Convergence rate of a new Bezier variant of Chlodowsky operators to bounded variation functions

Harun Karsli, Ertan Ibikli*

Department of Mathematics, Science Faculty, Ankara University, Tandogan 06100, Ankara, Turkey

Received 21 July 2006; received in revised form 22 November 2006

Abstract

In the present paper, we estimate the rate of pointwise convergence of the Bézier Variant of Chlodowsky operators $C_{n,x}$ for functions, defined on the interval extending infinity, of bounded variation. To prove our main result, we have used some methods and techniques of probability theory.

© 2007 Elsevier B.V. All rights reserved.

MSC: 41A25; 41A35; 41A36

Keywords: Approximation; Bounded variation; Chlodowsky polynomials; Bezier basis; Total variation; Rate of convergence

1. Introduction

Named after the French engineer-mathematician Pierre Bézier, a Bézier curve is a curved line defined by mathematical formulas. Bézier used these curves for the body design of the Renault car in 1970s. Bézier basis functions [1] play an important role in computer aided design which have many applications in applied mathematics and computer sciences.

Consider $n + 1$ control points $p_k, (k = 0–n).$ If we choose $p_k = f(k/n),$ we get Bernstein–Bézier parametric curve function or Bernstein–Bézier operator which is of the form

$$B_n(f; x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k} \quad (0 \leq x \leq 1)$$

and in general, $n \geq 1$ Bernstein–Bézier operator is of the form

$$B^{(z)}_n(f; x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) Q_{n,k}^{(z)}(x) \quad (0 \leq x \leq 1)$$

where $Q_{n,k}^{(z)}(x) = J_{n,k}^{(z)}(x) - J_{n,k+1}^{(z)}(x)$ and $J_{n,k}(x) = \sum_{j=k}^{n} P_{n,j}(x)$ be the Bézier basis functions.

* Corresponding author. Tel.: +90 312 2126720/1241; fax: +90 312 2235000. E-mail address: ibikli@science.ankara.edu.tr (E. Ibikli).

0377-0427/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
Let $BV(I)$ denotes the class of functions, which are bounded variation on a set $I \subset R$. Some authors studied on some linear positive operators to obtain the rate of convergence for functions in $BV(I)$. Some of the important papers on this topic are due to Bojanic and Vuilleumier [2], Cheng [3] and Zeng and Chen [15]. Very recently, several researchers have studied on some Bézier type operators for functions in $BV(I)$. For example, Zeng and Gupta [16] estimated the rate of convergence of Baskakov–Bézier type operators for locally bounded functions. Zeng and Piriou [17] estimated the rate of convergence of two Bernstein–Bézier type operators for bounded variation functions, Li and Gong [11] rate of convergence of Baskakov–Bézier type operators for locally bounded functions. Zeng and Piriou [17] estimated have studied on some Bézier type operators for functions in $BV(I)$ operators (1).

In this paper, by means of the techniques of probability theory, we shall estimate the rate of convergence for the Chlodowsky–Bézier operators $C_{n,x}$ for functions of bounded variation on $[0, \infty)$ at points $x$ where $f(x+)$ and $f(x-)$ exist.

Chlodowsky polynomials are given [4] by

$$C_n(f; x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) P_{n,k}\left(\frac{x}{b_n}\right) \quad (0 \leq x \leq b_n)$$

(1)

where $P_{n,k}(x/b_n) = \binom{n}{k} (x/b_n)^k (1-x/b_n)^{n-k}$ ($0 \leq x \leq b_n$) and $(b_n)$ is a sequence of increasing positive numbers, with the properties $\lim_{n \to \infty} b_n = \infty$ and $\lim_{n \to \infty} b_n/n = 0$.

For $x \geq 1$, we now introduce Chlodowsky–Bézier operators $C_{n,x}$ as follows:

$$C_{n,x}(f, x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) Q_{n,k}^{(x)}\left(\frac{x}{b_n}\right) \quad (0 \leq x \leq b_n),$$

(2)

where $Q_{n,k}^{(x)}(x/b_n) = (J_{n,k}(x/b_n))^2 - (J_{n,k+1}(x/b_n))^2$ and $J_{n,k}(x/b_n) = \sum_{j=k}^{n} P_{n,j}(x/b_n)$ be the Bézier basis functions. Obviously, $C_{n,x}$ is a positive linear operator and $C_{n,x}(1, x) = 1$. In particular when $x = 1$, the operators (2) reduce the operators (1).

