ODEs and Wiman–Valiron theory in the unit disc

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ABSTRACT

An asymptotic equality of Wiman–Valiron type is proved for the derivatives of analytic functions in the unit disc and applied to ODEs with analytic coefficients.

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1. Introduction

For a power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), the maximum term is \( \mu(r) = \mu(r, f) = \max_{n \geq 0} |a_n| r^n \) for \( r \geq 0 \), and the central index, denoted by \( N(r) = N(r, f) \), is the integer \( n \) for which the maximum is attained. (In case of ambiguity we pick the largest such \( n \).) We recall that \( N \) is non-decreasing and piecewise constant [7, p. 318]. We say that \( f \) is fully indexed if \( N \) assumes every non-negative integer value, and in that case \( R_n \) is an indexing sequence if \( N(R_n, f) = n \) and the maximum term at \( R_n \) is unique, for all \( n \).

If \( f \) is entire and \( \zeta \) is such that \( |f(\zeta)| = M(1, f) \), then for every positive integer \( q \),

\[
  f^{(q)}(\zeta) = (1 + o(1))(N(1, f))^{\frac{q}{2}} f(\zeta)
\]

(1)
as \( |\zeta| \to \infty \) outside a set of finite logarithmic measure [7, p. 341]. Here as usual \( M(1, f) = \max_{|z|=1} |f(z)| \). The relation (1) provides a rather powerful means for estimating the order of growth of solutions to linear differential equations

\[
f^{(n)} + b_{n-1} f^{(n-1)} + \cdots + b_0 f = 0
\]

(2)

with polynomial or entire coefficients \( b_0, \ldots, b_{n-1} \) [13,10]. There has been interest recently [1–3,8,9,12] in the growth of solutions of (2) near the boundary when the coefficients are assumed to be analytic in the unit disc but, as has been pointed out [2, pp. 285–286], [12, Section 2], the analysis is constrained by the lack of anything like (1) in the unit disc.

Our intention here is threefold: to show that for functions in the unit disc, results of the first author and Strumia [6] can be used to establish (1) for \( |\zeta| \) in a relatively thick subset of the interval \((0,1)\); to illustrate the effectiveness of (1) in the unit disc by giving quick proofs of results that otherwise require detailed argument; and to obtain growth estimates for solutions of (2) when the coefficients are analytic in the unit disc and behave, in a certain sense, as polynomials do in the plane.

Our results involve the order of \( f \). The order of a positive, increasing, real-valued function \( \Phi \) on \([0,1)\) is

\[
\lim_{r \to 1^-} \frac{\log \Phi(r)}{-\log(1-r)}.
\]

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and the order of $f$ is defined to be the order of $\log M(r, f)$. It is known [11, pp. 43, 45] that if $\rho$, $\rho'$ and $\rho''$ are the orders of $\log M(r, f)$, $\log \mu(f, r)$ and $N(r, f)$ respectively, then
\[
\rho = \rho' = \max(0, \rho'' - 1).
\] (3)

Thus in particular if $f$ has order $\rho > 0$ there is an increasing sequence $(T_j)$ for which
\[
\lim_{j \to \infty} T_j = 1 \quad \text{and} \quad \lim_{j \to \infty} \frac{\log N(T_j, f)}{\log(1 - T_j)} = \rho + 1.
\] (4)

From now on the sequence $(T_j)$ is fixed.

Let us note incidentally that (4) may fail if $\rho = 0$. For example, for $f(z) = \sum_{n=1}^{\infty} (1 - n^{-\alpha})e^{in}$, where $\alpha > 0$, we have $\log N(r) \sim -(1 + \alpha)^{-1}\log(1 - r)$.

We will prove:

**Theorem 1.** Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disc, of order $\rho > 0$. Given $\gamma$ satisfying
\[
0 < \gamma < \frac{\rho}{2(\rho + 1)},
\] (5)

let $\zeta$ be such that
\[
|f'(\zeta)| \geq N^{-\gamma}M(|\zeta|, f),
\] (6)

where $N = N(|\zeta|, f)$. Let $(T_j)$ be a sequence satisfying (4). Then, for every positive integer $q$,
\[
f^{(q)}(\zeta) = (1 + o(1)) \left(\frac{N}{\zeta}\right)^q f(\zeta)
\] (7)
as $|\zeta| \to 1^-$ outside a set $E$ such that
\[
\lim_{j \to \infty} m(E \cap (T_j, 1))/\log(1 - T_j) = 0.
\] (8)

**Corollary 1.** Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disc, of order $\rho > 0$. There is a sequence $\zeta_j$, with $|\zeta_j| \to 1^-$ as $j \to \infty$, such that $|f(\zeta_j)| = M(|\zeta_j|, f)$ for all $j$, and
\[
\lim_{j \to \infty} \frac{\log N(|\zeta_j|, f)}{\log(1 - |\zeta_j|)} = \rho + 1.
\] (9)

If $F$ is a set such that
\[
\lim_{t \to 1^-} m(F \cap (r, 1))/\log(1 - r) > 0.
\] (10)

then $\zeta_j$ can be chosen so that, for all $j$, $|\zeta_j| \in F \setminus E$, and therefore (7) holds at $\zeta = \zeta_j$.

