Friedrichs extension of operators defined by symmetric banded matrices

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Abstract

We consider the operator \( \mathcal{A} : l^2 \to l^2 \) defined by a \((2n+1)\)-diagonal infinite symmetric matrix. Using the recessive system of solutions of a certain associated \( 2n \)-order Sturm–Liouville difference equation we characterize the domain of the Friedrichs extension of \( \mathcal{A} \).

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1. Introduction

We consider the infinite symmetric banded matrix with the bandwidth \((2n+1)\)

\[
A = (a_{\mu,\nu}), \quad a_{\mu,\nu} = a_{\nu,\mu} \in \mathbb{R}, \quad \mu, \nu \in \mathbb{N} \cup \{0\}, \quad a_{\mu,\nu} = 0 \quad \text{for} \quad |\mu - \nu| > n.
\] (1)

If we set \( a_{\mu,\nu} = 0 \) (and \( y_\nu = 0 \)) for \( \mu < 0, \nu < 0 \), we associate with \( A \) the operator \( \mathcal{A} : l^2 \to l^2 \) defined for \( y = (y_k)_{k=0}^{\infty} \in l^2 \) by

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\[(\mathcal{A}y)_k = \sum_{j=k-n}^{k+n} a_{k,j} y_j, \quad k \in \mathbb{N}. \quad (2)\]

We are motivated by the papers [7,19], where the authors investigated the Friedrichs extension of operators defined by infinite Jacobi matrices and by singular 2n-order differential expressions, respectively. It was shown there that the domain of the Friedrichs extension of these operators can be characterized by the so-called recessive and principal solutions of certain associated difference and differential equations.

Here we associate with (2) a 2n-order Sturm–Liouville difference equation and using the concept of the recessive system of solutions of this Sturm–Liouville equation we characterize the domain of the Friedrichs extension of these operators. It was shown there that the domain of the Friedrichs extension of these operators can be characterized by the so-called recessive and principal solutions of certain associated difference and differential equations.

The paper is organized as follows: in the next section we recall elements of the theory of symmetric operators in Hilbert spaces and their self-adjoint extensions. In Section 3 we discuss the relationship between banded symmetric matrices, Sturm–Liouville difference operators, and linear Hamiltonian difference systems. We also present elements of the spectral theory of symmetric difference operators in this section. The main result of the paper is given in Section 4.

2. Friedrichs extension of a symmetric operator

First let us briefly recall the concept of the Friedrichs extension of a symmetric operator. Let \( H \) be a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \) and let \( \mathcal{A} \) be a densely defined symmetric operator in \( H \) with the domain \( \mathcal{D}(\mathcal{A}) \). Suppose also that \( \mathcal{A} \) is bounded below, i.e., there exists a real constant \( \gamma \) such that

\[ \langle \mathcal{A} x, x \rangle \geq \gamma \langle x, x \rangle, \quad x \in \mathcal{D}(\mathcal{A}). \]

Friedrichs [10] showed that there exists a self-adjoint extension \( \mathcal{A}_F \) of \( \mathcal{A} \), later named Friedrichs extension of \( \mathcal{A} \), which preserves the lower bound of \( \mathcal{A} \). The domain \( \mathcal{D}(\mathcal{A}_F) \) of this extension can be characterized as follows. The sesquilinear form

\[ T(x,y) := \langle \mathcal{A} x, y \rangle - \gamma (x,y), \quad \varepsilon > 0, \]

defines an inner product on \( H \), denote by \( \langle \cdot, \cdot \rangle_\mathcal{A} \) this inner product, and by \( H_\mathcal{A} \) the completion of \( \mathcal{D}(\mathcal{A}) \) in this product. Then the domain of \( \mathcal{A}_F \) is

\[ \mathcal{D}(\mathcal{A}_F) = H_\mathcal{A} \cap \mathcal{D}(\mathcal{A}^*), \]

where \( \mathcal{A}^* \) is the adjoint operator of \( \mathcal{A} \). It can be shown (see, e.g. [16, p. 352]) that for any \( x \in \mathcal{D}(\mathcal{A}_F) \) there exists a sequence \( x_n \in \mathcal{D}(\mathcal{A}) \) such that

\[ T(x-x_n,x-x_n) \to 0 \quad \text{as} \quad n \to \infty, \]

where \( T \) denotes the closure of \( T \). Another characterization of \( \mathcal{D}(\mathcal{A}_F) \) comes from Freudenthal [11]:

\[ \mathcal{D}(\mathcal{A}_F) = \{ x \in \mathcal{D}(\mathcal{A}^*) : \exists x_k \in \mathcal{D}(\mathcal{A}) \text{ such that } x_k \to x \text{ in } H \text{ and } T(x_j-x_k,x_j-x_k) \to 0 \quad \text{as} \quad j,k \to \infty \}. \quad (3) \]

The construction of the sequence \( x_n \) in our particular case, when \( \mathcal{A} \) is the operator defined by the infinite matrix in (1), is based on the so-called Reid’s construction of the recessive solution of linear Hamiltonian difference systems (see, e.g. [1,2]) and the resulting concept of the recessive system of solutions of even-order Sturm–Liouville equations introduced in [8]. The concept of the recessive solution of difference equations is the discrete version of the concept of the principal solution of differential equations and systems.

