The homotopy type of the complement of a coordinate subspace arrangement

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Abstract

The homotopy type of the complement of a complex coordinate subspace arrangement is studied by utilising some connections between its topological and combinatorial structures. A family of arrangements for which the complement is homotopy equivalent to a wedge of spheres is described. One consequence is an application in commutative algebra: certain local rings are proved to be Golod, that is, all Massey products in their homology vanish.

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1. Introduction

In this paper we study connections between the topology of the complements of certain complex arrangements, and algebraic and combinatorial objects associated to them.

Let

$$\mathcal{A} = \{L_1, \ldots, L_r\}$$

be a *complex subspace arrangement* in $\mathbb{C}^n$, that is, a finite set of complex linear subspaces in $\mathbb{C}^n$. For such

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an arrangement $\mathcal{A}$, define its *support* $|\mathcal{A}|$ as $|\mathcal{A}| = \bigcup_{i=1}^{r} L_i \subset \mathbb{C}^n$ and its *complement* $U(\mathcal{A})$ as

$$U(\mathcal{A}) = \mathbb{C}^n \setminus |\mathcal{A}|.$$

Arrangements and their complements play a pivotal role in many constructions of combinatorics, algebraic and symplectic geometry, etc.; they also arise as configuration spaces for different classical mechanical systems. Special problems connected with arrangements and their complements arise in different areas of mathematics and mathematical physics. The multidisciplinary nature of the subject results in ongoing theoretical improvements, a constant source of new applications and the penetration of new ideas and techniques in each of the component research areas. It is the interplay of methods from seemingly disparate areas that makes the theory of subspace arrangements a vivid and appealing field of research.

In the study of arrangements it is important to get a detailed description of the topology of their complements, including properties such as homology groups, cohomology rings, homotopy type, and so on. In this paper we are concerned with the homotopy type of the complement of a complex coordinate subspace arrangement. A *complex coordinate subspace* of $\mathbb{C}^n$ is given by

$$L_\sigma = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_{i_1} = \cdots = z_{i_k} = 0\}$$

where $\sigma = \{i_1, \ldots, i_k\}$ is a subset of $[n] = \{1, \ldots, n\}$, allowing us to define a *complex coordinate subspace arrangement* $\mathcal{CA}$ in $\mathbb{C}^n$ as a family of coordinate subspaces L_σ for $\sigma \subset [n]$. The main topological space we study, naturally associated to the complex coordinate subspace arrangement $\mathcal{CA}$, is the complement $U(\mathcal{CA})$ in $\mathbb{C}^n$. Our results are obtained by studying the topological and combinatorial structures of $U(\mathcal{CA})$ with the help of commutative and homological algebra, combinatorics and homotopy theory.

It has been known for some time that hyperplane arrangements have a torsion-free cohomology ring. Recently it was proved [13] that after suspending the complement of a hyperplane arrangement it becomes homotopy equivalent to a wedge of spheres. The case of complex coordinate subspace arrangements is much more complicated. Already at the cohomology level, there is a more intricate structure. The Buchstaber–Panov formula for $H^*(U(\mathcal{CA}))$ [1] detects torsion in special cases, implying that even stably $U(\mathcal{CA})$ cannot always be homotopy equivalent to a wedge of spheres. Even when $H^*(\mathcal{Z}_K)$ is torsion-free, $\mathcal{Z}_K$ may not decompose as a wedge of spheres, due to the presence of nontrivial cup products or Massey products in $H^*(\mathcal{Z}_K)$ [2,4,11]. That makes the question of when the complement of a coordinate subspace arrangement is homotopy equivalent to a wedge of spheres more difficult and therefore more interesting. The main goal of this paper is to describe a family of coordinate subspace arrangements for which the complement is homotopy equivalent to a wedge of spheres.

The basic connections between the topology, combinatorics and commutative algebra of coordinate subspace arrangements are as follows.

Let K be a simplicial complex on the vertex set $[n]$. We shall consider only complexes that are finite, abstract simplicial complexes represented by their collection of faces. Every simplicial complex K on the vertex set $[n]$ defines a complex arrangement of coordinate subspaces in $\mathbb{C}^n$ via the correspondence

$$K \ni \sigma \mapsto \mathrm{span}\{e_i : i \notin \sigma\}$$

where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{C}^n$. Equivalently, for each simplicial complex K on the set $[n]$, we associate the complex coordinate subspace arrangement

$$\mathcal{CA}(K) = \{L_\sigma \mid \sigma \notin K\}$$

and its complement

$$U(K) = \mathbb{C}^n \setminus \bigcup_{\sigma \notin K} L_\sigma. \tag{1}$$

On the other hand, to K and a commutative ring R with unit there is an associated algebraic object, the *Stanley–Reisner ring* $R[K]$, also known in the literature as the *face ring* of K. Denote by $R[v_1, \ldots, v_n]$ the graded polynomial algebra on n variables, where $\deg(v_i) = 2$ for each i over R. The Stanley–Reisner ring of a simplicial complex K on the vertex set $[n]$ is the quotient ring

$$R[K] = R[v_1, \ldots, v_n]/\mathcal{I}_K$$

where $\mathcal{I}_K$ is the homogeneous ideal generated by all square free monomials $v^\sigma = v_{i_1} \cdots v_{i_s}$ such that $\sigma = \{v_{i_1}, \ldots v_{i_s}\} \notin K$.

Coming back to topology and following the Buchstaber–Panov approach [1] to toric topology, there are two other topological spaces associated to a simplicial complex K and its Stanley–Reisner ring $R[K]$. The first space arises as a topological realisation of the Stanley–Reisner ring. It is the *Davis–Januszkiewicz space* $DJ(K)$, whose cohomology ring is isomorphic to the Stanley–Reisner ring $R[K]$. The Davis–Januszkiewicz space maps by an inclusion into the classifying space of the n-dimensional torus. The homotopy fibre of this inclusion can be identified with another torus space, the *moment-angle complex* $\mathcal{Z}_K$, which has as a deformation retract the complement $U(K)$ of the complex coordinate subspace arrangement [1, 8.0]. Different models of $DJ(K)$ and $\mathcal{Z}_K$ as well as their additional properties will be addressed later on in Section 2. These identifications show that the problem of determining the homotopy type of the complement of complex coordinate subspace arrangements is equivalent to determining the homotopy type of the moment-angle complex $\mathcal{Z}_K$. To do this we need to closely examine the homotopy fibration sequence

$$\mathcal{Z}_K \longrightarrow DJ(K) \xrightarrow{\text{incl}} BT^n.$$

The main technique employed for understanding this filtration is Mather's Cube Lemma [9], which relates homotopy pullbacks and homotopy pushouts in a cubical diagram. This is applied iteratively as K is built up one face at a time, in a prescribed order. An analysis of the component homotopy fibration and cofibration sequences produces our main result, Theorem 1.2 (see below).

To find a suitable simplicial complex K whose $U(K)$ will be homotopy equivalent to a wedge of spheres, we first look at its cohomology ring. As $U(K)$ is homotopy equivalent to $\mathcal{Z}_K$, this is the same as looking at the cohomology ring of $\mathcal{Z}_K$. The integral cohomology of $\mathcal{Z}_K$ has been calculated in [1, 7.6 and 7.7]. If $\mathcal{Z}_K$ is to be homotopy equivalent to a wedge of spheres then we need to consider simplicial complexes K for which all Massey products in $H^*(\mathcal{Z}_K)$ vanish. Note that by all Massey products we mean all cup-products and all higher Massey products. The vanishing of these Massey products will not imply that $\mathcal{Z}_K$ is itself homotopy equivalent to a wedge of spheres but at least on the cohomological level there will be no obstructions to such a claim. Combinatorists, from their point of view, have studied simplicial complexes and associated to them certain Tor algebras that correspond to the cohomology of $\mathcal{Z}_K$ as in our case. They have determined several classes of complexes for which it can be shown that all Massey products in the associated Tor algebras vanish. One such class is that consisting of shifted complexes.

Definition 1.1. A simplicial complex K is *shifted* if there is an ordering on its set of vertices such that whenever $\sigma \in K$ and $v' < v$, then $(\sigma - v) \cup v' \in K$.

The most elementary examples of shifted complexes are sets of vertices and any full ith skeleton $\Delta^i(n)$ of the standard simplicial complex $\Delta(n)$ on n vertices (also denoted by Δ^{n-1}). More complicated examples exist and will be illustrated in Section 9. Gasharov, Peeva and Welker [6] showed that when K is a shifted complex, then all Massey products in $H^*(\mathcal{Z}_K)$ are trivial. In this case we obtain a much stronger result by determining the homotopy type of $\mathcal{Z}_K$.

Theorem 1.2. *Let K be a shifted complex. Then $U(K)$, and therefore $\mathcal{Z}_K$, is homotopy equivalent to a wedge of spheres.*

Previously, the only known cases of simplicial complexes K for which the complement $U(K)$ has the homotopy type of a wedge of spheres occurred when K was a disjoint union of n vertices. When $n = 2$ or $n = 3$, these are classical results of low-dimensional topology, while the general case was proved by the authors [7]. The result in Theorem 1.2 is much more general.

Our next theorem describes the influence that combinatorial operations on simplicial complexes have with respect to the homotopy type of the moment-angle complex.

Theorem 1.3. *Let K_1 and K_2 be simplicial complexes such that $\mathcal{Z}_{K_1}$ and $\mathcal{Z}_{K_2}$ are homotopy equivalent to wedges of spheres. Then the following hold:*

(1) *if $K = K_1 \bigcup_\sigma K_2$ is obtained by gluing along a common face, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres;*
(2) *if $K = K_1 \coprod K_2$ is the disjoint union of simplicial complexes, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres;*
(3) *if $K = K_1 * K_2$ is the join of simplicial complexes, then $\mathcal{Z}_K$ is not homotopy equivalent to a wedge of spheres but $\Sigma \mathcal{Z}_K$ is.*

Note that if K_1 and K_2 are shifted complexes then their disjoint union is not a shifted complex. Also, if two shifted complexes are glued together along a common face, the resulting complex is not necessarily shifted. Therefore Theorem 1.3 extends Theorem 1.2 to a larger family of complexes for which $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres.

The information we have obtained on complex subspace arrangements has an application in commutative algebra. Let R be a local ring. One of the fundamental aims of commutative algebra is to describe the homology ring of R, that is $\mathrm{Tor}_R(k, k)$, where k is a ground field. The first step in understanding $\mathrm{Tor}_R(k, k)$ is to obtain information about its Poincaré series $P(R)$; more specifically, whether $P(R)$ is a rational function. A certain class of rings behaves well in this regard.

Definition 1.4. A local ring R is *Golod* if all Massey products in $\mathrm{Tor}_{k[v_1,...,v_n]}(R, k)$ vanish.

As an example, if K is a shifted complex then its associated Stanley–Reisner ring (or face ring) $k[K]$ is Golod. Golod [8] proved that if a local ring is Golod, then its Poincaré series is a rational function and it is determined by $P(\mathrm{Tor}_{k[v_1,...,v_n]}(R, k))$. Although being Golod is an important property, not many Golod rings are known. Using our results on the homotopy type of the complement of a coordinate subspace arrangement, we are able to use homotopy theory to gain some insight into these difficult homological–algebraic questions. The main results are as follows.

Theorem 1.5. *For a simplicial complex K,*

$$P(k[K]) \leq \frac{t(1+t)^n}{t - P(H^*(U(K); k))},$$

where " $\leq$ " stands for coefficient-wise inequality of power series. Equality is obtained when $k[K]$ is Golod.

Theorem 1.6. *If $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres then $k[K]$ is a Golod ring.*

Combining Theorems 1.5 and 1.6, we obtain the following result.

