H$_\alpha$-stability of modified Runge–Kutta methods for nonlinear neutral pantograph equations

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Abstract

In this paper, we investigate $H_\alpha$-stability of algebraically stable Runge–Kutta methods with a variable stepsize for nonlinear neutral pantograph equations. As a result, the Radau IA, Radau IIA, Lobatto IIIC method, the odd-stage Gauss–Legendre methods and the one-leg $\theta$-method with $\frac{1}{2} \leq \theta \leq 1$ are $H_\alpha$-stable for nonlinear neutral pantograph equations. Some experiments are given.

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1. Introduction

We consider the nonlinear neutral pantograph equation

$$y'(t) = F(t, y(t), y(qt), y'(qt)), \quad t > 0,$$
$$y(0) = y_0,$$  \hspace{1cm} (1.1)

where $0 < q < 1$, $y_0 \in \mathbb{C}^d$ and $F : \mathbb{R}^+ \times \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^d$ is continuous.

These systems arise in a variety of modelling phenomena such as in the electrodynamics and in nonlinear dynamical systems, and so on (see [3,4,13]).
In recent years, the stability properties of numerical methods for this kind of equations have been studied by numerous authors (see [1, 5–7, 9–13, 15–18]).

It is well known that we encountered the storage problem when applying the numerical method to solve Eq. (1.1) because of its unbounded delay. To avoid the storage problem, Bellen [1], J. Liang [10], Y. Liu [12, 13] and Xu [15] use the numerical method with a variable stepsize to solve
\[ y'(t) = \alpha y(t) + \beta y(qt), \]
and it is proved that the Runge–Kutta method with a regular matrix A is asymptotically stable if and only if \(|1 - b^T A^{-1} e| < 1\) in [15].

For (1.2), we can see that the Radau IA, Radau IIA, Lobatto IIIC method and the one-leg \(\theta\)-method with \(\frac{1}{2} < \theta \leq 1\) are asymptotically stable. But the Gauss–Legendre methods and one-leg \(\theta\)-method with \(\theta = \frac{1}{2}\) are not asymptotically stable (see [15]).

In [11], the modified Runge–Kutta method is constructed, which preserves the order of accuracy of the original one. The sufficient and necessary conditions under which the modified Runge–Kutta method with the variable mesh are asymptotically stable for the linear pantograph equations are given. It is proved that the odd-stage Gauss–Legendre methods, even-stage Lobatto IIIA and even-stage Lobatto IIIB are asymptotically stable, in addition to the Radau IA, Radau IIA and Lobatto IIIC methods. The one-leg \(\theta\)-method and the linear \(\theta\)-method are asymptotically stable when \(\frac{1}{2} \leq \theta \leq 1\).

In the present paper, we investigate the numerical stability of the algebraical stable Runge–Kutta methods for the following system:
\[ y'(t) = f(t, y(t)) + g(t, y(qt)) + C(t)y'(qt), \quad t > 0, \]
\[ y(0) = y_0, \] (1.3)
where \(f, g : [0, \infty) \times \mathbb{C}^d \to \mathbb{C}^d\) are continuous functions and \(C(t)\) is a continuous \(d \times d\) matrix function in \([0, \infty)\).

Equation (1.3) is a kind of special nonlinear equation of (1.1), the stability properties of numerical methods have been obtained in [6, 7]. The sufficient conditions for the asymptotical stability of numerical solution of the nonlinear equation (1.1) with assumptions of Theorem 5.3 in [7] are given. But in this paper the results are different from those in [7]. It turns out to be that the Radau IA, Radau IIA, Lobatto IIIC method, the odd-stage Gauss–Legendre methods and the one-leg \(\theta\)-method with \(\frac{1}{2} \leq \theta \leq 1\) are \(H_\alpha\)-stable.

2. The modified Runge–Kutta method

In this section, we consider the modified Runge–Kutta method \((A, b, c)\) with the form
\[ y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i f(t_n^i, y_{n+1}^i), \]
\[ y_{n+1}^i = y_n + \tilde{h}_n \sum_{j=1}^{s} a_{ij} f(t_n^i, y_{n+1}^j), \quad i = 1, 2, \ldots, s, \] (2.1)
where \(y_0 \in \mathbb{C}^d, \quad T > 0, \quad f : [0, \infty) \times \mathbb{C}^d \to \mathbb{C}^d\) is a continuous function. Let \(\Delta = \{0 = t_0 < t_1 < \cdots < t_n = T\}\) be a mesh and \(h_n = t_{n+1} - t_n\) is the stepsize, \(t_n^i = t_n + c_i h_n\) and \(\tilde{h}_n = (1 + \alpha_n(h_n)) h_n\) with \(\alpha_n(\eta)\) such that
(H1) $\alpha_n(\eta) = O(\eta^p)$ as $\eta \to 0$,
(H2) $\alpha_n(\eta) > 0$, for all $\eta$.

In view of [11] we can obtain that the order of the method (2.1) is $p$ for the $p$th-order Runge–Kutta method $(A, b, c)$ provided with $\bar{p} \geq p - 1$. In this paper, we assume that $0 \leq c_i \leq 1$ ($i = 1, 2, \ldots, s$) and $\sum_{i=1}^{s} b_i = 1$.

