CONCURRENCY MEASURE IN COMMUTATION MONOIDS

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The phenomenon of concurrency is modelled by commutativity. We define for any nonempty
word over the action alphabet its concurrency degree, a real number which measures the rate of
simultaneity, when the sequence of actions represented by the word is performed. We study
this degree and we show, in particular, that when a word of length \( n \) is chosen at random, as \( n \)
increases the degree tends with probability 1 to a fixed limit which is completely determined by
the commutation system.

1. Introduction

Cartier and Foata \([2,8]\) introduced monoids generated by an alphabet \( A \), equipped
with a binary relation \( \theta \subset A \times A \), to study combinatorial problems related to
rearrangements. Mazurkiewicz \([10]\) proposed the notion of trace language in such
monoids for modelling concurrent systems. These monoids have been the subject
of intensive studies, see \([3,4,7,11]\). As a matter of fact, they appear to provide a
powerful tool in studying parallelism and synchronization problems. More recently,
in his dissertation, Viennot \([12]\) discussed some related combinatorial properties and
showed their relationship with other known problems in statistical physics.

François's quantitative approach \([1,9]\) is based on the possibility of shuffling
words representing behaviours of the concurrent processes.

It seems quite natural to us that, if the commutativity offers the possibility of in-
terchanging (and thus that of making concurrent) actions, then it should be possible
to measure numerically the concurrency in terms of the commutation structure
defined by the system. It should be noted that in this first study we are not taking
into account the capacity of the executing agent. In fact we are only concerned with
the possibility of scheduling actions in a correct ordering that produces a maximal
concurrency.
In Section 2 we introduce preliminaries and notations used in the paper. The concurrency degree of a word is introduced in Section 3. We introduce and study the average degree for words of length $n$ in Section 4. Section 5 is devoted to an asymptotic study, where we prove that almost all sufficiently long words have the same concurrency degree. We conclude finally with the study of some simple examples in Section 6.

2. Preliminaries and notations

We recall briefly the basic definitions and properties. For a more detailed introduction the reader is referred to [3,4,7,10,11].

A commutation monoid (or system) is a finite alphabet $A$ together with a binary relation $\theta \subseteq A \times A$ which is supposed to be symmetric and irreflexive. We refer to $A$ as the action alphabet. Whenever $(a,b) \in \theta$, we write $ab = ba$ and we say that $a$ and $b$ commute.

Intuitively, when $a$ and $b$ commute the outcome is not altered if one changes their ordering in a sequence of actions involving an occurrence of $ab$ or $ba$. Thanks to this property, we are authorized to consider the commuting $a$ and $b$ as concurrent; indeed, if the result of performing $a$ does not affect that of $b$ and vice versa, $a$ and $b$ can be performed simultaneously. According to this view, the irreflexivity of $\theta$ is imposed by the fact that no two absolutely identical actions can be performed concurrently.

The empty word is denoted by $\lambda$. $A^*$, $A^+$ and $A^\omega$ denote respectively the set of words, the set of nonempty words and the set of infinite words over $A$. For any $w \in A^*$, $|w|$ denotes its length. If $B$ is a subset of $A$, then $(w)|_B$ is the word obtained from $w$ by dropping the letters which are not in $B$ and $|w|_B$ is the length of $(w)|_B$. The same notations hold for a letter $a$ instead of the subset $B$.

$\theta$ generates a congruence (see [3,4,10]). Whenever two words $w_1$ and $w_2$ are equivalent with respect to this congruence, we write $w_1 \equiv w_2$. This means that $w_1$ and $w_2$ are the same in an operational sense.

Throughout this paper we have a given commutation monoid $S = (A, \theta)$. We suppose that $A$ is of cardinality $N$.

3. Concurrency degree

Let $w$ be any word over $A$. To $w$ corresponds, in a topological-sort-way a directed graph $G_w$, in the following way (see [3,4]). Let $w = x_1x_2...x_n$ with $x_i \in A$, $i = 1, ..., n$. The graph $G_w$ has $n$ nodes labelled by $1, 2, ..., n$. There will be an edge from the node $i$ to the node $j$ iff $i < j$ and $x_i$ does not commute with $x_j$.