To explain the role of these concepts in the theory of Friedrichs extensions of differential and difference operators, let us start with the regular Sturm–Liouville differential operator

\[ L(y) := -(r(t)y')' + p(t)y, \quad (4) \]
where \( t \in (a, b) \), \(-\infty < a < b < \infty \), \( r^{-1}, p \in \mathcal{L}(a, b) \). It is well known that the domain of the Friedrichs extension \( \mathcal{D}(L) \) of the minimal operator defined by \( L \) is given by the Dirichlet boundary condition
\[
\mathcal{D}(L) = \{ y \in \mathcal{L}^2(a, b) : L(y) \in \mathcal{L}^2(a, b), y(a) = 0 = y(b) \}.
\]
If the operator \( L \) is singular at one or both endpoints \( a, b \), it was discovered by Rellich [21] that functions in \( \mathcal{D}(L) \) behave near \( a \) and \( b \) like the principal solution of a certain nonoscillatory differential equation associated with \( (4) \). This fact is a natural extension of \( (5) \) since the principal solution (at a singular point) of a second order differential equation is a solution which is less in a certain sense, than any other solution of this equation. We refer to the paper [18] where the concept of the principal solution had been introduced and to books [14, 22] for properties of the principal solution of \( (4) \) and the extension of this concept to linear Hamiltonian systems. Note also that the results of Rellich had been later extended in various directions, let us mention here at least the papers [15, 19, 20].

Concerning the Friedrichs extension of difference operators, the discrete counterpart of the concept of the principal solution is the so-called recessive solution. This concept for the second order Sturm–Liouville difference equation
\[
\Delta (t_k \Delta x_k) + p_k x_{k+1} = 0, \quad \Delta x_k := x_{k+1} - x_k,
\]
appears explicitly for the first time in [12], even if it is implicitly contained in a series of earlier papers. The fact that this solution of \( (6) \) plays the same role in the theory of second order difference operators and Jacobi matrices as the principal solution for differential operators had been established in [4, 7, 13]. In our paper we extend some results of these papers to matrix operators defined by \( (2) \).

### 3. Sturm–Liouville difference operators and symmetric banded matrices

We start this section with the relationship between banded symmetric matrices and Sturm–Liouville difference operators as established in [17]. Consider the \( 2n \)-order Sturm–Liouville difference operator
\[
L(y)_k := \sum_{\mu=0}^{n} (-\Delta)^\mu [r_k^{[\mu]} \Delta^\mu y_{k-\mu}], \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad r_k^{[n]} \neq 0,
\]
where \( \Delta y_k = y_{k+1} - y_k \) and \( \Delta^\nu y_k = \Delta^{\nu-1} y_k \). Expanding the forward differences in \( (7) \), with the convention that \( a_{\mu, \nu} = 0 \) for \( \mu, \nu < 0 \), we get the recurrence relation \( (2) \) with \( a_{ij} \) given by the formulas
\[
a_{k,j} = (-1)^j \sum_{\mu,j} \mu \left( \begin{array}{c} \mu \\ v - j \end{array} \right) r_{k,v}^{[\mu]}, \quad a_{k,-j} = (-1)^j \sum_{\mu,j} \mu \left( \begin{array}{c} \mu \\ v + j \end{array} \right) r_{k,v}^{[\mu]},
\]
for \( k \in \mathbb{N}_0 \) and \( j \in \{0, \ldots, n\} \). Consequently, one can associate with the difference operator \( L \) the matrix operator \( \mathcal{A} \) defined via an infinite matrix \( A \) by the formula
\[
(\mathcal{A}y)_k := L(y)_k, \quad k \in \mathbb{N}_0,
\]
where \( L \) is related to \( \mathcal{A} \) by \( (2) \) and \( (8) \). Conversely, having a symmetric banded matrix \( A = (a_{\mu, \nu}) \) with the bandwidth \( 2n + 1 \), one can associate with this matrix the Sturm–Liouville operator \( (7) \) with \( r_k^{[n]} \), \( \mu = 0, \ldots, n \), given by the formula
\[
r_{k+\mu}^{[n]} = (-1)^\mu \sum_{s=\mu}^{n} \sum_{l=1}^{s-\mu} \left( \begin{array}{c} s \\ \mu \end{array} \right) a_{k,k+s} + \sum_{l=1}^{s-\mu} \left( \begin{array}{c} s + l - 1 \\ l - 1 \end{array} \right) \left( \begin{array}{c} s - l - 1 \\ s - \mu - l \end{array} \right) a_{k-k-l+s},
\]
where \( k \in \mathbb{N}_0, 0 \leq \mu \leq n \).