In the similar way in [14], applying (2.1) to (1.1), we have

\begin{align*}
y_{n+1} &= y_n + h_n \sum_{i=1}^{s} b_i F(t_n, y_{i}^{n+1}, \gamma y_{i}^{n+1}, \gamma y_{i}^{n+1}),
\end{align*}

\begin{align*}
y_{i}^{n+1} &= y_n + \bar{h}_n \sum_{j=1}^{s} a_{ij} F(t_n, y_{j}^{n+1}, \gamma y_{j}^{n+1}, \gamma y_{j}^{n+1}),
\end{align*}

\begin{align*}
y_{i}^{n+1} &= F(t_n, y_{i}^{n+1}, \gamma y_{i}^{n+1}, \gamma y_{i}^{n+1}), \quad i = 1, 2, \ldots, s, \quad (2.2)
\end{align*}

where $0 \leq c_i \leq 1$ ($i = 1, 2, \ldots, s$), $b_1 + b_2 + \cdots + b_s = 1$, $y_{j}^{n+1}$, $\gamma y_{j}^{n+1}$, $Y_{i}^{n+1}$ and $\gamma Y_{i}^{n+1}$ denote the approximations to $y(t_n)$, $y(q(t_n))$, $y'(t_n)$ and $y'(q(t_n))$, respectively.

Here, the mesh $H = \{m; t_0, t_1, \ldots, t_n, \ldots\}$ is defined as follows. Let $t_0 > 0$ be given and $t_m = q^{-1}t_0$. We choose $m - 1$ grid points $t_1 < t_2 < \cdots < t_m$ in $(t_0, t_m)$ and define the other points by

\begin{align*}
t_{km+i} = q^{-k}t_i \quad \text{for} \quad k = -1, 0, \ldots, i = 0, 1, \ldots.
\end{align*}

(2.3)

It is easy to see that the grid point $t_n$ is such that $qt_n = t_{n-m}$ and the stepsize $h_n = t_{n+1} - t_n$ satisfies

\begin{align*}
h_n = q^{-1}h_{n-m}, \quad n = 0, 1, 2, \ldots,
\end{align*}

\begin{align*}
\lim_{n \to +\infty} h_n = \infty. \quad (2.4)
\end{align*}

We suppose to have the numerical solution available till the point $t_0$, which is called the initial data. In view of (2.3) and (2.4), the application of the above Runge–Kutta method to (1.3) yields the recurrence relation:

\begin{align*}
y_{n+1} &= y_n + h_n \sum_{i=1}^{s} b_i \{ f(t_n, y_{i}^{n+1}) + g(t_n, y_{i}^{n+1-m}) + C(t_n)Y_{i}^{n+1-m} \},
\end{align*}

\begin{align*}
y_{i}^{n+1} &= y_n + \bar{h}_n \sum_{j=1}^{s} a_{ij} \{ f(t_n, y_{j}^{n+1}) + g(t_n, y_{j}^{n+1-m}) + C(t_n)Y_{j}^{n+1-m} \},
\end{align*}

\begin{align*}
y_{i}^{n+1} &= f(t_n, y_{i}^{n+1}) + g(t_n, y_{i}^{n+1-m}) + C(t_n)Y_{i}^{n+1-m}, \quad i = 1, 2, \ldots, s. \quad (2.5)
\end{align*}

In order to study the stability of the Runge–Kutta method (2.5), we also consider the nonlinear pantograph equation

\begin{align*}
z'(t) = f(t, z(t)) + g(t, z(qt)) + C(t)z'(qt), \quad t > 0, \quad z(0) = z_0.
\end{align*}

(2.6)
Similarly, the Runge–Kutta method applied to the problem (2.6) leads to the following process:

\[
zn+1 = zn + h_n \sum_{i=1}^{s} bi \{ f(t^n_i, zn+1_i) + g(t^n_i, zn+1-m_i) + C(t^n_i) Z^n_{i+1-m} \},
\]

\[
z^n_{i+1} = zn + h_n \sum_{j=1}^{s} a_{ij} \{ f(t^n_j, zn+1_j) + g(t^n_j, zn+1-m_j) + C(t^n_j) Z^n_{j+1-m} \},
\]

\[
Z^n_{i+1} = f(t^n_i, zn+1_i) + g(t^n_i, zn+1-m_i) + C(t^n_i) Z^n_{i+1-m}, \quad i = 1, 2, \ldots, s. \quad (2.7)
\]

The following theorem gives the asymptotical stability conditions of the analytic solutions of (1.3).

**Theorem 2.1.** (See [8].) Consider the pantograph equations (1.3) and (2.6). If \( f \) satisfies

\[
\text{Re}(u_1 - u_2, f(t, u_1) - f(t, u_2)) \leq a \|u_1 - u_2\|^2, \quad (2.8)
\]

then for any given \( y_0 \) and \( z_0 \),

\[
\lim_{t \to \infty} \|y(t) - z(t)\| = 0,
\]

if

(i) \( a < 0 \),

(ii) \( c := \sup_{t \geq 0} \|C(t)\| < 1 \),

(iii) \( \omega^*(t) + \kappa a (1 - c) \leq 0 \) for \( t \geq 0 \) and some \( \kappa \in (0, 1) \), \( \quad (2.9) \)

where \( \omega^*(t) \) satisfies for all \( y, z \in \mathbb{C}^d \) and \( t \geq 0 \)

\[
\sup_{\tau \in [0, t]} \|g(\tau, y) - g(\tau, z) + C(\tau) (f(q\tau, y) - f(q\tau, z))\| \leq \omega^*(t) \|y - z\|. \quad (2.10)
\]

**Remark 2.1.** Assumption (2.9) in Theorem 2.1 is different from assumption (5.12) in [7]. For example, the following nonlinear neutral pantograph equation

\[
y'(t) = -y(t) - 4y^3(t) + qy^3(qt) + \frac{q}{4} y'(qt), \quad (2.11)
\]

satisfies assumption (2.9) in Theorem 2.1 but does not satisfy assumption (5.12) in [7].