Corollary 1.7. *If K is a simplicial complex with the property that $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, then*

$$P(k[K]) = \frac{t(1+t)^n}{t - P(H^*(U(K);k))}.$$

To close, let us remark that all the techniques used in this paper can be also applied to real and quaternionic coordinate subspace arrangements by changing the ground ring from complex numbers to real or quaternionic numbers. In those cases Theorem 1.2 describes the homotopy type of the complement of real or quaternionic coordinate subspace arrangements. For real arrangements instead of torus spaces and $\mathbb{C}P^\infty$, we look at spaces with an action of $\mathbb{Z}/2$ (also considered as S^0) and $\mathbb{R}P^\infty$, respectively, while in the case of quaternionic arrangements we deal with S^3 spaces and $\mathbb{H}P^\infty$.

The disposition of the paper is as follows. Section 2 catalogues the main objects of study and states various properties they satisfy. Section 3 through 9 build up to and deal with the primary focus of the paper, Theorem 1.2. Section 3 through 6 establish the preliminary homotopy theory. Included are identifications of the homotopy types of various pushouts, a review of homotopy actions, the general statement of Mather's Cube Lemma and a finer analysis of a special case involving homotopy actions, and several properties of the fat wedge. Section 7 considers a particular pattern of successive inclusions of one coordinate subspace into another which we term a regular sequence. Such a sequence need not always exist, but when it does we show there is a measure of control over the homotopy types of the successive homotopy fibres obtained from including the coordinate subspaces into the full coordinate space $X_1 \times \cdots \times X_n$. Section 8 gives conditions guaranteeing the existence of regular sequences, which are based on the properties of a shifted complex. Section 9 puts together all the material in Section 3 through 8 to prove Theorem 1.2. At this point, the class of simplicial complexes for which $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres includes the shifted complexes. Section 10 shows that there are other simplicial complexes K which have $\mathcal{Z}_K$ homotopy equivalent (or stably homotopy equivalent) to a wedge of spheres by proving Theorem 1.3. Finally, Section 11 turns to commutative algebra considering Golod rings and their properties, and proves Theorems 1.5 and 1.6.

2. The main objects: Their definitions and properties

As mentioned in the introduction, the main objective of this paper is the study of arrangements and their complements from a topological point of view. To pass from the combinatorial concept of arrangements to a topological one, we use different topological models associated to simplicial complexes K and their algebraic counterparts, the Stanley–Reisner ring $\mathbb{Z}[K]$ (or the face ring) of K.

The purpose of this section is to present the main objects which we are going to use and to set the notation. We rely heavily on constructions in toric topology introduced and studied by Buchstaber and Panov [1].

2.1. The Davis–Januszkiewicz space

A topological realisation of the Stanley–Reisner ring $\mathbb{Z}$ was given by Davis and Januszkiewicz [3]. Their model was a Borel-type construction. For our purposes we use another model, denoted $DJ(K)$, given by Buchstaber and Panov [1]. In what follows, we identify the classifying space of the circle S^1 with the infinite-dimensional projective space $\mathbb{C}P^\infty$, and therefore the classifying space BT^n of the n-torus with the n-fold product of $\mathbb{C}P^\infty$. For an arbitrary subset $\sigma \subset [n]$, define the σ-power of BT as

$$BT^\sigma = \{(x_1, \ldots, x_n) \in BT^n \mid x_i = * \text{ if } i \notin \sigma\}.$$

Definition 2.1. Let K be a simplicial complex on the index set $[n]$. The *Davis–Januszkiewicz* space is given as the cellular subcomplex

$$DJ(K) = \bigcup_{\sigma \in K} BT^\sigma \subset BT^n.$$

Buchstaber and Panov proved that there is a deformation retraction from Davis and Januszkiewicz's original model to $DJ(K)$.

It is an open question as to whether the homotopy type of $DJ(K)$ is determined by $H^*(DJ(K); \mathbb{Z})$. Notbohm and Ray [10] showed that this is true rationally, that is, the rational homotopy type of $DJ(K)$ is determined by $H^*(DJ(K); \mathbb{Q})$.

2.2. The moment-angle complex

Realise the torus T^n as a subspace of $\mathbb{C}^n$

$$T^n = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| = 1, \text{ for } i = 1, \ldots, n\right\}$$

contained in the unit polydisc

$$(D^2)^n = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leq 1, \text{ for } i = 1, \ldots, n\right\}.$$

For an arbitrary subset $\sigma \subset [n]$, define

$$B_\sigma = \left\{(z_1, \ldots, z_n) \in (D^2)^n \mid |z_i| = 1 \quad i \notin \sigma\right\}.$$

Definition 2.2. Let K be a simplicial complex on the index set $[n]$. Define the *moment-angle complex* $\mathcal{Z}_K$ by

$$\mathcal{Z}_K = \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^n.$$

Observe that since each B_σ is invariant under the action of T^n, the moment-angle complex $\mathcal{Z}_k$ is a T^n-space. Buchstaber and Panov showed that the moment-angle complex is another topological model of the Stanley–Reisner ring $\mathbb{Z}[K]$ by proving that the T^n-equivariant cohomology $H^*_{T^n}(\mathcal{Z}_K)$ is isomorphic to $\mathbb{Z}[K]$.

The following description of the moment-angle complex $\mathcal{Z}_K$ together with its relation to the complement of an arrangement plays the pivotal role in our approach to determine the homotopy type of the complement of a complex coordinate subspace arrangement.

Proposition 2.3 (*Buchstaber–Panov [1]*). *The moment-angle complex $\mathcal{Z}_K$ is the homotopy fibre of the embedding*

$$i\colon DJ(K) \longrightarrow BT^n.$$

Recall from (1) that $U(K)$ denotes the complement of the complex coordinate subspace arrangement associated to a simplicial complex K.

Theorem 2.4 (*Buchstaber–Panov [1]*). *There is an equivariant deformation retraction*

$$U(K) \longrightarrow \mathcal{Z}_K.$$

Theorem 2.4 ensures that the homotopy type of the complement $U(K)$ of a complex coordinate subspace arrangement can be obtained by finding the homotopy type of the moment-angle complex Z_K.

Buchstaber and Panov [1] described the cohomology algebra of Z_K by proving that there is an isomorphism

$$H^*(\mathcal{Z}_K; k) \cong \mathrm{Tor}_{k[v_1,\ldots,v_n]}(k[K], k)$$

as graded algebras.

3. Preliminary homotopy decompositions

The purpose of this section is to identify the homotopy type of several pushouts. Throughout this section and the remainder of the paper, we work in the category of based, connected topological spaces and continuous maps. We begin by stating Mather's Cube Lemma [9], which relates homotopy pullbacks and homotopy pushouts in a cubical diagram.

Lemma 3.1. *Suppose there is a homotopy commutative diagram*
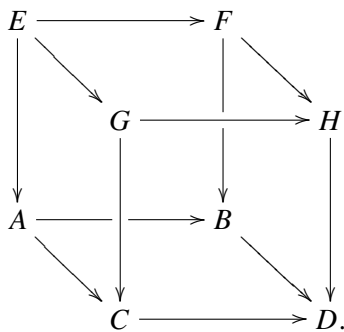
Suppose the bottom face A–B–C–D is a homotopy pushout and the sides E–G–A–C and E–F–A–B are homotopy pullbacks.

(a) *If the top face E–F–G–H is also a homotopy pushout then the sides G–H–C–D and F–H–B–D are homotopy pullbacks.*
(b) *If the sides G–H–C–D and F–H–B–D are also homotopy pullbacks then the top face E–F–G–H is a homotopy pushout.* $\square$

We next set some notation. Let $*$ denote the basepoint. For spaces X and Y, let $X \rtimes Y = (X \times Y)/(* \times Y)$, $X \wedge Y = (X \rtimes Y)/(X \times *)$, and $X * Y = \Sigma X \wedge Y$. The latter space is called the *join* of X and Y. To illustrate in the case of spheres, for which these operations will appear frequently in what follows, it is well known that $S^m \wedge S^n \simeq S^{m+n}$ and $S^m * S^n \simeq S^{m+n+1}$, and using the fact that S^m is a co-H space, we have $S^m \rtimes S^n \simeq S^m \vee (S^m \wedge S^n) \simeq S^m \vee S^{m+n}$. Fix spaces X_1 and X_2 and let $1 \leq j \leq 2$. Let $\pi_j : X_1 \times X_2 \longrightarrow X_j$ be the projection onto the jth factor and let $i_j : X_j \longrightarrow X_1 \times X_2$ be the inclusion into the jth factor. Let $q_j : X_1 \vee X_2 \longrightarrow X_j$ be the pinch map onto the jth wedge summand. Unless otherwise specified, we adopt the Milnor–Moore notation of denoting the identity map on a space X by X. Denote the map which sends all points to the basepoint by $*$.

Now we turn to the homotopy types of certain pushouts. The statements are organised in two pairs, Lemmas 3.2 and 3.3, and Lemmas 3.4 and 3.5. For each pair, the first statement is a special case of the second, and is used to help prove the more general statement.

Lemma 3.2. *Let A, B and C be spaces. Define Q as the homotopy pushout*

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ *\times B\ } & C \times B \\
\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\
A & \longrightarrow & Q.
\end{array}
$$

Then $Q \simeq (A * B) \vee (C \rtimes B)$. $\square$

Proof. See [7]. $\square$

Lemma 3.3. *Let A, B, C and D be spaces. Define Q as the homotopy pushout*

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ *\times B\ } & C \times B \\
\ \downarrow{\scriptstyle A\times *} & & \downarrow \\
A \times D & \longrightarrow & Q.
\end{array}
$$

Then $Q \simeq (A * B) \vee (C \rtimes B) \vee (A \ltimes D)$.

Proof. Let Q_1 be the homotopy pushout of the maps $A \times D \longrightarrow Q$ and $A \times D \xrightarrow{\pi_1} A$. Then there is a diagram of iterated homotopy pushouts

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ *\times B\ } & C \times B \\
\ \downarrow{\scriptstyle A\times *} & & \downarrow \\
A \times D & \longrightarrow & Q \\
\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\
A & \longrightarrow & Q_1.
\end{array}
$$

Observe that the outer rectangle is also a homotopy pushout, so by Lemma 3.2 we have $Q_1 \simeq (A * B) \vee (C \rtimes B)$. Further, the outer rectangle shows that the map $A \longrightarrow Q_1$ is null homotopic. Since $A \times B \xrightarrow{A \times *} A \times D$ is homotopic to the composite $A \times B \xrightarrow{\pi_1} A \xrightarrow{i_1} A \times D$, there is an iterated

homotopy pushout diagram

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\;*\times B\;} & C \times B \\
\downarrow{\scriptstyle \pi_1} & & \downarrow \\
A & \longrightarrow & Q_1 \\
\downarrow{\scriptstyle i_1} & & \downarrow \\
A \times D & \longrightarrow & Q.
\end{array}
$$

Since $A \longrightarrow Q_1$ is null homotopic, we can pinch out A in the lower pushout to obtain a homotopy pushout

$$
\begin{array}{ccc}
* & \longrightarrow & Q_1 \\
\downarrow & & \downarrow \\
A \ltimes D & \longrightarrow & Q.
\end{array}
$$

Hence $Q \simeq Q_1 \vee (A \ltimes D) \simeq (A * B) \vee (C \rtimes D) \vee (A \ltimes D)$. $\qquad\square$

Lemma 3.4. *Let A, B and C be spaces. Define Q as the homotopy pushout*

$$
\begin{array}{ccc}
A \times (B \vee C) & \xrightarrow{\;\pi_2\;} & B \vee C \\
\downarrow{\scriptstyle A \times q_2} & & \downarrow \\
A \times C & \longrightarrow & Q.
\end{array}
$$

Then $Q \simeq (A * B) \vee C$. Further, the composite $B \vee C \longrightarrow Q \xrightarrow{\;\simeq\;} (A * B) \vee C$ is homotopic to $* \vee C$.

