Some further results on ideal convergence in topological spaces

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**Abstract**

In this paper we make some further investigations on ideal convergence and in particular we concentrate on $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points. We try to establish the characterization of the set of $\mathcal{I}$-limit points (which has not been done in any structure so far) and show that this set can be characterized as an $F_\sigma$-set for a large class of ideals, namely analytic $P$-ideals and then make certain interesting observations on $\mathcal{I}$-cluster points.

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1. Introduction

We start by recalling the definition of asymptotic density as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subseteq \mathbb{N}$, then $K_n$ denotes the set \( \{ k \in K : k \leq n \} \) and $|K_n|$ stands for the cardinality of the set $K_n$. The asymptotic density of the subset $K$ is defined by

\[
d(K) = \lim_{n \to \infty} \frac{|K_n|}{n}
\]

provided the limit exists.

Using this idea of asymptotic density, the idea of convergence of a real sequence had been extended to statistical convergence by Fast [10] (see also [17]) as follows: A sequence $(x_n)_{n \in \mathbb{N}}$ of points in a metric space $(X, \rho)$ is said to be statistically convergent to $\ell$ if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{ k \in \mathbb{N} : d(x_k, \ell) \geq \epsilon \}$ has asymptotic density zero. A lot of investigations have been done on this very interesting convergence and its topological consequences after the initial works by Fridy [11,12]. For more references one can see [2,15] and in particular [1,7,8] for more recent studies and applications of statistical convergence in topological spaces where the reference of the other breakthrough work of Šalát can also be found.

In particular in [12] Fridy introduced the notions of statistical limit and statistical cluster points and showed that the set of all statistical cluster points is closed but the set of statistical limit points is neither open nor closed. In [14] this set was finally characterized as an $F_\sigma$-set (the same was established for double sequences in [3]).

On the other hand, in [13] an interesting generalization of the notion of statistical convergence was proposed. Namely it is easy to check that the family $\mathcal{I}_d = \{ A \subseteq \mathbb{N} : d(A) = 0 \}$ forms a non-trivial admissible (or free) ideal of $\mathbb{N}$ (recall that...
We then make certain interesting observations on large class of ideals, namely analytic (which has not been done in any structure till now) and show that this set can also be characterized as an \( -\)cluster points. In the main results of this paper we establish the characterization of the set of \( -\)cluster points in line of [7]. Throughout the paper we follow the usual notation and terminology (see for example [9]).

2. Main results

We first recall the following definitions and results from [15].

**Definition.** Let \( x = (x_n)_{n \in \mathbb{N}} \) be a sequence of elements in a topological space \( (X, \tau) \).

(a) \( y \in X \) is called an \( I \)-limit point of \( x \) if there exists a set \( M = \{m_1 < m_2 < m_3 < \cdots \} \subset \mathbb{N} \) such that \( M \notin I \) and \( \lim_{n \to \infty} x_{m_n} = y \).

(b) \( y \in X \) is called an \( I \)-cluster point of \( x \) if for every open set \( U \) containing \( y \), \( (n \in \mathbb{N}: x_n \in U) \notin I \).

We denote by \( I(C_x) \) and \( I(L_x) \) the collections of all \( I \)-cluster points and all \( I \)-limit points of \( x \), respectively. For an admissible ideal \( I \), it is known that \( I(L_x) \subset I(C_x) \) [13,15].

**Theorem 1.** Let \( I \) be an admissible ideal of \( \mathbb{N} \). Then:

1. \( I(C_x) \) is closed for each sequence \( x \) in \( X \).
2. Conversely if \( X \) is such that \( \text{hcl}(X) = \omega \) (i.e. every closed subset of \( X \) is separable [7]) and if there exists a disjoint sequence of sets \( (M_n)_{n \in \mathbb{N}} \) such that \( M_n \subset \mathbb{N}, M_n \notin I \forall n \), then for each non-void closed set \( F \subset X \), there exists a sequence \( x \) in \( X \) such that \( F = I(C_x) \).

Proof of (1) is given in [15]. Though proof of (2) is given in [15] taking \( X \) as hereditarily separable, one can easily observe that the condition \( \text{hcl}(X) = \omega \) is sufficient to prove the result.

