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Note

On the Planes of Narayana Rao and Satyanarayana

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The construction of the spread sets defining the Narayana Rao–Satyanarayana planes is generalized to odd powers of arbitrary primes p, $p \equiv 5 \pmod 6$. A second family of spread sets of a similar kind is introduced for odd powers of primes p, $p \equiv \pm 2 \pmod 5$. The translation complements corresponding to the first are determined and some properties of that corresponding to the second are indicated.

Translation planes of order 5^{2r} with r odd have been constructed in [1] by Narayana Rao and Satyanarayana. In this note we show that the construction in [1] can be generalized to produce translation planes of order p^{2r} with r odd for any prime number $p \equiv 5 \pmod 6$. Also along the same lines, we construct other spread sets defining translation planes of the same order for any prime number $p \equiv \pm 2 \pmod 5$. We also exhibit the translation complement of the planes of the first type.

1. Construction of Spread Sets

Let p be a prime number, $p \equiv 5 \pmod 6$ and let $q = p^r$ with r odd. Then -3 is a non-square in $GF(q)$ and for any $\beta \neq 0$, $\beta \in GF(q)$, there is a non-square $\gamma \in GF(q)$ such that

$$\beta^2 + 3\gamma = 0. \tag{1}$$

Let β, γ be fixed elements of $GF(q)$ subject to (1). For each $a, b \in GF(q)$ with $b \neq 0$ define $f_{a,b}$ and $g_{a,b}$ in $GF(q)$ by

$$f_{a,b} = -a^2 b + \beta a b^2 + \gamma b^3, \qquad g_{a,b} = -a + \beta b. \tag{2}$$

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Finally, for $a, b \in GF(q)$ with $b \neq 0$ define a matrix $B_{a,b}$ and define sets of matrices $\mathfrak{B}$, $\mathfrak{U}$, Σ over $GF(q)$ as follows:

$$B_{a,b} = \begin{pmatrix} a & b^{-1} \\ f_{a,b} & g_{a,b} \end{pmatrix}, \qquad \mathfrak{B} = \{B_{a,b} : a, b \in GF(q), b \neq 0\}$$
$$\mathfrak{U} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in GF(q), a \neq 0 \right\}, \qquad \Sigma = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \mathfrak{U} \cup \mathfrak{B}. \tag{3}$$

Taking $p = 5$, $\beta = 3$, $\gamma = 2$ in (1), (2), and (3) the set Σ is the spread set defined by Narayana Rao and Satyanarayana in [1]. In the general case we have

PROPOSITION 1.1. *Σ is a spread set over $GF(q)$.*

Proof. Since $\det B_{a,b} = -\gamma b^2 \neq 0$ it follows that $\mathfrak{U} \cup \mathfrak{B}$ consists of invertible matrices. Since trace $B_{a,b} = \beta b$ it follows that the characteristic polynomial of $B_{a,b}$ is $x^2 - \beta b X - \gamma b^2$. This polynomial is irreducible over $GF(q)$ because its discriminant γb^2 is a non-square by (1). Thus $B_{a,b}$ has no eigenvalues in $GF(q)$ and therefore the difference between $B_{a,b}$ and any scalar in $\mathfrak{U}$ is a non-singular matrix.

To end the proof it is enough to see that for $b \neq 0$, $d \neq 0$ and $(a, b) \neq (c, d)$ the matrix $B_{a,b} - B_{c,d}$ is also non-singular. To see this we follow [1]: The expression $\det(B_{a,b} - B_{c,d}) = 0$ is of the form $\lambda a^2 + \mu a + \nu = 0$ where λ, μ, ν depend on b, c, d only. Since $a \in GF(q)$ this equality implies that $\mu^2 - 4\lambda\nu$ is a square in $GF(q)$; but replacing λ, μ, ν in terms of b, c, d gives, using (1) that

$$\mu^2 - 4\lambda\nu = \gamma d^2(1 - bd^{-1})^4, \tag{4}$$

where if $b \neq d$ we get a contradiction as γ is a non-square, while if $b = d$ the condition $\det(B_{a,b} - B_{c,d}) = 0$ leads to $a = c$ contradicting $(a, b) \neq (c, d)$. This ends the proof as the cardinality of Σ is q^2.

A straightforward application of (1), (2), (3) together with the facts $\det B_{a,b} = -\gamma b^2$ and trace $B_{a,b} = \beta b$ yields that the spread set Σ satisfies also

$$M \in \mathfrak{U} \cup \mathfrak{B} \Rightarrow -M \in \mathfrak{U} \cup \mathfrak{B} \tag{5}$$

$$M \in \mathfrak{U} \cup \mathfrak{B} \Rightarrow \exists N \in \mathfrak{U} \cup \mathfrak{B} : M + N \notin \Sigma \tag{6}$$

$$M \in \mathfrak{U} \cup \mathfrak{B} \Rightarrow \exists N \in \mathfrak{U} \cup \mathfrak{B} : N \neq -M \quad \text{and} \quad (M^{-1} + N^{-1})^{-1} \notin \Sigma. \tag{7}$$

Assume next that p is a prime number, $p \equiv \pm 2 \pmod 5$, $q = p^r$ with r

odd. Then 5 is a non-square in GF(q) and we can proceed as before changing (1) through (4) by

$$\beta^2 - 5\gamma = 0 \tag{1'}$$

$$f_{a,b} = -a^2b + \beta ab^3 - \gamma b^5, \qquad g_{a,b} = -a + \beta b^2 \tag{2'}$$

and (3′) being identical to (3).