3. The stability analysis

In this section, we will discuss the stability of the modified Runge–Kutta method.

**Definition 3.1.** (See [2].) A Runge–Kutta method \((A, b, c)\) is said to be algebraically stable if

\[
M = BA + A^T B - bb^T \quad \text{and} \quad B
\]

are nonnegative definite, where \( B = \text{diag}\{b_1, b_2, \ldots, b_s\} \).
After some simple calculations, we can obtain that (2.5) is equivalent to the following form:

\[
y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i \left\{ f(t_n^i, y_{i}^{n+1}) + \sum_{p=1}^{k(n)} \prod_{l=1}^{p-1} C(\tau_{n-(l-1)m}^i) g(\tau_{n-(p-1)m}^i, y_{i}^{n+1-pm}) \right\} \\
+ \prod_{l=1}^{p} C(\tau_{n-(l-1)m}^i) f(\tau_{n-pm}^i, y_{i}^{n+1-pm}) \\
+ \prod_{l=1}^{k(n)} C(\tau_{n-(l-1)m}^i) g(\tau_{n-k(n)m}^i, y_{i}^{n+1-(k(n)+1)m}) \\
+ \prod_{l=1}^{k(n)+1} C(\tau_{n-(l-1)m}^i) Y_{i}^{n+1-(k(n)+1)m} \},
\]

\[
y_{i}^{n+1} = y_n + \tilde{h}_n \sum_{j=1}^{s} a_{ij} \left\{ f(t_n^j, y_{i}^{n+1}) + \sum_{p=1}^{k(n)} \prod_{l=1}^{p-1} C(\tau_{n-(l-1)m}^j) g(\tau_{n-(p-1)m}^j, y_{i}^{n+1-pm}) \right\} \\
+ \prod_{l=1}^{k(n)} C(\tau_{n-(l-1)m}^j) f(\tau_{n-pm}^j, y_{j}^{n+1-pm}) \\
+ \prod_{l=1}^{k(n)} C(\tau_{n-(l-1)m}^j) g(\tau_{n-k(n)m}^j, y_{j}^{n+1-(k(n)+1)m}) \\
+ \prod_{l=1}^{k(n)+1} C(\tau_{n-(l-1)m}^j) Y_{j}^{n+1-(k(n)+1)m} \},
\]

\[
Y_{i}^{n+1} = f(t_n^i, y_{i}^{n+1}) + g(t_n^i, y_{i}^{n+1-m}) + C(\tau_n^i) Y_{i}^{n+1-m}, \quad i = 1, 2, \ldots, s, \quad (3.1)
\]

where \(\prod_{l=0}^{p-1} C(\tau_{n-(l-1)m}) = 1\) for \(p - 1 < 0\), \(k(n) = \left\lfloor \frac{n+1}{m} \right\rfloor\) is the integer part of \(\frac{n+1}{m}\), \(\sum_{l=1}^{k(n)} a_{il} = 0\) for \(k(n) = 0\).

Similarly, (2.7) is equivalent to the following form:

\[
z_{n+1} = z_n + h_n \sum_{i=1}^{s} b_i \left\{ f(t_n^i, z_{i}^{n+1}) + \sum_{p=1}^{k(n)} \prod_{l=1}^{p-1} C(\tau_{n-(l-1)m}^i) g(\tau_{n-(p-1)m}^i, z_{i}^{n+1-pm}) \right\} \\
+ \prod_{l=1}^{p} C(\tau_{n-(l-1)m}^i) f(\tau_{n-pm}^i, z_{i}^{n+1-pm}) \\
+ \prod_{l=1}^{k(n)} C(\tau_{n-(l-1)m}^i) g(\tau_{n-k(n)m}^i, z_{i}^{n+1-(k(n)+1)m}) \\
+ \prod_{l=1}^{k(n)+1} C(\tau_{n-(l-1)m}^i) Z_{i}^{n+1-(k(n)+1)m} \},
\]
$$z_{n+1}^i = z_n + h_n \sum_{j=1}^s a_{ij} \left\{ f(t_{n}^j, z_{n+1}^j) + \sum_{p=1}^{k(n)} \left\{ \prod_{l=1}^{p-1} C(t_{n-(l-1)m}^j, z_{n+1-pm}^j) \\
+ \prod_{l=1}^{p} C(t_{n-(l-1)m}^j, f(t_{n-pm}^j, z_{n+1-pm}^j)) \right\} \\
+ \prod_{l=1}^{k(n)+1} C(t_{n-(l-1)m}^j, g(t_{n-k(n)m}^j, z_{n+1-(k(n)+1)m}^j)) \right\},$$

$$Z_{n+1}^i = f(t_{n}^i, z_{n+1}^i) + g(t_{n}^i, z_{n+1-m}^i) + C(t_{n}^i) Z_{n+1-m}^i, \quad i = 1, 2, \ldots, s.$$  (3.2)