The left-hand side of (10) is the *lower final density* of $F$ [6, p. 479]. The *upper final density* is the same except that the lower limit is replaced by the upper limit; if the upper and lower limits agree, the common value is the *final density*.

Assuming Theorem 1 for the moment, let us prove the corollary. By (3) and the fact that $\rho > 0$, there is an increasing sequence $T_j$ satisfying (4). Also, from (8) and (10), there is a number $\epsilon_0$, with $0 < \epsilon_0 < 1$, and a sequence $\epsilon_j$ satisfying $0 < \epsilon_j \leq \epsilon_0$ for all $j$, such that $T_j + \epsilon_j(1 - T_j) \in F \setminus E$ for all large $j$. Since $N$ is increasing, a simple calculation shows that (4) holds with $T_j$ replaced by $T_j + \epsilon_j(1 - T_j)$. We choose $\zeta_j$ such that $|\zeta_j| = T_j + \epsilon_j(1 - T_j)$ and $|f(\zeta_j)| = M(|\zeta_j|, f)$, and the conclusion follows.

Our next theorem on functions of order zero is of independent interest, in that its proof does not rely on the lemma of the logarithmic derivative.

**Theorem 2.** Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disc, of order zero, and that $\gamma : (0, 1) \to \mathbb{R}$ is positive and such that $\gamma(t) \to 0$ as $t \to 1^-$. If $\zeta$ is such that $|f(\zeta)| \geq N^{-\gamma(|\zeta|)}M(|\zeta|, f)$, then for every positive integer $q$ and positive number $\eta$,
\[
\frac{f^{(q)}(\zeta)}{f(\zeta)} = O\left(\frac{1}{1 - |\zeta|}\right)^{q + \eta}
\] (11)
as $|\zeta| \to 1^-$ outside a set of zero final density.
2. Proof of Theorem 1: A lemma

We adapt the argument for entire functions ([7, p. 341ff; see also [5]) to the unit disc. We will prove:

**Lemma 3.** Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is analytic in the unit disc, of order \( \rho > 0 \). Let \((T_j)\) be an increasing sequence satisfying (4) and define

\[
k_N = \lfloor \sqrt{(1 - r)^{-1} N (\log 3N)^2} \rfloor,
\]

where \( N = N(r, f) \) and \( \lfloor \cdot \rfloor \) denotes integer part. Then, for every positive integer \( q \) and every positive number \( \eta \),

\[
N^\eta \sum_{|n-N|>k_N} n^\eta |a_n| r^n \frac{1}{\mu(r, f)} \to 0
\]
as \( r \to 1^- \) outside a set \( E \) such that (8) holds.

We need the following result.

**Theorem A.** (See [6, Theorem 1 (with \( C = 2, \kappa = 1 \))].) Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is analytic in the unit disc and that \( F(z) = \sum_{n=0}^{\infty} A_n z^n \) is a fully indexed power series with indexing sequence \( R_n \). Suppose also that \( \epsilon \) satisfies \( 0 < \epsilon < 1 \) and that \( \psi_{\epsilon} \) is a positive, non-increasing function on \((0, 1)\) such that, for some \( R \in (0, 1) \),

\[
\psi_{\epsilon}(R) \log(R_{2N}/R_0) \leq \epsilon (1 - R),
\]

where \( N' = N(R', f) \) and \( R' = e^{-\epsilon(1-R)} \). Then

\[
\frac{|a_n| r^n}{\mu(r, f)} \leq \frac{|A_n| R_n^\epsilon}{\mu(R_n, F)} \psi_{\epsilon}(r), \quad 0 \leq n \leq 2N,
\]

\[
\frac{|a_n| r^n}{\mu(r, f)} \leq \max \left\{ \frac{|A_n| R_n^{\epsilon \eta}}{\mu(R_n, F)}, \left( \frac{R_n}{R_{2N}} \right)^{\psi_{\epsilon}(r)/2} \right\}, \quad n > 2N,
\]

where \( N = N(r, f) \), for all \( r \in (0, 1) \) outside a set \( E \) such that

\[
\log\text{meas}(E \cap (R, 1)) \leq 2\epsilon (1 - R).
\]