Sturm–Liouville difference equations are closely related to linear Hamiltonian difference systems (see, e.g. [5]). Let \( y \) be a solution of the equation
\[
L(y)_k = 0, \quad k \in \mathbb{N}_0.
\]
and let
\[
x_k = \begin{pmatrix} y_{k-1} \\ \Delta y_{k-2} \\ \vdots \\ \Delta^{n-1} y_{k-n} \end{pmatrix}, \quad u_k = \begin{pmatrix} \sum_{\mu=1}^{n}(-1)^{\mu-1} \Delta^{\mu-1} (r^{[\mu]}_k \Delta^{\mu} y_{k-\mu}) \\ \vdots \\ -\Delta (r^{[n]}_k \Delta^{n} y_{k-n}) + r^{(n-1)}_k \Delta^{n-1} y_{k-n+1} \end{pmatrix},
\]

(11)

where we extend \( y = (y_k)_{k=0}^{\infty} \) by \( y_{-1} = \cdots = y_{-n} = 0 \). Then \( \begin{pmatrix} x \\ u \end{pmatrix} \) solves the linear Hamiltonian difference system
\[
\Delta x_k = Ax_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A^T u_k,
\]

(12)

with
\[
B_k = \text{diag} \left\{ 0, \ldots, 0, \frac{1}{r^{[n]}_k} \right\}, \quad C_k = \text{diag} \left\{ r^{[0]}_k, \ldots, r^{(n-1)}_k \right\},
\]

(13)

and
\[
A = a_{ij} = \begin{cases} 1 & \text{if } j = i + 1, \ i = 1, \ldots, n - 1, \\ 0 & \text{elsewhere}, \end{cases}
\]

(14)

(of course, this \( A \) is different from \( A \) given by (1), we have used here the standard notation for linear Hamiltonian difference systems). Next we recall Reid’s construction of the recessive solution of (12) as it is introduced in [11] for three terms matrix recurrence relations. This construction naturally extends to (12) (see, e.g., [9]) and the important role is played there by the following concepts introduced in [5]. A \( 2n \times n \) matrix solution \( \begin{pmatrix} X \\ U \end{pmatrix} \) of (12) is said to be a conjoined basis if \( X^T U \) is symmetric and rank
\[
\begin{pmatrix} X \\ U \end{pmatrix} = n. \text{System (12) is said to be disconjugate in a discrete interval } [l, m], \ l, m \in \mathbb{N}, \text{if the } 2n \times n \text{ matrix solution } \begin{pmatrix} X \\ U \end{pmatrix} \text{ given by the initial condition } X_l = 0, \ U_l = I \text{ satisfies}
\]
\[
\text{Ker} X_{k+1} \subseteq \text{Ker} X_k \quad \text{and} \quad X_{k+1} (I - A)^{-1} B_k \geq 0
\]

(15)

for \( k = l, \ldots, m \). Here Ker, \( \dagger \) and \( \geq \) stand for the kernel, Moore–Penrose generalized inverse, and non-negative definiteness of a matrix indicated, respectively. System (12) is said to be nonoscillatory if there exists \( N \in \mathbb{N} \) such that this system is disconjugate on \([N, \infty)\) and it is said to be oscillatory in the opposite case. System (12) is said to be eventually controllable if there exist \( N, \kappa \in \mathbb{N} \) such that for any \( m \geq N \) the trivial solution \( \begin{pmatrix} X \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is the only solution for which \( x_m = x_{m+1} = \cdots = x_{m+k} = 0 \). Note that Hamiltonian system (12) corresponding to Sturm–Liouville equation (10) is controllable with the constant \( \kappa = n \), see [5, Remark 9].

A conjoined basis \( \begin{pmatrix} \tilde{X} \\ \tilde{U} \end{pmatrix} \) of (12) is said to be the recessive solution if \( \tilde{X}_k \) are nonsingular, \( \tilde{X}_k \tilde{X}_k^{-1} (I - A)^{-1} B_k \geq 0 \), both for large \( k \), and for any other conjoined basis \( \begin{pmatrix} X \\ U \end{pmatrix} \) for which the (constant) matrix \( X^T \tilde{U} - \tilde{U}^T \tilde{X} \) is nonsingular (such a solution is usually called dominant) we have
\[
\lim_{k \to \infty} X_k^{-1} \tilde{X}_k = 0.
\]

(16)

The recessive solution is determined uniquely up to a right multiple by a nonsingular \( n \times n \) matrix. The equivalent characterization of the recessive solution \( \begin{pmatrix} \tilde{X} \\ \tilde{U} \end{pmatrix} \) of eventually controllable Hamiltonian difference systems (12) is
\[
\lim_{k \to \infty} \left( \sum_{j=1}^{k} \tilde{X}_j^{-1} (I - A)^{-1} B_j \tilde{X}_j^{-1} \right)^{-1} = 0.
\]
Note that the existence of a conjoined basis \((\mathbf{X} \quad \mathbf{U})\) such that its first component \(\mathbf{X}\) is nonsingular and the second condition in (15) holds for large \(k\) implies that the first component of any other conjoined basis has the same property, see [6].