Here we point out that, the rates of convergence in the case $x \in (0, 1)$ for functions of bounded variation were also obtained by different operators. For example see [8].

The main theorem of this paper is as follows.

**Theorem.** Let $f$ be a function of bounded variation on every finite subinterval of $[0, \infty)$ and $\lim_{x \to \infty} f(x)$ exists, i.e. $f \in BV[0, \infty)$. Then for every $x \in (0, \infty)$ and for sufficiently large values of $n$, we have

$$\left| C_{n,x}(f; x) - \left[ \frac{1}{2^x} f(x+) + \left(1 - \frac{1}{2^x}\right) f(x-), \right] \right|$$

$$\leq x \cdot \frac{3B_n(x)b_n^2}{x^2(b_n-x)^2} \left\{ \sum_{k=1}^{n} \sqrt{x+(b_n-x)/\sqrt{k}} (g_x) \right\}$$

$$+ \frac{x}{\sqrt{2enx/b_n(1-x/b_n)}} \left[ |f(x+) - f(x-)| + |f(x) - f(x-)|e_n\left(\frac{x}{b_n}\right) \right],$$

(3)

where

$$B_n(x) = \frac{x(b_n-x)}{n}, \quad e_n\left(\frac{x}{b_n}\right) = \begin{cases} 1, & \frac{x}{b_n} = \frac{k'}{n} \text{ for some } k' \in IN \\ 0, & \frac{x}{b_n} \neq \frac{k'}{n} \text{ for all } k' \in IN \end{cases}$$
and $\sqrt{b}(g_x)$ is the total variation of $g_x$ on $[a, b]$,

$$g_x(t) = \begin{cases} 
  f(t) - f(x^+), & x < t \leq b, \\
  0, & t = x, \\
  f(t) - f(x^-), & 0 \leq t < x.
\end{cases}$$

2. Auxiliary results

In this section we give certain results, which are necessary to prove our main theorem.

Lemma 1. For every $x \in [0, b_n]$ and $n \geq 1$, we have

$$C_n(1; x) = 1, \quad C_n(t; x) = x, \quad C_n(t^2; x) = x^2 + \frac{x(b_n - x)}{n}.$$ (4)

From (4), by direct calculation, we find the following equality:

$$C_n((t - x)^2; x) = \frac{x(b_n - x)}{n}.$$ (5)

For proof of this lemma see [5].

Using the fact that $|a^2 - b^2| \leq x|a - b|$ with $0 \leq a, b \leq 1$ and $x \geq 1$, we obtain $Q^{(x)}_{n, k}(x/b_n) \leq xP_{n, k}(x/b_n)$. By this inequality and (5), we have

$$C_{n, x}((t - x)^2; x) \leq xC_{n, 1}((t - x)^2; x) = x \left[ \frac{x(b_n - x)}{n} \right].$$ (6)

Lemma 2. For all $x \in (0, \infty)$, we have

$$\lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) = \int_0^t K_{n, x} \left( \frac{x}{b_n}, \frac{u}{b_n} \right) \, du, \quad 0 \leq t < x$$

$$\leq \frac{x}{(x-t)^2} \frac{x(b_n - x)}{n},$$ (7)

where

$$K_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) = \begin{cases} 
  \sum_{kb_n \leq nt} Q^{(x)}_{n, k} \left( \frac{x}{b_n} \right), & 0 < t \leq b_n, \\
  0, & t = 0.
\end{cases}$$

Proof.

$$\lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) = \int_0^t K_{n, x} \left( \frac{x}{b_n}, \frac{u}{b_n} \right) \, du$$

$$\leq \int_0^t K_{n, x} \left( \frac{x}{b_n}, \frac{u}{b_n} \right) \left( \frac{x - u}{x - t} \right)^2 \, du$$

$$= \frac{1}{(x-t)^2} \int_0^t K_{n, x} \left( \frac{x}{b_n}, \frac{u}{b_n} \right) (x-u)^2 \, du$$

$$= \frac{1}{(x-t)^2} C_{n, x}((u - x)^2; x).$$
From (6), we have
\[
\lambda_n, \alpha \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \leq \frac{x(b_n - x)}{(x - t)^2 n}.
\]

**Lemma 3.** If \( y \) is a positive valued random variable with a non-degenerate probability distribution then by Schwarz inequality
\[
E(y^3) \leq (E(y^2)E(y^4))^{1/2},
\]
provided \( E(y^2), E(y^3) \) and \( E(y^4) < \infty \).