Here logarithmic measure is \( dr/r \), so that \( \log\text{meas}(E \cap (R, 1)) \geq m(E \cap (R, 1)) \). We let

\[
A_n = \exp \left( \int_0^n \alpha(t) \, dt \right), \quad R_n = \exp(-\alpha(n)), \quad n = 0, 1, 2, \ldots,
\]

where \( \alpha(t) = (\log(t + 1))^{-1} \), and apply Theorem A with \( F(z) = \sum_{n=0}^{\infty} A_n z^n \), which is fully indexed with indexing sequence \( R_n \), as can be easily checked. Given an increasing sequence \( T_j \to 1 \) as \( j \to \infty \), we define

\[
\epsilon_j = (\log(1 - T_j))^{-1/2}
\]

and \( \psi_{\epsilon_j}(r) = \epsilon_j (1 - r) \), noting that

\[
\psi_{\epsilon_j}(R) \log(R_{2N}/R_0) \leq \psi_{\epsilon_j}(R) \log(1/R_0) = \epsilon_j (1 - R),
\]

so that (14) is satisfied for all \( R \). With \( R = T_j \), we obtain from Theorem A,

\[
\frac{|a_n| r^n}{\mu(r, f)} \leq \frac{A_n R_n^\epsilon}{A_n R_n^{\epsilon \eta}} \epsilon_j^{1-r}, \quad 0 \leq n \leq 2N,
\]

\[
\frac{|a_n| r^n}{\mu(r, f)} \leq \max \left\{ \frac{A_n R_n^{\epsilon \eta}}{A_n R_n^\epsilon}, \left( \frac{R_n}{R_{2N}} \right)^{\epsilon_j (1-r)/2} \right\}, \quad n > 2N,
\]

for all \( r \in (T_j, 1) \) outside a set of measure at most \( 2\epsilon_j (1 - T_j) \) (taking account of the earlier remark on logarithmic measure). It is useful in what follows to make an additional restriction on \( r \), that \( r < T'_j \), where \( T'_j = 1 - \epsilon_j (1 - T_j) \). With this restriction, (19) and (20) hold for \( r \in (T_j, T'_j) \) outside a set \( E_j \) such that \( E_j \subseteq (T_j, T'_j) \) and \( m(E_j) \leq 2\epsilon_j (1 - T_j) \).

We estimate the left-hand side of (13) using (19) and (20). Without loss of generality, we assume that \( N(T_j, f) \geq 3 \) for all \( j \). We have
\[
R_{N}/R_{2N} = \exp\left(-\int_{N}^{2N} \frac{dt}{(t+e)(\log(t+e))^2}\right) \leq e^{-(3/10)(\log 3N)^{-2}}
\]  
(21)

and, for \( n > 2N \),

\[
\frac{A_nR_{N}^n}{A_N R_{N}^n} = \exp\left(\int_{N}^{n} \alpha(t) dt - (n - N)\alpha(N)\right) = \exp\left(\int_{N}^{n} (n - t)\alpha^\prime(t) dt\right)
\]

\[
\leq \exp\left(-\int_{N}^{n} \frac{n - t}{(t+e)(\log(t+e))^2} dt\right)
\]

\[
\leq \exp\left(-\frac{n}{3} \int_{N}^{n} \frac{dt}{(t+e)(\log(t+e))^2}\right)
\]

\[
\leq \exp\left(-\frac{2n/3}{4N/3} \int_{N}^{n} \frac{dt}{(t+e)(\log(t+e))^2}\right)
\]

\[
\leq e^{-(1/21)n(\log(7N/3))^{-2}} \leq e^{-(1/25)n(\log 3N)^{-2}},
\]

(22)

using the fact that \( N \geq 3 > e \). From (20) then, we certainly have

\[
|a_n|^{-n} \leq \mu(r, f)e^{-\nu j n}, \quad n > 2N,
\]  
(23)

where \( \nu_j = \nu_j(r) = (1/25)e_j(1-r)(\log 3N)^{-2} \), for \( r \in (T_j, T_j^\prime) \ \setminus E_j \). This holds for any increasing sequence \( T_j \) and any function \( f \), whatever its order.