Example 3.1. Let $A = \{a, b, c, d\}$ and $\theta$ defined by $ab \equiv ba$, $bc \equiv cb$, $ac \equiv ca$, $ad \equiv da$. For $w = dcabadbac$ we have the graph (see Fig. 1) from which we have dropped un-
necessary edges, such as (4, 7), which do not add any information on the execution ordering.

The depth of any $w \in A^*$, denoted by $\text{dep}(w)$, is defined to be the number of nodes on a longest path in $G_w$. The concurrency degree of a nonempty $w$, denoted by $d(w)$, is defined to be $|w|/\text{dep}(w)$. In Example 3.1, we have four longest paths and $\text{dep}(w) = 4$. We thus have $d(w) = \frac{9}{4}$. This measurement is justified by the fact that along 4 actions, it is possible to perform 9 actions and therefore, on the average $\frac{9}{4}$ actions can be performed simultaneously.

The graph $G_w$ is a suitable schema which illustrates in an intuitive way the ordering on the actions of $w$. There is, however, another characterization of $\text{dep}(w)$ which seems more convenient for our purpose. This characterization is related to the Foata normal form, which we recall briefly. For more details, see [2, 10].

A word $w$ is said to be in normal form if it is factorized up to the congruence $\equiv$ into $u_1 u_2 \ldots u_n$, $n \geq 0$, such that

(i) each $u_i$ is a nonempty product of letters commuting pairwisely, and
(ii) if $a$ is a letter in $u_{j-1}$ not appearing in $u_j$, then there is a letter $b$ in $u_j$ which does not commute with $a$.

For a formal proof of the existence and uniqueness of such factorization, see [10]. We refer to each $u_i$ as a factor. Thus a factor is a nonempty word $u$ whose letters commute pairwisely. A letter $a$ is said to commute with a factor $u$ if it commutes with each of its letters.

The normal form for $w$ can be obtained in the following inductive way:

(i) if $w$ is a letter, then its normal form is itself, and
(ii) if $w$ has the normal form $u_1 u_2 \ldots u_k$, then the normal form of $wa$, $a \in A$, is either $u_1 u_2 \ldots u_k a$ if $a$ does not commute with $u_k$ or else $u_1 u_2 \ldots u_i a \ldots u_k$ with $u_i \not= u_j a$, where $i$ is such that $a$ commutes with $u_{i-1}, \ldots, u_i$ but not with $u_{i-1}$.

In Example 3.1, we have $w = (da)(cba)(da)(bc)$. We have the fact that the actions inside a factor may be performed in any ordering but no change of ordering is possible between factors. It is not difficult to see that $\text{dep}(w)$ is the same as the number of factors in the normal form of $w$.

Some properties are studied in the following proposition.
Proposition 3.2. For any \( w, w_1, \ldots, w_k \in A \) the following statements hold:

(i) \( 1 \leq d(w) \leq N = \text{card}(A) \),
(ii) \( d(w_1 \ldots w_k) = \sum_{i=1}^{k} \text{dep}(w_i) d(w_i)/\text{dep}(w_1 \ldots w_k) \),
(iii) \( d(w_1 \ldots w_k) \) is lower-bounded by the (weighted) average of \( d(w_1), \ldots, d(w_k) \) weighted by \( \text{dep}(w_1), \ldots, \text{dep}(w_k) \) respectively,
(iv) \( d(w_1 \ldots w_k) \geq \min\{d(w_1), \ldots, d(w_k)\} \).

Proof. (i) \( \text{dep}(w) \) is upper-bounded by \( |w| \) and lower-bounded by \( w_1/N \).

(ii) Obvious by the fact \( |w_1 \ldots w_k| = w_1 + \ldots + w_k \).

(iii) Obtained by using the fact
\[
\text{dep}(w_1 \ldots w_k) \leq \text{dep}(w_1) + \ldots + \text{dep}(w_k).
\]

(iv) Easy consequence of (iii). \( \square \)

4. Average degree

For any positive integer \( n \) let
\[
d(n) = \frac{n}{N^n} \sum_{w \in A^n} \frac{1}{\text{dep}(w)}. \tag{1}
\]
\( d(n) \) is then the average concurrency degree of words of length \( n \). It is the expected concurrency degree when all words of length \( n \) have the same probability of being chosen.