**Lemma 4.** Let \( \zeta_1 \) be the random variables with two point binomial distribution
\[
P(\zeta_1 = k) = \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{1-k} \quad (k = 0, 1, \text{ and } x \in [0, b_n] \text{ is a parameter}).
\]
Then
\[
\begin{align*}
a_1 &= E, \quad \zeta_1 = \frac{x}{b_n}, \\
E(\zeta_1 - a_1)^2 &= \frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right)
\end{align*}
\]
and
\[
E(\zeta_1 - a_1)^4 = \left( \frac{x}{b_n} \right) - 4 \left( \frac{x}{b_n} \right)^2 + 6 \left( \frac{x}{b_n} \right)^3 - 3 \left( \frac{x}{b_n} \right)^4.
\]

**Proof.** Let \( \{\zeta_k\}_{k=1}^{\infty} \) be a sequence of independent random variables identically distributed with \( \zeta_1 \). For
\[
M_i \left( \frac{x}{b_n} \right) = \sum_{k=0}^{1} k^i \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{1-k},
\]
we find that
\[
\begin{align*}
M_0 \left( \frac{x}{b_n} \right) &= \sum_{k=0}^{1} k^0 \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{1-k} = 1 \quad \text{(here } 0^0 := 1) \\
M_1 \left( \frac{x}{b_n} \right) &= \sum_{k=0}^{1} k^1 \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{1-k} = \frac{x}{b_n}, \\
M_2 \left( \frac{x}{b_n} \right) &= \sum_{k=0}^{1} k^2 \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{1-k} = \frac{x}{b_n}, \\
M_3 \left( \frac{x}{b_n} \right) &= \sum_{k=0}^{1} k^3 \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{1-k} = \frac{x}{b_n}, \\
M_4 \left( \frac{x}{b_n} \right) &= \sum_{k=0}^{1} k^4 \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{1-k} = \frac{x}{b_n}.
\end{align*}
\]
From the definition of expectation, we get \( E(\xi_1 - a_1)^2 = M_1(x/b_n) = x/b_n \). Also

\[
E(\xi_1 - a_1)^4 = \sum_{j=0}^{4} \binom{4}{j} (-1)^j M_{4-j} \left( \frac{x}{b_n} \right) M_1^j \left( \frac{x}{b_n} \right)
\]

\[
= M_4 \left( \frac{x}{b_n} \right) - 4M_3 \left( \frac{x}{b_n} \right) M_1 \left( \frac{x}{b_n} \right) + 6M_2 \left( \frac{x}{b_n} \right) M_1^2 \left( \frac{x}{b_n} \right)
\]

\[
- 4M_1 \left( \frac{x}{b_n} \right) M_1 \left( \frac{x}{b_n} \right)^3 + M_1 \left( \frac{x}{b_n} \right)^4
\]

\[
= \left( \frac{x}{b_n} \right) - 4 \left( \frac{x}{b_n} \right)^2 + 6 \left( \frac{x}{b_n} \right)^3 - 3 \left( \frac{x}{b_n} \right)^4.
\]

Thus in view of Lemma 3, we have

\[
E(\xi_1 - a_1)^3 \leq \left( E(\xi_1 - a_1)^2 E(\xi_1 - a_1)^4 \right)^{1/2}
\]

\[
= \left( \frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right) \left[ \left( \frac{x}{b_n} \right) - 4 \left( \frac{x}{b_n} \right)^2 + 6 \left( \frac{x}{b_n} \right)^3 - 3 \left( \frac{x}{b_n} \right)^4 \right] \right)^{1/2}
\]

\[
\leq \sqrt{\frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right)} 0.288675.
\]

Such result can be found in [9]. □

Following lemma is the well-known Berry–Esseen bound for the classical central limit theorem of probability theory.