For the remainder of the proof we assume that \( f \) has order \( \rho > 0 \), as in the hypotheses of Lemma 3, and that \( T_j \) satisfies (4). Given \( \rho_0 \) satisfying \( 0 < \rho_0 < \rho \), we have, for \( r \in (T_j, T_j^\prime) \),

\[
\frac{\log N(r, f)}{-\log(1-r)} \geq \frac{\log N(T_j, f)}{-\log(1-T_j)} = \frac{\log N(T_j, f)}{-\log(e_j(1-T_j))} \geq 1 + \rho_0
\]

(24)

for all large \( j \), from (18). Let us note too that, from (18) and (4),

\[
\epsilon_j \geq (\log N(T_j, f))^{-1/2} \geq (\log N(r, f))^{-1/2},
\]

(25)

for \( r \in (T_j, T_j^\prime) \) for all large \( j \). Thus, from (24) and (25),

\[
\nu_j > (1/25)\epsilon_j N^{-1/(\rho_0+1)}(\log 3N)^{-2} > (1/25)N^{-1/(\rho_0+1)}(\log 3N)^{-5/2},
\]  
(26)

for \( r \in (T_j, T_j^\prime) \ \setminus E_j \) for all large \( j \).

Now, for \( t \in (0, 1) \),

\[
\sum_{n=2N+1}^{\infty} n^q |a_n|^{n} \leq K \frac{N^{q}t^{2N}}{1-t} \left(1 + \frac{1}{2(1-t)N}\right)^{q+1},
\]

(27)

where \( K = K(q) \) [4, Lemma 9]. We take \( t = e^{-\nu_j} \). Since \( (1 - e^{-\nu_j})N \to \infty \) as \( j \to \infty \), from (26), and also

\[
\frac{N^{q}e^{-2\nu_j/N}}{1-e^{-\nu_j}} = \frac{N^{q}e^{-\nu_j(2N-1)}}{e^{\nu_j}-1} \leq N^{q}e^{-\nu_j}N^{q+1/(\rho_0+1)} \exp\left(-(1/25)N^{\rho_0/(\rho_0+1)}(\log 3N)^{-5/2}\right),
\]

we have using (23)

\[
\sum_{n=2N+1}^{\infty} \frac{n^q |a_n|^{n}}{\mu(r, f)} \leq 50K (\log 3N)^{5/2} N^{q+1/(\rho_0+1)} \exp\left(-(1/25)N^{\rho_0/(\rho_0+1)}(\log 3N)^{-5/2}\right),
\]

(28)

for \( r \in (T_j, T_j^\prime) \ \setminus E_j \) for all large \( j \).
Finally, for \( k_N < |n - N| \leq N \),
\[
\frac{A_nR_N^n}{A_NR_N^n} = \exp \left( - \int \frac{(n - t) \, dt}{N \, (t + e)(\log(t + e))^2} \right)
\leq \exp\left(-\frac{6N^{-1}(\log 3N)^{-2}(n - N)^2}{(k_N + 1)^2(6N^{-1}(\log 3N)^{-2})} \right)
\leq \exp\left(-\frac{1/6(1 - r)^{-1}(\log 3N)^2}{1} \right),
\]
from (12), and therefore
\[
\left( \frac{A_nR_N^n}{A_NR_N^n} \right)^{(1-r)} \leq e^{-(1/10)(\log 3N)^2}.
\]
From this, (19) and (25),
\[
\sum_{k_N < |n - N| \leq N} n^q |a_n|^r \mu(r, f) \leq 2N e^{-(1/10)(\log 3N)^{3/2}}
\]
for \( r \in (T_j, T'_j) \setminus E_j \) for all large \( j \).

It follows from (28) and (29) that (13) holds as \( r \rightarrow 1^+ \) in \( \bigcup_{j=1}^{\infty} (T_j, T'_j) \setminus E_j \). If \( E \) is the complement of this set in \( (0, 1) \), then \( E \cap (T_j, 1) \subseteq E_j \cup (T'_j, 1) \) and therefore \( m(E \cap (T_j, 1)) \leq m(E_j) + (1 - T'_j) < 3\epsilon_j(1 - T'_j) \), which completes the proof of Lemma 3.