On the other hand, the average relative depth of words of length \( n \), or the expected value of \( \text{dep}(w)/n \) is defined similarly by
\[
l(n) = \frac{1}{n^{N^n}} \sum_{w \in A^n} \text{dep}(w). \tag{2}
\]

We have obviously
\[
1 \leq d(n) \leq N \tag{3}
\]
and
\[
1/N \leq l(n) \leq 1. \tag{4}
\]
Moreover if \( N \geq 2 \), the second \( \leq \) in (3) and the first \( \leq \) in (4) become \(<\).

Further properties are studied in the following proposition.

Proposition 4.1. Let \( n, n_1, \ldots, n_k \) be positive integers; we have

(i) \( l(\sum_{i=1}^{k} n_i) \) is upper-bounded by the (weighted) average of \( l(n_1), \ldots, l(n_k) \) weighted by \( n_1, \ldots, n_k \) respectively; in other words \( l \) is a convex function,
(ii) \( l(kn) \leq l(n) \),
(iii) \( d(n) \geq 1/l(n) \).
Proof. (i) Let $w_1, \ldots, w_k$ be words of length $n_1, \ldots, n_k$ respectively. We have \[ \text{dep}(w_1 \cdots w_k)/(\sum_{i=1}^{k} n_i) \leq \sum_{i=1}^{k} \text{dep}(w_i)/(\sum_{i=1}^{k} n_i). \] Taking expectations we get \[ l(w_1 \cdots w_k) \leq \sum_{i=1}^{k} n_i \text{dep}(w_i)/(n_1 + \cdots + n_k). \]

(ii) Apply (i) to $n_1 = n_2 = \cdots = n_k = n$.

(iii) The function $f$ defined by $f(\text{dep}) = 1/\text{dep}$ is convex. By Jensen’s inequality [6, pp. 153–154], we have

\[ d(w) = E(1/\text{dep}(w)) \geq 1/E(\text{dep}(w)) = 1/l(n). \]

Example 4.2. Let $A = \{a, b\}$ and $ab = ba$. The monoid is then commutative and \[ \text{dep}(w) = \max\{|w|_a, |w|_b\} \] and consequently

\[ d(n) = \frac{n}{2^n} \sum_{i=0}^{n} \binom{n}{i} \max\{i, n-i\}, \quad (5) \]

and

\[ l(n) = \frac{1}{n} \sum_{i=0}^{n} \binom{n}{i} \max\{i, n-i\}. \quad (6) \]

By a truncation on the binomial sum $l(n)$ simplifies to

\[ l(n) = \frac{1}{2} + \left( \frac{n-1}{\lfloor \frac{1}{2} n \rfloor} \right) / 2^n. \quad (7) \]

When $n$ grows the second term on the left tends to 0. Whence

\[ \lim_{n \to \infty} l(n) = \frac{1}{2}. \quad (8) \]

$d(n)$ upper-bounded by 2 (inequality (3)) and lower-bounded by $1/l(n)$ (clause (iii) of Proposition 4.1) tends thus to 2 as $n \to \infty$.

This simple example shows the difficulty of getting simple expressions without use of summations. Actually it does not seem that there exist such expressions for $d(n)$ except in trivial cases. The asymptotic study of degree thus appears to become a necessity.

In the following section, it will be shown that every commutation monoid possesses a “concurrency degree” which is nearly common to all words of sufficient length and, thus, the notion of average behaviour will appear of less importance.

5. Asymptotic behaviour

Before undertaking a systematic study of the asymptotic behaviour, let us examine the continuity of $d$ in a sense which will be discussed in the sequel. This study, as we shall see, leads us to an interesting result which characterizes entirely the concurrency of a commutation system from a purely quantitative point of view.
Consider the system of Example 4.2. Is it possible to extend the notion of concurrency degree to infinite words? Let \( \{w_n\} \) be the sequence of increasing words defined by Table 1. If \( n \) is odd, then \( d(w_n) = 2 \) and we set \( w_{n+1} = w_n a^k \) where the integer \( k \) is chosen such that \( |w_n a^k|_a = 3 |w_n|_b \) and, therefore, \( d(w_{n+1}) = \frac{1}{3} \). If \( n \) is even, then \( d(w_n) = \frac{4}{3} \) and we set \( w_{n+1} = w_n b^{k^*} \) where the integer \( k^* \) is chosen such that \( |w_n b^{k^*}|_b = |w_n|_a \) and, therefore, \( d(w_{n+1}) = \frac{2}{3} \). \( \{w_n\} \) is an increasing sequence of words and has a limit \( x \in A^\omega \). A reasonable definition for \( d(x) \) would be the limit of \( \{d(w_n)\} \) which does not exist, since \( d(w_n) \) fluctuates between \( \frac{1}{3} \) and \( 2 \).

