A self-stabilizing algorithm for cut problems in synchronous networks

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ABSTRACT
Consider a synchronized distributed system where each node can only observe the state of its neighbors. Such a system is called self-stabilizing if it reaches a stable global state in a finite number of rounds. Allowing two different states for each node induces a cut in the graph. In each round, every node decides whether it is (locally) satisfied with the current cut. Afterwards all unsatisfied nodes change sides independently with a fixed probability p. Using different notions of satisfaction enables the computation of maximal and minimal cuts, respectively. We analyze the expected time until such cuts are reached on several graph classes and study the impact of the parameter p and the initial cut.

1. Introduction
1.1. Motivation

In the language of distributed computing a system is called self-stabilizing if it reaches a global state with some desired property in finite time, regardless of the initialization. This implies that the system is able to stabilize even in the presence of faults [2,3]. Such self-stabilizing processes have been investigated for various graph problems like maximal matchings [4,5], independent sets [6], and domination [7]. A lot of research effort has been spent on self-stabilizing vertex coloring algorithms [8–12], motivated by code assignment problems in wireless networks.

In this work we consider self-stabilizing algorithms for maximal and minimal cuts in a synchronized distributed system. The network is given by an undirected graph \( G = (V, E) \). As we do not make use of IDs for the nodes, we assume that the network is anonymous. However, we assume that there is a central clock synchronization. In each round every node has one out of two possible states, which induces a cut of the network. In every round every node decides whether it is satisfied with the current cut, judging from a local perspective, i.e., the state of its neighbors. Unsatisfied nodes strive to (locally) improve the cut by changing sides. In order to break symmetries, we investigate a randomized algorithm where in each round every unsatisfied node changes sides with a fixed probability p.

By different notions of satisfaction different types of cuts can be produced. We say that a node is max-satisfied if at least half of its neighbors are on the other side of the cut. If all nodes are max-satisfied, the current cut cannot be increased by changing a single node. Hence the current cut is maximal, i.e., locally optimal with respect to the cut size (as opposed to maximum cuts representing global optima). From a global perspective, the system may be viewed as a self-stabilizing algorithm for maximal cuts.

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The system may also be regarded from a local perspective. For example, the problem can be seen as a relaxed code assignment problem where nodes are forced to use different codes to communicate. In a cut where all nodes are max-satisfied every node can communicate with a majority of neighbors, even if only two codes are available.

On the other hand, a node is min-satisfied if at least half of its neighbors are on the same side of the cut. This notion of satisfaction results in minimal cuts (as opposed to minimum cuts). Finding a minimum cut in a graph is an important task in computer science with applications to clustering, chip design, and network reliability. In our distributed and anonymous setting, however, we content with minimal cuts.

Using the above-mentioned two notions of satisfaction, we show that the system self-stabilizes and then focus on the expected time until a stable cut is obtained. We prove for both satisfaction models that planar graphs stabilize in linear time for appropriate constant values of $p$. The choice of $p$ is crucial since using constant $p$ on dense graphs may result in exponential stabilization times for the max-satisfaction model, with high probability. Finally, we investigate classes of sparse graphs like rings, torus graphs, and hypercubes. On rings the expected stabilization time is logarithmic for constant $p$. For some torus graphs, the choice of the initial cut decides between linear and logarithmic expected stabilization times.

1.2. Related work

Our work is related to the design of distributed approximation algorithms [13] since our algorithm approximates maximum and minimum cuts. This is especially interesting as Elkin [13] concludes in his survey that the distributed approximability of maximum and minimum cuts is still unsolved. However, the focus on this work is different; due to the restrictions in our distributed model we only settle for maximal and minimal cuts, i.e., local optima.

Gradinariu and Tixeuil [9] investigated a self-stabilizing coloring algorithm that is similar to our model. In their work, a node agrees with its neighborhood if it is colored with the maximal color value that is not used by any of its neighbors. In their distributed setting a node that disagrees with its neighborhood changes its color with probability 1/2. It is shown that this strategy stabilizes with a $(B + 1)$-coloring in expected time $O((B - 1) \log n)$ where $B$ is a bound on the maximal degree and $n$ is the number of nodes. This work loosely relates to our work as every 2-coloring represents a maximum cut. However, as typically $B + 1 > 2$ colors may be used, vertex coloring and cut problems are quite different.

The self-stabilizing algorithm may also be regarded from the perspective of evolutionary computation. The operator applied to each node can be seen as a combination of a mutation flipping bits with a fixed mutation probability $p$ and a selection mechanism which decides whether mutation should be applied, judging from the node’s satisfaction. This is equivalent to first creating a mutated state for every node (regardless of its satisfaction) and then selecting among the original and the mutated state, according to the node’s satisfaction at the time of mutation. In fact, several methods and tools used in this work are taken from the analysis of evolutionary algorithms [e.g. 14–17].

1.3. Our results

In the following, we estimate the expected stabilization time on different graph classes like planar graphs, dense random graphs and regular graphs like cycles, torus graphs and hypercubes. After presenting necessary definitions in Section 2, we start with general upper bounds on the expected stabilization time in both min-satisfaction and max-satisfaction models in Section 3. In particular, we derive an upper bound $O(n/p)$ for all planar graphs with $n$ nodes if $p ≤ 1/2$. This bound suggests to choose $p$ large ($p = 1/12$), but for dense graphs this may lead to exponential stabilization times. Section 4 presents such examples for the max-satisfaction process on the complete graph $K_n$ and dense random graphs in the $g(n, 1/2)$-model. For instance, on $K_n$ the expected stabilization time is exponential for $p = 1/2$, but polynomial if $p = Ω((\log n)/n)$ (and $p ≥ n^{-Ω(1)}$). For sparse graphs the choice of $p$ is less important. As shown in Section 5, rings stabilize in expected time $O((\log n)/p(1−p))$. Moreover, the investigation of torus graphs shows that the initialization can be crucial. With a worst-case initialization torus graphs stabilize in expected time $Ω(n/p)$, while random initialization yields a bound of $O((\log n)/p^2)$ on certain torus graphs if $p ≤ 1/2$. Section 6 finishes with conclusions and remarks on future work.

2. Definitions

Let $G = (V, E)$ be an undirected graph with $n = |V|$ nodes. For $U, W ⊆ V$ let $E(U, W)$ be the set of all edges between any node of $U$ and any node of $W$ and $E(U) = E(U, U)$. For $v ∈ V$ let $\deg(v)$ denote the degree of $v$. Let $\Delta(G) = \max_{v∈V} \deg(v)$ be the maximum degree in $G$ and $a(G) = \max_{U⊆V, |U|>1} \left[ \frac{|E(U)|}{|U|-1} \right]$ be the (edge) arboricity of $G$ (see [18]). We use $a(G)$ as a measure of local density in the graph and observe that $a(G)$ is small iff $G$ is “nowhere dense”. We remark that for every graph $G$, $a(G) ≤ \max_{U⊆V, |U|>1} \left[ \frac{\left(\sum_{v∈V} \deg(v)\right) - 2|E(U)|}{2(|U|-1)} \right] ≤ \lfloor n/2 \rfloor$.

At each point of time all nodes are either in state 0 or in state 1. In round $t$ let $V_t(1) ⊆ V$ denote the set of nodes in state 1; $V_t(0) = V \setminus V_t(1)$ is the corresponding complementing set. We synonymously use the term coloring and say that a node $v$ is c-colored if $v ∈ V_t(c), c ∈ \{0, 1\}$. In this case for a vertex $v ∈ V_t(c)$ we denote $\deg_t^c(v) = |E(\{v\}, V_t(1-c))|$ and $\deg_t(v) = \deg(v) - \deg_t^c(v)$. We define two notions of satisfaction mentioned before.