3. Proof of Theorem 1

We write \( \sum_{|n - N| \leq k_N} a_n z^n = z^{N-k_N} P_N(z) \), where \( P_N \) is a polynomial of degree at most \( 2k_N \). From (6) and Lemma 3 with \( q = 0 \),
\[
f(\zeta) = (1 + o(1))\zeta^{N-k_N} P_N(\zeta)
\]
as \( |\zeta| \rightarrow 1^- \) outside \( E \); also, again using Lemma 3 with \( q = 0 \), we have
\[
M(\zeta, P_N) = (1 + o(1))|\zeta|^{k_N-N} M(|\zeta|, f)
\]
as \( |\zeta| \rightarrow 1^- \) outside \( E \). Now,
\[
f^{(q)}(\zeta) = \frac{d^q}{d\zeta^q} \left( \zeta^{N-k_N} P_N(\zeta) \right) + O \left( \sum_{|n - N| > k_N} n^q |a_n||\zeta|^p \right)
\]
\[
= \frac{d^q}{d\zeta^q} \left( \zeta^{N-k_N} P_N(\zeta) \right) + o(f(\zeta))
\]
as \( |\zeta| \rightarrow 1^- \) outside \( E \), from Lemma 3. Also, with \( C^q_\ell \) the usual binomial coefficient,
\[
\frac{d^q}{d\zeta^q} \left( \zeta^{N-k_N} P_N(\zeta) \right) = \sum_{\ell=0}^q C^q_\ell (N - k_N) \ldots (N - k_N - q + \ell + 1) \zeta^{N-k_N-q+\ell} P_N^{(\ell)}(\zeta)
\]
\[
= (1 + o(1))N^q \zeta^{N-k_N-q} P_N(\zeta) + O \left( \sum_{\ell=1}^q N^q \zeta^{N-k_N} P_N^{(\ell)}(\zeta) \right)
\]
\[
= (1 + o(1))(N/\zeta)^q f(\zeta) + O \left( \sum_{\ell=1}^q N^q \zeta^{N-k_N} P_N^{(\ell)}(\zeta) \right)
\]
as \( |\zeta| \rightarrow 1^- \) outside \( E \), using (30). From (31), Lemma 6.1 of [5] and (6),
\[
P_N^{(\ell)}(\zeta) = O \left( k_N^{\ell} z^{N-k_N-N} M(|\zeta|, f) \right) = O \left( k_N^{\ell} z^{N-k_N-N} f(\zeta) \right)
\]
as \( |\zeta| \rightarrow 1^- \) and thus, from (30),
\[
\sum_{\ell=1}^q N^q \zeta^{N-k_N} P_N^{(\ell)}(\zeta) = O \left( f(\zeta) \sum_{\ell=1}^q N^q \zeta^{N-k_N} P_N^{(\ell)}(\zeta) \right)
\]
as $|\zeta| \to 1^-$ outside $E$. Since, from (12) and (24),
\[
N^{-1}k_N \leq \sqrt{(1 - |\zeta|)^{-1}N^{-1}(\log 3N)^2} \leq N^{-\rho_0/(2(\rho_0 + 1))}(\log 3N)^2 < 1, \tag{36}
\]
for all $|\zeta| \in (T_j, T'_j)$ for all large $j$,
\[
\sum_{\ell=1}^{q} N^{q-\ell+\gamma}k_{N}^{j} \leq qN^{q+\gamma}k_{N}. \tag{37}
\]
We choose $\rho_0$ sufficiently close to $\rho$ that $\rho_0/(2(\rho_0 + 1)) > \gamma'$, which is possible from (5), and conclude from (36) and (37) that $\sum_{\ell=1}^{q} N^{q-\ell+\gamma}k_{N} = o(N^q)$ as $|\zeta| \to 1^-$ outside $E$. Theorem 1 follows from this, (32), (33), (34) and (35).