Let

$$\omega_n = y_n - z_n,$$

$$W_{n+1}^j = y_{n+1}^i - z_{n+1}^i,$$

$$R_{n+1}^j = f(t_{n}^i, y_{n+1}^i) - f(t_{n}^j, z_{n+1}^j) + g(t_{n}^i, y_{n+1-m}^i) - g(t_{n}^j, z_{n+1-m}^j) + C(t_{n}^j) Y_{n+1-m}^i - C(t_{n}^i) Z_{n+1-m}^i,$$

$$p_{n+1}^i = y_{n+1}^i - Z_{n+1}^i,$$

$$Q_{n+1}^j = f(t_{n}^i, y_{n+1}^i) - f(t_{n}^j, z_{n+1}^j) + \sum_{p=1}^{k(n)} \left\{ \prod_{l=1}^{p-1} C(t_{n-(l-1)m}^j, g(t_{n-(p-1)m}^j, y_{n+1-pm}^j) - g(t_{n-(p-1)m}^j, z_{n+1-pm}^j)) \right\} \\
+ \prod_{l=1}^{k(n)} C(t_{n-(l-1)m}^j, f(t_{n-pm}^j, y_{n+1-pm}^j) - f(t_{n-pm}^j, z_{n+1-pm}^j)) \right\},$$

$$Q_{n+1}^j = f(t_{n}^i, y_{n+1}^i) - f(t_{n}^j, z_{n+1}^j) + \sum_{p=1}^{k(n)} \left\{ \prod_{l=1}^{p-1} C(t_{n-(l-1)m}^j, g(t_{n-(p-1)m}^j, y_{n+1-pm}^j) - g(t_{n-(p-1)m}^j, z_{n+1-pm}^j)) \right\} \\
+ \prod_{l=1}^{k(n)+1} C(t_{n-(l-1)m}^j, y_{n+1-(k(n)+1)m}^j - Z_{n+1-(k(n)+1)m}^i), \quad j = 1, 2, \ldots, s.$$  (3.3)

Then it follows from (3.1) and (3.2) that

$$\omega_{n+1} = \omega_n + h_n \sum_{j=1}^s b_j Q_{n+1}^j,$$

$$W_{n+1}^i = \omega_n + h_n \sum_{j=1}^s a_{ij} Q_{n+1}^j,$$

$$R_{n+1}^i = P_{n+1}^i, \quad i = 1, 2, \ldots, s.$$  (3.3)
Let
\[ W_n = \left( (W_{n1}^T, (W_{n2}^T, (W_{n3}^T, \ldots, (W_{ns}^T)T, \right), \]
\[ Q_n = \left( (Q_{n1}^T, (Q_{n2}^T, (Q_{n3}^T, \ldots, (Q_{ns}^T)T, \right), \]
\[ P_n = \left( (P_{n1}^T, (P_{n2}^T, (P_{n3}^T, \ldots, (P_{ns}^T)T, \right), \]
\[ R_n = \left( (R_{n1}^T, (R_{n2}^T, (R_{n3}^T, \ldots, (R_{ns}^T)T, \right), \]
\[ e = (1, 1, 1, \ldots, 1)^T. \tag{3.4} \]

Then (3.3) can be written as
\[ W_{n+1} = e \otimes \omega_n + \bar{h}_n (A \otimes I_d) Q_{n+1}, \]
\[ \omega_{n+1} = \omega_n + h_n (b^T \otimes I_d) Q_{n+1}, \]
\[ R_{n+1} = P_{n+1}, \] \tag{3.5}

where \( I_d \) denotes the identity matrix.

Lemma 3.1. If \( c < q < 1 \), then
\[ \sum_{i=0}^{\infty} c^{k(i)} h_i \leq t_0 \frac{1}{q - c}. \]

Proof. In view of (2.3) and (2.4), we have
\[ \sum_{i=0}^{\infty} c^{k(i)} h_i = h_0 + h_1 + \cdots + h_m^2 + ch_m + \cdots + ch_{2m-2} + c^2 h_{2m-1} + \cdots \]
\[ \leq (h_0 + \cdots + h_m) \left( 1 + \frac{c}{q} + \frac{c^2}{q^2} + \cdots + \frac{c^l}{q^l} + \cdots \right) \]
\[ \leq \frac{1}{q} (h_m + \cdots + h_{1}) \left( \frac{q}{q - c} \right) \leq t_0 \frac{1}{q - c}. \]

Definition 3.2. The Runge–Kutta method is said to be \( H_\alpha \)-stable if for any \( q \in (0, 1) \), any initial data and any mesh \( H \), the application of the method (2.2) to (1.3) and (2.6) with assumption (3.7) holding and \( \alpha_n(\eta) \) satisfying (H1) and (H2) generates the approximations \( y_n \) and \( z_n \) satisfying \( \| y_n - z_n \| \to 0 \) as \( n \to \infty \).

Lemma 3.2. Assume that the Runge–Kutta method is algebraically stable, \( f \) satisfies (2.8), \( \alpha_n(\eta) \) satisfies (H1), (H2) and (3.7) hold. Then
\[ \| \omega_{n+1} \|^2 \leq \| \omega_0 \|^2 + \sum_{j=1}^{s} b_j \left\{ \sum_{i=0}^{n} \left[ (2 - \kappa) + 2c^{k(i)} + \frac{a\rho}{q - c} \right] h_i \| W_{j+1} \|^2 \right\} \]
\[ + (M_1 + M_2) \frac{t_0}{q - c} + \frac{\rho a}{q - c} \sum_{j=1}^{s} b_j \| W_0 \|^2, \] \tag{3.6}

provided with
(i) $a < 0,$
(ii) $c := \sup_{t \geq 0} \| C(t) \| < q,$  
(iii) $\omega^*(t) - a \rho \leq 0$ for $t \geq 0,$

where $\rho = -\kappa (1 - c), \kappa \in (0, \frac{2q - 2c}{q + 1 - 2c})$, $M_1 = \max_{-m \leq i \leq 0} \| g(t^j_{i+m-1}, y^i_j) - g(t^j_{i+m-1}, z^i_j) \|^2$ and $M_2 = c \max_{i \leq 0} \| Y^i_j - Z^i_j \|^2$.