**Definition 1.** A node $v$ is max-satisfied at time $t$ if $\deg_t^+(v) ≥ \deg_t^-(v)$. A node $v$ is min-satisfied at time $t$ if $\deg_t^+(v) ≤ \deg_t^-(v)$.
If it is clear from the context whether we consider the max-satisfaction model or the min-satisfaction model, we simply speak of nodes being satisfied. The following holds for both models. Fixing one satisfaction model, let \( V_{t}^{\text{sat}} \) denote the set of all nodes that are satisfied at time \( t \) and \( V_{t}^{\text{unsat}} := V \setminus V_{t}^{\text{sat}} \) denote the set of unsatisfied nodes. Given \( 0 < p < 1 \), the self-stabilizing cut algorithm is formally defined as follows.

**Self-stabilizing cut algorithm**

1. In round \( t \) execute the following rule simultaneously for all nodes \( v \):
   2. if \( v \in V_{t}^{\text{unsat}} \) then
   3. invert state of \( v \) for round \( t + 1 \) with probability \( p \).

A cut where all nodes are satisfied is called *stable*. The stabilization time is defined as the first round with a stable cut. We are interested in the expected stabilization time, where the initial cut may be chosen uniformly at random or by an adversary. In the latter case, we speak of the worst-case expected stabilization time.

Observe that for bipartite graphs one can easily switch between the two models of satisfaction. Given a bipartition \( V = U \cup W \) of the graph \( G = (V, E) \), flipping (inverting) all nodes in \( U \) turns every cut edge into a non-cut edge and vice versa. Thereby, the meaning of \( \deg^{+}_G(v) \) and \( \deg^{-}_G(v) \) is interchanged and a node becomes min-satisfied if it has been max-satisfied before. In particular, a stable cut for one model becomes a stable cut for the other model after this transformation.

More precisely, let the function \( h \) on the state space \( \{0, 1\}^n \) be such a transformation, then the following holds. Consider the algorithm applied to both models. If the max-satisfaction model starts in state \( x_0 \) and the min-satisfaction model starts in state \( y_0 = h(x_0) \), then at any point of time \( t \) for any state \( x_t \) the probability that the max-satisfaction model is in state \( x_t \) equals the probability that the min-satisfaction model is in state \( y_t = h(x_t) \). This symmetrical behavior implies that the random stabilization times for the two models have the same probability distribution. It therefore suffices to focus on one model when dealing with bipartite graphs.

In the max-satisfaction model, shortly max-model, a stable configuration represents a maximal cut, i.e., a cut that cannot be enlarged by changing a single node. This is because a local improvement implies an unsatisfied node. The same holds for the min-model and minimal cuts. In a non-distributed setting one may easily obtain maximal and minimal cuts by local search, simply changing a single unsatisfied node in each round. The number of cut edges is then strictly increasing over time, implying that at most \(|E|\) iterations are needed in order to find a maximal or minimal cut. The self-stabilizing cut algorithm can simulate an iteration of local search if exactly one specific unsatisfied node is flipped, which happens with probability \( p \cdot (1 - p)^{|V_{\text{sat}}| - 1} > 0 \). Hence, there is a positive probability that the algorithm simulates a whole run of local search and therefore, it will eventually end up with a stable cut.

**Proposition 1.** In both the max-model and the min-model, the self-stabilizing cut algorithm stabilizes in finite time with probability 1.

In the following we present more precise results; we prove bounds between logarithmic, polynomial, and exponential orders for different graph classes. As we are especially interested in the impact of the parameter \( p \), we state our results with respect to \( n \) and \( p \).

### 3. A general upper bound

In this section we derive general upper bounds for both the max-model and the min-model. Thereby, we exploit that under certain conditions there is a probabilistic tendency to increase the cut size in the max-model and to decrease the cut size in the min-model, respectively.

Recall that \( a(G) \) denotes the arboricity of \( G \). The main result in this section says that if \( p \leq 1/(4a(G)) \) then the expected stabilization time in both models is upper-bounded by \( 2|E|/p \). Instead of the arboricity of a graph, one may also consider its treewidth. As shown in [19, Proposition 2], the arboricity of a graph with treewidth \( k \) is at most \( k \). Hence this result also holds when the arboricity is replaced by the treewidth of \( G \).

In order to prove the upper bound, we exploit the following fact. For, say, the max-model the number of cut edges is not necessarily monotone increasing. However, we will show that the cut size for any non-stable cut increases in expectation during one iteration. Such a tendency is often called drift and it can be turned into an upper bound on the expected time until the cut size has increased to the maximum size of \(|E|\) or a stable cut is reached beforehand. The following lemma has been presented by [20]; similar results were derived independently by other authors, see [21, Lemma 6] for an upper bound and [22, Lemma 12] for a lower bound with drift estimates.

**Lemma 2 (Upper Bounds with Drift Analysis [20]).** Consider a Markov process \( \{X_t\}_{t \in \mathbb{N}} \) with finite state space \( S \) and a function \( g: S \rightarrow \mathbb{R}^+_0 \). Let \( T := \inf\{t \geq 0: g(X_t) = 0\} \). If there exists \( \delta > 0 \) such that for any time \( t \geq 0 \) and any state \( X_t \) with \( g(X_t) > 0 \) the condition \( E[g(X_t) - g(X_{t+1}) \mid X_t] \geq \delta \) holds, then

\[
E[T \mid X_0] \leq \frac{g(X_0)}{\delta}.
\]

**Theorem 3.** On any graph \( G = (V, E) \), if \( p \leq 1/(4a(G)) \), the expected stabilization time for both the max-model and the min-model is upper-bounded by \( 2|E|/p \).
Proof. Let $\mathcal{P}_t = (V_t(0), V_t(1))$ and let $f(\mathcal{P}_t) := |E(V_t(0), V_t(1))|$ be the number of cut edges in $\mathcal{P}_t$. We first focus on the max-model. Consider one round of the algorithm and let $V^\text{flip}_t$ be the set of nodes changing sides (flipping) in round $t$. If $v$ is the only node to be flipped in round $t$, this operation increases the cut size by $\deg^-_t(v) - \deg^+_t(v) \geq 1$. If $V^\text{flip}_t$ is an independent set, the total increase of the cut size is $\sum_{v \in V^\text{flip}_t} (\deg^-_t(v) - \deg^+_t(v)) \geq |V^\text{flip}_t|$. However, if two changing nodes share an edge, this edge is counted wrongly for both nodes. This implies

$$f(\mathcal{P}_{t+1}) - f(\mathcal{P}_t) \geq \sum_{v \in V^\text{flip}_t} (\deg^-_t(v) - \deg^+_t(v)) - 2|E(V^\text{flip}_t)|$$

The expected gain in cut size is at least

$$\mathbb{E}[f(\mathcal{P}_{t+1}) - f(\mathcal{P}_t)] \geq p|V^\text{unsat}_t| - 2p^2|E(V^\text{unsat}_t)|.$$ 

Observe $|E(V^\text{unsat}_t)| \leq a(G) \cdot (|V^\text{unsat}_t| - 1) < a(G) \cdot |V^\text{unsat}_t|$ by definition of $a(G)$. Along with the assumption $p \leq 1/(4a(G))$, we arrive at

$$\mathbb{E}[f(\mathcal{P}_{t+1}) - f(\mathcal{P}_t)] \geq p|V^\text{unsat}_t| - 2p^2 \cdot a(G) \cdot |V^\text{unsat}_t| \geq p/2 \cdot |V^\text{unsat}_t|.$$ 

As long as the current cut is not stable, $|V^\text{unsat}_t| \geq 1$, hence the expected increase in cut size is at least $p/2$. We now apply Lemma 2 to our process with $g(\mathcal{P}_t) := |E| - f(\mathcal{P}_t)$ and $\delta := p/2$. Thereby, we finish our considerations prematurely if a stable cut is found before a $g$-value of 0 is reached. This allows us to use the bound $p/2$ on the drift, which holds under the assumption that the cut is not yet stable. By Lemma 2 the expected time until a cut of size $|E|$ is reached or a maximal cut is found beforehand is bounded by $|E|/\delta = 2|E|/p$.