4. Proof of Theorem 2

Given a positive integer $l$, write $Q_l(z) = \sum_{n=0}^{2l} a_n z^n$, so that
\[
f^{(q)}(z) = Q^{(q)}_l(z) + O \left( \sum_{n=2l+1}^{\infty} n^q|a_n||\zeta|^n \right). \tag{38}
\]
As we noted earlier, (23) holds for $r \in (T_j, T'_j) \setminus E_j$ for any increasing sequence $T_j$, even when $f$ has order 0. Thus, given $l \geq N$, we obtain, from (23) and (27) with $t = e^{-v_j}$,
\[
\sum_{n=2l+1}^{\infty} n^q|a_n||\zeta|^n \leq \sum_{n=2l+1}^{\infty} n^q e^{-v_j n} \\
\leq K \frac{n^q e^{-2v_j l}}{1 - e^{-v_j}} \left( 1 + \frac{1}{2l(1 - e^{-v_j})} \right)^{q+1} \\
\leq K l^q v_j^{-1} e^{-lv_j} \left( 1 + \frac{e^{v_j}}{2lv_j} \right)^{q+1}, \tag{39}
\]
for $|\zeta| \in (T_j, T'_j) \setminus E_j$ for all $j$. Now, for $|\zeta| \in (T_j, T'_j) \setminus E_j$,
\[
v_j = (1/25)(1 - |\zeta|) \left( \log \frac{1}{1 - T_j} \right)^{-1/2} (\log 3N)^{-2} \\
= (1/25 + o(1))(1 - |\zeta|) \left( \log \frac{1}{1 - |\zeta|} \right)^{-1/2} (\log 3N)^{-2}
\]
as $j \to \infty$. Also, since $f$ has order 0, we have, from (3), $N(|\zeta|, f) \leq (1 - |\zeta|)^{-1+o(1)}$, and therefore
\[
v_j = (1 - |\zeta|)^{1+o(1)} \tag{40}
\]
as $j \to \infty$, for $|\zeta| \in (T_j, T'_j) \setminus E_j$. Given a positive number $\eta$, we let $l = v_j^{-1-\eta}$ and note that $l \geq N$ for all large $j$, so that (39) holds. Thus, for any $\eta > 0$,
\[
N^q \sum_{n=2l+1}^{\infty} n^q|a_n||\zeta|^n = o(\mu(|\zeta|, f)) \quad \tag{41}
\]
as $j \to \infty$, for $|\zeta| \in (T_j, T'_j) \setminus E_j$. It follows from (41) with $q = 0$ that $f(\zeta) = (1 + o(1)) Q_l(\zeta)$, and also that $M(|\zeta|, Q_l) = (1 + o(1)) M(|\zeta|, f)$ as $j \to \infty$, for $|\zeta| \in (T_j, T'_j) \setminus E_j$. Further [5, Lemma 6.1], $M_l(r, Q^{(q)}_l) = O(\rho M(r, Q_l))$ as $r \to 1^-$, and therefore
\[
Q^{(q)}_l(\zeta) = O(\rho^q M^{(q)}(\zeta) f(\zeta))
\]
as $j \to \infty$, for $|\zeta| \in (T_j, T'_j) \setminus E_j$. Combining this with (38) and (41), we conclude that
\[
f^{(q)}(\zeta)/f(\zeta) = O(\rho^q M^{(q)}(\zeta) = O(v_j^{-(1+\eta)q+o(1)})
\]
as $j \to \infty$, for $|\zeta| \in (T_j, T'_j) \setminus E_j$. From (40), we obtain (11) — with a different $\eta$ — as $|\zeta| \to 1^-$ outside $E$, the complement in $(0, 1)$ of $\bigcup_{j=1}^{\infty} (T_j, T'_j) \setminus E_j$. We now choose $T_j = j/(j+1)$. For $r$ satisfying $T_j \leq r < T_{j+1}$,
\[ m(E \cap (r, 1)) \leq m(E \cap (T_j, 1)) \leq m(E \cap (T_j, T'_j)) + 1 - T'_j \]
\[ \leq m(E_j) + \epsilon_j(1 - T_j) \leq 3\epsilon_j(1 - T_j) \leq 5\epsilon(1 - r), \]
so that \( E \) has final density 0. This concludes the proof of Theorem 2.

5. Applications to ODEs

Consider the equation
\[ f'' + bf = 0, \tag{42} \]
where \( b \) is analytic in the unit disc. Following the notation in [8], we define \( H_q^\infty = \{ b : \sup_{0 \leq r < 1} M(r, b)(1 - r)^q \leq \infty \} \), for any \( q \geq 0 \), and \( \mathcal{H} = \bigcup_{q \geq 0} H_q^\infty \). If, for \( b \in \mathcal{H} \),
\[ p = \inf\{q \geq 0 : b \in H_q^\infty \}, \]
we say that \( b \in \mathcal{G}_p \). A result of Heittokangas [8, Theorem 3.1.4] shows that if \( f \) is a solution of (42), with \( b \in \mathcal{G}_p \) for some \( p \geq 0 \), then \( \rho \), the order of \( f \), is at most \( p/2 - 1 \).

We give a quick proof of this result. Without loss of generality we may assume that \( \rho > 0 \). Let \( \zeta_j \) be the sequence of Theorem 1, Corollary 1, with \( F = (0, 1) \). Then (7) holds with \( \zeta = \zeta_j \) and we obtain, from (42) and the fact that \( b \in \mathcal{G}_p \),
\[ N(|\zeta_j|, f)^2 \leq (1 - |\zeta_j|)^{-p + o(1)} \tag{43} \]
as \( j \to \infty \). Thus, from (9),
\[ -2(\rho + 1 + o(1)) \log(1 - |\zeta_j|) = 2 \log N(|\zeta_j|, f) \leq -(p + o(1)) \log(1 - |\zeta_j|), \]
and the conclusion follows.