2.1. Characterization of the set \( I(L_x) \)

An ideal \( I \) is said to be a \( P \)-ideal (or said to satisfy condition (AP)) if for every sequence \( (A_n)_{n \in \mathbb{N}} \) of elements of \( I \) there exists \( A_\infty \in I \) such that \( A_n \cap A_\infty \) is a finite set for every \( n \in \mathbb{N} \). Also recall that after identifying the power set \( P(\mathbb{N}) \) of \( \mathbb{N} \) with the Cantor space \( C = \{0, 1\}^\mathbb{N} \) in a standard manner we may consider an ideal as a subset of \( C \). An ideal is called an analytic ideal if it corresponds to an analytic subset of \( C \). Solecki [18,19] proved that every analytic \( P \)-ideal is determined by some lower semicontinuous submeasure on \( \mathbb{N} \).

**Definition.** Let \( S \) be a set. We say that a map \( \varphi: P(S) \to [0, \infty) \) is a submeasure on \( S \) if it satisfies the following conditions:

- \( \varphi(\emptyset) = 0 \) and \( \varphi(s) < \infty \) for every \( s \in S \).
- \( \varphi \) is monotone: if \( A \subset B \subset S \), then \( \varphi(A) \leq \varphi(B) \).
- \( \varphi \) is subadditive: if \( A, B \subset S \), then \( \varphi(A \cup B) \leq \varphi(A) + \varphi(B) \).

A submeasure \( \varphi \) on \( \mathbb{N} \) is lower semicontinuous if for every \( A \subset \mathbb{N} \) we have \( \varphi(A) = \lim_{n \to \infty} \varphi(A \cap [1, n]) \).

Note that a submeasure on \( \mathbb{N} \) is lower semicontinuous if and only if it is lower semicontinuous as a function from \( P(\mathbb{N}) \) to \( [0, \infty) \). Solecki in [19] proved that every analytic \( P \)-ideal \( I \) can be presented as

\[
I = \left\{ A \subset \mathbb{N} : \lim_{n \to \infty} \varphi(A \cap [1, n]) = 0 \right\}
\]

(*) for some lower semicontinuous submeasure \( \varphi \) on \( \mathbb{N} \).
Theorem 2. Let $X$ be a first countable space. For any sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ the set $\mathcal{I}(L_x)$ is an $F_\sigma$-set provided $\mathcal{I}$ is an analytic $P$-ideal.

Proof. Since $\mathcal{I}$ is an analytic $P$-ideal, there exists a lower semicontinuous submeasure $\varphi$ satisfying $(\ast)$. For any $r \in \mathbb{N}$ let

$$F_r = \left\{ p \in X : \exists A = \{n_1 < n_2 < n_3 < \cdots \} \subset \mathbb{N}, \lim_{n \to \infty} x_{a_n} = p \text{ and } \lim_{n \to \infty} \varphi(A \setminus \{n, 1, n, n, \ldots \}) > \frac{1}{r} \right\}.$$  

We shall now show that each $F_r$ is a closed subset of $X$. Let $\alpha \in F_r$ and let $U$ be a neighborhood of $\alpha$. Since $X$ is first countable, there is a sequence $(\alpha_j)_{j \in \mathbb{N}}$ in $F_r$ converging to $\alpha$. For each $\alpha_j$, we can find $A_j \subset \mathbb{N}$ with $\lim_{n \to \infty} x_{\alpha_n} = \alpha_j$ and $\lim_{n \to \infty} \varphi(A_j \setminus \{n, 1, n, n, \ldots \}) > \frac{1}{r}$. Let $(\epsilon_j)_{j \in \mathbb{N}}$ be a monotonically decreasing sequence of positive real numbers converging to 0. We now proceed as follows: First choose $n_1 \in \mathbb{N}$ such that $\varphi(A_1 \setminus \{n_1, 1\}) > \frac{1}{r} - \frac{\epsilon_1}{2}$. Now lower semicontinuity of $\varphi$ implies that $\varphi(A_1 \setminus \{n_1, 1\}) = \lim_{n \to \infty} \varphi((A_1 \setminus \{n_1, 1\}) \cap [1, n])$. Choose $m_1 \in \mathbb{N}$ such that $\varphi((A_1 \setminus \{n_1, 1\}) \cap [1, n]) > \varphi(A_1 \setminus \{n_1, 1\}) - \frac{\epsilon_1}{2}$ for all $n \geq m_1$. Again there exists $m_1 \in \mathbb{N}$ such that $\varphi(A_2 \setminus \{n_2, 1\}) > \frac{1}{r} - \frac{\epsilon_2}{2}$ and so on. Thus we can construct a sequence $(n_1 < n_2 < n_3 < \cdots)$ of positive integers such that