The recessive solution \((\tilde{\mathbf{X}} \quad \tilde{\mathbf{U}})\) of (12) can be constructed as follows. Let \(l, m \in \mathbb{N}\), \(l > m\), be such that (12) is disconjugate on \([m, \infty)\), and consider the solution \((\mathbf{X}^l \quad \mathbf{U}^l)\) of (12) given by the condition \(\mathbf{X}^l_m = I\), \(\mathbf{X}^l_l = 0\), where \(I\) is the identity matrix. Such a solution exists because of disconjugacy of (12) on \([m, l]\) and for every \(k \in [m, l]\) we have

\[
\lim_{l \to \infty} (\mathbf{X}^{(l)}_k \quad \mathbf{U}^{(l)}_k) = (\tilde{\mathbf{X}}_k \quad \tilde{\mathbf{U}}_k).
\]

If (12) is rewritten Sturm–Liouville equation (7), i.e., the entries in the first row of the matrix \(\tilde{\mathbf{X}}\) are solutions \(\tilde{y}^{(1)}, \ldots, \tilde{y}^{(n)}\) of (10), we call these solutions the recessive system of solutions of (7). Nonoscillation and disconjugacy of (10) is defined via nonoscillation and disconjugacy of the associated linear Hamiltonian difference system, hence recessive system of solutions of (10) exists whenever this equation is nonoscillatory.

Next we recall the result (see [17, Lemma 2]) which relates the quadratic form associated with the matrix \(A\), the quadratic functional associated with (7), and the quadratic functional associated with (12). Let \(y = \{y_k\} \in l^2\) and suppose that there exists \(N \in \mathbb{N}\) such that \(y_k = 0\) for \(k \geq N\). If we extend \(y\) by \(y_{-1} = \cdots = y_{-n} = 0\), we have the identity

\[
\langle Ay, y \rangle = \mathcal{F}(y) = \mathcal{I}(x, u),
\]

where

\[
\mathcal{F}(y) := \sum_{k=0}^{n} \sum_{\mu=0}^{n} r_k^{(\mu)} (\Delta^\mu y_{k-\mu})^2,
\]

and

\[
\mathcal{I}(x, u) = \sum_{k=0}^{\infty} [u_k^T B_k u_k + x^T_{k+1} C_k x_{k+1}]
\]

with \(x, u\) in \(\mathcal{I}\) related to \(y\) by (11) and the matrices \(A, B, C\) are given by (13) and (14). According to [5], the quadratic functional \(\mathcal{I}\) is positive for all \((x, u)\) satisfying

\[
\Delta x_k = Ax_{k+1} + B_k u_k
\]

with \(x_0 = 0, x_k = 0\) for large \(k, x \neq 0\), if and only if system (12) is disconjugate on \([0, \infty)\). Moreover, for such \((x, u)\) we have

\[
\mathcal{I}(x, u) = \sum_{k=0}^{\infty} x^T_{k+1} [-\Delta u_k + C_k x_{k+1} - A^T u_k].
\]

Note that a pair \((x, u)\) satisfying (18) is said to be admissible for \(\mathcal{I}\).

We finish this section with some results of the general theory operators defined by even order (formally) symmetric difference expressions or by symmetric banded matrices. The maximal operator associated with the infinite matrix \(A\) is defined by

\[
(\mathcal{A}_{\text{max}} y)_k = (Ay)_k, \quad k \in \mathbb{N}_0
\]

on the domain

\[
\mathcal{D} := \mathcal{D}(\mathcal{A}_{\text{max}}) = \{y \in l^2 : Ay \in l^2\}.
\]
The minimal operator $\mathcal{A}_{\text{min}}$ is the closure of the so-called preminimal operator which is the restriction of $\mathcal{A}_{\text{max}}$ to the domain
\[ \mathcal{D}_0 := \{ y = (y_k)_{k=0}^\infty \text{, only finitely many } y_k \neq 0 \}. \]

Denote the so-called deficiency indices by
\[ q_\pm := \dim(\ker \mathcal{A}_{\text{max}} \mp iI). \]

We also denote by $L_{\text{max}}, L_{\text{min}}$ the corresponding Sturm–Liouville difference operators related to $\mathcal{A}_{\text{max}}, \mathcal{A}_{\text{min}}$ by (9). If we suppose that $\mathcal{A}$ is bounded below (and since we suppose that the entries $a_{i,j}$ of $A$ are real), we have $q := q_+ = q_-$. Moreover, $q \in \{0, \ldots, n\}$. This is due to the fact that we extended $y = (y_k)_{k=0}^\infty \in \ell^2$ to negative integers as $y_k = 0$, so we implicitly suppose the boundary conditions $y_{-1} = 0 = \Delta y_{-2} = \cdots = \Delta^{n-1} y_{-n}$. This corresponds to the situation when a $2n$-order symmetric differential operator is considered on an interval $(a, b)$ with the boundary conditions $y(a) = y(n)(a) = \cdots = y(n-1)(a)$ at the regular left endpoint.