Proof. First consider the homotopy pushout

$$
\begin{array}{ccc}
B & \longrightarrow & B \vee C \\
\downarrow & & \downarrow{\scriptstyle q_2} \\
* & \longrightarrow & C.
\end{array}
$$

In general, if M is the homotopy pushout of maps $X \xrightarrow{\;f\;} Y$ and $X \xrightarrow{\;g\;} Z$ then an easy application of the Cube Lemma (Lemma 3.1) shows that $N \times M$ is the homotopy pushout of $N \times X \xrightarrow{\;N \times f\;} N \times Y$ and $N \times X \xrightarrow{\;N \times g\;} N \times Z$. In our case, taking the product with A gives a homotopy pushout

$$
\begin{array}{ccc}
A \times B & \longrightarrow & A \times (B \vee C) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle A \times q_2} \\
A & \xrightarrow{\;i_1\;} & A \times C.
\end{array}
$$

Now consider the diagram of iterated homotopy pushouts

$$
\begin{array}{ccccccc}
A \times B & \longrightarrow & A \times (B \vee C) & \xrightarrow{\pi_2} & B \vee C & \xrightarrow{q_1} & B \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle A \times q_2} & & \downarrow & & \downarrow \\
A & \xrightarrow{i_1} & A \times C & \longrightarrow & Q & \longrightarrow & Q'
\end{array}
$$

where the right pushout defines Q'. Because the squares are all homotopy pushouts so is the outermost rectangle. Thus, as the top row is homotopic to the projection π_2, we see that $Q' \simeq A * B$. The right pushout then implies there is a homotopy cofibration $C \longrightarrow Q \longrightarrow Q' \simeq A * B$.

On the other hand, the composite $A \times B \longrightarrow A \times (B \vee C) \xrightarrow{\pi_2} B \vee C$ is homotopic to the composite $A \times B \xrightarrow{\pi_2} B \xrightarrow{j_1} B \vee C$, where j_1 is the inclusion. Thus there is an iterated homotopy pushout diagram

$$
\begin{array}{ccccc}
A \times B & \xrightarrow{\pi_2} & B & \xrightarrow{j_1} & B \vee C \\
\downarrow{\scriptstyle \pi_1} & & \downarrow & & \downarrow \\
A & \longrightarrow & A * B & \longrightarrow & Q.
\end{array}
$$

As $q_1 \circ j_1$ is homotopic to the identity map on B, the composite $A * B \longrightarrow Q \longrightarrow Q' \simeq A * B$ is homotopic to the identity map. Hence the homotopy cofibration $C \longrightarrow Q \longrightarrow A * B$ splits as $Q \simeq (A * B) \vee C$.

Further, this decomposition of Q implies that the restriction of $B \vee C \longrightarrow Q$ corresponds to the inclusion $C \longrightarrow (A * B) \vee C$. The right square in the previous diagram shows that the restriction of the map $B \vee C \longrightarrow Q$ to B is null homotopic as this restriction factors through the map $B \longrightarrow A * B$ which is null homotopic. Thus the composite $B \vee C \longrightarrow Q \simeq (A * B) \vee C$ is homotopic to $* \vee C$. $\square$

Lemma 3.5. *Let A, B, C and D be spaces. Define Q as the homotopy pushout*

$$
\begin{array}{ccc}
A \times (B \vee C) & \xrightarrow{\pi_2} & B \vee C \\
\downarrow{\scriptstyle A \times (* \vee C)} & & \downarrow \\
A \times (D \vee C) & \longrightarrow & Q.
\end{array}
$$

Then $Q \simeq (A * B) \vee (A \ltimes D) \vee C$. Further, letting $M = (A * B) \vee (A \ltimes D)$, the composite $B \vee C \longrightarrow Q \xrightarrow{\simeq} M \vee C$ is homotopic to $* \vee C$.

Proof. Observe that the map $* \vee C$ is homotopic to the composite $B \vee C \xrightarrow{q} C \xrightarrow{i} D \vee C$, where q is the pinch map and i is the inclusion. Then there is a diagram of iterated homotopy pushouts

$$
\begin{array}{ccccc}
A \times (B \vee C) & \xrightarrow{A \times q} & A \times C & \xrightarrow{A \times i} & A \times (D \vee C) \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
B \vee C & \longrightarrow & Q' & \longrightarrow & Q
\end{array} \tag{2}
$$

which defines the space Q' and the maps f and g. By Lemma 3.4, $Q' \simeq (A * B) \vee C$. We will show that there is a homotopy cofibration $Q' \longrightarrow Q \longrightarrow A \ltimes D$ for which the second map has a right homotopy

inverse. If so then $Q \simeq Q' \vee (A \ltimes D) \simeq (A * B) \vee (A \ltimes D) \vee C$, and the additional statement identifying the composite $B \vee C \longrightarrow Q \xrightarrow{\simeq} M \vee C$ as $* \vee C$ follows from Lemma 3.4, proving the lemma.

Consider the homotopy cofibration $C \xrightarrow{i} D \vee C \longrightarrow D$. Regard D as the homotopy pushout of $C \xrightarrow{i} D \vee C$ and $C \longrightarrow *$. Then taking the product with A gives a homotopy pushout

$$
\begin{array}{ccc}
A \times C & \xrightarrow{A \times i} & A \times (D \vee C) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow \\
A & \xrightarrow{i_1} & A \times D
\end{array}
$$

where i_1 is the inclusion. The homotopy cofibre of i_1, and therefore of $A \times i$, is $A \ltimes D$. Thus the right homotopy pushout in (2) shows that there is a homotopy cofibration $Q' \longrightarrow Q \longrightarrow A \ltimes D$.

Next, the projection in the left square of (2) implies that the restrictions of f and g to A are null homotopic. So A can be pinched out to give a homotopy pushout diagram

$$
\begin{array}{ccccc}
A \ltimes C & \xrightarrow{A \ltimes i} & A \ltimes (D \vee C) & \longrightarrow & A \ltimes D \\
\downarrow & & \downarrow & & \| \\
Q' & \longrightarrow & Q & \longrightarrow & A \ltimes D.
\end{array}
$$

The inclusion $A \times D \longrightarrow A \times (D \vee C)$ induces an inclusion $A \ltimes D \longrightarrow A \ltimes (D \vee C)$ which is a right homotopy inverse of $A \ltimes (D \vee C) \longrightarrow A \ltimes D$. Thus the composite $A \ltimes D \longrightarrow A \ltimes (D \vee C) \longrightarrow Q$ is a right homotopy inverse of $Q \longrightarrow A \ltimes (D \vee C)$. This completes the proof. $\square$

Lemma 3.6. *Suppose there is a homotopy pushout*

$$
\begin{array}{ccc}
A \times B & \xrightarrow{f} & D \\
\downarrow{\scriptstyle * \times B} & & \downarrow \\
C \times B & \xrightarrow{g} & E
\end{array}
$$

where the restriction of f to B is null homotopic. Then g factors through a map $g' : C \rtimes B \longrightarrow E$ and g' has a left homotopy inverse.

Proof. As the restriction of f to B is null homotopic, the homotopy commutativity of the diagram in the statement of the lemma implies that the restriction of g to B is also null homotopic. Pinching B out on the left side results in a homotopy pushout

$$
\begin{array}{ccc}
A \rtimes B & \xrightarrow{f'} & D \\
\downarrow{\scriptstyle * \rtimes B} & & \downarrow \\
C \rtimes B & \xrightarrow{g'} & E \\
\downarrow & & \downarrow \\
Y & = & Y
\end{array}
$$

for maps f' and g'. Since $* \rtimes B$ is null homotopic, we have $Y \simeq (C \rtimes B) \vee \Sigma(A \rtimes B)$, implying that g' has a left homotopy inverse.   □

4. A review of homotopy actions

This section is a brief reminder of some properties of homotopy actions. Suppose there is a homotopy fibration

$$F \longrightarrow E \longrightarrow B.$$

Let $\partial : \Omega B \longrightarrow F$ be the connecting map in the homotopy fibration sequence. Then there is a canonical homotopy action $\theta : F \times \Omega B \longrightarrow F$ such that:

(a) θ restricted to F is homotopic to the identity map,
(b) θ restricted to ΩB is homotopic to ∂, and
(c) there is a homotopy commutative diagram

$$
\begin{array}{ccc}
\Omega B \times \Omega B & \xrightarrow{\ \mu\ } & \Omega B \\
{\scriptstyle \partial \times \Omega B}\downarrow & & \downarrow{\scriptstyle \partial} \\
F \times \Omega B & \xrightarrow{\ \theta\ } & F.
\end{array}
$$

A special case is given by the path-loop fibration $\Omega B \longrightarrow \mathcal{P}B \longrightarrow B$. Here, the homotopy action $\theta : \Omega B \times \Omega B \longrightarrow \Omega B$ is homotopic to the loop multiplication.

Next, the homotopy action is natural for maps of homotopy fibration sequences. If there is a homotopy fibration diagram

$$
\begin{array}{ccc}
F & \longrightarrow E & \longrightarrow B \\
{\scriptstyle f}\downarrow & {\scriptstyle g}\downarrow & \downarrow{\scriptstyle h} \\
F' & \longrightarrow E' & \longrightarrow B'
\end{array}
$$

then there is a homotopy commutative diagram of actions

$$
\begin{array}{ccc}
F \times \Omega B & \xrightarrow{\ \theta\ } & F \\
{\scriptstyle f \times \Omega h}\downarrow & & \downarrow{\scriptstyle f} \\
F' \times \Omega B' & \xrightarrow{\ \theta'\ } & F'.
\end{array}
$$

One example of this that we will make use of is the following.

Lemma 4.1. *Suppose* $F \longrightarrow E \xrightarrow{\ f\ } B$ *is a homotopy fibration with homotopy action* $\theta : F \times \Omega B \longrightarrow F$. *Then the homotopy fibration* $F \longrightarrow E \times X \xrightarrow{f \times X} B \times X$ *has a homotopy action* $\theta' : F \times (\Omega B \times \Omega X) \longrightarrow F$ *which factors as*

$$
\begin{array}{ccc}
F \times (\Omega B \times \Omega X) & \xrightarrow{\ \theta'\ } & F \\
{\scriptstyle F \times \pi_1}\downarrow & & \| \\
F \times \Omega B & \xrightarrow{\ \theta\ } & F
\end{array}
$$

where π_1 is the projection.

Proof. Projecting, we obtain a homotopy pullback

$$
\begin{array}{ccc}
F \longrightarrow E \times X & \xrightarrow{f \times X} & B \times X \\
\| \qquad\quad \downarrow{\pi_1} & & \downarrow{\pi_1} \\
F \longrightarrow E & \xrightarrow{\ f\ } & B.
\end{array}
$$

The asserted homotopy commutative diagram now follows from the naturality of the homotopy action. $\square$

5. A special case of the Cube Lemma

This section describes a particular case of the Cube Lemma which involves a homotopy action in the homotopy pushout of fibres. Suppose there is a homotopy pushout

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D.
\end{array}
$$

Suppose there is a space Z and a map $D \longrightarrow Z$. Map each of A, B, C and D into Z and take homotopy fibres; name these E, F, G and H respectively. Then there is a homotopy commutative cube

in which the bottom face is a homotopy pushout and all four sides are homotopy pullbacks. Lemma 3.1 implies that the top face is also a homotopy pushout. In practice, we will have $Z = C \times Y$ for some space Y, together with two additional conditions, described in the following proposition.