**Lemma 5 (Berry–Esseen).** Let \( \{\xi_k\}_{k=1}^{\infty} \) be a sequence of independent and identically distributed random variables with finite variance such that the expectation \( E(\xi_1) = a_1 \in \mathbb{R} \), the variance \( \text{Var}(\xi_1) = E(\xi_1 - a_1)^2 = b_1^2 > 0 \) and \( E|\xi_1 - E(\xi_1)|^3 < \infty \). Then there exists a constant \( C, 1/\sqrt{2\pi} \leq C < 0.82 \), such that for all \( n \) and \( t \)

\[
| P \left( \frac{1}{b_1 \sqrt{n}} \sum_{k=1}^{n} (\xi_k - a_1) \leq t \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du | \leq C \frac{E|\xi_1 - E(\xi_1)|^3}{b_1^3 \sqrt{n}}.
\]

Its proof can be found in Shiryayev [12, p. 432].
Lemma 6. For all $x \in [0, b_n]$, we have

$$\left| \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right) - \left( \frac{1}{2} \right) \right| \leq 0.82 \ast 0.288675 \leq \frac{1}{\sqrt{2}enx/b_n(1-x/b_n)}.$$ (8)

Proof. By direct calculation, one has from Lemmas 4 and 5 the desired result. □

Lemma 7. For all $x \in (0, b_n)$ and $x \geq 1$, we have

$$C_{n,2}(\text{sgn}(t-x); x) = 2^x \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right)^2 - 1 + e_n \left( \frac{x}{b_n} \right) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right).$$ (9)

Proof. Since,

$$C_{n,2}(\text{sgn}(t-x); x) = \sum_{k \leq n} \text{sgn} \left( \frac{kb_n}{n} - x \right) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right)$$

$$= \sum_{n \geq k > nx/b_n} (2^x - 1) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right) - \sum_{0 \leq k < nx/b_n} Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right)$$

and from (4), we can write

$$C_{n,2}(1; x) = \sum_{n \geq k > nx/b_n} Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right) + \sum_{0 \leq k < nx/b_n} Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right) + e_n \left( \frac{x}{b_n} \right) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right).$$

Thus

$$C_{n,2}(\text{sgn}(t-x); x) = \sum_{n \geq k > nx/b_n} (2^x - 1) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right)$$

$$- \left[ 1 - \sum_{n \geq k > nx/b_n} Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right) - e_n \left( \frac{x}{b_n} \right) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right) \right]$$

$$= 2^x \sum_{n \geq k > nx/b_n} Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right) - 1 + e_n \left( \frac{x}{b_n} \right) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right)$$

$$= 2^x \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right)^2 - 1 + e_n \left( \frac{x}{b_n} \right) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right).$$

This completes the proof of (9). □

Lemma 8. If the conditions of Theorem hold, we have for all $x \in (0, b_n)$ and $x \geq 1$

$$\left| \frac{f(x) - f(x-)}{2^x} C_{n,2}(\text{sgn}(t-x); x) + \left[ f(x) - \frac{1}{2^x} f(x+) - \left( 1 - \frac{1}{2^x} \right) f(x-) \right] C_{n,2}(\delta_x; x) \right|$$

$$\leq \frac{x}{\sqrt{2}enx/b_n(1-x/b_n)} \left[ |f(x+) - f(x-)| + |f(x) - f(x-)|e_n \left( \frac{x}{b_n} \right) \right].$$ (10)
Proof. One has
\[
\frac{f(x^+) - f(x^-)}{2^x} C_{n,2}(\text{sgn}(t - x); x) + \left[ f(x) - \frac{1}{2^x} f(x^+) - \left( 1 - \frac{1}{2^x} \right) f(x^-) \right] C_{n,2}(\delta_x; x)
\]
\[
\leq \frac{f(x^+) - f(x^-)}{2^x} C_{n,2}(\text{sgn}(t - x); x)
\]
\[
+ \left| f(x) - \frac{1}{2^x} f(x^+) - \left( 1 - \frac{1}{2^x} \right) f(x^-) \right| C_{n,2}(\delta_x; x)
\]
\[
\leq \frac{f(x^+) - f(x^-)}{2^x} \left[ 2^x \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right)^{\alpha} - 1 \right] - e_n \left( \frac{x}{b_n} \right) Q^{(2)}_{n,k} \left( \frac{x}{b_n} \right)
\]
\[
+ \left| f(x) - \frac{1}{2^x} f(x^+) - \left( 1 - \frac{1}{2^x} \right) f(x^-) \right| e_n \left( \frac{x}{b_n} \right) Q^{(2)}_{n,k} \left( \frac{x}{b_n} \right)
\]
\[
= \frac{f(x^+) - f(x^-)}{2^x} \left[ 2^x \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right)^{\alpha} - 1 \right]
\]
\[
+ \left| f(x) - f(x^-) \right| e_n \left( \frac{x}{b_n} \right) Q^{(2)}_{n,k} \left( \frac{x}{b_n} \right).
\]