The statement can be proven for the min-model in exactly the same way. The expected decrease of the cut size is at least $p/2$ if the cut is not yet stable. Applying Lemma 2 with $g(\mathcal{P}_t) := f(\mathcal{P}_t)$ proves the claim for the min-model. \(\square\)

Section 5 contains examples where the bound from Theorem 3 is asymptotically tight. Note that the simple strategy of choosing $p = 1/(4n)$ is oblivious of the graph at hand and, nevertheless, yields a polynomial bound of $8|E|/n$ rounds since $a(G) \leq \left\lceil \frac{n}{2} \right\rceil \leq n$ for every graph. This also proves that the expected stabilization time can be polynomial for all graphs if the parameter $p$ is chosen appropriately.

From Theorem 3 one can easily derive a handy upper bound for all planar graphs. The arboricity of a planar graph is known to be at most 3. A proof follows by contradiction. If there is a set $U \subseteq V$ with $|U| > 1$ such that $\frac{|E(U)|}{|U|-1} > 3$, this implies $|E(U)| > 3|U| - 3$. However, this contradicts the fact that the number of edges in a planar graph with $k$ nodes is at most $3k - 6$ (see, e.g., [23]). Therefore $a(G) \leq 3$ holds if $G$ is planar.

Corollary 4. On any planar graph $G = (V, E)$, if $p \leq 1/12$, the expected stabilization time for the max-model and the min-model is bounded by $2|E|/p \leq 6n/p$.

4. Dense graphs

The upper bounds from the previous section grow with $1/p$, suggesting to always choose $p$ large. In this section, however, we prove for the max-model that in dense graphs large values for $p$ may result in exponentially large stabilization times.

The complete graph $K_n$ is the simplest dense graph. For even $n$, a cut is maximal (and maximum in this case) if $|V_t(0)| = n/2$. However, if $p$ is chosen too large, it may happen that too many nodes change sides simultaneously and a majority of 0-nodes is turned into a similarly large majority of 1-nodes, and so forth. This may result in a non-stable equilibrium that is hard to overcome. The following result shows that for large $p$ the max-model needs exponential time to stabilize.

Theorem 5. Consider the complete graph $K_n$, $n$ even, with $n^{-1/3} \leq p \leq 1/2$ and an arbitrary, non-stable initialization. Then the stabilization time of the max-model is at least $\frac{1}{2} \exp\left(\frac{np^2}{192}\right)$ with probability $1 - o(1)$.

First, we state the following technical lemma which is proven in the Appendix. Here, $\text{Bin}(x, p)$ refers to the binomial distribution with $x$ trials and success probability $p$.

Lemma 6. $\Pr[\text{Bin}(x, p) = \frac{x}{2}] = o(1)$ if $\frac{p}{2} \leq x \leq n$ and $n^{-1/3} \leq p \leq \frac{1}{2}$.

Proof of Theorem 5. Due to the symmetry of $K_n$, we have $V^\text{unsat}_t \in \{\emptyset, V_t(0), V_t(1)\}$. If $V^\text{unsat}_t \neq \emptyset$, then $|V^\text{unsat}_t| = \max(|V_t(0)|, |V_t(1)|)$. We will reduce the analysis to a simpler Markov chain $\mathbb{M}$ which consists of only four different states $A$, $B$, $C$, and $D$ defined on a scale of $|V^\text{unsat}_t|$, the number of unsatisfied nodes. The point behind these four states is as follows. The state $D$ represents all stable cuts with $|V^\text{unsat}_t| = n/2$. State $B$ contains a range of values for $|V^\text{unsat}_t|$ where there is a good chance to reach the stable state $D$ in one round. State $A$ contains cuts with large values of $|V^\text{unsat}_t|$ where too many nodes are required to flip in order to reach $D$ in one round with reasonable probability. State $C$ corresponds to an equilibrium state with small values of $|V^\text{unsat}_t|$ (slightly larger than $n/2$) that is hard to overcome.

If we disregard transitions with probabilities exponentially small in $n$, the following holds (see Fig. 1 for an illustration). If $\mathbb{M}$ is in state $A$, it cannot reach the stable state $D$. If $\mathbb{M}$ is in state $B$, there is a probability of $o(1)$ to hit $D$; however, in all
Theorem

Overview on the transitions between the four states. The dashed line represents a transition which occurs with probability $o(1)$, all other transitions (solid lines) may occur with higher probability. Transitions between states with exponentially small probability are not drawn.

other cases will go to the equilibrium state C. If $\mathcal{M}$ is in state C, $\mathcal{M}$ will stay in C with very high probability. The states are formally defined in the following table.

| state | $|V_{t}^{\text{unsat}}|$ ∈ |
|-------|------------------|
| A     | $[\alpha + 1, n]$ |
| B     | $[\beta + 1, \alpha]$ |
| C     | $[\frac{n}{2} + 1, \beta]$ |
| D     | $\{\frac{n}{2}\}$ |

with $\alpha := \frac{n}{2} + \left\lceil \frac{-4np + 2np^2 + 8}{8 - 8p - 2p^2} \right\rceil$, $\beta := \frac{n}{2} + \left\lceil \frac{-np + np^2}{8}\right\rceil + p + \frac{p^2}{4} - 1$.

Note that the state A might be empty if $p$ is close to 1/2. We further observe that $\beta - 1 > \frac{n}{2}$ for large enough $n$, since by assumption $p \geq n^{-1/3}$. To see that the remaining states are well defined, we observe that $\alpha > \beta + 1$ is equivalent to

$$\left\lceil \frac{4np + 4np^2 + 8}{8 - 8p - 2p^2} \right\rceil > \left\lceil \frac{np}{2} + \frac{np^2}{2} + p + \frac{p^2}{4} \right\rceil$$

and this inequality holds since

$$\left\lceil \frac{4np + 2np^2 + 8}{8 - 8p - 2p^2} \right\rceil > \frac{4np + 2np^2 + 8}{8} \geq \frac{np}{2} + \frac{np^2}{8} + p + \frac{p^2}{4} + 1,$$

provided that $n$ is sufficiently large.