Proposition 5.1. *Suppose there is a decomposition $Z = C \times Y$ such that:*

 (i) *the composite $C \longrightarrow D \longrightarrow C \times Y$ is homotopic to the inclusion of the first factor;*
(ii) *the composite $B \longrightarrow D \longrightarrow C \times Y$ has a right homotopy inverse when looped.*

Let M be the homotopy fibre of the map $A \longrightarrow C$. Then:

(a) *$E \simeq M \times \Omega Y$ and $G \simeq \Omega Y$;*
(b) *the homotopy pushout of fibres becomes*

$$
\begin{array}{ccc}
M \times \Omega Y & \xrightarrow{\ g\ } & F \\
\downarrow{\scriptstyle \pi} & & \downarrow \\
\Omega Y & \longrightarrow & H
\end{array}
$$

where π is the projection and the restriction of g to ΩY is null homotopic;

(c) *the map g is homotopic to the composite*

$$
M \times \Omega Y \xrightarrow{\ g|_M \times \Omega Y\ } F \times \Omega Y \xrightarrow{\ F \times i\ } F \times (\Omega C \times \Omega Y) \xrightarrow{\ \theta\ } F
$$

where $g|_M$ is the restriction of g to M, i is the inclusion into the second factor, and θ is the homotopy action of $\Omega C \times \Omega Y$ on F.

Proof. First consider the effect of condition (i) on the cube, in particular, on the face $E - G - A - C$. Let $\mathcal{P}Y$ be the path space of Y. The inclusion $C \longrightarrow C \times Y$ can be replaced up to homotopy equivalence by the product $C \times \mathcal{P}Y \longrightarrow C \times Y$. The map $A \longrightarrow C$ is then replaced by the product map $A \times * \longrightarrow C \times \mathcal{P}Y$. Composing into $C \times Y$ then gives a homotopy pullback

$$
\begin{array}{ccccc}
N & \longrightarrow & A \times * & \longrightarrow & C \times Y \\
\downarrow & & \downarrow & & \| \\
* \times \Omega Y & \longrightarrow & C \times \mathcal{P}Y & \longrightarrow & C \times Y
\end{array}
$$

which defines the space N. Since the maps defining the homotopy pullback are all product maps, N is homotopy equivalent to the product $N_1 \times N_2$, where N_1 is the homotopy pullback of the maps $A \longrightarrow C$ and $* \longrightarrow C$, and N_2 is the homotopy pullback of the maps $* \longrightarrow \mathcal{P}Y$ and $\Omega Y \longrightarrow \mathcal{P}Y$. That is, $N_1 \simeq M$ and $N_2 \simeq \Omega Y$. Further, the map $N \longrightarrow \Omega Y$ is homotopic to the projection $M \times \Omega Y \longrightarrow \Omega Y$. This proves part (a), that $E \simeq M \times \Omega Y$ and $G \simeq \Omega Y$, and also shows in part (b) that the map $E \longrightarrow G$ corresponds to the projection.

Next, consider the cube face $E - F - A - B$. Observe that the connecting map $\Omega C \times \Omega Y \longrightarrow N$ for the fibration along the top row of the pullback defining N corresponds to the product map $\Omega C \times \Omega Y \xrightarrow{\ \delta \times \Omega Y\ } M \times \Omega Y$, where δ is the connecting map in the homotopy fibration sequence $\Omega C \xrightarrow{\ \delta\ } M \longrightarrow A \longrightarrow C$. Using the homotopy equivalence $E \simeq M \times \Omega Y$ we are considering the homotopy pullback diagram

$$
\begin{array}{ccccccc}
\Omega C \times \Omega Y & \xrightarrow{\ \delta \times \Omega Y\ } & M \times \Omega Y & \longrightarrow & A & \longrightarrow & C \times Y \\
\| & & \downarrow{\scriptstyle g} & & \downarrow & & \| \\
\Omega C \times \Omega Y & \xrightarrow{\ \ \gamma\ \ } & F & \longrightarrow & B & \longrightarrow & C \times Y
\end{array}
$$

where γ is the connecting map. Condition (ii) implies that γ is null homotopic. The homotopy commutativity of the left square then immediately implies that the restriction of g to ΩY is null homotopic. This completes the proof of part (b).

The naturality of the homotopy action applied to the homotopy pullback in the previous paragraph gives a homotopy commutative diagram

$$
\begin{array}{ccc}
(M \times \Omega Y) \times (\Omega C \times \Omega Y) & \xrightarrow{\theta'} & M \times \Omega Y \\
\downarrow{\scriptstyle g \times (\Omega C \times \Omega Y)} & & \downarrow{\scriptstyle g} \\
F \times (\Omega C \times \Omega Y) & \xrightarrow{\quad \theta \quad} & F,
\end{array}
$$

where θ' and θ are the respective actions. Since the homotopy fibration sequence $\Omega C \times \Omega Y \xrightarrow{\delta \times \Omega Y} M \times \Omega Y \longrightarrow A \times * \longrightarrow C \times Y$ is a product of fibration sequences, θ' is homotopic to the product of the actions of the individual fibrations. That is, the homotopy fibration sequence $\Omega C \xrightarrow{\delta} M \longrightarrow A \longrightarrow C$ has a homotopy action $\theta'' : M \times \Omega C \longrightarrow M$, while the homotopy fibration sequence $\Omega Y \longrightarrow \Omega Y \longrightarrow PY \longrightarrow Y$ has a homotopy action $\mu : \Omega Y \times \Omega Y \longrightarrow \Omega Y$ given by the loop multiplication. The map θ' is then the composite

$$
\theta' : (M \times \Omega Y) \times (\Omega C \times \Omega Y) \xrightarrow{M \times T \times \Omega Y} (M \times \Omega C) \times (\Omega Y \times \Omega Y) \xrightarrow{\theta'' \times \mu} M \times \Omega Y
$$

where T is the map which interchanges factors. Precomposing with the inclusion of factors 1 and 4, $M \times \Omega Y \xrightarrow{j \times i} (M \times \Omega Y) \times (\Omega C \times \Omega Y)$, we have $\theta' \circ (j \times i)$ homotopic to the identity map. The homotopy commutative diagram of actions above then results in a string of homotopies

$$
g \simeq g \circ \theta' \circ (j \times i) \simeq \theta \circ (g \times 1_{\Omega Y \times \Omega Y}) \circ (j \times i) \simeq \theta \circ (g|_M \times i)
$$

which proves part (c). □

Corollary 5.2. *There is a homotopy cofibration*

$$
M \rtimes \Omega Y \xrightarrow{g'} F \longrightarrow H
$$

where g' is an extension of g to $M \rtimes \Omega Y$.

Proof. Consider the homotopy pushout of fibres in Proposition 5.1. We know that the restriction of g to ΩY is null homotopic. Since the projection π has a right inverse, the map $\Omega Y \longrightarrow H$ is also null homotopic. Thus the factor ΩY in the left column of the homotopy pushout can be pinched out, resulting in a new homotopy pushout

$$
\begin{array}{ccc}
M \rtimes Y & \xrightarrow{g'} & F \\
\downarrow & & \downarrow \\
* & \longrightarrow & H
\end{array}
$$

which is exactly the asserted homotopy cofibration. □

6. Proper coordinate subspaces of the fat wedge

Let $X_1, \ldots, X_n$ be path-connected spaces. In this section we investigate properties of the homotopy fibre of the inclusion of the fat wedge $FW(1, \ldots, n)$ into the product $X_1 \times \cdots \times X_n$. Here,

$$
FW(1, \ldots, n) = \{(x_1, \ldots, x_n) \mid \text{at least one } x_i \text{ is } *\}.
$$

Including the fat wedge into the product gives a homotopy fibration

$$F^n \longrightarrow FW(1, \ldots, n) \longrightarrow X_1 \times \cdots \times X_n$$

which defines the space F^n. Porter [12] showed that F^n is homotopy equivalent to $\Omega X_1 * \cdots * \Omega X_n$ by examining certain subspaces of contractible spaces. Doeraene [5] reproduced this result in a more general setting by using the Cube Lemma. We include a proof using the Cube Lemma for the sake of completeness.

Lemma 6.1. *There is a homotopy equivalence $F^n \simeq \Omega X_1 * \cdots * \Omega X_n$.*

Proof. We induct on n. When $n = 1$, we have $FW(1) = *$ and so $F^1 = \Omega X_1$. Assume $F^{n-1} \simeq \Omega X_1 * \cdots * \Omega X_{n-1}$. Observe that there is a topological pushout

$$
\begin{array}{ccc}
FW(1, \ldots, n-1) & \xrightarrow{\ i\ } & FW(1, \ldots, n-1) \times X_n \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_{n-1} & \longrightarrow & FW(1, \ldots, n)
\end{array}
$$

where i is the inclusion into the first factor. Mapping all four corners into $X_1 \times \cdots \times X_n$ and taking homotopy fibres gives homotopy fibrations

$$F^n \longrightarrow FW(1, \ldots, n) \longrightarrow X_1 \times \cdots \times X_n \tag{3}$$

$$F^{n-1} \longrightarrow FW(1, \ldots, n-1) \times X_n \longrightarrow X_1 \times \cdots \times X_n \tag{4}$$

$$\Omega X_n \longrightarrow X_1 \times \cdots \times X_{n-1} \longrightarrow X_1 \times \cdots \times X_n \tag{5}$$

$$F^{n-1} \times \Omega X_n \longrightarrow FW(1, \ldots, n-1) \longrightarrow X_1 \times \cdots \times X_n. \tag{6}$$

Note that homotopy fibration (5) is the product of the identity fibration $* \longrightarrow X_1 \times \cdots \times X_{n-1} \longrightarrow X_1 \times \cdots \times X_{n-1}$ and the path-loop fibration $\Omega X_n \longrightarrow * \longrightarrow X_n$. This relates to both homotopy fibrations (4) and (6). Homotopy fibration (4) is the product of the fibration $F^{n-1} \longrightarrow FW(1, \ldots, n-1) \longrightarrow X_1 \times \cdots \times X_{n-1}$ and the identity fibration above. Hence the inclusion i induces a map of fibres $F^{n-1} \times \Omega X_n \longrightarrow F^{n-1}$ which is the projection onto the first factor. Homotopy fibration (6) is the product of the fibration $F^{n-1} \longrightarrow FW(1, \ldots, n-1) \longrightarrow X_1 \times \cdots \times X_{n-1}$ and the path-loop fibration. Hence the inclusion $FW(1, \ldots, n-1) \longrightarrow X_1 \times \cdots \times X_{n-1}$ induces a map of fibrations $F^{n-1} \times \Omega X_n \longrightarrow \Omega X_n$ which is the projection onto the second factor. Collecting all this information on the homotopy fibres, Lemma 3.1 says that there is a homotopy pushout of fibres

$$
\begin{array}{ccc}
F^{n-1} \times \Omega X_n & \xrightarrow{\ \pi_1\ } & F^{n-1} \\
{\scriptstyle \pi_2}\downarrow & & \downarrow \\
\Omega X_n & \longrightarrow & F^n.
\end{array}
$$

It is well known that in general the homotopy pushout of the projections $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$ is homotopy equivalent to $A * B$. Thus, in our case, $F^n \simeq F^{n-1} * \Omega X_n$. The inductive hypothesis on F^{n-1} then implies that $F^n \simeq \Omega X^1 * \cdots * \Omega X_n$. $\square$

For $1 \le i \le n$, let $X_1 \times \cdots \times \widehat{X}_i \times \cdots \times X_n$ be the subspace of $X_1 \times \cdots \times X_n$ in which the ith coordinate is fixed as $*$. Let $FW(1, \cdots, \hat{i}, \cdots n)$ be the fat wedge in $X_1 \times \cdots \times \widehat{X}_i \times \cdots \times X_n$. Let

$B_i = X_i \times FW(1, \ldots, \hat{i}, \ldots n)$. Observe that each B_i is a subspace of $FW(1, \ldots, n)$ and there is a topological pushout

$$
\begin{array}{ccc}
FW(1, \ldots, \hat{i}, \ldots n) & \longrightarrow & B_i \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times \widehat{X}_i \times \cdots \times X_n & \longrightarrow & FW(1, \ldots, n).
\end{array}
\tag{7}
$$

Consider the sequence of inclusions $B_i \longrightarrow FW(1, \ldots, n) \longrightarrow X_1 \times \cdots \times X_n$. Using Lemma 6.1, we obtain a homotopy pullback

$$
\begin{array}{ccc}
F_i & \xrightarrow{\ h_i\ } & \Omega X_1 * \cdots * \Omega X_n \\
\downarrow & & \downarrow \\
B_i & \longrightarrow & FW(1, \ldots, n) \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & =\!=\!= & X_1 \times \cdots \times X_n
\end{array}
$$

which defines the map h_i.