Now we estimate
\[
2^x \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right)^{\alpha} - 1.
\]

Using the fact that \(|a^x - b^x| \leq |a - b|^{x}a - b| \) with \(0 \leq a, b \leq 1 \) and \(x \geq 1 \), we obtained
\[
2^x \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right)^{\alpha} - \left( \frac{1}{2} \right)^{\alpha} \leq 2^\alpha \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right) - \frac{1}{2},
\]
and hence from (8), we have
\[
2^x \left( \sum_{n \geq k > nx/b_n} P_{n,k} \left( \frac{x}{b_n} \right) \right)^{\alpha} - \left( \frac{1}{2} \right)^{\alpha} \leq \frac{2^\alpha}{\sqrt{2enx/b_n(1 - x/b_n)}},
\]
and replacing the variable \(x \) with \(x/b_n \) in [14, p. 365, Eq. (5)] we get,
\[
P_{n,k} \left( \frac{x}{b_n} \right) \leq \frac{1}{\sqrt{2e} \sqrt{nx/b_n(1 - x/b_n)}} \leq \frac{1}{\sqrt{2enx/b_n(1 - x/b_n)}}.
\]

Consequently from (11) we get (10). \(\square\)

3. Proof of main theorem

Proof. For any \(f \in BV[0, \infty)\), we can decompose \(f \) into four parts on \([0, b_n] \) for sufficiently large \(n\),
\[
f(t) = \frac{1}{2^x} f(x^+) + \left( 1 - \frac{1}{2^x} \right) f(x^-) + g_x(t) + \frac{f(x^+) - f(x^-)}{2^x} \text{sgn}(t - x)
\]
\[
+ \delta_x(t) \left[ f(x) - \frac{1}{2^x} f(x^+) - \left( 1 - \frac{1}{2^x} \right) f(x^-) \right],
\]
(12)
where
\[
\delta_x(t) = \begin{cases} 
1, & x = t, \\
0, & x \neq t
\end{cases}
\]
and
\[
\operatorname{sgn}(t) = \begin{cases} 
2^x - 1, & t > 0, \\
0, & t = 0, \\
-1, & t < 0.
\end{cases}
\]
If we apply the operator $C_{n,2}$ on both sides of equality (12), we have
\[
C_{n,2}(f; x) = \left[ \frac{1}{2^x} f(x+) + \left( 1 - \frac{1}{2^x} \right) f(x-) \right] C_{n,2}(1; x) + C_{n,2}(g_s; x) \\
+ \frac{f(x+) - f(x-)}{2^x} C_{n,2}(\operatorname{sgn}(t-x); x) \\
+ \left[ f(x) - \frac{1}{2^x} f(x+) - \left( 1 - \frac{1}{2^x} \right) f(x-) \right] C_{n,2}(\delta_x; x).
\]
It follows that:
\[
\left\| \left[ \frac{1}{2^x} f(x+) + \left( 1 - \frac{1}{2^x} \right) f(x-) \right] C_{n,2}(1; x) \right\| = \left| C_{n,2}(g_s; x) \right| + \left\| \frac{f(x+) - f(x-)}{2^x} C_{n,2}(\operatorname{sgn}(t-x); x) \right\| \\
+ \left\| f(x) - \frac{1}{2^x} f(x+) - \left( 1 - \frac{1}{2^x} \right) f(x-) \right\| C_{n,2}(\delta_x; x) \right\|. (13)
\]
Firstly we estimate $|C_{n,2}(g_s; x)|$ as below. Since Chlodowsky Polynomial may be written in the form of a Lebesgue–Stieltjes integral, we can rewrite $C_{n,2}(g_s; x)$ as follows:
\[
|C_{n,2}(g_s; x)| = \left| \sum_{k=0}^{n} g_s \left( \frac{kb_n}{n} \right) Q_{n,k}^{(2)} \left( \frac{x}{b_n} \right) \right| \\
= \left| \int_0^{b_n} g_s(t) \frac{\partial}{\partial t} K_{n,x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right|. (14)
\]
To estimate (14), we decompose it into three parts, as follows:
\[
= \left| \int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+(b_n-x)/\sqrt{n}} + \int_{x+(b_n-x)/\sqrt{n}}^{b_n} g_s(t) \frac{\partial}{\partial t} K_{n,x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right| \\
\leq \left| \int_0^{x-x/\sqrt{n}} g_s(t) \frac{\partial}{\partial t} K_{n,x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right| + \left| \int_{x-x/\sqrt{n}}^{x+(b_n-x)/\sqrt{n}} g_s(t) \frac{\partial}{\partial t} K_{n,x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right| \\
+ \left| \int_{x+(b_n-x)/\sqrt{n}}^{b_n} g_s(t) \frac{\partial}{\partial t} K_{n,x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right| \\
= |I_{1,2}(n, x)| + |I_{2,2}(n, x)| + |I_{3,2}(n, x)|. (15)
\]
We shall evaluate \(I_{1,2}(n, x), I_{2,2}(n, x)\) and \(I_{3,2}(n, x)\). To do this we first observe that \(I_{1,2}(n, x), I_{2,2}(n, x)\) and \(I_{3,2}(n, x)\) can be written as a Lebesque–Stieltjes integral