In order to analyze $\mathcal{M}$, we shall lower bound and upper bound the number of nodes which flip in one round. Recall that $V_{t}^{\text{flip}}$ is the set of changing nodes in step $t$. We call a step $t$ good if $(1 - \frac{p}{4}) p |V_{t}^{\text{unsat}}| \leq |V_{t}^{\text{flip}}| \leq (1 + \frac{p}{4}) p |V_{t}^{\text{unsat}}|$ holds. Using the Chernoff bounds (cf. Theorem 17)

$$\Pr\left[(1 - \delta) \mu \leq |V_{t}^{\text{flip}}| \leq (1 + \delta) \mu\right] \leq 2 \exp\left(-\frac{\delta^2 \mu}{3}\right)$$

with $\delta = \frac{p}{4}$ and $\mu = |V_{t}^{\text{unsat}}| \cdot p$ yields

$$\Pr\left[(1 - \frac{p}{4}) p |V_{t}^{\text{unsat}}| \leq |V_{t}^{\text{flip}}| \leq (1 + \frac{p}{4}) p |V_{t}^{\text{unsat}}|\right] \leq 2 \exp\left(-\frac{|V_{t}^{\text{unsat}}| p^3}{3 \cdot 16}\right) \leq 2 \exp\left(-\frac{np^3}{96}\right).$$

By the union bound, the probability that the first $\frac{1}{2} \exp\left(-\frac{np^3}{192}\right)$ steps are all good is at least

$$1 - \frac{1}{2} \exp\left(-\frac{np^3}{192}\right) \cdot 2 \exp\left(-\frac{np^3}{96}\right) = 1 - \exp\left(-\frac{np^3}{96}\right).$$

In all following calculations we only consider good steps. Assume w.l.o.g. that at time $t$, $|V_{t}(0)| > |V_{t}(1)|$ holds (the case $|V_{t}(1)| > |V_{t}(0)|$ is done in exactly the same way). Since $V_{t+1}(0) = V_{t}(0) \setminus V_{t}^{\text{flip}}$, we have

$$|V_{t+1}(0)| \leq |V_{t}(0)| - \left(1 - \frac{p}{4}\right) p \cdot |V_{t}(0)| = |V_{t}(0)| \cdot \left(1 - p + \frac{p^2}{4}\right)$$

and

$$|V_{t+1}(0)| \geq |V_{t}(0)| - \left(1 + \frac{p}{4}\right) p \cdot |V_{t}(0)| = |V_{t}(0)| \cdot \left(1 - p - \frac{p^2}{4}\right).$$
First, we will show that the Markov chain \( \mathcal{M} \) will stay in the equilibrium state \( C \), if it is located in \( C \).

\[
|V_{t+1}(0)| \leq |V_t(0)| \cdot \left(1 - p + \frac{p^2}{4}\right)
\]

\[
\leq \beta \cdot \left(1 - p + \frac{p^2}{4}\right)
\]

\[
\leq \left(\frac{n}{2} + \frac{np}{2} + \frac{np^2}{8} + p + \frac{p^2}{4}\right) \cdot \left(1 - p + \frac{p^2}{4}\right)
\]

\[
= \frac{n}{2} - \frac{np}{4} + \frac{np^4}{32} + O(1)
\]

\[
\leq \frac{n}{2} - 1.
\]

for large enough \( n \). On the other hand, we have

\[
|V_{t+1}(0)| \geq \left(1 - p - \frac{p^2}{4}\right) \cdot |V_t(0)|
\]

\[
\geq \left(1 - p - \frac{p^2}{4}\right) \cdot \left(\frac{n}{2} + 1\right)
\]

\[
= \frac{n}{2} - \frac{np}{2} - \frac{np^2}{8} + 1 - p - \frac{p^2}{4}.
\] (1)

Hence \( \frac{n}{2} + 1 \leq |V_{t+1}(1)| \leq \frac{n}{2} + \frac{np}{4} + \frac{np^2}{8} + p + \frac{p^2}{4} - 1 \leq \beta \), and \( \mathcal{M} \) stays in \( C \). Next, we show that in good rounds it is not possible to reach the stable state \( D \) from \( A \):

\[
|V_{t+1}(0)| \geq \left(1 - p - \frac{p^2}{4}\right) \cdot |V_t(0)|
\]

\[
\geq \left(1 - p - \frac{p^2}{4}\right) \cdot \alpha
\]

\[
\geq \left(1 - p - \frac{p^2}{4}\right) \cdot \left(\frac{n}{2} + 4np + 2np^2 + 8\right)
\]

\[
= \left(1 - p - \frac{p^2}{4}\right) \cdot \left(\frac{4n - 4np - np^2}{8 - 8p - 2p^2} + \frac{4np + 2np^2 + 8}{8 - 8p - 2p^2}\right)
\]

\[
= \left(1 - p - \frac{p^2}{4}\right) \cdot \frac{4n + np^2 + 8}{8 - 8p - 2p^2}
\]

\[
= \frac{n}{2} + \frac{np^2}{8} + 1 > \frac{n}{2}.
\]

Furthermore, we show that from state \( B \) we reach either the equilibrium state \( C \) or the stable state \( D \) in a good step, using \( p \leq 1/2 \) and \( 8 - 8p - 2p^2 \geq 7/2 \):

\[
|V_{t+1}(0)| \leq \left(1 - p + \frac{p^2}{4}\right) \cdot |V_t(0)|
\]

\[
\leq \left(1 - p + \frac{p^2}{4}\right) \cdot \alpha
\]

\[
\leq \left(1 - p + \frac{p^2}{4}\right) \cdot \left(\frac{n}{2} + 4np + 2np^2 + 8\right)
\]

\[
= \left(1 - p + \frac{p^2}{4}\right) \cdot \frac{4n + np^2 + 8}{8 - 8p - 2p^2} + O(1)
\]

\[
= \frac{4n - 4np + 2np^2 - np^3 - \frac{1}{4}np^4}{8 - 8p - 2p^2} + O(1)
\]

\[
= \frac{4n - 4np - np^2 + np^3 - \frac{1}{4}np^4}{8 - 8p - 2p^2} + \frac{2np^2 + \frac{1}{2}np^4}{8 - 8p - 2p^2} + O(1)
\]
Obviously, every cut with Theorem $A$ every node in the minority will be satisfied. Recall that for any subset construction of $G$ and the randomized self-stabilizing cut algorithm).

stabilization time of the max-model with $p$ among all graphs with $n$ occurs independently with probability $p$ probability space $1$.

If $|V_t(1)| \leq n - \left(1 - p - \frac{p^2}{4}\right) \cdot |V_t(1)|$

\[\leq n - \left(1 - p - \frac{p^2}{4}\right) \cdot \alpha\]

\[\leq n - \left(1 - p - \frac{p^2}{4}\right) \cdot \left(\frac{n}{2} + 1\right)\]

\[\leq \beta.\]

Hence, either state $C$ or $D$ is reached from $B$. Using Lemma 6, $D$ is only reached with probability at most $o(1)$ (in good steps). Consequently, $M$ will go to state $C$ from $B$ and will stay in state $C$ for the next $\frac{1}{2} \exp(\frac{np^3}{192})$ steps with probability $1 - o(1) - \exp(-\frac{np^3}{192}) = 1 - o(1)$. The claim follows. □

On the other hand, the effect of too many flipping nodes decreases with decreasing $p$. The following result shows that if $p = O((\log n)/n)$ (and, of course, $p \geq n^{-o(1)})$ the expected stabilization time is polynomial.

**Theorem 7.** Consider the complete graph $K_n$, $n$ even, with an arbitrary initialization. Then the expected stabilization time of the max-model is bounded above by $1/p \cdot (1 - p)^{-n/2}$.

**Proof.** Obviously, every cut with $|V_t(0)| = n/2$ is stable. As long as the system has not yet stabilized, there is a unique majority of either 0- or 1-nodes. The minority nodes are satisfied, while all majority nodes are unsatisfied. As long as the majority does not change from 0 to 1 or vice versa, the number of majority nodes is decreasing. Without loss of generality, we start with a majority of 0-nodes, $|V_0(0)| \geq n/2$. If at time $t$ the 0-nodes form a majority, the expected decrease in the number of 0-nodes is $p|V_t(0)| \geq pn/2$. By drift arguments from Lemma 2, the expected time until a cut with at most $n/2$ 0-nodes is obtained is bounded by $n/2 \cdot 2/(pn) = 1/p$.