This time we have det $B_{a,b} = \gamma b^4$, trace $B_{a,b} = \beta b^2$, the discriminant of the characteristic polynomial of $B_{a,b}$ is the non-square γb^4. A little tedious but elementary algebra shows that in the proof of Proposition 1, with the same notation, we get

$$\mu^2 - 4\lambda v = \gamma d^{-2}(d-b)^2 \, [(d+b)^2 + bd]^2, \tag{4'}$$

where $\mu^2 - 4\lambda v$ can only be zero if $d = b$ as the second factor leads to the polynomial $X^2 + 3X + 1$, which is irreducible in $GF(q)[X]$. If $d = b$ then $a = c$ is forced again and otherwise $\mu^2 - 4\lambda v$ is non-square. We have then sketched the proof of

PROPOSITION 1.2. *If $q = p^r$ with r odd and $p \equiv \pm 2 \pmod 5$ and if*

$$\Sigma' = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in GF(q) \right\} \cup \left\{ \begin{pmatrix} a & b^{-1} \\ f_{a,b} & g_{a,b} \end{pmatrix} : a, b \in GF(q), b \neq 0 \right\},$$

where $f_{a,b}$ and $g_{a,b}$ are defined by (1′) *and* (2′), *then Σ' is a spread set over $GF(q)$.*

Observe that, since (2′) defines non-homogeneous polynomials, our spread set Σ' does not satisfy a property like (5). The very useful Lemma 3.4 in [1] cannot be used then in the calculation of the translation complement of the plane defined by Σ'.

2. TRANSLATION COMPLEMENTS

We choose our notation following [2] as follows. Let X denote the vector space of rows over $GF(q)$ of dimension 2, V be the external direct sum $X + X$ and consider the 2-dimensional subspaces of V: $V(0) = \{(x, 0): x \in X\}$, $V(\infty) = \{(0, x): x \in X\}$, $V(M) = \{(x, xM): x \in X\}$ for $M \in \mathfrak{U} \cup \mathfrak{B} \subseteq \Sigma$. Let $\pi = \{V(0), V(\infty)\} \cup \{V(M): M \in \mathfrak{U} \cup \mathfrak{B}\}$. Then [2, Theorem 2.3] π is a spread defining a translation plane with kernel $GF(q)$.

The translation complement of this plane is the set stabilizer of π in $\Gamma L(4, q)$ under its action by multiplication on the right. Denote by G_0 this translation complement.

Since the field automorphisms of $GF(q)$ leave $\mathfrak{U}$ and $\mathfrak{B}$ invariant it follows that $G_0 = G \cdot \operatorname{aut} GF(q)$, where G is the set stabilizer of π in $GL(4, q)$.

We write elements of $GL(4, q)$ in 2×2 block form, denote by I the identy 2×2 block, and let

$$Z = \left\{ \begin{pmatrix} xI & 0 \\ 0 & xI \end{pmatrix} : x \in GF(q) \right\},$$

$$H = \left\{ \begin{pmatrix} P & 0 \\ 0 & cP \end{pmatrix} : P = \begin{pmatrix} c & 0 \\ a & c^{-1} \end{pmatrix}; a, c \in GF(q), c \neq 0 \right\}.$$

Then Z and H are subgroups of $GL(4, q)$ which normalize each other and $Z \cap H = \{\text{id.}\}$. Hence $Z \times H$ is a subgroup of $GL(4, q)$ of order $(q-1)^2 q$.

The proof used by Narayana Rao and Satyanarayana in [1] can now be followed almost step by step to show that $G = Z \times H$ and the orbits of G on π are $\{V(0)\}$, $\{V(\infty)\}$, $\{V(M): M \in \mathfrak{U}\}$, $\{V(M): M \in \mathfrak{B}\}$. We have then

PROPOSITION 2.1. *The translation complement of the translation plane defined by the spread set Σ is the group $G_0 = (Z \times H) \cdot \operatorname{aut} GF(q)$, of order $(q-1)^2 qr$ and its orbits of special points are $\{V(0)\}$, $\{V(\infty)\}$, $\{V(M): M \in \mathfrak{U}\}$, $\{V(M): M \in \mathfrak{B}\}$.*

For our second spread set Σ', Pomareda [3] has shown, using very lengthy calculations, that the translation complement has the order $(q-1)^2 qr$ and it has five orbits of special points indicating that these translation planes are possibly new.

We remark finally that the spread set Σ can be defined also over an infinite field in which -3 is a non-square if all cubic equations have solutions in the field. The field $\mathbb{R}$ is an example. We do not know if the spread Σ' can be defined over infinite fields also.

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