\[
I_{1,2}(n, x) = \int_0^{x-x/\sqrt{n}} g_x(t) d_I \left( \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right),
\]

\[
I_{2,2}(n, x) = \int_0^{x+(b_n-x)/\sqrt{n}} g_x(t) d_I \left( \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right)
\]

and

\[
I_{3,2}(n, x) = \int_{x+(b_n-x)/\sqrt{n}}^{b_n} g_x(t) d_I \left( \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right),
\]

where \(\lambda_{n, x}(x/b_n, t/b_n) = \int_t^x K_{n, x}(x/b_n, u/b_n) du\).

First we estimate \(I_{2,2}(n, x)\). For \(t \in [x-x/\sqrt{n}, x+(b_n-x)/\sqrt{n}]\), we have

\[
|I_{2,2}(n, x)| = \left| \int_{x-x/\sqrt{n}}^{x+(b_n-x)/\sqrt{n}} (g_x(t) - g_x(x)) d_I \left( \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right|
\]

\[
\leq \int_{x-x/\sqrt{n}}^{x+(b_n-x)/\sqrt{n}} |g_x(t) - g_x(x)| d_I \left( \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right)
\]

\[
\leq \sum_{k=2}^{n} \frac{1}{n-1} \sqrt{\sum_{x-x/\sqrt{n}}^{x+(b_n-x)/\sqrt{n}} (g_x)}.
\]

Next, we estimate \(I_{1,2}(n, x)\). Using partial Lebesque–Stieltjes integration, we obtain

\[
I_{1,2}(n, x) = \int_0^{x-x/\sqrt{n}} g_x(t) d_I \left( \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \right)
\]

\[
= g_x \left( x - \frac{x}{\sqrt{n}} \right) \lambda_{n, x} \left( \frac{x}{b_n}, \frac{x-x/\sqrt{n}}{b_n} \right) - g_x(0) \lambda_{n, x} \left( \frac{x}{b_n}, \frac{0}{b_n} \right)
\]

\[
- \int_0^{x-x/\sqrt{n}} \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) d_I (g_x(t)).
\]

Since \(|g_x (x-x/\sqrt{n})| = |g_x (x-x/\sqrt{n}) - g_x(x)| \leq \sqrt{\int_{x-x/\sqrt{n}}^{x-x/\sqrt{n}} (g_x)}\), it follows that

\[
|I_{1,2}(n, x)| \leq \sqrt{\int_{x-x/\sqrt{n}}^{x-x/\sqrt{n}} (g_x)} \lambda_{n, x} \left( \frac{x}{b_n}, \frac{x-x/\sqrt{n}}{b_n} \right) + \int_0^{x-x/\sqrt{n}} \lambda_{n, x} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) d_I \left( -\sqrt{\frac{1}{x} (g_x)} \right).
\]