Let $y^{[1]}, y^{[2]} \in \ell^2$ and let $(x^{[1]}, u^{[1]}), (x^{[2]}, u^{[2]})$ be the associated (via (11)) sequences of $2n$-dimensional vectors. We define
\[ [y^{[1]}, y^{[2]}]_k := \left( \begin{array}{c} [x^{[1]}_k]^T \\ [u^{[2]}_k] \end{array} \right), \quad J = \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right). \]

Using the fact that
\[ \Delta \begin{pmatrix} x^{[1]}_k \\ u^{[2]}_k \end{pmatrix} J \begin{pmatrix} x^{[2]}_k \\ u^{[2]}_k \end{pmatrix} = (x^{[1]}_{k+1} \Delta u^{[2]}_{k+1} + A^T u^{[2]}_{k}) - (x^{[2]}_{k+1} \Delta u^{[1]}_{k+1} + A^T u^{[1]}_{k}) = y^{[2]}_k L (y^{[1]}_k) - y^{[1]}_k L (y^{[2]}_k) \]
we obtain Green’s formula (see also [2])
\[ \sum_{k=0}^{N} [y^{[2]}_k L (y^{[1]}_k) - y^{[1]}_k L (y^{[2]}_k)] = [y^{[1]}, y^{[2]}]_{N+1}. \]

In particular, if $y, z \in \mathcal{D}$, there exists the limit $[y, z]_\infty = \lim_{k \to \infty} [y, z]_k$, and, since $\mathcal{A}_{\text{max}} = \mathcal{A}_{\text{min}}$, the domain of $\mathcal{A}_{\text{min}}$ is given by
\[ \mathcal{D}(\mathcal{A}_{\text{min}}) = \{ y \in \mathcal{D} : [y, z]_\infty = 0 \ \forall y \in \mathcal{D} \}. \]

Similarly as for even order symmetric differential operators, self-adjoint extensions of the minimal operator $L_{\text{min}}$ are defined by boundary conditions at $\infty$. More precisely, if the operator $\mathcal{A}_{\text{min}}$ is not self-adjoint, i.e., $q \geq 1$, let $y^{[1]}, \ldots, y^{[q]} \in \mathcal{D}$ be such that
\[ [y^{[i]}, y^{[j]}]_\infty = 0, \quad i, j = 1, \ldots, q, \]
and that $y^{[1]}, \ldots, y^{[q]}$ are linearly independent modulo $\mathcal{D}(\mathcal{A}_{\text{min}})$ (i.e., no nontrivial combination of $y^{[1]}, \ldots, y^{[q]}$ belongs to $\mathcal{D}(\mathcal{A}_{\text{min}})$). Then a self-adjoint extension of $L_{\text{min}}$ (and hence also of $\mathcal{A}_{\text{min}}$) is defined as the restriction of $\mathcal{A}_{\text{max}}$ to the domain
\[ \mathcal{D}_\varepsilon := \{ y \in \mathcal{D} : [y, y^{[j]}]_\infty = 0, j = 1, \ldots, q \}. \]

4. Friedrichs extension of symmetric matrix operators

Throughout this section we suppose that there exists $\varepsilon > 0$ such that the minimal operator associated with the matrix $A$ given in (1) satisfies
\[ \langle Ay, y \rangle \geq \varepsilon \langle y, y \rangle, \quad \text{for } 0 \neq y \in \mathcal{D}(\mathcal{A}_{\text{min}}). \]
This means, in view of the previous section, that the associated Sturm–Liouville operator (7) and linear Hamiltonian difference system (12) are disconjugate on \([0, \infty)\). We also suppose that
\[
a_{k+n} \neq 0 \quad \text{for } k \in \mathbb{N}.
\]

This assumption (which is the typical assumption for tridiagonal matrices which are the special case \(n = 1\)) means that \(A\) given by (1) is a “real” \(2n + 1\) diagonal matrix, i.e., the bandwidth is really \(2n + 1\) in each row of \(A\). This assumption is equivalent to the assumption \(a_{k}^{(n)} \neq 0\) in (7).