This simple example seems to give a serious blow to any attempt to define a concurrency degree for infinite words. Since an infinite word \( x \) is approximated by an increasing sequence of finite words, \( d(x) \) must be defined as the limit of the degree sequence. In other words \( d \) must be continuous. Unfortunately this is not possible in general, since there are sequences for which the limit degree does not exist. Nevertheless, as we shall see this remark does not constitute any drawback toward the study of concurrency degree for infinite words.

It is well known that the real interval \([0, 1]\) is one-to-one mapped onto \( A^\omega \) up to a set of measure 0, by \( f(x) = N\)-ary representation of \( x \). And thus we hope that the set \( \{x \in A^\omega | d(x) \text{ undefined}\} \) is of measure 0.

We now turn into a more careful analysis of the example. By the weak law of large numbers [5, Chapter X], we have with probability 1

\[
\frac{|w_a|}{|w|} \rightarrow \frac{1}{3} \quad \text{and} \quad \frac{|w_b|}{|w|} \rightarrow \frac{1}{3} \quad \text{as} \quad |w| \rightarrow \infty.
\]  

Whence with probability 1

\[
d(w) = \frac{|w|}{\text{dep}(w)} = \frac{1}{\max\{|w_a|/|w|, \frac{|w_b|}{|w|}\}} \rightarrow 2.
\]  

**Remark 5.1.** Note that we have implicitly supposed that \( a \) and \( b \) have the same probability of appearing, an assumption which holds in this paper. The study is not, however, limited by this assumption. Suppose that \( a \) and \( b \) appear with probabilities \( \alpha \) and \( \beta \) respectively. Then, with probability 1

\[
\max\{|w_a|/|w|, \frac{|w_b|}{|w|}\} \rightarrow \max\{\alpha, \beta\}.
\]  

Whence

\[
d(w) \rightarrow \frac{1}{\max\{\alpha, \beta\}}.
\]  

The maximal concurrency is thus realized when \( \alpha = \beta = \frac{1}{3} \).

| \( n \) | 1 | 2 | 3 | 4 | 5 | ...  
|---|---|---|---|---|---|---|
| \( w_n \) | \( ab \) | \( abab \) | \( ababab \) | \( abababab \) | \( ababababab \) | ...  
| \( d(w_n) \) | 2 | \( \frac{1}{3} \) | 2 | \( \frac{4}{3} \) | 2 | ...  

Table 1
This convergence is more promising indeed; not only the probability of significant fluctuations tends to 0, the concurrency degree tends to a certain limit with probability 1. The rest of this section is devoted to a proof on the validity of this claim for any commutation monoid.

5.1. Markov chain model

Returning to the method of factorization introduced in Section 3, the process may be regarded as a random process in the following sense. At the $n$th step the state of the normal form is $u_1 u_2 \ldots u_k$. A letter $a$ is added with probability $1/N$, the next state will be

(i) $u_1 u_2 \ldots u_k a$ if $a$ does not commute with $u_k$, or else
(ii) $u_1 u_2 \ldots u_{k-1} u_k$ if $a$ commutes with $u_k$.

It is a Markov chain whenever we take for the set of states the countably infinite set of all possible factorizations. The parameter of interest is the number of factors; its value is $k$ at the $n$th step and it changes into $k + 1$ (case (i)) or else remains the same (case (ii)).

Thus Markov chains seem to be very promising in a statistical evaluation of $d$ and subsequently $d$. It should, however, be noted that generally neither the parameter $k$ nor the last factor $u_k$ has sufficient information to evaluate probabilistically the future transitions of the Markov chain. It means that we cannot reduce the study to that of the last factor. The hugeness and complexity of the set of all possible factorizations and the nonhomogeneity of the commutation structure involve a quite difficult computation in the Markov chain model. It would, therefore, be desirable to develop some more tools.