Lemma 6.2. *The map h_i is null homotopic.*

Proof. Consider the homotopy pushout in diagram (7). We wish to apply Proposition 5.1 with $A = FW(1, \ldots, \hat{i}, \ldots, n)$, $B = B_i$, $C = X_1 \times \cdots \times \widehat{X}_i \times \cdots \times X_n$, $D = FW(1, \ldots, n)$, and $Z = X_1 \times \cdots \times X_n$. We need to check that the two conditions in Proposition 5.1 hold. Observe that $Z = C \times X_i$ and $C \longrightarrow Z$ is the inclusion of the first factor so condition (i) is satisfied. Since $B_i = X_i \times FW(1, \ldots, \hat{i}, \ldots n)$ and the map $FW(1, \ldots, \hat{i}, \ldots, n) \longrightarrow X_1 \times \cdots \widehat{X}_i, \ldots \times X_n$ has a right homotopy inverse when looped, the map $B_i \longrightarrow X_1 \times \cdots \times X_n$ also has a right homotopy inverse when looped, and so condition (ii) is satisfied. Proposition 5.1 then says that when the four corners of the pushout in diagram (7) are mapped into $X_1 \times \cdots \times X_n$ and homotopy fibres are taken, there is a homotopy pushout of fibres

$$
\begin{array}{ccc}
(\Omega X_1 * \cdots * \widehat{\Omega X}_i * \cdots * \Omega X_n) \times \Omega X_i & \xrightarrow{\ g\ } & F_i \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle h_i} \\
\Omega X_i & \longrightarrow & \Omega X_1 * \cdots * \Omega X_n
\end{array}
\tag{8}
$$

where π is the projection, the restriction of g to ΩX_i is null homotopic, and g is determined by the action of $\Omega X_1 \times \cdots \times \Omega X_n$ on F_i.

We next examine how g is determined by this action. Since $B_i = X_i \times FW(1, \ldots, \hat{i}, \ldots, n)$, we can project to obtain a homotopy pullback

$$
\begin{array}{ccc}
F_i & = = = = = & F_i \\
\downarrow & & \downarrow \\
B_i & \longrightarrow & FW(1, \ldots, \hat{i}, \ldots, n) \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & \stackrel{\pi}{\longrightarrow} & X_1 \times \cdots \times \widehat{X}_i \times \cdots \times X_n.
\end{array}
$$

Lemma 4.1 says that g factors through a projection,

$$
\begin{array}{ccc}
(\Omega X_1 * \cdots * \widehat{\Omega X}_i * \cdots * \Omega X_n) \times \Omega X_i & \stackrel{g}{\longrightarrow} & F_i \\
\downarrow{\scriptstyle \pi} & & \| \\
(\Omega X_1 * \cdots * \widehat{\Omega X}_i * \cdots * \Omega X_n) & \longrightarrow & F_i.
\end{array}
$$

The projection of g lets us define a composite

$$
g' : (\Omega X_1 * \cdots * \widehat{\Omega X}_i * \cdots * \Omega X_n) \rtimes \Omega X_i \stackrel{\pi}{\longrightarrow} \Omega X_1 * \cdots * \widehat{\Omega X}_i * \cdots * \Omega X_n \longrightarrow F_i.
$$

We can use g' to pinch out the factor of ΩX_i in diagram (8) in order to obtain a homotopy cofibration

$$
(\Omega X_1 * \cdots * \widehat{\Omega X}_i * \cdots * \Omega X_n) \rtimes \Omega X_i \stackrel{g'}{\longrightarrow} F_i \stackrel{h_i}{\longrightarrow} \Omega X_1 * \cdots * \Omega X_n.
$$

To simplify notation, let $Y = \Omega X_1 * \cdots * \widehat{\Omega X}_i * \cdots * \Omega X_n$. Since Y is a suspension, $Y \rtimes \Omega X_i \simeq Y \vee (Y \wedge \Omega X_i)$. So g' can alternatively be described by the composite $Y \rtimes \Omega X_i \stackrel{\simeq}{\longrightarrow} Y \vee (Y \wedge \Omega X_i) \stackrel{q}{\longrightarrow} Y \longrightarrow F_i$, where q is the pinch map. Thus $Y \wedge \Omega X_i$ is sent trivially into F_i by g' and so $\Sigma Y \wedge \Omega X_i$ retracts off the homotopy cofibre $\Omega X_1 * \cdots * \Omega X_n$ of g'. But $\Sigma Y \wedge \Omega X_i \simeq \Omega X_1 * \cdots * \Omega X_n$. Thus in the homotopy cofibration sequence $Y \rtimes \Omega X_i \stackrel{g'}{\longrightarrow} F_i \stackrel{h_i}{\longrightarrow} \Omega X_1 * \cdots * \Omega X_n \stackrel{\delta}{\longrightarrow} \Sigma(Y \rtimes \Omega X_i)$, the map δ has a left homotopy inverse and hence h_i is null homotopic. $\square$

In what follows a coordinate subspace denotes an arbitrary union of $X_{i_1} \times \cdots \times X_{i_j}$ for some $1 \le i_1 < \cdots < i_j \le n$. We now use the spaces B_i and Lemma 6.2 to generalise to the case of any proper coordinate subspace of $FW(1, \ldots, n)$.

Proposition 6.3. *Suppose A is a proper coordinate subspace of $FW(1, \ldots, n)$. Include A into FW and then include into $X_1 \times \cdots \times X_n$ to obtain a homotopy pullback*

$$
\begin{array}{ccc}
F & \stackrel{h}{\longrightarrow} & \Omega X_1 * \cdots * \Omega X_n \\
\downarrow & & \downarrow \\
A & \longrightarrow & FW(1, \ldots, n) \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & = = = = & X_1 \times \cdots \times X_n
\end{array}
$$

which defines the map h. Then h is null homotopic.

Proof. First observe that the inclusion of A into $FW(1, \ldots, n)$ factors through B_i for some i. This statement really just follows from the definitions. In terms of coordinates,

$$B_i = \{(x_1, \ldots, x_m) \mid \text{at least one of } x_1, \ldots, \hat{x}_i, \ldots, x_m \text{ is } *\}.$$

If the inclusion of A into $FW(1, \ldots, n)$ does not factor through B_i, then A must contain a sequence of the form $(x_1, \ldots, x_n)$ in which each of $x_1, \ldots, \hat{x}_i, \ldots, x_n$ is not $*$. Since A is a coordinate subspace, every sequence of this form must be in A. (Note that A is a subspace of $FW(1, \ldots, n)$ so this forces x_i to be $*$ in each such sequence.) If this is true for $1 \le i \le n$, then all of $FW(1, \ldots, n)$ is contained in A, contradicting the hypothesis that A is a proper coordinate subspace of $FW(1, \ldots, n)$.

The factorisation of $A \longrightarrow FW(1, \ldots, n)$ through B_i results in a diagram of iterated homotopy pullbacks

$$
\begin{array}{ccccc}
F & \longrightarrow & F_i & \xrightarrow{h_i} & \Omega X_1 * \cdots * \Omega X_n \\
\downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & B_i & \longrightarrow & FW(1, \ldots, n) \\
\downarrow & & \downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & = & X_1 \times \cdots \times X_n & = & X_1 \times \cdots \times X_n.
\end{array}
$$

The outer rectangle is the homotopy pullback defining h, so h factors through h_i. But h_i is null homotopic by Lemma 6.2, and so h is null homotopic. $\square$

7. Homotopy fibres associated to regular sequences

Let $X_1, \ldots, X_n$ be path-connected spaces. Let A and B be two coordinate subspaces of $X_1 \times \cdots \times X_n$, where $B \subseteq A$. Let F_A and F_B be the homotopy fibres of the inclusions of A and B respectively into $X_1 \times \cdots \times X_n$. Observe that there is a map of fibres $F_B \longrightarrow F_A$. The purpose of this section is to consider the homotopy types of F_A and F_B and how these are related by the map of fibres. In general, not much could be expected to be said. We show that if A is built up from B by what we call a regular sequence, and if the homotopy type of F_B is of a certain description, then the homotopy type of F_A is of the same description and there is control over the map of fibres. All this is made concrete in Theorem 7.2 and Proposition 7.5.

We begin by defining what is meant by a regular sequence. Let $\{i_1, \ldots, i_m\}$ be a subset of $\{1, \ldots, n\}$, where $i_1 < \cdots < i_m$. Let $\{j_1, \ldots, j_{n-m}\}$ be the complement of $\{i_1, \ldots, i_m\}$ in $\{1, \ldots, n\}$, where $j_1 < \cdots < j_{n-m}$. Let $FW(i_1, \ldots, i_m)$ be the fat wedge in $X_{i_1} \times \cdots \times X_{i_m}$. Let A_0 and A be coordinate subspaces of $X_1 \times \cdots \times X_n$ such that $X_1 \vee \cdots \vee X_n \subseteq A_0$ and $A_0 \subseteq A$. Then A can be built up iteratively from A_0 by a sequence of topological pushouts

$$
\begin{array}{ccc}
FW(i_1, \ldots, i_m) & \longrightarrow & A_{k-1} \\
\downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} & \longrightarrow & A_k
\end{array}
\tag{9}
$$

where $1 \leq k \leq l$, and $A_l = A$. There may be many choices of sequences of pushouts which realise A in this way. A particular type of sequence, if it exists, is well suited to identifying the homotopy fibre of the inclusion $A \longrightarrow X_1 \times \cdots \times X_n$.

Definition 7.1. Let $X_1, \ldots, X_n$ be path-connected spaces. A coordinate subspace A of $X_1 \times \cdots \times X_n$ is *regular* if the sequence

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = A$$

has the following property for each $1 \leq k \leq l$. Let $\{s_1, \ldots, s_r\}$ be the largest subset of $\{j_1, \ldots, j_{n-m}\}$ for which A_{k-1} can be written as a product $A_{k-1} = A'_{k-1} \times X_{s_1} \times \cdots \times X_{s_r}$ (permuting the coordinates if necessary). Then there is a topological pushout

$$
\begin{array}{ccc}
M_{k-1} & \longrightarrow & N_{k-1} \\
\downarrow & & \downarrow \\
FW(i_1, \ldots, i_m) & \longrightarrow & A_{k-1}
\end{array}
$$

where M_{k-1} is a proper coordinate subspace of $FW(i_1, \ldots, i_m)$.

The definition of a regular sequence may seem on first reading to be a bit mystifying, but it arises naturally when considering coordinate subspaces associated to shifted complexes. It might be useful at this point to momentarily skip ahead to Examples 8.2 and 8.3 in order to see the connection.

To go along with the definition, we establish some notation. Let $\{t_1, \ldots, t_{n-m-r}\}$ be the complement of $\{s_1, \ldots, s_r\}$ in $\{j_1, \ldots, j_{n-m}\}$. Let $S = X_{s_1} \times \cdots \times X_{s_r}$ and $T = X_{t_1} \times \cdots \times X_{t_{n-m-r}}$, so $S \times T = X_{j_1} \times \cdots \times X_{j_{n-m}}$ and $A_{k-1} = A'_{k-1} \times S$.