$$\varphi(A_j \cap (n_j, n_{j+1}]) > \frac{1}{r} - \epsilon_j, \quad j \in \mathbb{N}.$$  

Let us define $A = \bigcup_{j} (A_j \cap (n_j, n_{j+1}])$. Then clearly $\lim_{n \to \infty} \varphi(A \setminus \{n, 1, n, n, \ldots \}) > \frac{1}{r}$ and so $A \not\in \mathcal{I}$. Let $A = \{l_1 < l_2 < l_3 < \cdots \}$. Since $\lim_{n} x_n = \alpha$ and $\alpha \in U$ so $\alpha_j \in U \forall j \geq j_0$ for some $j_0 \in \mathbb{N}$. This implies that $x_n \in U$ for all but a finite number of indices $n$ of the set $A$. Therefore $\alpha \in F_r$. Hence $F_r$ is a closed subset of $X$. The assertion now immediately follows from the fact that $\mathcal{I}(L_x) = \bigcup_{i=1}^{\infty} F_i$. □

For the next theorem we shall need the following result.

Lemma 1. ([7]) Let $X$ be a space with $\text{hdcl}(X) = \omega$. Then for any closed set $F \subset X$ there is a sequence $x = (x_n)_{n \in \mathbb{N}}$ in $X$ such that $F = L(x) \{z \in L(x) \text{ if for each neighborhood } W \text{ of } z, \{n \in \mathbb{N} : x_n \in W\} \text{ is infinite} \}$.

Theorem 3. Let $X$ be a space with $\text{hdcl}(X) = \omega$. Then for each $F_\sigma$-set $A$ in $X$ there exists a sequence $x = (x_n)_{n \in \mathbb{N}}$ in $X$ such that $A = \mathcal{I}(L_x)$ provided $\mathcal{I}$ is an analytic $P$-ideal.

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$ where each $A_i$ is a closed subset of $X$. By Lemma 1, for each $i$, we can find a sequence $(y_{i,j})_{j \in \mathbb{N}} \subset A_i$ (see the proof of Lemma 1 from [7]) such that $A_i = L(y_{i,j})$.

Before we proceed, we first observe that if $K \not\in \mathcal{I}$ then $\lim_{n \to \infty} \varphi(K \setminus \{n, 1\}) = \beta$ (say) $\neq 0$. Then $\beta > 0$ (possibly $\beta = \infty$). From the lower semicontinuity of $\varphi$, there are finite pairwise disjoint sets $C_j, j \in \mathbb{N}$ with $C_j \subset K$ and $\lim_{j \to \infty} \varphi(C_j) = \beta$. Let $\mathbb{N} = \bigcup_{i=1}^{\infty} D_i$ be a decomposition of $\mathbb{N}$ into pairwise disjoint subsets of $\mathbb{N}$. Put $K_1 = (K \setminus \bigcup_{j} C_j) \cup (\bigcup_{i \in D_i} C_i)$ and for $i > 1$ $K_i = \bigcup_{j \in D_i} C_j$. Then one can check that the sets $K_i$‘s are pairwise disjoint subsets of $K$, $K = \bigcup_{i=1}^{\infty} K_i$. Further it follows that

$$\lim_{n \to \infty} \varphi(K_i \setminus \{n, 1\}) = \beta \quad \forall i \in \mathbb{N}.$$  

Clearly then $K_i \not\in \mathcal{I} \forall i$.