5.2. Saturated factors and renewal theory

When the factorization ends with a factor $u_k$, its future depends generally not only on $u_k$ but also on the previous factors. There are, however, exceptions to this rule. There are ending factors which "renew" the factorization process.

Since the letters inside a factor are pairwisely commutable, we may identify factors with commutative nonempty subsets of $A$. A factor $u$ is said to be saturated if it is maximal in the inclusion sense. Thus the nonempty subset $u$ of $A$ is saturated iff

(i) $x, y \in u$, $x \neq y = xy = yx$,

(ii) $x \in A$, $x \in u = \exists y \in u$ such that $xy \neq yx$.

We say that the factorization process reaches a saturation if the last obtained factor is saturated. For the sake of homogeneity we assume that the first saturation is reached at the beginning of the factorization.
Remark 5.2. (i) If $A$ is commutative, the only saturated factor is $A$ itself. If $\emptyset = \emptyset$, then saturated factors are singletons.

(ii) It is possible to get a saturated factor $B$ containing a given letter $b$ by the following algorithm.

\[
\begin{align*}
B := \{b\}; & \quad \text{while } \exists x \in A - B \text{ s.t. } \forall y \in B, xy = yx \nonumber \\text{do} \nonumber \\
& \quad B := B + \{x\} \text{ od} \nonumber
\end{align*}
\]

(iii) When the factorization process reaches a saturation, it is regenerated, i.e. its future does not depend on the past. For a more formal discussion, let $T_i$ be the number of inserted letters between the $(i-1)$th and the $i$th saturations, $i = 1, 2, 3, \ldots$. Then $\{T_i\}$ is a sequence of mutually independent random variables with a common distribution. If we set $S_n = \sum_{i=1}^{n} T_i$, the sequence $\{S_n\}$ is a renewal process [6, pp. 184-190]. The event of reaching a saturation will be considered as a recurrent event in the factorization process. In the sequel we shall use terms borrowed from renewal theory such as *waiting time, interarrival*, etc., see [5, Chapter XIII].

(iv) According to (iii), when a saturation is reached the future of the factorization process does not depend on what has happened in the past. If we wish to decompose $\text{dep}$ into the sum of the growth on the interarrivals, it will be necessary to know whether a saturation is persistent and if so whether the waiting time (i.e. $T_i$ introduced in (iii)) is of finite expectation.

In Example 4.2, the only saturated factor is $\{a, b\}$; the factorization reaches a saturation whenever it has met the same number of $a$ and $b$. This event is persistent but the expected waiting time is infinite, see [5, p. 315]. Thus the study is not reducible to a finitely expected interval in this case.

Example 5.3. Let $(A, \emptyset)$ be a commutation monoid with a letter $a$ which commutes with no other. The appearance of $a$ produces a saturation. At each step of the factorization, the probability of reaching this event is $1/N$. This recurrent event is thus persistent and the expected waiting time is $N$.

Let $w, u \in A^*$, $u$ is said to regenerate $w$ if the insertion of the last letter of $wu$ produces a saturation; we suppose that $1$ (the empty word) regenerates $1$.

We need also the following definition in the sequel:

A commutation system $(A, \emptyset)$ can be characterized by its conflict graph, which is an undirected graph $G$ defined in the following way: $A$ is the set of nodes of $G$; there will be an edge between $a$ and $b$, $a \neq b$ if and only if $a$ and $b$ do not commute.

The following lemma provides a sufficient condition for the persistency of the recurrent event. The condition can informally be interpreted as the absence of a "strong commutation".

Lemma 5.4. If the conflict graph $G$ of a commutation system is connected, then the recurrent event is aperiodic and persistent and has a finite waiting time expectation.
Proof. The keypoint is that for each node there exists a "path" starting with that node and visiting all nodes (by passing possibly several times through a node); the length of the path is bounded by $2N$. One may use the D.F.S. algorithm to obtain this path.