From (7), it is clear that

\[
\lambda_{n, x} \left( \frac{x}{b_n}, \frac{x-x/\sqrt{n}}{b_n} \right) \leq \frac{B_n(x)}{(x/\sqrt{n})^\frac{\alpha}{2}}.
\]

It follows that

\[
|I_{1,2}(n, x)| \leq \sqrt{\int_{x-x/\sqrt{n}}^{x-x/\sqrt{n}} (g_x) \frac{B_n(x)}{(x/\sqrt{n})^\frac{\alpha}{2}}} + \int_0^{x-x/\sqrt{n}} \frac{\alpha}{(x-t)^2} B_n(x) d_I \left( -\sqrt{\frac{1}{x} (g_x)} \right)
\]

\[
= \sqrt{\int_{x-x/\sqrt{n}}^{x-x/\sqrt{n}} (g_x) \frac{B_n(x)}{(x/\sqrt{n})^\frac{\alpha}{2}}} + \alpha \int_0^x \frac{1}{(x-t)^2} d_I \left( -\sqrt{\frac{1}{x} (g_x)} \right).
\]
Furthermore, since

\[
\int_0^{x-x/\sqrt{n}} \frac{1}{(x-t)^2} dt \left( -\sqrt{t} (g_x) \right) = -\frac{1}{(x-t)^2} \sqrt{t} (g_x) \bigg|_0^{x-x/\sqrt{n}} + \int_0^{x-x/\sqrt{n}} 2 \frac{x}{(x-t)^3} \sqrt{t} (g_x) dt
\]

\[
= -\frac{1}{(x/\sqrt{n})^2} \sqrt{x} (g_x) + \frac{1}{x^2} \sqrt{0} (g_x)
\]

\[
+ \int_0^{x-x/\sqrt{n}} 2 \frac{x}{(x-t)^3} \sqrt{t} (g_x) dt.
\]

Putting \( t = x - x/\sqrt{u} \) in the last integral, we get

\[
\int_0^{x-x/\sqrt{n}} \frac{2}{(x-t)^3} \sqrt{t} (g_x) dt = \frac{1}{x^2} \int_0^n \sqrt{x} (g_x) du = \frac{1}{x^2} \sum_{k=1}^n \sqrt{x} (g_x).
\]

Consequently,

\[
|I_{1,2}(n, x)| \leq \sqrt{x} (g_x) \frac{x B_n(x)}{(x/\sqrt{n})^2}
\]

\[
+ \frac{x B_n(x)}{(x/\sqrt{n})^2} \left\{ -\frac{1}{(x/\sqrt{n})^2} \sqrt{x} (g_x) + \frac{1}{x^2} \sqrt{x} (g_x) + \frac{1}{x^2} \sum_{k=1}^n \sqrt{x} (g_x) \right\}
\]

\[
= \frac{x B_n(x)}{x^2} \left\{ \sqrt{x} (g_x) + \frac{1}{x^2} \sum_{k=1}^n \sqrt{x} (g_x) \right\}
\]

\[
= \frac{x B_n(x)}{x^2} \left\{ \sqrt{0} (g_x) + \sum_{k=1}^n \sqrt{x} (g_x) \right\}.
\]

(17)

Using the similar method for estimating \( |I_{3,2}(n, x)| \), we get

\[
|I_{3,2}(n, x)| \leq \frac{x B_n(x)}{(b_n - x)^2} \left\{ \sqrt{0} (g_x) + \sum_{k=1}^n \sqrt{x} (g_x) \right\}
\]

\[
\leq \frac{x B_n(x)}{(b_n - x)^2} \left\{ \sqrt{0} (g_x) + \sum_{k=1}^n \sqrt{x} (g_x) \right\}.
\]

(18)
Putting (16)–(18) in (15), we get

\[ |C_{n,2}(g_x; x)| \leq |I_{1,2}(n, x)| + |I_{2,2}(n, x)| + |I_{3,2}(n, x)| \]

\[ \leq \frac{B_n(x)}{x^2} \left\{ \frac{x}{x} \sqrt{g_x} + \sum_{k=1}^{n} \left\{ \frac{x}{x} \sqrt{g_x} \right\} \right\} \]

\[ + \frac{B_n(x)}{(b_n - x)^2} \left\{ \frac{b_n}{x} \sqrt{g_x} + \sum_{k=1}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} \right\} \]

\[ + \frac{1}{n-1} \sum_{k=2}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x). \quad (19) \]