We now come back to the main proof. First decompose $\mathbb{N}$ into pairwise disjoint sets $\mathbb{N} = \bigcup_{i \in \mathbb{N}} M_i$ where $M_i \not\in \mathcal{I}$ for any $i$. Further for each $i \in \mathbb{N}$, decompose $M_i = \bigcup_{j \in \mathbb{N}} B_{i,j}$ where

$$\lim_{n \to \infty} \varphi(B_{i,j} \setminus \{n, 1\}) = \lim_{n \to \infty} \varphi(M_i \setminus \{n, 1\}) = \beta_i \quad \forall j \in \mathbb{N}$$  

and $B_{i,j} \not\in \mathcal{I} \forall j \in \mathbb{N}$. Define a sequence $x = (x_n)_{n \in \mathbb{N}}$ as follows:

$$x_n = y_{i,j} \quad \text{for each } n \in B_{i,j}.$$  

We shall show that $A = \mathcal{I}(L_x)$. The following two cases may arise.

(i) First suppose that there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $x$ converging to $y$ and $y \not\in A$. Then for every $i$, $\bigcup_{j=1}^{i} M_j$ contains only finite number of terms $n_1, n_2, \ldots, n_k$, but $[n_1, n_2, \ldots, n_k] \not\in \mathcal{I}$ (since $\mathcal{I}$ is admissible). So $y \not\in \mathcal{I}(L_x)$.

(ii) Now take $y \in A$. Then $y \in A_i$ for some $i \in \mathbb{N}$. Then there is a subsequence $(x_{n_{k,i}})_{k \in \mathbb{N}}$ of $(x_{n_{k,i}})_{i \in \mathbb{N}}$ which converges to $y$. Choose an $\epsilon > 0$ such that $\epsilon < \beta_i$. Following the proof in Theorem 1 we can find a sequence of positive integers $t_1 < t_2 < t_3 < \cdots$ such that

$$\varphi(B_{i,k} \cap (t_k, t_{k+1}]) > \beta_i - \epsilon.$$
We now define \( B = \bigcup_{k \in \mathbb{N}} \{ B_{i,k} \cap (t_k, t_{k+1}) \} \). Then it readily follows that
\[
\lim_{n \to \infty} \varphi(B \setminus [1, n]) \geq \beta_1 - \epsilon.
\]
Therefore \( B \notin \mathcal{I} \) and it is easy to observe that \( \lim_{n \to \infty, n \notin B} x_n = y \). This shows that \( y \in \mathcal{I}(L_y) \). This completes the proof of the theorem. \( \square \)

2.2. Further properties of \( \mathcal{I} \)-cluster points

We now present the following additional properties related to the set \( \mathcal{I}(C_y) \) (the first theorem is also true for \( \mathcal{I}(L_y) \)).

**Theorem 4.** If \( x = (x_n)_{n \in \mathbb{N}} \) and \( y = (y_n)_{n \in \mathbb{N}} \) are two sequences in \( X \) such that \( \{ n \in \mathbb{N} : x_n \neq y_n \} \notin \mathcal{I} \) then \( \mathcal{I}(C_x) = \mathcal{I}(C_y) \),
\( \mathcal{I}(L_x) = \mathcal{I}(L_y) \).

**Proof.** Let \( w \in \mathcal{I}(C_x) \). Let \( U \) be an open set with \( w \in U \). Then from the definition, \( \{ n \in \mathbb{N} : x_n \in U \} \notin \mathcal{I} \). But note that if \( \{ n \in \mathbb{N} : y_n \in U \} \notin \mathcal{I} \) then since
\[
\{ n \in \mathbb{N} : x_n \in U \} \cap \{ n \in \mathbb{N} : x_n \neq y_n \} \cup \{ n \in \mathbb{N} : y_n \in U \}
\]
so by our assumption \( \{ n \in \mathbb{N} : x_n \in U \} \notin \mathcal{I} \) which is a contradiction. Hence \( \{ n \in \mathbb{N} : y_n \in U \} \notin \mathcal{I} \) and so \( w \in \mathcal{I}(C_y) \). Similarly we can show this \( \mathcal{I}(C_y) \subseteq \mathcal{I}(C_x) \) and so the equality is proved.

