorname{gr}((x, y), ((y, x), x))$$

and Labute [10, Theorem 4′] can be applied. We conclude that

$$\operatorname{cd} F(x, y)/(r) = 2.$$

Case 2: $p = 3$.

In this case Labute's method fails when applied to r. However, rewriting r in terms of the conjugates $x_j = y^{-j}xy^j$, we obtain

$$r_0 = x_0{}^3(x_0^{-1}x_1, (x_1^{-1}, x_0)^{x_0}).$$

Let $\{s_n\}$ be a strictly decreasing sequence of rational numbers tending to $1/2 = 1/(p-1)$, and let ω_n be the (x, τ, p)-filtration on $F(x, y)$, where $\tau = (s_n, 1/2)$. Keep n fixed for the time being. Then the image $\operatorname{gr} r_0$ of r_0 in the corresponding free mixed Lie algebra $\operatorname{gr} F(x_0, x_1)$ is of the form

$$\operatorname{gr} r_0 = [\operatorname{gr} x_0, [\operatorname{gr} x_1, \operatorname{gr} x_0]] - [\operatorname{gr} x_1, [\operatorname{gr} x_1, \operatorname{gr} x_0]].$$

By Labute [10, Theorem 4′], $\operatorname{cd} F(x_0, x_1)/(r_0) = 2$ and $\operatorname{gr}((r_0))$ is the ideal $(\operatorname{gr} r_0)$ of $\operatorname{gr} F(x_0, x_1)$ generated by $\operatorname{gr} r_0$. Let $M(\xi_0)$ be the free mixed Lie algebra generated by one symbol ξ_0, and let

$$\alpha: M(\xi_0) \to \operatorname{gr}(F(x_0, x_1)/(r_0)) \simeq \operatorname{gr} F(x_0, x_1)/(\operatorname{gr}(r_0))$$

be the map that sends ξ_0 to $\operatorname{gr} \bar{x}_0$, where $\operatorname{gr}(F(x_0, x_1)/(r_0))$ is the mixed Lie algebra corresponding to the quotient filtration $\bar{\omega}_n$. Clearly α has a left inverse and

$$\bar{\omega}_n(\bar{x}_0^{p^k}) = \varphi^k(\bar{\omega}_n(\bar{x}_0)) = \varphi^k(s_n),$$

where φ is as in Lazard [11] or Gildenhuys [5, Section 3]. Similarly, $\bar{\omega}_n(\bar{x}_1^{p^k}) = \varphi^k(1/2)$. Now

$$\bar{\omega}(\bar{x}_0^{p^k}) = \lim_{n\to\infty} \varphi^k(s_n) = \varphi^k(1/2) = \bar{\omega}(\bar{x}_1^{p^k}),$$

where $\bar{\omega}$ is the $(1/2, p)$-filtration on $F(x_0, x_1)/(r_0)$. (See Gildenhuys [5, Section 3], where a similar argument is used.) Hence condition (iii)(b) of Theorem 4.4 is satisfied. From the fact that $\operatorname{gr}(F(x_0, x_1)/(r_0))$ is embedded in $\operatorname{gr}(\mathcal{O}\mathcal{U}(F(x_0, x_1)/(r_0)))$, which has no zero-divisors (see Labute, [10, Theorem 4′]), we can deduce, as in Gildenhuys [5, Section 3], that condition (iii)(c) of Theorem 4.4 is satisfied. Conditions (i) and (iii)(a) of Theorem 4.4 are trivially satisfied, and (ii) has already been proved, for $q = 2$. Hence

$$\operatorname{cd}(N/R) = 2, \qquad \operatorname{cd} F(x, y)/(r) \leqslant 3.$$

Case 3: $p = 2$.
We can write r in the form

$$r = x^2((x, y), (y, x^2)),$$

and by an argument similar to the one used in Gildenhuys [4] we deduce that r and x^2 generate the same closed normal subgroup $F(x, y)$, and hence $\mathrm{cd}(F(x, y)/(r)) = \infty$.

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