Let $t + 1$ be the first round where $|V_{t+1}(0)| \leq n/2$. We investigate the random decisions in round $t$ in more detail. The unconditional probability that in this round a stable cut is reached equals

\[\Pr[|V_{t+1}(0)| = n/2] = \left(\frac{|V_t(0)|}{n/2}\right)^p|V_t(0)|^{-n/2} (1 - p)^{n/2}.\]

On the other hand,

\[\Pr[|V_{t+1}(0)| \leq n/2] \leq \left(\frac{|V_t(0)|}{n/2}\right)^p|V_t(0)|^{-n/2}.\]

This implies for the conditional probability of hitting a stable cut, provided $|V_{t+1}(0)| \leq n/2$.

\[\Pr[|V_{t+1}(0)| \mid |V_{t+1}(0)| \leq n/2] = \frac{\Pr[|V_{t+1}(0)| = n/2]}{\Pr[|V_{t+1}(0)| \leq n/2]} \geq (1 - p)^{n/2}.\]

If $|V_{t+1}(0)| = n/2$, we are done. Otherwise $|V_{t+1}(0)| < n/2$ and we repeat the argumentation with symmetric roles for $V_t(0)$ and $V_t(1)$. The number of such trials needed to find a stable cut can be estimated by a geometric distribution with parameter $(1 - p)^{n/2}$, hence the expected number of trials is $(1 - p)^{-n/2}$ and the total expected stabilization time is at most $1/p \cdot (1 - p)^{-n/2}$. □

With a similar approach, the negative result for an unlucky initialization can also be extended to random graphs of a probability space $\mathcal{G}(n, p')$ defined as follows. The random graph consists of $n$ nodes and between any pair of nodes an edge occurs independently with probability $p'$. The case $p' = 1/2$ is especially interesting as $G \in \mathcal{G}(n, 1/2)$ is a uniform sample among all graphs with $n$ nodes.

**Theorem 8.** Consider a graph $G = (V, E)$ in $\mathcal{G}(n, 1/2)$, $n$ even, and assume that initially $\frac{2p}{32} n \leq |V_0(0)| \leq \frac{2p}{32} n$. Then the stabilization time of the max-model with $p = \frac{1}{2}$ is $\exp(\Omega(n))$ with probability $1 - \exp(-\Omega(n))$ (with respect to the randomized construction of $G$ and the randomized self-stabilizing cut algorithm).

In order to prove Theorem 8, we require the following combinatorial lemma. This lemma says that if the size of one side is by a constant factor larger than the size of the other one, then almost every node of the majority will be unsatisfied, while every node in the minority will be satisfied. Recall that for any subset $A \subseteq V$, $\deg_A(v)$ is the number of neighbors of $v$ in $A$. 

\[= n - \frac{np^2}{8} + \frac{2np^2 + \frac{1}{2}np^4}{8 - 8p - 2p^2} + O(1)\]

\[\leq n - \frac{np}{16} + \frac{np + \frac{1}{16}np}{7/2} + O(1)\]

\[= n - \frac{41np}{112} + O(1) \leq \beta\]
Lemma 9. Consider a graph \( G = (V, E) \) in \( g(n, \frac{1}{2}) \) and fix a constant \( \varepsilon > 0 \). With probability \( 1 - e^{-\Omega(n)} \) it holds for every \( A \subseteq V \) with \( |A| \geq \left( \frac{1}{2} + \varepsilon \right) \cdot n \) that for every node \( v \in V \) (except at most \( \log n \) nodes), \( \deg_{\text{in}}(v) > \deg_{\text{out}}(v) \).

A proof for Lemma 9 is given in the Appendix.

Proof of Theorem 8. Call a graph \( G \) of \( g(n, 1/2) \) good if the condition of Lemma 9 holds. By Lemma 9, \( G \) is good with probability \( 1 - \exp(-\Omega(n)) \). Consider now some round \( t \) where \( \frac{20}{32} n \leq |V_t(0)| \leq \frac{23}{32} n \) (the symmetric case \( \frac{20}{32} n \leq |V_t(1)| \leq \frac{23}{32} n \) is done in the same way). Since \( G \) is good, at least \( |V_t(0)| - \log n \geq \frac{n}{2} \) nodes are unsatisfied. Call round \( t \) good if a portion of at least \( \frac{20}{32} \) and at most \( \frac{23}{32} \) of the unsatisfied nodes flips. Using the Chernoff bounds (Theorem 17), we conclude that round \( t \) is good with conditional probability \( 1 - \exp(-\Omega(n)) \). In the case of a good round we have

\[
V_{t+1}(0) \leq |V_t(0)| - \left( \frac{31}{64} \cdot (|V_t(0)| - \log n) \right)
\]

\[
\leq \frac{33}{64} |V_t(0)| + \frac{31}{64} \log n
\]

\[
\leq \frac{33}{64} \cdot \frac{23}{32} n + \frac{31}{64} \log n \leq \frac{12}{32} n.
\]

for large enough \( n \). On the other hand we have

\[
V_{t+1}(0) \geq |V_t(0)| - \left( \frac{33}{64} \cdot (|V_t(0)| + \log n) \right)
\]

\[
= \frac{31}{64} |V_t(0)| - \frac{33}{64} \log n
\]

\[
\geq \frac{31}{64} \cdot \frac{20}{32} n - \frac{33}{64} \log n \geq \frac{9}{32} n.
\]

Hence, \( \frac{20}{32} n \leq |V_{t+1}(1)| \leq \frac{23}{32} n \). By the union bound, the first \( \exp(cn) \) consecutive steps, \( c > 0 \) a sufficiently small constant, are all good with probability \( 1 - \exp(-\Omega(n)) \) and the claim follows. \( \square \)

5. Ring graphs, torus graphs, and hypercubes

We now consider commonly used network topologies like ring graphs (and other graphs with maximum degree 2), torus graphs, and hypercubes.

5.1. Ring graphs

Consider any graph \( G = (V, E) \) with maximum degree 2, so ring graphs are contained in this class as a special case. Theorem 3 yields an upper bound \( O(n/p) \) if \( p \leq 1/8 \). We improve upon this result by exploiting that on these topologies satisfied nodes cannot become unsatisfied again.

Definition 2. A set of nodes \( S \subseteq V \) is called stable with respect to the current cut \( \mathcal{P} \) if all nodes in \( S \) are satisfied and will remain so in all future rounds regardless of all nodes’ decisions afterwards. A node \( v \) is called stable if it is contained in a stable set; otherwise, \( v \) is called unstable.

Isolated nodes are trivially stable, hence we assume that \( G \) does not contain isolated nodes. Then in the max-model (min-model) a node \( v \) is satisfied iff it has at least one neighbor \( w \) on the other side of the cut (on the same side of the cut, respectively). This condition also implies that \( w \) is satisfied. Even stronger, \( v \) and \( w \) will remain satisfied forever since the edge \( \{v, w\} \) will never be touched again. Therefore, on graphs with maximum degree 2 all satisfied nodes are stable.

Theorem 10. The expected stabilization time for the max-model and the min-model on any graph \( G = (V, E) \) with \( \Delta(G) \leq 2 \) is \( O((\log n)/(p(1-p))) \).