The proof for \( \mathcal{I} \)-limit points is similar. \( \square \)

**Theorem 5.** For any sequence \( x = (x_n)_{n \in \mathbb{N}} \) in a hereditarily Lindelöf space \( X \), there exists a sequence \( y = (y_n)_{n \in \mathbb{N}} \) in \( X \) such that \( \{ n \in \mathbb{N} : x_n \neq y_n \} \notin \mathcal{I} \) and \( \mathcal{I}(C) = L(y) \) provided \( \mathcal{I} \) satisfies the condition (AP) or in other words is a \( P \)-ideal.

**Proof.** The proof is finished if \( \mathcal{I}(C_x) = L(x) \). If not, then \( \mathcal{I}(C_x) \subseteq L(x) \). For each \( z \in L(x) \setminus \mathcal{I}(C_x) \) we can find an open set \( U_z \) containing \( z \) such that \( \{ n \in \mathbb{N} : x_n \in U_z \} \notin \mathcal{I} \). Now \( \{ U_z : z \in L(x) \setminus \mathcal{I}(C_x) \} \) forms an open cover of \( L(x) \setminus \mathcal{I}(C_x) \) and so it has a countable subcover \( \{ U_{z_i} : i \in \mathbb{N} \} \). Put \( A_i = \{ n \in \mathbb{N} : x_n \in U_{z_i} \} \). Then \( (A_i) \) is a sequence of sets in \( \mathcal{I} \) and since \( \mathcal{I} \) has property (AP), there exists an \( A \in \mathcal{I} \) such that \( A \setminus A_i \) is finite for each \( i \). If \( \mathbb{N} \setminus A = \{ k_n : n \in \mathbb{N} \} \), then construct \( y = (y_n)_{n \in \mathbb{N}} \) as follows:
\[
y_n = x_{k_n} \quad \text{if} \quad n \in A,
\]
\[
y_n = x_n \quad \text{if} \quad n \notin A.
\]
Clearly then \( \{ n \in \mathbb{N} : x_n \neq y_n \} \subseteq A \) and so belongs to \( \mathcal{I} \). By Theorem 4 we have \( \mathcal{I}(C_y) = \mathcal{I}(C_x) \). But note that the subsequence \( \{ x_{k_n} \} \) of \( (y_n)_{n \in \mathbb{N}} \) has no accumulation point in \( L(x) \setminus \mathcal{I}(C_x) \) and so has no \( \mathcal{I} \)-limit point of \( (y_n) \) (since \( \mathcal{I} \) is admissible). So \( L(y) = \mathcal{I}(C_y) \) and consequently we have \( L(y) = \mathcal{I}(C_x) \). \( \square \)

**Theorem 6.** For any compact subset \( K \) of \( X \) and any sequence \( x = (x_n)_{n \in \mathbb{N}} \) in \( X \), \( \{ n \in \mathbb{N} : x_n \in K \} \notin \mathcal{I} \) implies that \( K \cap \mathcal{I}(C_x) \neq \emptyset \).

**Proof.** If possible let \( K \cap \mathcal{I}(C_x) = \emptyset \). Now for each \( z \in K \) there exists an open set \( U_z \) containing \( z \) such that \( M_z = \{ n \in \mathbb{N} : x_n \in U_z \} \notin \mathcal{I} \). Now \( \{ U_z : z \in K \} \) forms an open cover of the compact set \( K \) and so has a finite subcover \( \{ U_{z_1}, U_{z_2}, U_{z_3}, \ldots, U_{z_k} \} \). But then
\[
\{ n \in \mathbb{N} : x_n \in K \} \subseteq M_{z_1} \cup M_{z_2} \cup \cdots \cup M_{z_k}.
\]
Since the set on the right hand side belongs to \( \mathcal{I} \) so is then the set on the left hand side which is a contradiction. \( \square \)

2.3. Uniform spaces

For the rest of the section our underlying structure is a Hausdorff uniform space \((X, \mathcal{U})\). By \( \text{CL}(X) \) and \( \text{K}(X) \) we denote the set of all nonempty closed subsets of \( X \) and all nonempty compact subsets of \( X \), respectively. Recall that a sequence \( x = (x_n)_{n \in \mathbb{N}} \in (X, \mathcal{U}) \) is called bounded if \( \{ n \in \mathbb{N} : U(x) = q \in \mathcal{U} \} \) for some point \( p \in X \) and some \( U \in \mathcal{U} \). \((X, \mathcal{U})\) is called boundedly compact if each bounded set in \( X \) is compact. \((X, \mathcal{U})\) is said to have nice closed sections if for each \( U \in \mathcal{U} \) and \( x \in X \), \( \mathcal{U}[x] \) is compact provided \( \mathcal{U}[x] \neq X \).