Obviously, \(1/x^2 + 1/(b_n - x)^2 = b_n^2/x^2(b_n - x)^2\), for \(x/b_n \in [0, 1]\) and \(\sqrt{x-x/\sqrt{k}} \leq \sqrt{x+(b_n-x)/\sqrt{k}} (g_x)\). Hence, we obtain from (19)

\[ |C_{n,2}(g_x; x)| \leq \frac{B_n(x)}{x^2} \left\{ \frac{x}{x} \sqrt{g_x} + \sum_{k=1}^{n} \left\{ \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} \right\} \right\} \]

\[ + \frac{1}{n-1} \sum_{k=2}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x) \]

\[ = \frac{B_n(x)b_n}{x^2(b_n - x)^2} \left\{ \frac{b_n}{x} \sqrt{g_x} + \sum_{k=1}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x) \right\} + \frac{1}{n-1} \sum_{k=2}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x). \]

On the other hand, note that

\[ \sqrt{g_x} \leq \frac{n}{\sum_{k=1}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x)}, \quad \text{for } x/b_n \in [0, 1]. \quad (20) \]

By (20), we have

\[ |C_{n,2}(g_x; x)| \leq \frac{2B_n(x)b_n}{x^2(b_n - x)^2} \left\{ \sum_{k=1}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x) \right\} + \frac{1}{n-1} \sum_{k=2}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x). \]

Note that \(1/(n-1) \leq (B_n(x)b_n^2/x^2(b_n - x)^2), \) for \(n > 1, x/b_n \in [0, 1]\). Consequently

\[ |C_{n,2}(g_x; x)| \leq \frac{3B_n(x)b_n}{x^2(b_n - x)^2} \left\{ \sum_{k=1}^{n} \frac{x+(b_n-x)/\sqrt{k}}{x-x/\sqrt{k}} (g_x) \right\}. \quad (21) \]

Combining (10) and (21) in (13), we get the required result. Thus, the proof is completed. □
We can give an example.

**Example.** If we choose function \( f(x) = |x| \exp(-x) \) sequence \( b_n = \sqrt{n} \) control points \( n = 36 \) and \( n = 1, 4, 30 \). We get following graphics of approximating function \( C_{n,x} \) in Fig. 1, respectively.

(i) \( x = 1 \), control points \( n = 36 \), Interval \( I = [0, 6] \) Jump points: \( x = 1, 2, 3, 4, 5 \), \((1/2^k) f(x) + (1 - 1/2^k) f(x^-) = 1/2 e^{-1}, 5/2 e^{-2}, 9/2 e^{-3} \ldots \) respectively. (See blue curve.)

(ii) \( x = 4 \), control points \( n = 36 \), Interval \( I = [0, 6] \) Jump points: \( x = 1, 2, 3, 4, 5 \), \((1/2^k) f(x) + (1 - 1/2^k) f(x^-) = 1/8 e^{-1}, 9/8 e^{-2}, 17/8 e^{-3} \ldots \), respectively (See black curve.)

(iii) \( x = 30 \), control points \( n = 36 \), Interval \( I = [0, 6] \) Jump points: \( x = 1, 2, 3, 4, 5 \), \((1/2^k) f(x) + (1 - 1/2^k) f(x^-) = 1/8 e^{-1}, 9/8 e^{-2}, 17/8 e^{-3} \ldots \), respectively. (See red curve.)

Notice that we computed the expressions above by using Maple 9 in computer.

It is seen in Fig. 1 if we change \( x \) then the position of jump points change up or down. Hence we get maximum pointwise difference between \( C_{n,x} \) and control points \( n \) to be made small.

Also we can see that if control points \( n \) increase then the domain intervals increase.

<table>
<thead>
<tr>
<th>Control points ( n )</th>
<th>Interval ( = [0, b_n] = [0, \sqrt{n}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>([0, 2])</td>
</tr>
<tr>
<td>9</td>
<td>([0, 3])</td>
</tr>
<tr>
<td>16</td>
<td>([0, 4])</td>
</tr>
<tr>
<td>225</td>
<td>([0, 15])</td>
</tr>
<tr>
<td>10000</td>
<td>([0, 100])</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>
References