We now prove that any $w \in A^*$ can be regenerated by some word $u$ of length $\leq 2N + N^2$. If $w$ is empty, let $u = 1$. Otherwise let $f$ be the last factor. $f$ contains a letter $b$. Let $u_1$ be the word corresponding to the above stated path starting with the node $b$ in $G$. The letters of $u_1$ added one-by-one to the end of $w$, introduce $|u_1|$ new factors. $u_1$ may be viewed as a "bottle-neck" in $wu_1$; for the new steps of the factorization process cannot modify the factors of $w$. If the last factor $v$ of $u_1$ is saturated, then set $u = u_1$, otherwise add an appropriate number of a letter commuting with $v$ until its right place of insertion is $v$, and repeat the insertions as long as there is a letter which commutes with $v$. Let $u$ be the obtained word from $u_1$ by adding occurrences of letters necessary to saturate $v$. Then $u$ regenerates $w$ and, since the remaining process of factorization does not involve the factors of $w$, it is possible to get an upper bound for $|u|$ independent of $w$. A rough evaluation of $|u|$ yields

$$|u| \leq R,$$  \hspace{1cm} (13)

where

$$R = 2N + N^2.$$

This means that at any step of the process, the probability $p$ of reaching a recurrent event within the $R$th forthcoming steps is not less than $1/N^R$ which is positive. The event is, therefore, persistent of finite expected waiting time (see [5, pp. 328–329], it is also easy to see that the distribution of the waiting time is dominated by a geometric distribution).

Note that it is always possible to increase the waiting time for the above-stated saturation by 1, since it is possible to increase the length of $u_1$ by repeating one of its letters. Consequently the recurrent event cannot be periodic. \hfill $\square$

The next proposition shows that if the commutation system has a persistent recurrent event with a finite expected waiting time, then with a high probability, the concurrency degree of a long word, chosen at random, is not far from a fixed real number which is determined by the system. We use in the sequel the subscript $n$ to emphasize on the dependency on $n$; thus $w_n$ is a random word of length $n$ and $M_n$ is the (random) recurrence number for $w_n$.

Proposition 5.5. Let $\mathcal{R}$ be a persistent recurrent event with a finite expected waiting time $\mu$. Denote by $h$ the expected depth in an interarrival. If the word $w_n$ is chosen at random from $A^n$, then as $n \rightarrow \infty$,

$$d(w_n) \rightarrow \mu/h$$  \hspace{1cm} (14)

with probability 1.
Proof. Let $w_n$ be a word of length $n$. Decompose $w_n$ into $v_1 \ldots v_{M_n}$, where $M_n$ is the number of occurrences of $\mathcal{A}$ in the factorization of $w_n$ into a normal form and $v_i$'s are chosen in a way such that each $v_i$ terminates with a letter which generates the event $\mathcal{A}$ in the factorization. Each $v_i$ thus contributes independently to the depth of $w$ and

$$\text{dep}(w_n) = \text{dep}(v_1) + \cdots + \text{dep}(v_{M_n}) + \text{dep}(v).$$  \hspace{1cm} (15)

For a fixed $M_n$, $\text{dep}(v_1) + \cdots + \text{dep}(v_{M_n})$ is the sum of $M_n$ mutually independent random variables with a common distribution and expectation $E(\text{dep}(v_i)) = \mu$. It should be noted that $\mu$ is finite since $\text{dep}(v_i)$ is bounded by $\nu_i$ which has the expectation $\mu$.

By the (strong) law of large numbers [5, p. 260], we get

$$\frac{1}{M_n} \sum_{1 \leq i \leq M_n} \text{dep}(v_i) \rightarrow \mu \text{ a.s. } \text{ as } M_n \rightarrow \infty.$$  \hspace{1cm} (16)

On the other hand, $M_n$ is a random function of $n$, and we have [5, p. 321]

$$E(M_n/n) = 1/\mu \quad \text{and} \quad \text{Var}(M_n/n) = O(1/n).$$

It follows that with probability 1

$$M_n/n \rightarrow 1/\mu, \quad \text{as } n \rightarrow \infty.$$  \hspace{1cm} (17)

Putting together (16)-(18), and using the fact that $\text{dep}(v)/n \rightarrow 0$ with probability 1, we get

$$\frac{\text{dep}(w_n)}{n} \rightarrow \frac{\mu}{\mu} \text{ with probability 1, } \text{ as } n \rightarrow \infty.$$  \hspace{1cm} (18)

Since $\mu > 0$, this yields (15).