Note that assumption (21) essentially means no loss of generality. Friedrichs extension can be constructed for operators bounded below only, i.e., for \(A\) satisfying (instead of (21)) the assumption \(\langle Ay, y \rangle \geq \gamma \langle y, y \rangle\) for some \(\gamma \in \mathbb{R}\). However, under this assumption we can apply our construction to the operator defined by \(A - (\gamma - \varepsilon)I\), \(I\) being the infinite identity matrix, \(\varepsilon > 0\), and the results remain unchanged.

The next statement is the main result of our paper. It reduces to [7, Theorem 4] for tridiagonal matrices \(A\) in (1) and it is a discrete counterpart of the main result of [19].

**Theorem 1.** Let \(y^{(1)}, \ldots, y^{(n)}\) be the recessive system of solutions of the equation \(L(y) = 0\), where \(L\) is associated with \(\mathcal{A}\) by (9). Then the domain of the Friedrichs extension \(\mathcal{A}_F\) of \(\mathcal{A}_{\min}\) is
\[
\mathcal{D}(\mathcal{A}_F) = \{ y \in \mathcal{D}(\mathcal{A}_{\max}) : [y, y^{(j)}]_\infty = 0, \quad j = 1, \ldots, n \}.
\]

**Proof.** The main part of the proof consists in proving that the sequences \(y^{(j)}, j = 1, \ldots, n\), in the recessive system of solutions are in \(\mathcal{D}(\mathcal{A}_F)\). Let \(\left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right)\) be the recessive solution of Hamiltonian system (12) whose columns are formed by \(2n\)-dimensional vectors \(\left(\begin{array}{c} \hat{x}^{(j)} \\ \hat{u}^{(j)} \end{array}\right)\), \(j = 1, \ldots, n\), related to \(y^{(1)}, \ldots, y^{(n)}\) by (11). Without loss of generality we may suppose that the matrix \(\hat{X}\) formed by the vectors \(\hat{x}^{(1)}, \ldots, \hat{x}^{(n)}\) is nonsingular because of (21). Indeed, (21) implies disconjugacy in \([0, \infty)\) of Hamiltonian system (12) associated with the equation \(L(y) = 0\), which means that \(\hat{X}_0\) is nonsingular by [6]. Further, let
\[
\hat{X}_k = \hat{X}_k \sum_{j=0}^{k-1} \hat{B}_j, \quad \hat{U}_k = \hat{U}_k \sum_{j=0}^{k-1} \hat{B}_j + \hat{X}_k^{T-1},
\]

where
\[
\hat{B}_j := \hat{X}_j^{-1}(I - A)^{-1} \hat{B}_j \hat{X}_j^{T-1}.
\]
is the so-called associated dominant solution of (12). The fact that \(\left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right)\) is really a solution of (12) is proved e.g. in [2, p.107]. Further, for a fixed \(m \in \mathbb{N}\), we denote
\[
X_k^{[m]} = \hat{X}_k - \hat{X}_k \hat{X}_m^{-1} \hat{X}_m, \quad U_k^{[m]} = \hat{U}_k - \hat{U}_k \hat{X}_m^{-1} \hat{X}_m.
\]

Then according to (16)
\[
\lim_{m \to \infty} \left(\begin{array}{c} X_k^{[m]} \\ U_k^{[m]} \end{array}\right) = \left(\begin{array}{c} \hat{X}_k \\ \hat{U}_k \end{array}\right)
\]