For $0 \leq k \leq l$, let F_k be the homotopy fibre of the inclusion $A_k \longrightarrow X_1 \times \cdots \times X_n$. Observe that if $A_{k-1} = A'_{k-1} \times S$ then there is a diagram of iterated homotopy pullbacks

$$
\begin{array}{ccccc}
F_{k-1} & = & F_{k-1} & = & F_{k-1} \\
\downarrow & & \downarrow & & \downarrow \\
A'_{k-1} & \xrightarrow{\ i\ } & A_{k-1} & \xrightarrow{\ \pi\ } & A'_{k-1} \\
\downarrow & & \downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} \times T & \xrightarrow{\ i\ } & X_{i_1} \times \cdots \times X_{i_m} \times S \times T & \xrightarrow{\ \pi\ } & X_{i_1} \times \cdots \times X_{i_m} \times T
\end{array}
$$

where i and π are the inclusion and projection respectively.

In Theorem 7.2 we make the seemingly odd assumption that the fibre F_0 is a co-H space. However, in the context of coordinate subspace arrangements, this condition arises naturally, as we are trying to show that certain homotopy fibres (labelled $\mathcal{Z}_K$) are homotopy equivalent to wedges of spheres for appropriate simplicial complexes K, in which case the fibres $\mathcal{Z}_K$ are co-H spaces.

Theorem 7.2. *Suppose there is a regular sequence of coordinate subspaces*

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = A.$$

Assume that the homotopy fibre F_0 of the inclusion $A_0 \longrightarrow X_1 \times \cdots \times X_n$ is a co-H space. Then the following hold:

(a) *for $1 \le k \le l$, there is a homotopy cofibration*

$$(\Omega X_{i_m} * \cdots * \Omega X_{i_m}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \longrightarrow F_{k-1} \longrightarrow F_k$$

and a homotopy decomposition

$$(\Omega X_{i_m} * \cdots * \Omega X_{i_m}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \simeq C_{k-1} \vee D_{k-1}$$

where C_{k-1} maps trivially into F_{k-1} and D_{k-1} retracts off F_{k-1};
(b) *there is a homotopy decomposition $F_{k-1} \simeq D_{k-1} \vee E_{k-1}$ for some space E_{k-1};*
(c) *F_k is a co-H space and there is a homotopy decomposition $F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$.*

Proof. As the proof of part (a) is lengthy, we begin by assuming that part (a) has been proved and show that parts (b) and (c) hold. With F_0 as the base case, we inductively assume that F_{k-1} is a co-H space. Let E_{k-1} be the homotopy cofibre of $D_{k-1} \longrightarrow F_{k-1}$. By part (a), this map has a left homotopy inverse $F_{k-1} \longrightarrow D_{k-1}$. Since F_{k-1} is a co-H space, we can add to obtain a composite $F_{k-1} \longrightarrow F_{k-1} \vee F_{k-1} \longrightarrow D_{k-1} \vee E_{k-1}$ which is a homotopy equivalence. This proves part (b). Next, including D_{k-1} into $C_{k-1} \vee D_{k-1}$ we obtain a homotopy pushout

$$
\begin{array}{ccccc}
D_{k-1} & \longrightarrow & C_{k-1} \vee D_{k-1} & \longrightarrow & C_{k-1} \\
\| & & \downarrow & & \downarrow \\
D_{k-1} & \longrightarrow & F_{k-1} & \longrightarrow & E_{k-1} \\
& & \downarrow & & \downarrow \\
& & F_k & = & F_k.
\end{array}
$$

By part (a) the map $C_{k-1} \longrightarrow F_{k-1}$ is null homotopic, so in the pushout the map $C_{k-1} \longrightarrow E_{k-1}$ is also null homotopic. Hence $F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$. Finally, E_{k-1} is a retract of F_{k-1} which has been inductively assumed to be a co-H space, so E_{k-1} is also a co-H space. Thus F_k is a wedge of two co-H spaces and so is itself a co-H space.

We now prove part (a).

Step 1. Setting up: Consider the pushout in diagram (9). We apply Proposition 5.1 with $A = FW(i_1, \ldots, i_m)$, $B = A_{k-1}$, $C = X_{i_1} \times \cdots \times X_{i_m}$, $D = A_k$ and $Z = X_1 \times \cdots \times X_n$. We need to check that conditions (i) and (ii) of Proposition 5.1 hold. Observe that $Z = C \times (X_{j_1} \times \cdots \times X_{i_{n-m}})$ and $C \longrightarrow Z$ is the inclusion of the first factor, so condition (i) is satisfied. Since $X_1 \vee \cdots \vee X_n$ is a subspace of A_{k-1} and the inclusion $X_1 \vee \cdots \vee X_n \longrightarrow X_1 \times \cdots \times X_n$ has a right homotopy inverse when looped, the inclusion $A_{k-1} \longrightarrow X_1 \times \cdots \times X_n$ also has a right homotopy inverse when looped, and so condition (ii) is also satisfied. Proposition 5.1 insures that when the four corners of the pushout in diagram (9) are mapped into $X_1 \times \cdots \times X_n$ and homotopy fibres are taken, there is a homotopy pushout of fibres

$$
\begin{array}{ccc}
(\Omega X_{i_1} * \cdots * \Omega X_{i_m}) \times (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) & \xrightarrow{\ g\ } & F_{k-1} \\
\downarrow{\scriptstyle \pi} & & \downarrow \\
\Omega X_{j_1} \times \cdots \Omega X_{j_{n-m}} & \longrightarrow & F_k
\end{array}
\tag{10}
$$

where π is the projection, the restriction of g to $\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}$ is null homotopic, and g is determined by the action of $\Omega X_1 \times \cdots \times \Omega X_n$ on F_{k-1}. As the restriction of g to $\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}$ is

null homotopic, we can pinch out this factor in diagram (10) and, as in Corollary 5.2, obtain a homotopy cofibration

$$(\Omega X_{i_1} * \cdots * \Omega X_{i_t}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-t}}) \overset{g'}{\longrightarrow} F_{k-1} \longrightarrow F_k \tag{11}$$

where g' is an extension of g to the half-smash.

Step 2. The summand C_{k-1}: The decomposition $A_{k-1} = A'_{k-1} \times S$ implies that there is a homotopy pullback

$$
\begin{array}{ccc}
F_{k-1} & =\!=\!=\!=\!= & F_{k-1} \\
\downarrow & & \downarrow \\
A_{k-1} & \overset{\pi}{\longrightarrow} & A'_{k-1} \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n & \overset{\pi}{\longrightarrow} & (X_{i_1} \times \cdots \times X_{i_m}) \times T
\end{array}
$$

where π is the projection. Lemma 4.1 says that the map g in diagram (10) factors through a projection,

$$
\begin{array}{ccc}
(\Omega X_{i_1} * \cdots * \Omega X_{i_m}) \times (\Omega S \times \Omega T) & \overset{g}{\longrightarrow} & F_{k-1} \\
{\scriptstyle 1 \times \pi} \downarrow & & \| \\
(\Omega X_{i_1} * \cdots * \Omega X_{i_m}) \times \Omega T & \overset{\tilde{g}}{\longrightarrow} & F_k
\end{array}
$$

where $\tilde{g}$ is the restriction of g to $(\Omega X_{i_1} * \cdots * \Omega X_{i_m}) \times \Omega T$.

Let $Y = \Omega X_{i_1} * \cdots * \Omega X_{i_m}$. Since the restriction of g to $(\Omega S \times \Omega T)$ is null homotopic, the restriction of $\tilde{g}$ to ΩT is null homotopic. Thus $\tilde{g}$ factors through $Y \rtimes \Omega T$. It was only necessary to choose some extension g' of g to the half-smash in (11) in order to obtain the homotopy cofibration, so we could have taken g' to be the composite $Y \rtimes (\Omega S \times \Omega T) \overset{1 \times \pi}{\longrightarrow} Y \rtimes \Omega T \longrightarrow F_{k-1}$. Since Y is a suspension,

$$Y \rtimes (\Omega S \times \Omega T) \simeq Y \vee (Y \wedge \Omega S) \vee (Y \wedge \Omega T) \vee (Y \wedge \Omega S \wedge \Omega T).$$

Let $C_{k-1} = (Y \wedge \Omega S) \vee (Y \wedge \Omega S \wedge \Omega T)$ and let $D_{k-1} = Y \vee (Y \wedge \Omega T)$. Then g' can alternatively be described by the composite $Y \rtimes (\Omega S \times \Omega T) \overset{\simeq}{\longrightarrow} C_{k-1} \vee D_{k-1} \overset{q}{\longrightarrow} D_{k-1}$, where q is the pinch map. Thus C_{k-1} is sent trivially into F_{k-1} by g', as asserted.

Step 3. The summand D_{k-1}: It remains to show that $D_{k-1} = Y \vee (Y \wedge \Omega T)$ is a retract of F_{k-1}. Again, we consider $A_{k-1} = A'_{k-1} \times S$, where $S = X_{s_1} \times \cdots \times X_{s_r}$. Observe that $\{i_1, \ldots, i_m\}$ and $\{s_1, \ldots, s_r\}$ are disjoint sets in $\{1, \ldots, n\}$ so the inclusion $FW(i_1, \ldots, i_m) \longrightarrow A_{k-1}$ of diagram (9) factors as a composite $FW(i_1, \ldots, i_m) \longrightarrow A'_{k-1} \longrightarrow A_{k-1}$. Define the space A''_k as the topological pushout

$$
\begin{array}{ccc}
FW(i_1, \ldots, i_m) & \longrightarrow & A'_{k-1} \\
\downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} & \longrightarrow & A''_k.
\end{array}
$$

Since A is regular, there is a topological pushout

$$
\begin{array}{ccc}
M_{k-1} & \longrightarrow & N_{k-1} \\
\downarrow & & \downarrow \\
FW(i_1, \ldots, i_m) & \longrightarrow & A'_{k-1}
\end{array}
\tag{12}
$$

where M_{k-1} is a proper coordinate subspace of $FW(i_1, \ldots, i_m)$. Note that all the spaces in diagram (12) are coordinate subspaces of $X_{i_1} \times \cdots \times X_{i_m} \times T$. We intend to map the four corners of the pushout into $X_{i_1} \times \cdots \times X_{i_m} \times T$, take homotopy fibres, and apply the Cube Lemma. Before doing so we identify the homotopy fibres. Let F_M be the homotopy fibre of the inclusion $M_{k-1} \longrightarrow X_{i_1} \times \cdots \times X_{i_m}$. By Lemma 6.1, the homotopy fibre of the inclusion $FW(i_1, \ldots, i_m) \longrightarrow X_{i_1} \times \cdots \times X_{i_m}$ is homotopy equivalent to $\Omega X_{i_1} * \cdots * \Omega X_{i_m}$. Including $X_{i_1} \times \cdots \times X_{i_m}$ into $X_{i_1} \times \cdots \times X_{i_m} \times T$ we obtain a homotopy pullback

$$
\begin{array}{ccccc}
F_M \times \Omega T & \longrightarrow & M_{k-1} & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times T \\
\downarrow{\scriptstyle h \times \Omega T} & & \downarrow & & \Vert \\
(\Omega X_{i_1} * \cdots * \Omega X_{i_m}) \times \Omega T & \longrightarrow & FW(i_1, \ldots, i_m) & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times T
\end{array}
$$

for some map h. Let F_N be the homotopy fibre of the inclusion $N_{k-1} \longrightarrow X_{i_1} \times \cdots \times X_{i_m} \times T$. The definition of a regular sequence includes the hypothesis that $X_1 \vee \cdots \vee X_n \subseteq A_0$, and so $X_1 \vee \cdots \vee X_n \subseteq A_{k-1}$. Having projected away from coordinates $s_1, \ldots, s_l$, we have $X_{i_1} \vee \cdots \vee X_{i_m} \vee X_{j_1} \vee \cdots \vee X_{j_t} \subseteq A'_{k-1}$. As diagram (12) is a homotopy pushout and $FW(i_1, \ldots, i_m)$ intersects $X_{j_1} \vee \cdots \vee X_{j_t}$ at a point, we must have $X_{j_1} \vee \cdots \vee X_{j_t} \subseteq N_{k-1}$. Thus $\Omega T = \Omega X_{j_1} \times \cdots \times \Omega X_{j_t}$ retracts off ΩN_{k-1}. Therefore, in the homotopy pullback

$$
\begin{array}{ccccc}
F_M \times \Omega T & \longrightarrow & M_{k-1} & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times T \\
\downarrow{\scriptstyle \overline{f}} & & \downarrow & & \Vert \\
F_N & \longrightarrow & N_{k-1} & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times T
\end{array}
$$