Proof. Since the Runge–Kutta method is algebraically stable and $bb^T$ is nonnegative definite, we have

$$\| \omega_{n+1} \|^2 = \left( \omega_n + h_n \sum_{i=1}^s b_i Q_i^{n+1}, \omega_n + h_n \sum_{i=1}^s b_i Q_i^{n+1} \right)$$

$$= \| \omega_n \|^2 + 2h_n \sum_{i=1}^s b_i \text{Re} \left( Q_i^{n+1}, \omega_n \right) + h_n^2 \sum_{i,j=1}^s b_i b_j \left( Q_i^{n+1}, Q_j^{n+1} \right)$$

$$+ h_n^2 \sum_{i,j=1}^s b_i b_j \left( Q_i^{n+1}, Q_j^{n+1} \right)$$

$$\leq \| \omega_n \|^2 + 2h_n \sum_{i=1}^s b_i \text{Re} \left( Q_i^{n+1}, W_i^{n+1} \right)$$

$$- h_n^2 \sum_{i,j=1}^s \left\{ (1 + \alpha_n(h_n)) (b_i a_{ij} + b_j a_{ji} - b_i b_j) \right\} \left( Q_i^{n+1}, Q_j^{n+1} \right)$$

$$\leq \| \omega_n \|^2 + 2h_n \sum_{j=1}^s b_j \text{Re} \left( Q_j^{n+1}, W_j^{n+1} \right).$$

(3.8)

It is easy to see that

$$2h_n \text{Re} \left( W_j^{n+1}, Q_j^{n+1} \right) = 2h_n (\beta_{n+1,1} + \beta_{n+1,2} + \beta_{n+1,3} + \beta_{n+1,4}),$$

(3.9)

where

$$\beta_{n+1,1} = \text{Re} \left( W_j^{n+1}, f \left( t^j_n, y_j^{n+1} \right) - f \left( t^j_n, z_j^{n+1} \right) \right).$$

$$\beta_{n+1,2} = \text{Re} \left( W_j^{n+1}, \sum_{p=1}^{k(n)} \left\{ \prod_{l=1}^{p-1} C(t^j_{n-(l-1)m}) \left\{ g(t^j_{n-(p-1)m}, y_j^{n+1-pm}) - g(t^j_{n-(p-1)m}, z_j^{n+1-pm}) \right\} \right\} + \prod_{l=1}^{p} C(t^j_{n-(l-1)m}) \left\{ f \left( t^j_{n-pm}, y_j^{n+1-pm} \right) - f \left( t^j_{n-pm}, z_j^{n+1-pm} \right) \right\} \right) \right).$$
\[ \beta_{n+1,k} = \text{Re} \left( W_j^{n+1} \prod_{l=1}^{k(n)} C(t_{n-(l-1)m}^j) \left( g(t_{n-k(n)m}^j, y_j^{n+1-(k(n)+1)m}) - g(t_{n-k(n)m}^j, z_j^{n+1-(k(n)+1)m}) \right) \right), \]

It follows from (2.8) and (iii) that

\[ \beta_{n+1,1} \leq a \| W_j^{n+1} \|^2, \]
\[ \beta_{n+1,2} \leq a \rho \sum_{p=1}^{k(n)} c^{p-1} \| W_j^{n+1} \| \| W_j^{n+1-pm} \|^2, \]
\[ \beta_{n+1,3} \leq c^{k(n)} \| W_j^{n+1} \| \| g(t_{n-k(n)m}^j, y_j^{n+1-(k(n)+1)m}) - g(t_{n-k(n)m}^j, z_j^{n+1-(k(n)+1)m}) \|^2, \]
\[ \beta_{n+1,4} \leq c^{k(n)+1} \| W_j^{n+1} \| \| y_j^{n+1-(k(n)+1)m} - Z_j^{n+1-(k(n)+1)m} \|^2. \]

Substituting (3.12) into (3.8) yields

\[ \| \omega_{n+1} \|^2 \leq \| \omega_n \|^2 + \sum_{j=1}^{s} b_j \left\{ (2 - \kappa) a + 2c^{k(n)} \right\} \| W_j^{n+1} \|^2 \]
\[ + \rho a h_n \sum_{p=1}^{k(n)} c^{p-1} \| W_j^{n+1-pm} \|^2 + c^{k(n)} h_n \| g(t_{n-k(n)m}^j, y_j^{n+1-(k(n)+1)m}) - g(t_{n-k(n)m}^j, z_j^{n+1-(k(n)+1)m}) \|^2 \]
\[ + c^{k(n)+1} h_n \| y_j^{n+1-(k(n)+1)m} - Z_j^{n+1-(k(n)+1)m} \|^2. \]