Remark 5.6. (i) In a first quantitative study of concurrency the use of a weak convergence (convergence with probability 1) seems to be adequate. We used, however, in the proof of Proposition 5.5 a strong convergence (convergence almost sure) to be in agreement with Feller [5, p. 260].

(ii) Note that if $G$ is connected, the persistent recurrent event is not unique. In fact, in this case there are infinitely many ways of defining a persistent recurrent event. It is important to see that Proposition 5.5 asserts that the ratio $\mu/h$ does not depend on a particular choice of a persistent recurrent event, since this ratio must be common to almost all infinite words. Therefore when $G$ is connected, it will be legitimate to refer to $\mu/h$ as the concurrency degree of the commutation system $S = (A, \theta)$ and since the monoid is completely characterized by the conflict graph $G$, we let

$$\text{DEG}(S) = \text{DEG}(G) = \mu/h.$$  \hspace{1cm} (19)
We are now ready to state our central theorem which asserts that in any commutation system there is a concurrency degree which is common to almost all infinite words.

**Theorem 5.7.** Suppose that the conflict graph $G$ of the commutation system $S=(A,\emptyset)$ has $k$ connected components $G_1=(A_1,E_1), \ldots, G_k=(A_k,E_k)$. Set $\alpha_i=\text{card}(A_i)/N$ for $i=1,\ldots,k$. Let $w_n$ be a word of length $n$ chosen at random. We have with probability 1

$$d(w_n) \rightarrow \text{DEG}(G), \quad \text{as } n \rightarrow \infty,$$

where $\text{DEG}(G)=\text{DEG}(S)$ is defined by

$$\text{DEG}(G)=\min\{\text{DEG}(G_1)/\alpha_1, \ldots, \text{DEG}(G_k)/\alpha_k\}.$$  

Note that $\text{DEG}(G_i)$ is defined by Remark 5.6(ii). It is the concurrency degree of $S_i=(A_i,\emptyset_i)$, where $\emptyset_i=\emptyset \cap (A_i \times A_i)$.

**Proof.** First we remark that any two letters chosen from $A_i$ and $A_j$, $i \neq j$, commute and whence any $w$ is congruent to any word of the shuffle product of $(w)_{A_1}, \ldots, (w)_{A_k}$. Consequently $\text{dep}(w)=\max_{i=1}^k \{\text{dep}((w)_{A_i})\}$ where $(w)_{A_i}$ is a word over the alphabet $A_i$ and its depth is defined in the commutation monoid $S_i=(A_i,\emptyset_i)$.

Since $G_i$ is connected, Proposition 5.5 applied to $(w_n)_{A_i}$ yields

$$\frac{\text{dep}((w_n)_{A_i})}{|w_n|_{A_i}} \rightarrow \frac{1}{\text{DEG}(G_i)} \quad \text{with probability 1, \quad as } n \rightarrow \infty.$$  

On the other hand

$$\frac{\text{dep}(w_n)}{n} = \frac{k}{n} \max_{i=1}^k \frac{\text{dep}((w_n)_{A_i})}{|w_n|_{A_i}}.$$  

The second factor on the right tends to $\alpha_i$ with probability 1 as $n \rightarrow \infty$. Combining (22) and (23) we get

$$d(w_n) \rightarrow \min_{i=1}^k \frac{\text{DEG}(G_i)}{\alpha_i}. \quad \square$$

### 6. Examples

The results obtained in Section 5 are important in that not only they show the existence of a concurrency degree for a commutation system, but they provide powerful tools (decomposition into connected components, reduction of statistics to an interarrival) as well for computing this degree.

We conclude with some examples.
Example 6.1. Let $A = \{a, b, c\}$ and $\theta$ defined by $ab = ba$, $bc = cb$. The conflict graph is shown in Fig. 2. It has two connected components, they are both of concurrency degree 1. Whence, by Theorem 5.7, the concurrency degree of the whole system is $\frac{1}{2}$. This informally means that for almost all sufficiently large sequences of actions, $\frac{1}{2}$ actions are being performed simultaneously on the average.