\( bs(X) \) will denote the set of all bounded sequences in \((X, \mathcal{U})\), while \( \mathcal{I} - cs(X) \) will denote the set of all sequences \( x = (x_n)_{n \in \mathbb{N}} \in X \) with \( \mathcal{I}(C_y) \neq \emptyset \). Since by Theorem 6, \( \mathcal{I}(C_y) \neq \emptyset \) for bounded sequences so \( bs(X) \subseteq \mathcal{I} - cs(X) \). If \((X, \mathcal{U})\) is boundedly compact then \( \mathcal{I}(C_y) \) is compact, so bounded and also is a closed set (by Theorem 1). Hence we can introduce the mapping
\[
\theta : bs(X) \to K(X) \quad \text{by} \quad \theta(x) = \mathcal{I}(C_x) \quad \text{for each} \quad x = (x_n)_{n \in \mathbb{N}} \in bs(X).
\]
We endow $\mathcal{I} = \text{cs}(X)$ with a uniformity $\tilde{U}$ defined as follows:

$$\tilde{U} = \{ \tilde{U} = (x_n, (y_n)); \forall n \in \mathbb{N} \mid x_n \in U[x_n] \text{ and } y_n \in U[y_n]; U \in \mathcal{U} \}.$$ 

On $K(X)$ we consider the Hausdorff–Bourbaki uniformity $U_H$ inherited from the space $\text{CL}(X)$, namely

$$U_H = \{ U = (A, B) \in \text{CL}(X) \times \text{CL}(X); U \in \mathcal{U}, A \subset U[B] \text{ and } B \subset U[A] \}.$$ 

**Theorem 7.** Let $(X, \mathcal{U})$ be a boundedly compact uniform space. Then the mapping $\theta_{\mathcal{I}} : (bs(X), \tilde{U}) \to (K(X), U_H)$ is uniformly continuous.

**Proof.** Let $U \in \mathcal{U}$. Choose a $V \in \mathcal{U}$ such that $V^3 \subset U$. Let $((x_n), (y_n)) \in \tilde{V}$ and $p \in \mathcal{I}(C_x)$ (where $x = (x_n)_{n \in \mathbb{N}}$). Then $B = \{ n \in \mathbb{N} : x_n \in V[p] \} \notin \mathcal{I}$. Now for each $n \in B$, $(x_n, y_n) \in V$ and $(x_n, p) \in V$ so that $(y_n, p) \in V^2$, i.e., $y_n \in V^2[p]$. Hence

$$\{ n \in \mathbb{N} : y_n \in V^2[p] \} \supset \{ n \in \mathbb{N} : x_n \in V[p] \}$$

which implies that the set on the left hand side also cannot belong to $\mathcal{I}$. Again as $\overline{V^2[p]}$ is compact, by Theorem 6, $\overline{V^2[p]} \cap \mathcal{I}(C_y) \neq \emptyset (y = (y_n)_{n \in \mathbb{N}})$. This implies that $p \in V^3[I(C_y)]$. Therefore

$$\mathcal{I}(C_x) \subset V^2[I(C_y)] \subset U[I(C_y)].$$

Similarly it can be shown that $\mathcal{I}(C_y) \subset U[I(C_x)]$. Hence $(\mathcal{I}(C_x), \mathcal{I}(C_y)) \in \mathcal{U} \in U_H$.  

Now we recall the following concepts.

$2^X$ will denote the collection of all closed subsets of a Hausdorff topological space $X$. For $A \subset X$ and any family $A$ of subsets of $X$ we define

$$A^C = X \setminus A, \quad A^c = \{ A^C : A \in A \},$$

$$A^- = \{ F \in 2^X : F \cap A \neq \emptyset \},$$

$$A^+ = \{ F \in 2^X : F \subseteq A \}.$$

Let $\Delta \subset 2^X$ and let $\Delta$ be closed under finite unions and contains all singletons.