By induction we can obtain

\[ \| \omega_{n+1} \|^2 \leq \| \omega_0 \|^2 + \sum_{j=1}^{s} b_j \left\{ \sum_{i=0}^{n} \left( (2 - \kappa) a + 2c^{k(i)} \right) \| W_j^{i+1} \|^2 \right\} \]
\[ + \rho a \sum_{i=0}^{n} h_i \sum_{p=1}^{k(i)} c^{p-1} \| W_j^{i+1-pm} \|^2 \]
\[ + \sum_{i=0}^{n} c^{k(i)} h_i \left\| g(t_j - k(i)m, y_j^{i+1-(k(i)+1)m}) - g(t_j^{i+1-(k(i)+1)m}) \right\|^2 \]
\[ + \sum_{i=0}^{n} c^{k(i)+1} h_i \left\| y_j^{i+1-(k(i)+1)m} - Z_j^{i+1-(k(i)+1)m} \right\|^2 \right\}. \tag{3.14} \]

It follows from Lemma 3.1 that
\[ \sum_{i=0}^{n} c^{k(i)} h_i \left\| g(t_j - k(i)m, y_j^{i+1-(k(i)+1)m}) - g(t_j^{i+1-(k(i)+1)m}) \right\|^2 \leq \max_{-m \leq i \leq 0} \left\| g(t_j^{i+1-(k(i)+1)m}) - g(t_j^{i+1-(k(i)+1)m}) \right\|^2 \sum_{i=0}^{\infty} c^{k(i)} h_i \leq M_1 t_0 \frac{1}{q - c}, \tag{3.15} \]
\[ \sum_{i=0}^{n} c^{k(i)+1} h_i \left\| y_j^{i+1-(k(i)+1)m} - Z_j^{i+1-(k(i)+1)m} \right\|^2 \leq M_2 t_0 \frac{1}{q - c}. \tag{3.16} \]

and
\[ \rho \alpha \sum_{i=0}^{n} h_i \sum_{p=1}^{k(i)} c^{p-1} \| W_j^{i+1-pm} \|^2 \]
\[ = \alpha \rho \left( \sum_{i=m-1}^{2m-2} h_i \| W_j^{i+1-m} \|^2 + \sum_{i=2m-1}^{3m-2} h_i (\| W_j^{i+1-m} \|^2 + c \| W_j^{i+1-2m} \|^2) + \ldots \right. \]
\[ + \sum_{i=k(n)m-1}^{n} h_i (\| W_j^{i+1-m} \|^2 + c \| W_j^{i+1-2m} \|^2 + \ldots + c^{k(n)-1} \| W_j^{i+1-k(n)m} \|^2) \)
\[ = \alpha \rho \sum_{l=1}^{k(n)} \sum_{i=l-1}^{n} c^{l-1} h_i \| W_j^{i+1-lm} \|^2 \leq \rho \alpha \sum_{l=1}^{\infty} c^{l-1} \left( \sum_{i=0}^{n} h_i \| W_j^{i+1} \|^2 + \| W_j^{0} \|^2 \right) \]
\[ \leq \frac{\rho \alpha}{q - c} \sum_{i=0}^{n} h_i \| W_j^{i+1} \|^2 + \frac{\rho \alpha}{q - c} \| W_j^{0} \|^2, \tag{3.17} \]
which implies that the lemma is true. \[ \square \]

In the following we assume that

(H3) there exists \( \alpha(H) > 0 \) such that \( \lim_{n \to \infty} \alpha_n(h_n) = \alpha(H) \).

**Theorem 3.1.** Assume that the conditions in Lemma 3.2 are satisfied, (H3) holds, \( \det A \neq 0 \) and
\[ |1 - \frac{b^T A^{-1} e}{1 + \alpha(H)}| < 1, \]
then the Runge–Kutta method is \( H_\alpha \)-stable.
Proof. It follows from Lemma 3.2 that
\[
\|\omega_{n+1}\|^2 \leq \|\omega_0\|^2 + \sum_{j=1}^s b_j \left\{ \sum_{i=0}^n \left( (2-\kappa) a + 2c^{k(i)} + \frac{\rho a}{q-c} \right) h_i \|W_{j+1}^i\|^2 \right\} + (M_1 + M_2) \frac{t_0}{q-c} + \frac{\rho a}{q-c} \sum_{j=1}^s b_j \|W_j^0\|^2.
\]
(3.18)

In view of $c < q < 1$ and
\[
(2-\kappa) a + \frac{\rho a}{q-c} < 0 \quad \text{for } \kappa \in \left(0, \frac{2q-2c}{q+1-2c}\right),
\]
there exists $N_1 > 0$ such that for all $n \geq N_1$
\[
(2-\kappa) a + 2c^{k(n)} + \frac{\rho a}{q-c} \leq \left( 1 - \frac{1}{2^k} \right) a + \frac{\rho a}{2(q-c)} < 0.
\]
Therefore since
\[
\|\omega_{n+1}\|^2 - \sum_{j=1}^s b_j \left\{ \sum_{i=N_1+1}^n \left( (2-\kappa) a + 2c^{k(i)} + \frac{\rho a}{q-c} \right) h_i \|W_{j+1}^i\|^2 \right\}
\leq \sum_{j=1}^s b_j \left\{ \sum_{i=0}^{N_1} \left( (2-\kappa) a + 2c^{k(i)} + \frac{\rho a}{q-c} \right) h_i \|W_{j+1}^i\|^2 \right\} + (M_1 + M_2) \frac{t_0}{q-c} + \frac{\rho a}{q-c} \sum_{j=1}^s b_j \|W_j^0\|^2,
\]
(3.19)

there exists a constant $M > 0$ such that $\|\omega_n\|^2 \leq M$ for all $n \geq 0$ and
\[
- \sum_{j=1}^s b_j \sum_{i=N_1+1}^n \left( (2-\kappa) a + 2c^{k(i)} + \frac{\rho a}{q-c} \right) h_i \|W_{j+1}^i\|^2 < \infty.
\]

Thus
\[
\lim_{i \to +\infty} \sqrt{h_i} \|W_{j+1}^i\| = 0,
\]
(3.20)

which implies that $\lim_{n \to \infty} \|W^n\| = 0$.