In the same way it can be shown that for solutions of \( f^{(k)} + bf = 0 \), where \( b \in \mathcal{G}_p \), we have \( \rho \leq p/k - 1 \), a result that originally appeared in [3]. More generally, upper bounds on the order of solutions of (2) may be obtained if \( b_k \in \mathcal{G}_{p_k} \), \( k = 0, 1, \ldots, n - 1 \). Indeed, assuming without loss of generality that \( \rho > 0 \), we conclude from (7) that
\[ N(|\zeta_j|, f)^n + (1 + o(1))\zeta_j N(|\zeta_j|, f)^{n-1}b_{n-1}(\zeta_j) + \cdots + (1 + o(1))\zeta_j^n b_0(\zeta_j) = 0, \]
where \( \zeta_j \) is the sequence of Theorem 1, Corollary 1. A simple proof by contradiction along the lines of [13, pp. 127–128] shows that
\[ \rho \leq \max\{p_j/(n-j) : 0 \leq j \leq n - 1\}, \]
an inequality that has been proved by other methods [12, Theorem 1].

In some ways, functions in \( \mathcal{H} \) are counterparts of polynomials in the plane, but, as pointed out in [3, p. 737], polynomials behave in the same way in every direction as \( \rho \to \infty \), whereas functions in \( \mathcal{H} \) may behave differently near different boundary points of the unit disc. With the idea of a disc analogue of a polynomial in mind, let us say that a function \( b \), analytic in the unit disc, is \( \alpha \)-polynomial regular, for some positive number \( \alpha \), if there is a set \( F \subseteq (0, 1) \) of positive lower final density such that
\[ |b(z)| = (1 - |z|)^{-\alpha + o(1)} \tag{44} \]
as \( |z| \to 1^- \) through \( F \). We denote by \( \mathcal{P} \) the set of functions which are \( \alpha \)-polynomial regular for some \( \alpha \).

Example. To construct an example of a polynomial regular function, consider \( b(z) = \sum_{j=3}^{\infty} \lambda^j z^j \), where \( \alpha > 0 \) and \( \lambda \) is a (large) positive integer. Given a positive integer \( n \), consider \( r \) satisfying \( 1 - D\lambda^{-n} = r_n \leq r \leq r_n' = 1 - D'\lambda^{-n} \), where
\[ D = \log((\lambda^\alpha - 1)/5), \quad D' = (\lambda - 1)^{-1} \log(6\lambda^\alpha). \tag{45} \]

Now,
\[ \sum_{j=3}^{n-1} \lambda^j r^j \leq \sum_{j=0}^{n-1} \lambda^j = \frac{\lambda^{an}}{\lambda^\alpha - 1} \]
while
\[ \lambda^{an} r^{\lambda n} \geq \lambda^{an} (1 - D\lambda^{-n})^{\lambda n} = (1 + o(1))\lambda^{an} e^{-D} \geq 4\frac{\lambda^{an}}{\lambda^\alpha - 1}, \]
for all large \( n \), and therefore
\[
\sum_{j=3}^{n-1} \lambda^j \zeta_j \leq \frac{1}{4} \lambda^j \zeta_j,
\]
for all large \( n \). Also
\[
\sum_{j=n+1}^{\infty} \lambda^j \zeta_j = \lambda^{j+1} (1 + \lambda^j + \lambda^{j+1} + \cdots)
\]
\[
\leq \lambda^{j+1} (1 + \lambda^j + \lambda^{j+1} + \cdots)
\]
\[
= \frac{\lambda^{j+1}}{1 - \lambda^j}
\]
so that, since \( r \leq r'_n \),
\[
\sum_{j=n+1}^{\infty} \frac{\lambda^j \zeta_j}{\lambda^j n^\alpha} = \frac{\lambda^{j+1}}{1 - \lambda^j} \leq 1 + o(1).
\]
As \( n \to \infty \), combining (46) and (47), we obtain
\[
(1/2) \lambda^j \zeta_j = |b(z)| \leq (3/2) \lambda^j \zeta_j,
\]
for \( r_n \leq r \leq r'_n \) for all large \( n \). Moreover, for \( r = 1 - \theta \lambda^j \), where \( D \leq \theta \leq D' \),
\[
\lambda^j (1 - \theta \lambda^j)^\alpha = (1 + o(1)) \exp(-\theta) \lambda^j = (1 + o(1)) \exp(-\theta) \lambda^j = (1 - \lambda^j)^\alpha.
\]
Thus we have (44) as \( r \to 1^- \) through \( F = \bigcup_{n=1}^{\infty} (1 - D \lambda^j, 1 - D' \lambda^j) \). Finally, \( F \) has positive lower final density since, for \( r_n \leq r \leq r'_n \),
\[
m(F \cap (r, 1)) \geq m(F \cap (r'_n, 1))
\]
\[
= (1 - r'_n)^{-1} \sum_{j=0}^{\infty} (D - D') \lambda^j
\]
\[
= (1 - r'_n)^{-1} \left( \frac{D - D'}{\lambda - 1} \right) \lambda^j = \frac{(D - D')}{D' \lambda - 1}.
\]
Notice that by (45) if \( \lambda \) is large enough, the lower final density of \( F \) can be made as close to 1 as we please.