(the pullback defines the map $\overline{f}$) the restriction of $\overline{f}$ to ΩT is null homotopic. Now recall from Step 2 that the homotopy fibre of the inclusion $A'_{k-1} \longrightarrow X_{i_1} \times \cdots \times X_{i_m} \times T$ is homotopy equivalent to F_{k-1}. Thus, when the four corners of the pushout in diagram (12) are mapped into $X_{i_1} \times \cdots \times X_{i_m} \times T$ and homotopy fibres are taken, Lemma 3.1 implies that there is a homotopy pushout of fibres

$$
\begin{array}{ccc}
F_M \times \Omega T & \xrightarrow{\overline{f}} & F_N \\
\downarrow{\scriptstyle h \times \Omega T} & & \downarrow \\
(\Omega X_{i_1} * \cdots * \Omega X_{i_m}) \times \Omega T & \xrightarrow{\overline{g}} & F_{k-1}
\end{array}
\tag{13}
$$

for some map $\overline{g}$. We can identify $\overline{g}$: it is the restriction of the map g in diagram (10) to $(\Omega X_{i_1} * \cdots * \Omega X_{i_m}) \times \Omega T$. This is because, as in the proof of Proposition 5.1(c), the map $\overline{g}$ is determined by the

action of $\Omega X_{i_1} \times \cdots \times \Omega X_{i_m} \times \Omega T$ on F_{k-1}. But the pullback

$$
\begin{array}{ccccc}
F_{k-1} & \longrightarrow & A'_{k-1} & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times T \\
\| & & \downarrow & & \downarrow \\
F_{k-1} & \longrightarrow & A_k & \longrightarrow & X_{i_1} \times \cdots \times X_{i_m} \times S \times T
\end{array}
$$

obtained from including A'_{k-1} into $A_{k-1} = A'_{k-1} \times S$ implies that the action of $\Omega X_{i_1} \times \cdots \times \Omega X_{i_m} \times \Omega T$ on F_{k-1} is the restriction of the action of $\Omega X_{i_1} \times \cdots \times \Omega X_{i_m} \times \Omega T \times \Omega S$ on F_{k-1}, that is, the action of $\Omega X_1 \times \cdots \times \Omega X_n$ on F_{k-1}, and the latter action determines g. Consequently, the factorisation of g through g' implies that the restriction $\overline{g}$ factors as a composite (using the notation from Step 2) $\gamma : Y \rtimes \Omega T \longrightarrow Y \rtimes (\Omega S \times \Omega T) \xrightarrow{g'} F_{k-1}$. Note that D_{k-1} was defined as $Y \vee (Y \wedge \Omega T) \simeq Y \rtimes \Omega T$, and we are trying to prove precisely that γ has a left homotopy inverse.

Consider diagram (13). Since M_{k-1} is a proper coordinate subspace of $FW(i_1, \ldots, i_m)$, Proposition 6.3 implies that h is null homotopic. Thus $h \times \Omega T$ is homotopic to $* \times \Omega T$. We have seen that the restriction of $\overline{f}$ to ΩT is null homotopic. Lemma 3.6 now applies, and shows that γ has a left homotopy inverse. $\square$

We now condense some of the information coming out of Theorem 7.2 by concentrating on how the fibre F_0 of the starting point A_0 of the regular sequence relates to the fibre F_l of the ending point A_l of the sequence. Let θ be the composite

$$
\theta : F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_l.
$$

In particular, we want to know how the homotopy type of F_0 influences that of F_l. This requires a suitable hypothesis on the homotopy type of F_0 to get going. We now define a class of spaces which will do the job.

Definition 7.3. Let $\mathcal{G}_1^n$ be the collection of spaces F which are homotopy equivalent to a wedge of summands of the form $\Omega X_{i_1} * \cdots * \Omega X_{i_m}$, where $1 \leq i_1 < \cdots < i_m \leq n$.

Consider Definition 7.3 in the case of primary interest, when $X_i = \mathbb{C}P^\infty$ for $1 \leq i \leq n$. Then $\Omega X_i \simeq S^1$ and so $\Omega X_{i_1} * \cdots * \Omega X_{i_m} \simeq S^{2m-1}$, in which case F is homotopy equivalent to a wedge of spheres. As spaces which are homotopy equivalent to wedges of spheres will appear repeatedly, it will be convenient to introduce an abbreviated way of saying this.

Definition 7.4. Let $\mathcal{W}$ be the collection of spaces F which are homotopy equivalent to a wedge of spheres.

Proposition 7.5. *Assume the hypotheses of Theorem 7.2. Suppose in addition that (for all path-connected spaces $X_1, \ldots, X_n$) the homotopy fibre F_0 of the inclusion $A_0 \longrightarrow X_1 \times \cdots \times X_n$ is such that $F_0 \in \mathcal{G}_1^n$. Consider the map of fibres $\theta : F_0 \longrightarrow F_l$. The following hold:*

(a) $F_l \in \mathcal{G}_1^n$, and

(b) *there is a homotopy decomposition $F_0 \simeq F_0^1 \vee F_0^2$, where $F_0^1, F_0^2 \in \mathcal{G}_1^n$, the restriction of θ to F_0^1 is null homotopic, and the restriction of θ to F_0^2 has a left homotopy inverse.*

Proof. Theorem 7.2 gives that for $1 \leq k \leq l$, there is a homotopy cofibration

$$(\Omega X_{i_m} * \cdots * \Omega X_{i_m}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \xrightarrow{f_k} F_{k-1} \xrightarrow{g_k} F_k$$

and there are homotopy decompositions

$$(\Omega X_{i_m} * \cdots * \Omega X_{i_m}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \simeq C_{k-1} \vee D_{k-1},$$
$$F_{k-1} \simeq D_{k-1} \vee E_{k-1},$$
$$F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$$

where the restriction of f_k to C_{k-1} is null homotopic and the restriction of f_k to D_{k-1} has a left homotopy inverse. This implies that the restriction of g_k to D_{k-1} is null homotopic and the restriction of g_k to E_{k-1} has a left homotopy inverse.

First observe that, in general, there are homotopy decompositions $(\Sigma A) \rtimes B \simeq \Sigma A \vee (\Sigma A \wedge B)$ and $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B)$. Noting that the join is a suspension, by using the first decomposition and iterating on the second we see that

$$(\Omega X_{i_m} * \cdots * \Omega X_{i_m}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \in \mathcal{G}_1^n.$$

The fact that Theorem 7.2 holds for all path-connected spaces $X_1, \ldots, X_n$ means that the decompositions are independent of the particular choices of those spaces. This lets us make an advantageous choice of $X_1, \ldots, X_n$, observe how the decompositions behave in this special case, and then infer the general decompositions.

The advantageous choice is to take $X_i = \mathbb{C}P^\infty$ for $1 \leq i \leq n$. Then $\Omega X_i \simeq S^1$ and

$$(\Omega X_{i_m} * \cdots * \Omega X_{i_m}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}}) \in \mathcal{W}.$$

Thus $C_{k-1}, D_{k-1} \in \mathcal{W}$. The hypothesis on F_0 implies that in this case $F_0 \in \mathcal{W}$. Thus the homotopy equivalence $F_0 \simeq D_0 \vee E_0$ implies that $E_0 \in \mathcal{W}$. Hence $F_1 \in \mathcal{W}$. Inductively, we see that $C_{k-1}, D_{k-1}, E_{k-1}, F_k \in \mathcal{W}$ for all $1 \leq k \leq l$. In particular, $F_l \in \mathcal{W}$. We next describe the decomposition $F_0 \simeq F_0^1 \vee F_0^2$. The decomposition $F_k \simeq \Sigma C_{k-1} \vee E_{k-1}$ gives a retraction of E_{k-1} off F_k. Consider how this relates to the decomposition $F_k \simeq D_k \vee E_k$. Since $D_k, E_k \in \mathcal{W}$, we can choose subwedges of spheres E_{D_k}, E_{E_k} of D_k, E_k respectively such that $E_{k-1} \simeq E_{D_k} \vee E_{E_k}$. Let $F_0^1 = D_1 \vee E_{D_1} \vee \cdots \vee E_{D_{l-1}}$, and let $F_0^2 = E_{E_{l-1}}$. Then $F_0^1 \vee F_0^2 \simeq F_0$. The condition that g_k is null homotopic when restricted to D_k then implies that it is null homotopic when restricted to E_{D_k}, and so collectively we see that the restriction of θ to F_0^1 is null homotopic. The condition that g_{l-1} has a left homotopy inverse when restricted to E_{l-1} implies that the restriction of θ to $F_0^2 = E_{E_{l-1}}$ has a left homotopy inverse.

Now consider the general case. Observe that, by keeping track of the indices i_s and j_t on each copy of $\Omega X_{i_s} \simeq S^1$ and $\Omega X_{j_t} \simeq S^1$ in the special case, we can discern which wedge summands of $(\Omega X_{i_m} * \cdots * \Omega X_{i_m}) \rtimes (\Omega X_{j_1} \times \cdots \times \Omega X_{j_{n-m}})$ are in C_{k-1} and which are in D_{k-1}. In particular, we see that $C_{k-1}, D_{k-1} \in \mathcal{G}_1^n$ for each $1 \leq k \leq l$. The same index bookkeeping on the successive decompositions in the special case then implies that $E_{k-1}, F_k \in \mathcal{G}_1^n$ for $1 \leq k \leq l$ – in particular, $F_l \in \mathcal{G}_1^n$, proving part (a) – and there is a decomposition $F_0 \simeq F_0^1 \vee F_0^2$ such that $F_0^1, F_0^2 \in \mathcal{G}_1^n$, the restriction of θ to F_0^1 is null homotopic and the restriction of θ to F_0^2 has a left homotopy inverse, which proves part (b). $\quad \square$

8. The existence of regular sequences

In this section we give a general set of conditions which guarantees the existence of regular sequences. In Examples 8.2 and 8.3 we then give particular instances which will be used later in Section 9. The set of conditions is phrased in terms of shifted complexes. Recall that a simplicial complex K is *shifted* if there is an ordering on its set of vertices such that whenever $\sigma \in K$ and $v' < v$, then $(\sigma - v) \cup v' \in K$. Two additional definitions we need are the following. Let K be a simplicial complex. The *link* and the *star* of a simplex $\sigma \in K$ are the subcomplexes

$$\text{link}_K \sigma = \{\tau \in K \mid \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\};$$
$$\text{star}_K \sigma = \{\tau \in K \mid \sigma \cup \tau \in K\}.$$

One interpretation of these definitions is via ordered sequences. Let K be a simplicial complex on the index set $[n]$. The vertices are ordered by their integer labels. If σ is a simplex of K on vertices $\{i_1, \ldots, i_m\}$, where $1 \leq i_1 < \cdots < i_m \leq n$, identify σ with the sequence $(i_1, \ldots, i_m)$. The simplices of K are then ordered left lexicographically.

Now suppose K is a shifted complex. If $(i_1, \ldots, i_m)$ is an $(m-1)$-dimensional simplex of K, then K must contain every simplex of dimension $m-1$ which is lexicographically less than $(i_1, \ldots, i_m)$. Let rest$\{2, \ldots, n\}$ be the simplicial subcomplex of K which is defined as the collection of simplices $(i_1, \ldots, i_m) \in K$ with $i_1 \geq 2$. Observe that star(1) consists of those simplices $(i_1, \ldots, i_m)$ for which $(1, \ldots, i_1 - 1, i_1, \ldots, i_m)$ is also a simplex of K, and link(1) consists of those simplices which are in both star(1) and rest$\{2, \ldots, n\}$.