From (H3) and $|1 - \frac{B^T A^{-1} e}{1 + \alpha(H)}| < 1$, there exist constants $\delta > 0$ and $N_2$ such that for all $n > N_2$
\[
\left| 1 - \frac{B^T A^{-1} e}{1 + \alpha_n(h_n)} \right| \leq \left| 1 - \frac{B^T A^{-1} e}{1 + \alpha(H)} \right| + \delta < 1.
\]
(3.21)

Consequently, for any given $\varepsilon > 0$, there exists an integer $N \geq \max\{N_1, N_2\}$ such that
\[
\|W^n\| \leq 1 - \left( \frac{1}{2\|B^T A^{-1}\|} \right) \varepsilon, \quad n \geq N,
\]
\[
\left( \left| 1 - \frac{1}{1 + \alpha(H)} B^T A^{-1} e \right| + \delta \right)^n \leq \frac{1}{2M} \varepsilon, \quad n \geq N.
\]
(3.22)
It follows by (3.5) that
\[
\omega_{n+1} = \left(1 - \frac{1}{1 + \alpha_n(h_n)} b^T A^{-1} e\right) \omega_n + \left(\frac{1}{1 + \alpha_n(h_n)} b^T A^{-1} \otimes I_d\right) W^{n+1}, \tag{3.23}
\]
which implies
\[
\|\omega_{n+1}\| \leq \left|1 - \frac{1}{1 + \alpha_n(h_n)} b^T A^{-1} e\right| \|\omega_n\| + \left|\frac{1}{1 + \alpha_n(h_n)} b^T A^{-1} \otimes Id\right| W^{n+1}. \tag{3.24}
\]
As a result, for \(n > N\),
\[
\|w_{n+N}\| \leq \left(1 - \frac{1}{1 + \alpha(H)} b^T A^{-1} e\right) \|w_{n+N-1}\| + \|b^T A^{-1}\| W^{n+N}
\]
\[
\leq \left(1 - \frac{1}{1 + \alpha(H)} b^T A^{-1} e\right) \|w_N\| + \sum_{i=0}^{n-1} \left(\left|1 - \frac{1}{1 + \alpha(H)} b^T A^{-1} e\right| + \delta\right)^i \|b^T A^{-1}\| W^{N+i+1}
\]
\[
\leq \left(1 - \frac{1}{1 + \alpha(H)} b^T A^{-1} e\right) \|w_N\| + \frac{1 - \left(\left|1 - \frac{1}{1 + \alpha(H)} b^T A^{-1} e\right| + \delta\right)}{2 \|b^T A^{-1}\|} \epsilon \sum_{i=0}^{n-1} \left(\left|1 - \frac{1}{1 + \alpha(H)} b^T A^{-1} e\right| + \delta\right)^i \|b^T A^{-1}\|
\]
\[
\leq \left(1 - \frac{1}{1 + \alpha(H)} b^T A^{-1} e\right) \|w_N\| + \frac{\epsilon}{2} \leq \epsilon, \tag{3.25}
\]
which implies that
\[
\lim_{n \to +\infty} \|y_n - z_n\| = 0. \square
\]

**Corollary 3.1.** Assume that the conditions (i)–(iii) in Lemma 3.2 are satisfied. Then:

1. The Radau IA, Radau IIA and Lobatto IIIC methods are \(H_\alpha\)-stable.
2. The Gauss–Legendre methods are \(H_\alpha\)-stable if and only if \(s\) is odd.
3. The one-leg \(\theta\)-methods are \(H_\alpha\)-stable if and only if \(\frac{1}{2} \leq \theta \leq 1\).

**Proof.** (i) Since these methods are algebraically stable, the matrix \(A\) is regular and \(b^T A^{-1} e = 1\),
\[
\alpha_n(\eta) = \frac{\eta^{p-1}}{1 + \eta^{p-n}} \text{ satisfies (H1)–(H3)} \quad \text{and} \quad \left|1 - \frac{b^T A^{-1} e}{1 + \alpha(H)}\right| = \frac{1}{2} < 1.
\]

(ii) It is obvious that the Gauss–Legendre methods are algebraically stable, the matrix \(A\) is regular and

\[
\left|1 - \frac{b^T A^{-1} e}{1 + \alpha(H)}\right| = \left|1 - \frac{2}{1 + \alpha(H)}\right| < 1.
\]

Therefore we also can choose \(\alpha_n(\eta) = \frac{\eta^{p-1}}{1 + \eta^{p-n}}\) and
\[
\left|1 - \frac{b^T A^{-1} e}{1 + \alpha(H)}\right| = \left|1 - \frac{2}{1 + \alpha(H)}\right| < 1.
\]
Conversely, $H_\alpha$-stable implies that $\lim_{n \to +\infty} \|y_n - z_n\| = 0$. We focus on the special case that Eq. (1.2) and $y(0) = y_0$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha, \beta$ satisfy

$$\text{Re}(\alpha) < \beta.$$ 

It is obvious that this case satisfying assumption (3.7). If $s$ is even, then

$$\left| 1 - \frac{b^T A^{-1} e}{1 + \alpha(H)} \right| = 1.$$

In view of [11], $\lim_{n \to \infty} y_n \neq 0$.