for every \(k \in \mathbb{N}\). Also, \(\left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right), \left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right)\) are conjoined basis of (12) for which \(\hat{X}_k^T \hat{U}_k - \hat{U}_k^T \hat{X} = -I\) holds, which means that the matrix \(\left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right)\) is symplectic, i.e.,
\[
\left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right)^T J \left(\begin{array}{c} \hat{X} \\ \hat{U} \end{array}\right) = J.
\]
This implies the identity
\[ \begin{pmatrix} \bar{X} & \tilde{X} \\ \bar{U} & \tilde{U} \end{pmatrix} \neq \begin{pmatrix} \bar{X} & \tilde{X} \\ \bar{U} & \tilde{U} \end{pmatrix}^T = \mathcal{J} \]
which, in terms of the matrices \( \bar{X}, \tilde{X}, \bar{U}, \tilde{U} \), reads as
\[ \bar{U}_k \bar{X}_k^T - \tilde{U}_k \tilde{X}_k^T = I, \quad \bar{X}_k \bar{X}_k^T = \tilde{X}_k \tilde{X}_k^T, \quad \bar{U}_k \tilde{U}_k^T = \tilde{U}_k \bar{U}_k^T. \]
Denote, for \( j \in \{1, \ldots, m\} \),
\[ x_k^{(j,m)} = \begin{cases} x_k^{(m)} e_j, & k \leq m, \\ 0, & k > m, \end{cases} \quad u_k^{(j,m)} = \begin{cases} U_k^{(m)} e_j, & k < m, \\ 0, & k \geq m, \end{cases} \]
where \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \), number 1 being the \( j \)-th entry, is the standard canonical basis in \( \mathbb{R}^n \). Then by (23) the first entry of \( x_k^{(j,m)} \), denote it by \( y_k^{(j)} := e_j^T x_k^{(j,m)} \), satisfies (for every \( k \in \mathbb{N} \))
\[ \lim_{m \to \infty} y_k^{(j,m)} = y_k^{(j)}, \]
y\( (j) = (y_k^{(j)}) \) being defined at the beginning of this proof. Then \( \Delta x_k^{(j,m)} = A x_k^{(j,m)} + B_k u_k^{(j,m)} \), i.e., \( \left( x_k^{(j,m)} \, u_k^{(j,m)} \right) \) is admissible for \( \mathcal{A} \) and for \( l > m \) we have using (19) (with the matrices \( B, C \) given by (13))
\begin{align*}
\mathcal{A} \left( x_k^{(j,m)} - x_k^{(j,l)} \right) & = \sum_{k=0}^{\infty} \left( (x_k^{(j,m)} - x_k^{(j,l)} \right)^T C_k (x_k^{(j,m)} - x_k^{(j,l)} + (u_k^{(j,l)} - u_k^{(j,m)})^T B_k (u_k^{(j,m)} - u_k^{(j,l)}) \right) \\
& = \sum_{k=0}^{m-1} (x_k^{(j,m)} - x_k^{(j,l)} \right)^T \left[ - \Delta u_k^{(j,m)} + C_k x_k^{(j,m)} - A^T u_k^{(j,m)} \right] \\
& \quad - \sum_{k=0}^{m-1} (x_k^{(j,m)} - x_k^{(j,l)} \right)^T \left[ - \Delta u_k^{(j,l)} + C_k x_k^{(j,l)} - A^T u_k^{(j,l)} \right] \\
& \quad + \sum_{k=0}^{l-1} (x_k^{(j,l)} \right)^T \left[ - \Delta u_k^{(j,l)} + C_k x_k^{(j,l)} - A^T u_k^{(j,l)} \right] \\
& \quad - \sum_{k=0}^{m-1} (x_k^{(j,l)} \right)^T \left[ - \Delta u_k^{(j,m)} + C_k x_k^{(j,m)} - A^T u_k^{(j,m)} \right].
\end{align*}
According to the definition of \( \left( x_k^{(j)}, u_k^{(j)} \right) \), only the last summand in the previous expression is non-zero, denote it by (\(*\)). For this expression we have (taking into account that \( - \Delta u_k^{(j,m)} + C_k x_k^{(j,m)} - A^T u_k^{(j,m)} = 0 \) for \( k = 0, \ldots, m-2 \))
\begin{align*}
(\ast) & = \sum_{k=0}^{m-1} \left( x_k^{(j,l)} \right)^T \left[ - \Delta u_k^{(j,l)} + C_k x_k^{(j,l)} - A^T u_k^{(j,l)} \right] \\
& = e_j^T \left( x_k^{(j,l)} \right)^T (I - A^T) u_k^{(j,m)}.
\end{align*}
Hamiltonian system (12) can be written in the recurrence form
\[ x_{k+1} = \tilde{A} x_k + \tilde{A} B_k u_k, \quad u_{k+1} = C_k \tilde{A} x_k + (C_k \tilde{A} B_k + I - A^T) u_k, \]
where \( \tilde{A} = (I - A)^{-1} \), whose matrix is symplectic as can be verified by a direct computation, see also [3] or [5], and hence
By the same computation we find that (for the domain given by the right-hand side of (22), we denote it by we essentially follow the idea introduced in [19].

\[
\begin{pmatrix}
\tilde{A}
& \tilde{A}B_k

& C_k\tilde{A}

& C_k\tilde{A}B_k + I - A^T
\end{pmatrix}^{-1} = 
\begin{pmatrix}
B_k\tilde{A}^T C_k + I - A

& -B_k\tilde{A}^T

& -\tilde{A}^T C_k

& -\tilde{A}^T
\end{pmatrix}
\]

This means that (12) can be written as the the so-called reversed symplectic system (in the matrix form)

\[
X_k = (B_k\tilde{A}^T C_k + I - A)X_{k+1} - \tilde{A}_k\tilde{A}^T U_{k+1}, \quad U_k = -\tilde{A}^T C_k X_{k+1} + \tilde{A}^T U_{k+1}.
\]

Using the second equation in this reversed system with \(k = m - 1\), \(X^m = (X^m)\), and taking into account that \(X_m^{[m]} = 0\), we have

\[
\langle X_m^{[l]} \rangle^T (I - \tilde{A}^T) U_m^{[m]} = (X_m^{[l]})^T U_m^{[m]} = (\tilde{X}_m - \tilde{X}_m\tilde{l}_j^{-1}\tilde{l}_j) (\tilde{U}_m - \tilde{U}_m\tilde{l}_m^{-1}\tilde{l}_m)

= (\tilde{X}_m^{-1}\tilde{X}_m - \tilde{X}_m^{-1}\tilde{l}_j) \tilde{l}_m (\tilde{U}_m\tilde{l}_m^{-1}\tilde{l}_m - \tilde{U}_m\tilde{l}_m^{-1}\tilde{l}_m^{-1})

= (\tilde{X}_m^{-1}\tilde{X}_m - \tilde{X}_m^{-1}\tilde{l}_j),
\]

here we have used identities (24).