Example 6.2. Let $A = \{a, b, c\}$ and $\theta$ defined by $ab = ba$. $G$ is then the graph shown in Fig. 3. $c$ is a saturated factor; the event of meeting an occurrence of $c$ is a persistent recurrent event. Then by Proposition 5.5, it suffices to study the parameters of interest on an interval between two occurrences of $c$ (the first $c$ is excluded and the second one is included in the interval). Let $M$ be the length of this interval, it has a geometric distribution

$$P(M = m) = \frac{1}{2} \left( \frac{1}{2} \right)^{m-1}, \quad m = 1, 2, 3, \ldots$$

with

$$\mu = E(M) = 3.$$  \hspace{1cm} (24)

It remains to find the expected depth in an interarrival. For an interarrival of a fixed length $m$, we are in the situation of Example 4.2 and from (6), we get the conditional expected depth

$$D_m = \frac{1}{2}(m+1) + (m-1)\left( \binom{m-2}{\frac{1}{2}(m-1)} \right) \frac{2^{m-1}}{2^{m-1}}, \quad m = 1, 2, \ldots$$

or in terms of generating series

$$D(z) = \sum_{m \geq 1} D_m z^m = \frac{z}{1 - z} + \frac{z^2}{2(1-z)^2} + z \sum_{m \geq 1} m \binom{m-1}{\frac{1}{2}m} \left( \frac{1}{2} z \right)^m. \hspace{1cm} (27)$$

And we have

$$h = \sum_{m \geq 1} \frac{1}{2}(\frac{1}{2})^{m-1} D_m = \frac{1}{2}D(\frac{1}{2}). \hspace{1cm} (28)$$
The last summation on the right of (27) is decomposed into \( A(z) + B(z) \) with

\[
A(z) = \sum_{n \geq 1} \frac{(2n)!}{n! (n-1)!} (\frac{1}{4}z^2)^n, \tag{29}
\]

and

\[
B(z) = z \sum_{n \geq 0} \frac{(2n+1)!}{n! n!} (\frac{1}{4}z^2)^n. \tag{30}
\]

These functions are calculated by elementary operations such as shifting, change of variable and derivations, on the well-known generating function of the Catalan numbers \( C(z) = (1 - \sqrt{1 - 4z^2})/2z^2 \). A straightforward calculation yields

\[
A(z) = \frac{1}{2} z^2 (1 - z^2)^{-3/2}, \tag{31}
\]

and

\[
B(z) = \frac{1}{2} z (1 - z^2)^{-1/2} + \frac{1}{2} z^3 (1 - z^2)^{-3/2}. \tag{32}
\]

Finally, we get from (26), (29), (30),

\[
h = 2 + \frac{1}{\sqrt{5}}, \tag{33}
\]

whence

\[
\text{DEG} = 3/(2 + 1/\sqrt{5}) = 1.22588403552664\ldots. \tag{34}
\]

**Example 6.3.** As the last example we compare the commutation system \( S_1 \) and \( S_2 \) defined by the conflict graphs shown in Fig. 4.

For \( G_1 \) we get, by Theorem 5.5,

\[
\text{DEG}_1 = \min \left\{ 4, \frac{4}{3} \cdot \frac{3}{2 + 1/\sqrt{5}} \right\} = \frac{4}{2 + 1/\sqrt{5}} = 1.634512004736886\ldots.
\]

Each component of \( G_2 \) corresponds to a noncommutative monoid and is thus of concurrency degree 1. By Theorem 5.7,

\[
\text{DEG}_2 = 2.
\]

Thus \( S_2 \) is appreciably more concurrent than \( S_1 \).

**Remark 6.4.** Proposition 5.5 and Theorem 5.7 can be restated with suitable changes whenever the assumption of equally likely letters is dropped. This remark, of
perhaps minor theoretical interest, provides an adaptability to the theory in view of its applications in realistic situations.

The computation of the concurrency degree of the commutation system $S = (A, \theta)$ with a given probability distribution seems to be of the same complexity as in the "usual" case. It seems also quite interesting to recompute $\text{DEG}$ in the above examples with respect to a given probability distribution $p_a, p_b, \ldots$ and then to discuss the obtained expressions in terms of $p_a, p_b, \ldots$ considered as parameters.

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References