(iii) Since one-leg $\theta$-methods with $\frac{1}{2} \leq \theta \leq 1$ are algebraically stable and $A = \theta$. Therefore let $\alpha_n(\eta) = \frac{1}{2\theta} \frac{\eta}{1+\eta}$, it is easy to see that $\alpha_n(\eta)$ satisfies (H1)–(H3) with $\alpha(H) = \frac{1}{2\theta}$ and

$$\left| 1 - \frac{b^T A^{-1} e}{1 + \alpha(H)} \right| = \left| 1 - \frac{1}{\theta} \frac{1}{1 + \alpha(H)} \right| < 1. \quad \Box$$

**Remark 3.1.** From Corollary 3.1, we can see that the odd-stage Gauss–Legendre method and the one-leg $\theta$-method with $\frac{1}{2} \leq \theta \leq 1$ are $H_\alpha$-stable, while the classical Gauss–Legendre method and the classical one-leg $\theta$-method with $\theta = \frac{1}{2}$ are not asymptotically stable (see [15]).

4. Numerical experiment

In this section, we give two numerical experiments to illustrate the results in our paper.

We consider the linear neutral pantograph equation

$$y'(t) = -ay(t) + \frac{qa}{2} \exp\{(q - 1)at\}y(qt) + \frac{q}{2} \exp\{(q - 1)at\}y'(qt), \quad t > 0,$$

$$y(0) = y_0, \quad (4.1)$$

and

$$z'(t) = -az(t) + \frac{qa}{2} \exp\{(q - 1)at\}z(qt) + \frac{q}{2} \exp\{(q - 1)at\}z'(qt), \quad t > 0,$$

$$z(0) = z_0. \quad (4.2)$$

It is easy to see that the solution of (4.1) is $y(t) = y_0 \exp(-at)$ and the solution of (4.2) is $z(t) = z_0 \exp(-at)$.

Let $H = \{-m; t_0, t_1, \ldots\}$ be a geometric mesh which is defined by

$$t_n = q^{-\frac{m}{2}} 2^{-10}, \quad n \geq -m, \quad (4.3)$$

and $h_n = t_{n+1} - t_n$.

In Fig. 1, $a = 10^{-4}$, $q = \frac{1}{2}$, $m = 30$, $y_0 = 1$ and $z_0 = 2$, we draw the difference $\|y_n - z_n\|$ of the modified one-leg $\theta$-method with $\theta = \frac{3}{4}$ and $\alpha_n(h_n) = \frac{1}{2\theta} \frac{h_n}{1+h_n}$. It can be seen that the difference $\|y_n - z_n\|$ tends to zero as $n \to \infty$, which is in agreement with Corollary 3.1.

In Table 1, $a = 10^{-4}$, $q = \frac{1}{2}$, $\theta = \frac{3}{4}$ and $\theta = \frac{1}{2}$ with $\alpha_n(h_n) = \frac{1}{2\theta} \frac{h_n}{1+h_n}$, we list the absolute errors (AE) and relative errors (RE) at $t = 1$ of the modified one-leg $\theta$-method with geometric mesh and the ratio of the errors of the case $m = 50$ over that of $m = 100$. From Table 1, we
Fig. 1. The difference $\|y_n - z_n\|$ for the modified one-leg $\theta$-method.

Table 1
The errors of the modified one-leg $\theta$-methods

<table>
<thead>
<tr>
<th>$\theta$ = $\frac{3}{4}$</th>
<th>$\theta$ = $\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ = 20</td>
<td>$\text{RE}$</td>
</tr>
<tr>
<td>4.3647E−010</td>
<td>4.3691E−010</td>
</tr>
<tr>
<td>3.3066E−010</td>
<td>3.3099E−010</td>
</tr>
<tr>
<td>2.6308E−010</td>
<td>2.6334E−010</td>
</tr>
<tr>
<td>2.1775E−010</td>
<td>2.1797E−010</td>
</tr>
<tr>
<td>1.1623E−010</td>
<td>1.1634E−010</td>
</tr>
<tr>
<td>Ratio</td>
<td>1.8734</td>
</tr>
</tbody>
</table>

can see that the modified one-leg $\theta$-method is of order 1 if $\theta = \frac{3}{4}$ and is of order 2 if $\theta = \frac{1}{2}$ for (4.1).

Let us consider the following nonlinear neutral pantograph equation

$$y'(t) = -y(t) - 4y^3(t) + qy^3(qt) + \frac{q}{4}y'(qt), \quad t > 0,$$

$$y(0) = 1,$$  \hspace{1cm} (4.4)

and

$$z'(t) = -z(t) - 4z^3(t) + qz^3(qt) + \frac{q}{4}z'(qt), \quad t > 0,$$

$$z(0) = 2.$$  \hspace{1cm} (4.5)

It is easy to see that (4.4) and (4.5) satisfy assumption (3.7) in Lemma 3.2 but not satisfy assumption (5.12) in [7].

In Fig. 2, $q = \frac{1}{2}$, $m = 50$, $\theta = \frac{3}{4}$, $y_0 = 1$ and $z_0 = 2$, we draw the difference $\|y_n - z_n\|$ of the modified one-leg $\theta$-method with $\alpha_n(h_n) = \frac{1}{20} \frac{h_n}{1+h_n}$. It can be seen that the difference $\|y_n - z_n\|$ tends to zero as $n \to \infty$, which is in agreement with Corollary 3.1.
Fig. 2. The difference $\|y_n - z_n\|$ for the modified one-leg $\theta$-methods.

References