Proof. Consider a node \( v \) that is unsatisfied in round \( t \) and the random decision whether to flip \( v \) or not. At least one decision makes \( v \) satisfied in round \( t + 1 \). The “right” random decision for \( v \) is made with probability at least \( q := \min[p, 1-p] \). In expectation \( q |V_{t}^{\text{unsat}}| \) nodes become satisfied (and therefore stable), hence for any \( V_{0}^{\text{unsat}} \subseteq V \)

\[
\mathbb{E}[|V_{t+1}^{\text{unsat}}| | |V_{0}^{\text{unsat}}|] \leq (1-q) \cdot |V_{0}^{\text{unsat}}|.
\]

We now show inductively that \( \mathbb{E}[|V_{t+1}^{\text{unsat}}| | |V_{0}^{\text{unsat}}|] \leq (1-q)^t \cdot |V_{0}^{\text{unsat}}| \). Clearly we have \( \mathbb{E}[|V_{0}^{\text{unsat}}| | |V_{0}^{\text{unsat}}|] = |V_{0}^{\text{unsat}}| \) and using the law of total expectation

\[
\mathbb{E}[|V_{t+1}^{\text{unsat}}| | |V_{0}^{\text{unsat}}|] = \mathbb{E}\left[\mathbb{E}[|V_{t+1}^{\text{unsat}}| | |V_{t}^{\text{unsat}}| | |V_{0}^{\text{unsat}}|] | |V_{0}^{\text{unsat}}|\right]
\]

\[
\leq \mathbb{E}[|V_{t}^{\text{unsat}}| | |V_{0}^{\text{unsat}}|] \cdot (1-q) \cdot |V_{0}^{\text{unsat}}| = (1-q)^t \cdot |V_{0}^{\text{unsat}}|.
\]

\[
\leq (1-q)^{t+1} \cdot |V_{0}^{\text{unsat}}|.
\]
Consider the min-model for $G$. Theorem 12.

The worst-case expected stabilization time for both the max-model and the min-model on $G_{r \times s}$ is $\Omega(n/p)$.
An upper bound can be shown using that unsatisfied nodes have a good chance to become part of a cycle of equally colored nodes.

**Lemma 13.** Consider the torus graph $G_{r \times s}$. If the current cut contains an unsatisfied node $v$, the probability that $v$ becomes stable within the next two rounds is at least $p^2(1 - p)^2$.

**Proof.** Without loss of generality, $v$ is 1-colored and we consider the min-model. We name nodes around $v$ according to their direction from $v$ and identify nodes with their corresponding colors. First consider the case $\deg^+(v) = 0$, implying $v_N = v_E = v_S = v_W = 0$. If any node from $\{v_{NW}, v_{NE}, v_{SE}, v_{SW}\}$ is 0-colored, say $v_{NW}$, flipping $v$ and not flipping $v_N, v_W, v_NW$ creates a cycle. As $v_{NW}$ is satisfied, the probability for such an event is at least $p(1 - p)^2$. Now, assume $v_{NW} = v_{NE} = v_{SE} = v_{SW} = 1$. Then flipping $v_N$ and not flipping $v$ and $v_{NW}$ creates a cycle of 1-nodes. The probability for this to happen is at least $p^2(1 - p)^2$.

Let $\deg^+(v) = 1$ and w.l.o.g. assume that $v_N$ is 1-colored. If $v_{SW}$ or $v_{SE}$ is 0-colored, a 0-cycle is created with probability at least $p(1 - p)^2$. Hence, assume $v_{SW} = v_{SE} = 1$. If $v_{NW}$ or $v_{NE}$ is 1-colored, say $v_{NW}$, then $v_N$ is unsatisfied and flipping it and not flipping $v_{NW}$ and $v$ creates a 1-cycle, with probability $p(1 - p)^2$. The only remaining case is $v_{NW} = v_{NE} = 0$. If the next round flips $v$ and does not flip $v_W, v_{NW}$, and $v_{NE}$, then $v_N$ becomes unsatisfied in the following round. Flipping $v_N$ and not flipping $v_{NW}$ creates a cycle. The probability for these two rounds to be successful is at least $p^2(1 - p)^2$. □

The expected time to decrease the number of unstable nodes is at most $1/(p^2(1 - p)^2) = O(1/p^2)$ if $p \leq 1/2$, hence the following theorem is immediate.

**Theorem 14.** The worst-case expected stabilization time for both the max-model and the min-model on $G_{r \times s}$ is $O(n/p^2)$ if $p \leq 1/2$.

We believe that with random initialization the expected stabilization time is much smaller. It is very unlikely that random initialization creates long paths of unstable 1-nodes. However, such paths of length $\Theta(\log n)$ are still quite likely. Using the same arguments leading to Theorem 12, a lower bound of $\Omega((\log n)/p)$ can be shown. An upper bound is more difficult. If the torus has side lengths, say, $\Theta(\sqrt{n}) \times \Theta(\sqrt{n})$, we can exclude paths of length $\omega(\log n)$ in the initial configuration with high probability. But we cannot exclude that in the following rounds by chance several paths of unstable 1-nodes merge to form a larger path. Only for the special case where the number of rows (or, symmetrically, the number of columns) is constant, we can prove a bound that is of order $O((\log n)/p^2)$ (if $p \leq 1/2$).

**Theorem 15.** After random initialization, the expected stabilization time for both the max-model and the min-model on $G_{r \times s}$ is $O((\log n) \cdot r^2/p^2)$ if $p \leq 1/2$.

**Proof.** Let $L_i := \{(x, i) \mid 0 \leq x < r - 1\}$, $1 \leq i \leq s$, be the nodes in the $i$th column in the graph and note $|L_i| = r$. The probability that all nodes in $L_i$ are initialized zero (or initialized one) is exactly $2^{-\gamma + 1}$. In this case, $L_i$ is a stable set. The probability that there is no stable set among the consecutive columns $L_i, L_{i+1}, \ldots, L_{i+\gamma-1}$, where $\gamma = 2 \cdot 2^{r-1} \cdot \ln n$ for a fixed $i$ is

$$1 - 2^{-r+1} = (1 - 2^{-r+1})^n \leq 2^{-2^{r-1} \cdot \ln n} \leq n^{-2}. $$

Dividing the torus into blocks containing $\gamma$ consecutive columns each, the probability that each block contains at least one stable column is at least $1 - n^{-1}$. Assume that every block contains a stable column and denote by $S$ the set of stable nodes after initialization. Then $G[S]$ consists of connected components, each of which consists of at most $2 \gamma r$ nodes. Consider one component $C$. We define a success (in $C$) as an event that there is a currently unsatisfied node $v \in C$ and $v$ becomes satisfied within the next two rounds. Unless $C$ is stable, there is at least one unsatisfied node in $C$ and by Lemma 13 the probability of a success is at least $q := p^2(1 - p)^2$. We now argue that with high probability $C$ becomes stable within $2T$ rounds, $T := 4\gamma r / q$. Imagine a sequence of $T$ coin flips where each coin shows heads with probability $q$. The expected number of coins showing heads is $qT = 4\gamma r$. By the Chernoff bound (cf. Theorem 17) with $\delta = 1/2$ the probability that less than $2\gamma r$ coins show heads is at most

$$\exp\left(-4\gamma r \cdot \frac{\delta^2}{2}\right) = \exp\left(-\frac{\gamma r}{2}\right) = \exp(-r \cdot 2^{1-\ln r} \leq n^{-2},$$

since $r \geq 2$. As $|C| \leq 2\gamma r$, the probability that $C$ does not become stable within $2T$ rounds is at most $n^{-2}$. Taking the union bound over at most $n$ components, the whole graph is stable after $2T$ rounds with probability at least $1 - n^{-1}$.

The unconditional probability that the bound $2T$ holds is by the union bound at least $1 - 2n^{-1}$. In case there is a block without stable column or in case the system has not stabilized after $2T$ rounds, we use the upper bound $O((\log n)/p^2)$ by Theorem 14 to estimate the remaining stabilization time. As this is only necessary with probability at most $2n^{-1}$, the unconditional expected stabilization time is bounded by $2T + O(1/p^2) = O((\log n) \cdot r^2/p^2)$.

The bound from Theorem 15 depends crucially on $r$. However, we do not believe that the stabilization time is significantly affected by the aspect ratio of the torus. Instead, we conjecture that an upper bound $O((\log n)^k/p^k)$ for some $k = O(1)$ holds for all torus graphs.