We now prove

**Theorem 4.** Suppose that \( f \) is an analytic solution of (42) of order \( \rho \), where \( b \) is \( \alpha \)-polynomial regular for some \( \alpha > 2 \). Then \( \rho = \alpha/2 - 1 \).

**Proof.** To prove Theorem 4, note first that, by (42) and (44),
\[
\frac{f''(z)}{f(z)} = (1 - |z|)^{-\alpha + o(1)}
\]
as \( |z| \to 1^- \) through \( F \). Since \( \alpha > 2 \), we deduce from Theorem 2 that \( \rho > 0 \). Let \( \zeta_j \) be the sequence of Theorem 1, Corollary 1, where \( F \) is the set of positive lower final density associated with \( b \). From (7) and (50) we obtain
\[
2(\rho + o(1)) \log \left( \frac{1}{1 - |\zeta_j|} \right) = 2 \log N(\zeta_j, f) = (\alpha + o(1)) \log \left( \frac{1}{1 - |\zeta_j|} \right),
\]
which implies that \( \rho = \alpha/2 - 1 \), and Theorem 4 is proved. \( \Box \)

We note that \( \alpha > 2 \) is best possible. Indeed by Theorem 3.1.4 in [8], \( \rho = 0 \) for all solutions of (42) if \( A \in G_p \), \( p \leq 2 \).

Theorem 4 is false if we assume only that \( b \in H \). Indeed, the bounded (and hence zero order) function \( f(z) = \exp((z + 1)/(z - 1)) \) is a solution of (42) in the unit disc with
\[
b(z) = 4 \left( \frac{1}{(z - 1)^4} + \frac{1}{(z - 1)^2} \right).
Finally, consider the differential equation
\[ f'' + b_1 f' + b_0 f(z) = 0, \]  
where \( b_0 \) and \( b_1 \) are \( \alpha_0 \) and \( \alpha_1 \)-polynomial regular respectively, with associated sets \( F_0 \) and \( F_1 \) such that \( F_0 \cap F_1 \) has positive lower final density. Suppose further that
\[ \min(\alpha_0 - \alpha_1, \alpha_0/2) > 1. \]  
Dividing (51) by \( f \) and using Theorem 2 together with (52), we deduce that \( f \) has positive order. Thus by Theorem 1, Corollary 1, there exists a sequence \( \zeta_j \) such that
\[ N(|\zeta_j|, f) = \left( \frac{1}{1 - |\zeta_j|} \right) \rho + o(1) \]  
and
\[ \left( \frac{N(|\zeta_j|, f)}{|\zeta_j|} \right)^2 + \frac{N(|\zeta_j|, f)}{|\zeta_j|} \left( \frac{1}{1 - |\zeta_j|} \right)^{\alpha_1 + o(1)} + \left( \frac{1}{1 - |\zeta_j|} \right)^{\alpha_0 + o(1)} = 0 \]  
as \( j \to \infty \). Solving this equation by the quadratic formula reveals the possible orders of solutions to (51). For example if \( \alpha_0/2 > \alpha_1 \), then clearly \( \rho = \alpha_0/2 - 1 \). On the other hand if \( \alpha_1 < \alpha_0 - 1 \) and \( \alpha_0 \leq 2\alpha_1 \), \( \rho \) could be \( \alpha_1 - 1 \) or \( \alpha_0 - \alpha_1 - 1 \).

An example in [12, p. 3] shows, at least when \( b_0 \) and \( b_1 \) are in \( \mathcal{H} \) and \( \alpha_0 > 5 \) and \( \alpha_1 > 3 \), that (51) can have solutions \( f_1 \) with \( \rho(f_1) = \alpha_1 - 1 \) and \( f_2 \) with \( \rho(f_2) = \alpha_0 - \alpha_1 - 1 \).

For linear differential equations (2) with coefficients in \( \mathcal{P} \), we obtain an algebraic equation like (53) of degree \( n \). The \( n \) possible orders and asymptotics of solutions mirror the Wittich, Newton–Puiseux results when the coefficients of (2) are polynomials (cf. [10, Section 22]).

References