Consequently, by (16)

\[
\mathcal{D}(\chi[x;m] - \chi[y;j], u[y;m] - u[y,j]) = e_j^T [\tilde{X}_m^{-1}\tilde{X}_m - \tilde{X}_m^{-1}\tilde{l}_j] e_j \to 0
\]
as \(m, l \to \infty\), i.e., by (17)

\[
\mathcal{A}(y[y;m] - y[y;j], y[y;m] - y[y,j]) \to 0.
\]

By the same computation we find that (for \(i = 1, \ldots, n\))

\[
Q(\tilde{X}[y;j] - \tilde{X}[y;m], u[y;j] - u[y,m]) = (\mathcal{A}(y[y;i] - y[y;m], y[y;j] - y[y,m]) \to 0
\]
as \(m \to \infty\), which means, by (21), that \(y[y;m] \to y[y]\) in \(\ell^2\) as \(m \to \infty\). Consequently, in view of (3), \(y[y] \in \mathcal{D}(\mathcal{A}, \mathcal{A})\), \(j = 1, \ldots, n\).

Now we prove that (22) really characterizes the domain of the Friedrichs extension of \(\mathcal{A}_{\min}\). Here we essentially follow the idea introduced in [19]. We have

\[
[y[y], y[y]] = e_j^T \left( \begin{array}{c} \tilde{X} \\ U \end{array} \right)^T f \left( \begin{array}{c} \tilde{X} \\ U \end{array} \right) e_j = e_j^T (\tilde{X}^T U - U^T \tilde{X}) e_j = 0,
\]

since the recessive system of solutions of \(L(y) = 0\) determines (via (11)) a conjoined basis of (12). Hence, the domain given by the right-hand side of (22), we denote it by \(\mathcal{D}\), is the domain of a self-adjoint realization of \(\mathcal{A}_{\min}\). Note that boundary conditions in (22) need not be linearly independent relative \(\mathcal{D}(l_{\min})\), see Remark 1 (ii) below. Now, let \(y \in \mathcal{D}(\mathcal{A}, \mathcal{A})\), then also (by self-adjointness) \(y \in \mathcal{D}(\mathcal{A}, \mathcal{A})\). Since \(y[y] \in \mathcal{D}(\mathcal{A}, \mathcal{A})\), we have \(y[y], y[y] = 0, j = 1, \ldots, n\), i.e., \(\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}\). This, together with the fact that \(\mathcal{D}\) is a domain of a self-adjoint extension of \(\mathcal{A}_{\min}\) shows that \(\mathcal{D}(\mathcal{A}) = \mathcal{D}\).

**Remark 1**

(i) In the previous theorem we have proved that sequences which are in the domain of the Friedrichs extension of the operator \(\mathcal{A}\) behave near \(\infty\) like sequences from the recessive system of solutions of the associated Sturm–Liouville operator (7). Consider now again this Sturm–Liouville operator and let \(\left( \begin{array}{c} X \\ U \end{array} \right)\) be a *dominant* solution of (12) associated with (7), i.e., (16) holds. Theorem 1, coupled with (16), suggests the conjecture that the domain \(\mathcal{D}(\mathcal{A})\) can be also described as follows

\[
\mathcal{D}(\mathcal{A}) = \left\{ y = (y_k)_{k=0}^\infty \in \mathcal{D}(\mathcal{A}_{\max}), \lim_{k \to \infty} X_k^{-1} x_k = 0, x_k \text{ given by (11)} \right\}.
\]

This conjecture is a subject of the present investigation.
(ii) Observe that similarly to higher order symmetric differential expressions, $n$ boundary conditions hidden in (22) need not be linearly independent, see [19]. The number of linearly independent conditions among them depends (again similarly to differential expressions) on the number $q = q_\pm$ defined in (20). In particular, if $q = 0$, i.e., the operator $A_{\text{min}}$ is self-adjoint and $A_F = A_{\text{min}}$, then boundary conditions (22) are implied by the assumption $Ay \in \ell^2$ involved in the definition of $\mathcal{D}$. A typical example of this situation is the operator $L(y) = \Delta^2 y_{k-1}$ where the recessive solution is $\tilde{y}_k = 1$, i.e., $0 = [y, \tilde{y}]_\infty = \lim_{k \to \infty} \Delta y_k$ and this condition is implied by $y \in \ell^2, \Delta^2 y \in \ell^2$ which define $\mathcal{D}$ in this case.

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