### 5.3. Hypercubes

Recall that the node set of a $d$-dimensional hypercube is given by $\{0, 1\}^d$ and edges are between nodes which differ in exactly one coordinate. We are interested in the worst-case expected stabilization time on hypercubes. For torus graphs
we identified paths of unstable 1-nodes that delay the stabilization process. As nodes in the $d$-dimensional hypercube have larger degree if $d > 4$, we identify larger structures of unstable nodes.

**Theorem 16.** The worst-case expected stabilization time for both the max-model and the min-model on a $d$-dimensional hypercube with $n = 2^d$ nodes, $d \geq 4$ even, is $\Omega((n^{1/2} + 1)/p)$.

**Proof.** As the hypercube is bipartite, it suffices to argue for the min-model. Given a graph $G' = (V', E')$, a `snake-in-box in $G'$ is a sequence of connected nodes $s_i', \ldots, s_j'$ such that $(s_i', s_j') \in E'$ implies $j = i \pm 1$ (identifying $s_{j+1}'$ with $s_1'$ and $s_0'$ with $s_j'$).

It is known how to construct a `snake-in-box of length $5/24 \cdot 2^d - 44$ in the $d$-dimensional hypercube [24]. Let $s_1', \ldots, s_k'$ be a `snake-in-box in the $(d/2)$-dimensional hypercube with $\ell \geq 5/24 \cdot 2^{d/2} - 44$ and let $S = \{s_1', \ldots, s_{k-1}'\}$ (notice that the last element of the `snake-in-box has been removed). Let $v[i] \in \{0, 1\}$ denote the value of the $i$th coordinate of $v$ and define an initial cut as follows:

$$ v \in V_0(1) \iff (v[1]v[2] \ldots v[d/2] \in S) \land (v[d - 1]v[d] = 00). $$

Each 0-node with $v[d - 1]v[d] = 00$ is satisfied since flipping one of the last $d/2$ bits results in a 0-neighbor. All other 0-nodes are satisfied since flipping one of the first $d - 2 \geq d/2$ bits leads to a 0-neighbor. We conclude that all 0-nodes are satisfied and, therefore, stable. Divide all 1-nodes into $\ell - 1$ layers, where Layer $i$ is the set of all 1-nodes $v$ with $v[1] \ldots v[d/2] = s_i$. For a 1-node $v$ flipping a bit at position $i \in \{d/2 + 1, \ldots, d - 2\}$ results in a 1-neighbor. Due to the `snake-in-box property of $S$, $v$ has at most two additional 1-neighbors obtained by flipping specific single bits among the first $d/2$ positions. More precisely, after initialization all 1-nodes in Layers 1 and $\ell - 1$ are unsatisfied with a 1-degree (i.e. number of 1-neighbors) of $d/2 - 1$ while every other 1-node has 1-degree $d/2$ and thus is satisfied.

Over time, spread cut can spread out to neighbors layers as follows, starting with Layers 1 and $\ell - 1$. If an unsatisfied 1-node flips, all its 1-neighbors with 1-degree $d/2$ become satisfied. A layer is called satisfied with respect to the current cut if it only contains satisfied 1-nodes. Observe that in every round all satisfied layers are connected in the subgraph of all 1-nodes. Satisfied layers form a “chain” of unsatable 1-nodes, similar to paths of unstable 1-nodes in the case of torus graphs. As for torus graphs, the chain can only be shortened step by step, starting from both ends simultaneously.

We focus on the outermost satisfied layers and define as potential the minimum difference $\alpha - \beta$ for $\alpha \leq \beta$ such that for every satisfied layer $i$ we have $\alpha < i < \beta$. Layers $\alpha$ and $\beta$ therefore represent the ends of the chain of satisfied layers and the potential $\alpha - \beta$ describes the length of the chain. The initial potential equals $\ell - 2$ and a potential of 0 is necessary for a stable cut. Layers $\alpha$ and $\beta$ both contain unsatisfied 1-nodes and a round flipping one of these nodes decreases $\beta$ or $-\alpha$ by 1, respectively. The probability of decreasing the potential by 1 or 2 in one round is at most $\delta := \min\{1, 2^{d/2 - 1}.p\}$, taking the union bound over at most $2^{d/2 - 1}$ unsatisfied 1-nodes in layers $\alpha$ and $\beta$. The expected waiting time for such an event is bounded below by $1/\delta$, hence the expected time until the potential has decreased to 0 is bounded below by $1/\delta \cdot (\ell - 2)/2 = \Omega(n^{1/2} + 1/p)$. \(\square\)

## 6. Conclusions and future work

We investigated a self-stabilizing algorithm for maximal and minimal cuts in a restricted distributed environment. The time until the system stabilizes depends on the model of satisfaction, the underlying network, the parameter $p$, and the initial cut. Surprisingly, the expected stabilization time can range from logarithmic to exponential values. While sparse graphs such as planar graphs, rings, and torus graphs stabilize in expected time $O(n/p^{0(1)})$ (or even in logarithmic time) for max- and min-models, on many dense graphs the stabilization time for the max-model is exponential with high probability if $p$ is constant. Moreover, we have seen for certain torus graphs that there is an exponential gap between random and worst-case initialization.

Several open questions remain, for example a tight bound on the expected stabilization time for all torus graphs and hypercubes with random initialization. Our models use a fixed probability $p$ for flipping unsatisfied nodes. One may also think of other, local strategies, for example flipping an unsatisfied node $v$ with probability proportional to $1/\deg(v)$ or depending on the degrees of $v$'s neighbors.

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**Appendix**

We recall the following concentration inequalities.

**Theorem 17 (The Chernoff Bounds, Cf. [25]).** Let $X_1, X_2, \ldots, X_n$ be independent $\{0, 1\}$-random variables. Let $X := \sum_{i=1}^n X_i$ and $\mu = E[X]$. Then for any number $0 \leq \delta \leq 1$,

$$ \Pr[X \leq (1 - \delta)\mu] \leq \exp\left(-\frac{\mu \cdot \delta^2}{2}\right), $$

$$ \Pr[X \geq (1 + \delta)\mu] \leq \exp\left(-\frac{\mu \cdot \delta^2}{2}\right). $$
and
\[
\Pr[X \geq (1 + \delta) \mu] \leq \exp\left(-\frac{\mu \cdot \delta^2}{3}\right).
\]

**Theorem 18** (Hoeffding’s Bound [26]). Let \(X_1, X_2, \ldots, X_n\) be independent random variables with \(X_i \in [a_i, b_i]\) for all \(1 \leq i \leq n\). Let \(X := \sum_{i=1}^n X_i\) and \(\mu = \mathbb{E}[X]\). Then for any number \(\delta \geq 0\),
\[
\Pr[|X - \mu| \geq \delta] \leq 2 \cdot \exp\left(-\frac{2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).
\]