The upper $\Delta$-topology denoted by $\tau_{\Delta^+}$ is the topology with the base

$$\{ (D^c)^+ : D \in \Delta \} \cup \{ \text{CL}(X) \}.$$

When $\Delta = \text{CL}(X)$, the resulting topology is known as the upper Vietoris topology $\tau_{\Delta^+}$ and for $\Delta = K(X)$, the resulting topology is known as upper Fell topology $\tau_{F^+}$.

The lower Vietoris topology $\tau_{\Delta^-}$ is generated by the subbase $\{ U^- : U \subset X \text{ is open} \}$. The $\Delta$-topology $\tau_{\Delta}$ is defined as $\tau_{\Delta^+} \cap \tau_{\Delta^+}$ and so has basic sets of the form

$$(D^c)^+ \cap \left( \bigcap_{i=1}^m V_i^- \right), \quad D \in \Delta, V_1, \ldots, V_m \text{ open in } X.$$ 

Similarly the Vietoris topology and Fell topology are defined as $\tau_V = \tau_{\Delta^+} \cup \tau_{\Delta^-}$, $\tau_F = \tau_{F^+} \cup \tau_{F^-}$.

**Theorem 8.** For a locally compact uniform space $(X, \mathcal{U})$, the mapping $\theta_{\mathcal{I}} : (\mathcal{I} = \text{cs}(X), \tilde{U}) \to (\text{CL}(X), \tau_{\Delta^-})$ is continuous.

**Proof.** Take a basic open set $G^-$ from $(\text{CL}(X), \tau_{\Delta^-})$ where $G$ is open in $X$. Let $x = (x_n)_{n \in \mathbb{N}} \in \theta_{\mathcal{I}}^{-1}(G^-)$. Then clearly $\mathcal{I}(C_x) \cap G \neq \emptyset$. Let $p \in \mathcal{I}(C_x) \cap G$. By the local compactness, we can find a $V \in \mathcal{U}$ such that $p \in \mathcal{I}[p] \subset G$ and $V[p]$ is compact. Choose $W \in \mathcal{U}$ such that $W^2 \subset V$. Since $p \in \mathcal{I}(C_x)$, $\{ n \in \mathbb{N} : x_n \in W[p] \} \notin \mathcal{I}$. Take $y = (y_n)_{n \in \mathbb{N}} \in W[x_n]$. Now $y_n \in W[x_n]$ $\forall n$ and $x_n \in W[p]$ imply that $y_n \in W^2[p] \subset V[p]$. Thus

$$\{ n \in \mathbb{N} : x_n \in W[p] \} \subset \{ n \in \mathbb{N} : y_n \in V[p] \} \subset \{ n \in \mathbb{N} : y_n \in \overline{V[p]} \}.$$

Then obviously $\{ n \in \mathbb{N} : y_n \in \overline{V[p]} \} \notin \mathcal{I}$. Since $\overline{V[p]}$ is compact, Theorem 6 implies that $\mathcal{I}(C_y) \cap \overline{V[p]} \neq \emptyset$, and so $\mathcal{I}(C_y) \cap G \neq \emptyset$. Hence $y \in \theta_{\mathcal{I}}^{-1}(G^-)$. This shows that $W[x_n] \subset \theta_{\mathcal{I}}^{-1}(G^-)$ and hence $\theta_{\mathcal{I}}$ is continuous.  

**Theorem 9.** Let $(X, \mathcal{U})$ be a non-compact uniform space with nice closed sections. Then the mapping $\theta_{\mathcal{I}} : (\mathcal{I} = \text{cs}(X), \tilde{U}) \to (\text{CL}(X), \tau_{F^+})$ is continuous.

We can prove this result following similar arguments like the last theorem and so the proof is omitted.

**Corollary 1.** Let $(X, \mathcal{U})$ be a non-compact uniform space with nice closed sections. Then the mapping $\theta_{\mathcal{I}} : (\mathcal{I} = \text{cs}(X), \tilde{U}) \to (\text{CL}(X), \tau_{F^+})$ is continuous.

The same type of observation can be derived if $\text{CL}(X)$ is endowed with the proximal topology [7] also.
Acknowledgement

The author is thankful to the referee for his/her several valuable suggestions which improved the presentation of the paper.

References