**Proof of Lemma 6.** Recall that \(\text{Var}[\text{Bin}(x, p)] = xp(1 - p)\). Define an auxiliary random variable \(Z := \frac{\text{Bin}(x, p) - xp}{\sqrt{xp(1 - p)}}\). By the central limit theorem, \(Z\) converges to the standard normal distribution [27, p. 79], i.e.,
\[
\lim_{x \to \infty} \Pr[Z \leq a] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{y^2}{2}} dy
\]
for all \(a \in \mathbb{R}\). Now,
\[
\Pr\left[\left|\text{Bin}(x, p) - \frac{n}{2}\right| < 1\right] \equiv \Pr\left[Z \cdot \sqrt{xp(1 - p)} + xp - \frac{n}{2} < 1\right]
\]
\[
= \Pr\left[-xp + \frac{n}{2} - 1 < \sqrt{xp(1 - p)} < \frac{xp + n}{2} + 1\right]
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \int_{-xp + \frac{n}{2} - 1}^{xp + \frac{n}{2} + 1} e^{-\frac{y^2}{2}} dy + o(1),
\]
using \(e^{-\frac{y^2}{2}} \leq 1\) yields
\[
\Pr\left[\left|\text{Bin}(x, p) - \frac{n}{2}\right| < 1\right] \leq \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{\sqrt{xp(1 - p)}} + o(1) \leq \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{\sqrt{\frac{1}{2} \cdot n^{-1/3} \cdot \frac{1}{2}}} + o(1) = o(1),
\]
and the claim follows. \(\square\)

**Proof of Lemma 9.** We have to prove that the number of vertices that have more neighbors in \(A^c\) than in \(A\) is bounded by \(\log n\). In order to show this, we have to bound the probability that a vertex \(v \in V\) has more neighbors in \(A^c\) than in \(A\), given the existence of \(x\) vertices with this property, where \(1 \leq x \leq \log n\). This is done in the following claim; note that the set \(B\) can be viewed as a set of vertices for which all incident edges have been exposed.

**Claim 19.** Consider a graph \(G = (V, E)\) in \(\mathcal{G}(n, \frac{1}{2})\) and fix a constant \(\varepsilon > 0\). With probability \(1 - \exp(-\Omega(n))\) it holds for every subset \(A \subseteq V, |A| \geq \left(\frac{1}{2} + \varepsilon\right) \cdot n\) and \(B \subseteq V\) with \(|B| \leq \log n\) that for every node \(v \in V\) (except at most \(\log n\) nodes),
\[
\text{deg}_{A^c,B}(v) > \text{deg}_{A,B}(v) + |B|.
\]

**Proof.** Fix a subset \(A \subseteq V\) of size at least \(\left(\frac{1}{2} + \varepsilon\right) \cdot n\) and fix a subset \(B \subseteq V\) of size at most \(\log n\). Consider an arbitrary vertex \(v \in V\). By linearity of expectations,
\[
\mathbb{E}[\text{deg}_{A,B}(v)] = \sum_{v' \in A \setminus \{v\}} \Pr[(v', v) \in E]
\]
\[
\geq \sum_{v' \in A \setminus \{v\}} \Pr[(v', v) \in E] - \log n
\]
\[
\geq \left(\left(\frac{1}{2} + \varepsilon\right) \cdot n - 1\right) \cdot \frac{1}{2} - \log n
\]
\[
= \left(\frac{1}{2} + \varepsilon\right) \cdot \frac{n}{2} - \frac{1}{2} - \log n,
\]
and
\[
\mathbb{E}[\text{deg}_{A^c,B}(v)] = \sum_{v' \in A^c \setminus \{v\}} \Pr[(v', v) \in E] \leq \left(n - \left(\frac{1}{2} + \varepsilon\right) \cdot n\right) \cdot \frac{1}{2} = \left(\frac{1}{2} - \varepsilon\right) \cdot \frac{n}{2}.
\]
Hence, by Hoeffding’s bound we get
\[
\Pr \left[ |\deg_{A \setminus B}(v) - E[\deg_{A \setminus B}(v)]| \geq \frac{\epsilon}{8} E[\deg_{A \setminus B}(v)] \right] \leq 2 \exp \left( - \frac{2 \left( \frac{\epsilon}{8} E[\deg_{A \setminus B}(v)] \right)^2}{n} \right) = \exp(-\Omega(n)).
\]

Similarly, we obtain
\[
\Pr \left[ |\deg_{A \setminus C}(v) - E[\deg_{A \setminus C}(v)]| \geq \frac{\epsilon}{8} E[\deg_{A \setminus C}(v)] \right] \leq 2 \exp \left( - \frac{2 \left( \frac{\epsilon}{8} E[\deg_{A \setminus C}(v)] \right)^2}{n} \right) = \exp(-\Omega(n)).
\]

By the union bound, we have with probability \(1 - \exp(-\Omega(n))\) that
\[
\deg_{A \setminus C}(v) + |B| \leq E[\deg_{A \setminus C}(v)] + \frac{\epsilon}{8} E[\deg_{A \setminus B}(v)] + |B|
\]
\[
\leq \left( \frac{1}{2} - \epsilon \right) \frac{n}{2} + \frac{\epsilon}{8} E[\deg_{A \setminus B}(v)] + \log n
\]
\[
\leq \left( \frac{1}{2} + \epsilon \right) \frac{n}{2} - \epsilon n + \frac{\epsilon}{8} n + \log n
\]
\[
= \left( \frac{1}{2} + \epsilon \right) \frac{n}{2} - \frac{7\epsilon}{8} n + \log n
\]
\[
\leq E[\deg_{A \setminus B}(v)] - \frac{7\epsilon}{8} n + 2 \log n + \frac{1}{2}
\]
\[
\leq \deg_{A \setminus B}(v) - \Theta(n) + 2 \log n + \frac{1}{2}
\]
\[
< \deg_{A \setminus B}(v). \quad \square
\]

Call a node \(v \in V\) bad if \(\deg_{A \setminus B}(v) \leq \deg_{A \setminus C}(v)\) and let \(X_v\) be the event indicating whether \(v\) is bad. As shown above, \(\Pr[X_v] \leq \exp(-\Omega(n))\).

Let us now assume that we know for a set of vertices \(\{v_1, v_2, \ldots, v_{k-1}\}\) with \(1 \leq k \leq \log n\) that each of them is bad. Consider a vertex \(v_k \notin \{v_1, v_2, \ldots, v_{k-1}\}\). Conditioned on the event \(\bigwedge_{i=1}^{k} X_{v_i}\), each edge \((u, v)\) with \(u \notin \{v_1, v_2, \ldots, v_{k-1}\}\) does still exist with probability \(1/2\). Together with the fact that \(|E(\{v_k, \{v_1, v_2, \ldots, v_{k-1}\})| \leq k - 1\), we obtain by Claim 19 with \(B = \{v_1, v_2, \ldots, v_{k-1}\}\) that
\[
\Pr \left[ \bigwedge_{i=1}^{k} X_{v_i} \right] \leq \Pr \left[ \deg_{A \setminus B}(v) \leq \deg_{A \setminus C}(v) + |B| \right] \leq \exp(-\Omega(n)).
\]

Using this, we can bound the probability for having at least \(\log n\) bad nodes by
\[
\sum_{\substack{\{v_1, \ldots, v_{\log n}\} \subseteq A \\mid |v_1, \ldots, v_{\log n}| = \log n}} \Pr \left[ \bigwedge_{i=1}^{k} X_{v_i} \right] = \sum_{\substack{\{v_1, \ldots, v_{\log n}\} \subseteq A \\mid |v_1, \ldots, v_{\log n}| = \log n}} \prod_{i=1}^{k} \Pr \left[ X_{v_i} \mid \bigwedge_{i=1}^{k-1} X_{v_i} \right]
\]
\[
\leq \sum_{\substack{\{v_1, \ldots, v_{\log n}\} \subseteq A \\mid |v_1, \ldots, v_{\log n}| = \log n}} \left( \exp(-\Omega(n)) \right)^{\log n}
\]
\[
= \left( \frac{n}{\log n} \right) \cdot \exp(-\Omega(n \log n))
\]
\[
\leq \exp(-\Omega(n \log n)).
\]

As the number of possible subsets for \(A\) is bounded by \(2^n\), the claim follows by taking the union bound over all subsets \(A\). \(\square\)

References