All this can now be formulated topologically in terms of coordinate subspaces. Assume that K is a simplicial complex on the index set $[n]$. Let $X_1, \ldots, X_n$ be path-connected spaces. Then we can associate a coordinate subspace A of $X_1 \times \cdots \times X_n$ to K by letting A be the union of all subspaces $X_{i_1} \times \cdots \times X_{i_m}$, where $(i_1, \ldots, i_m)$ is a simplex of K. Now suppose K is shifted. Let Star(1), Link(1), and Rest$\{2, \ldots, n\}$ be the coordinate subspaces of $X_1 \times \cdots \times X_n$ associated to star(1), link(1), and rest$\{2, \ldots, n\}$ respectively.

We now give a set of conditions on the inclusion of one shifted complex into another which guarantees the existence of a regular sequence between their corresponding coordinate subspaces.

Proposition 8.1. *Let L and K be two shifted complexes on the index set* [n], *where L is contained within* star(1) *of K and K has no disjoint points. Fix path-connected spaces* $X_1, \ldots, X_n$. *Let B and A be the coordinate subspaces of* $X_1 \times \cdots \times X_n$ *which correspond to L and K respectively. Let* Star$(1) \subseteq A$ *be the coordinate subspace which corresponds to* star$(1) \subseteq K$. *Then there is a sequence of coordinate subspaces*

$$B = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = \text{Star}(1)$$

which is regular.

Before beginning with the proof of Proposition 8.1 we give two examples which will be used subsequently. Observe that since all n vertices are in L, the coordinate subspace $X_1 \vee \cdots \vee X_n$ is contained in B.

Example 8.2. Let K be a connected shifted complex. Let L be the disjoint union of the n vertices of K. Consider star(1) in K. Then $B = X_1 \vee \cdots \vee X_n$, $A = \text{Star}(1)$, and Proposition 8.1 says that there exists

a sequence of coordinate subspaces

$$X_1 \vee \cdots \vee X_n = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = \text{Star}(1)$$

which is regular.

Example 8.3. Let K be a connected shifted complex. Let L be link(1). Let $\text{star}_R(2)$ be star(2) in rest$\{2, \ldots, n\}$. Then $B = \text{Link}(1)$ and $A = \text{Star}_R(2)$. To apply Proposition 8.1 we need to check that (within Rest$\{2, \ldots, n\}$) link(1) is contained in $\text{star}_R(2)$. Let $(i_1, \ldots, i_m)$ be a simplex of link(1). If $i_1 = 2$ then $(i_1, \ldots, i_m)$ is clearly in $\text{star}_R(2)$. If $i_1 \neq 2$, then as link(1) $\subseteq$ star(1) (in K), the definition of link(1) says there exists a simplex $(1, 2, \ldots, i_1 - 1, i_1, \ldots, i_m)$ in star(1). The restricted simplex $(2, \ldots, i_1 - 1, i_1, \ldots, i_m)$ is therefore in $\text{star}_R(2)$. Thus $\text{star}_R(2)$ contains all the simplices in link(1). Proposition 8.1 then implies that there is a sequence of coordinate subspaces

$$\text{Link}(1) = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_l = \text{Star}_R(1)$$

which is regular.

Proof of Proposition 8.1. We adjoin subspaces to B in two separate iterations. These adjunctions correspond to gluing simplices to L one at a time until star(1) in K is obtained.

Iteration 1: Since K is connected and shifted, every vertex in K is connected by an edge to the vertex 1, that is, the simplex $(1, j)$ is in K for every $2 \leq j \leq n$. Now L may contain disjoint points. If so, since L is shifted, the simplices $(1, j)$ will not be in L for $j \geq j_0$, where j_0 is the first vertex not connected to 1. In terms of coordinate subspaces, each X_j is a wedge summand of B, and B contains the coordinate subspaces $X_1 \times X_j$ for $j < j_0$. The point of this first iteration is to adjoin the coordinate subspaces $X_1 \times X_j$ for $j \geq j_0$. They will be adjoined in left lexicographical order. The adjunction is realised by a homotopy pushout

$$
\begin{array}{ccc}
X_1 \vee X_j & \longrightarrow & A_{k-1} \\
\downarrow & & \downarrow \\
X_1 \times X_j & \longrightarrow & A_k
\end{array}
$$

which defines the space A_k. Here, we begin with the j_0 case, where $A_0 = B$, so $k = j - 1 - j_0$. To show that this sequence is regular, we need to show that there is a homotopy pushout

$$
\begin{array}{ccc}
M_{k-1} & \longrightarrow & N_{k-1} \\
\downarrow & & \downarrow \\
X_1 \vee X_j & \longrightarrow & A_{k-1}.
\end{array}
$$

Take $M_{k-1} = X_1$. Observe that by the iteration to this point, A_{k-1} is the wedge $X_1 \vee \cdots \vee X_n$ with the coordinate subspaces $X_1 \times X_i$ adjoined for $2 \leq i \leq j - 1$. In particular, X_j is a wedge summand of A_{k-1}. Let N_{k-1} be the complementary wedge summand of A_{k-1}, so $A_{k-1} \simeq X_j \vee N_{k-1}$. Then it is clear that $M_{k-1} = X_1$ includes into N_{k-1}, the diagram above homotopy commutes, and it is in fact a homotopy pushout.

Iteration 2: First observe that at the end of Iteration 1, all the coordinate subspaces $X_1 \times X_j$ for $2 \leq j \leq n$ have been adjoined to Star(1). So $A_{n-j_0} = X_1 \times (X_2 \vee \cdots \vee X_n)$.

We now adjoin the remaining coordinate subspaces of Star(1) in a two-step process. The idea is to adjoin all the remaining coordinate subspaces corresponding to the two-dimensional simplices of star(1) in lexicographic order, then the coordinate subspaces corresponding to the three-dimensional simplices of star(1), and so on. Suppose all the coordinate subspaces corresponding to the $(m − 2)$-dimensional simplices in star(1) have been adjoined. Suppose $(1, i_2, \ldots, i_m)$ is the simplex of dimension $m − 1$ of least lexicographic order whose corresponding coordinate subspace has not already been adjoined. To perform the adjunction it is necessary that the coordinate subspaces corresponding to the boundary of $(1, i_2, \ldots, i_m)$ have already been adjoined. The boundary is composed of the simplices

$$(1, i_2, \ldots, i_{m-1}), (1, i_3, \ldots, i_m), \ldots, (1, i_2, \ldots, i_{m-2}, i_m), \quad \text{and} \quad (i_2, \ldots, i_m).$$

All the coordinate subspaces corresponding to boundary simplices starting with the vertex 1 have already been adjoined by the inductive hypothesis: all the simplices are of dimension $m − 2$ and are all clearly in star(1). The lexicographical ordering implies that the coordinate subspace corresponding to the simplex $(i_2, \ldots, i_m)$ has not yet been adjoined. So we first need to adjoin the coordinate subspace corresponding to $(i_2, \ldots, i_m)$ and then adjoin the coordinate subspace corresponding to $(1, i_2, \ldots, i_m)$. Note that the coordinate subspaces corresponding to the boundary simplices of $(i_2, \ldots, i_m)$ have already been adjoined because star(1) being shifted means that if τ is a simplex in the boundary of $(i_2, \ldots, i_m)$ then $(1, \tau)$ is also a simplex of star(1), and as its dimension is $m − 2$, the corresponding coordinate subspace has already been adjoined by the inductive hypothesis.

The two-step gluing process is realised by the homotopy pushouts

$$
\begin{array}{ccc}
FW(i_2, \ldots, i_m) & \longrightarrow & A_{k-1} \\
\downarrow & & \downarrow \\
X_{i_1} \times \cdots \times X_{i_m} & \longrightarrow & A_k
\end{array}
\qquad
\begin{array}{ccc}
FW(1, i_2, \ldots, i_m) & \longrightarrow & A_k \\
\downarrow & & \downarrow \\
X_1 \times X_{i_2} \times \cdots \times X_{i_m} & \longrightarrow & A_{k+1}
\end{array}
$$

where the pushouts define the spaces A_k and A_{k+1}. Observe that if we assume $A_{k-1} \simeq X_1 \times A'_{k-1}$ – this is true for the base case A_{n-j_0} as mentioned at the beginning of this iteration – then the two-step process in adjoining the coordinate subspace corresponding to the simplex $(1, i_2, \ldots, i_m)$ implies that $A_{k+1} \simeq X_1 \times A'_{k+1}$. Thus if we show that the two-step process is itself a regular sequence, then the entire iteration is a string of two-step regular sequences and so is a regular sequence, completing the proof.

For the $k − 1$ case, as $A_{k-1} \simeq X_1 \times A'_{k-1}$, the definition of a regular sequence forces us to project onto A'_{k-1} and look for a homotopy pushout

$$
\begin{array}{ccc}
M_{k-1} & \longrightarrow & N_{k-1} \\
\downarrow & & \downarrow \\
FW(i_2, \ldots, i_m) & \longrightarrow & A'_{k-1}
\end{array}
$$

where M_{k-1} is a proper coordinate subspace of $FW(i_2, \ldots, i_m)$. Having projected away from variable 1, this homotopy pushout is really a lower-dimensional case which builds up Star(2) within Rest$\{2, \ldots, n\}$. The inductive hypothesis on dimension means that we can assume that this homotopy pushout exists. For

the k case, we need to show that there is a homotopy pushout

$$
\begin{array}{ccc}
M_k & \longrightarrow & N_k \\
\downarrow & & \downarrow \\
FW(1, i_2, \ldots, i_m) & \longrightarrow & Ak
\end{array}
$$

where M_k is a proper coordinate subspace of $FW(1, i_2, \ldots, i_m)$. Let $M_{k-1} = X_{i_2} \times FW(i_3, \ldots, i_m)$. (Note that M_{k-1} equals Star(1) in $FW(i_2, \ldots, i_m)$.) Observe that if such a homotopy pushout exists, then N_k needs to contain all the coordinate subspaces of A_k except $X_{i_2} \times \cdots \times X_{i_m}$. But this is exactly the description of A_{k-1}, so by taking $N_k = A_{k-1}$ we obtain the desired homotopy pushout. $\quad\square$

9. The homotopy type of $\mathcal{Z}_K$ for shifted complexes

Recall that if K is a simplicial complex on the index set $[n]$, then there is a corresponding Davis–Januszkiewicz space $DJ(K)$ and a homotopy fibration

$$
\mathcal{Z}_K \longrightarrow DJ(K) \longrightarrow \prod_{i=1}^{n} BT.
$$

One of the main goals of the paper is to prove Theorem 1.2, which we restate as:

Theorem 9.1. *If K is a shifted complex, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres.*

Any ith skeleton $\Delta^i(n)$ of the standard simplex $\Delta(n)$ on n vertices is shifted. Other examples are easy to construct; we give two to illustrate.

Example 9.2. Let K be the simplicial complex consisting of vertices $\{1, 2, 3, 4\}$ and edges $\{12, 13, 14, 23, 24\}$. Then K is shifted.

Example 9.3. Let K be the simplicial complex consisting of vertices $\{1, 2, 3, 4, 5\}$ and edges $\{12, 13, 14, 15, 23, 24, 25, 34, 35\}$. Then K is shifted. Note that $K' = K \cup \{123\}$ is shifted, but $K'' = K \cup \{124\}$ is not shifted.

It is well known (and easy to prove) that if K is shifted then each of link(1), star(1), and rest$\{2, \ldots, n\}$ is shifted, star(1) = (1) $*$ link(1), and there is a topological pushout

$$
\begin{array}{ccc}
\text{link}(1) & \longrightarrow & \text{rest}\{2, \ldots, n\} \\
\downarrow & & \downarrow \\
\text{star}(1) & \longrightarrow & K.
\end{array}
$$

This results in a corresponding homotopy pushout of Davis–Januszkiewicz spaces

$$
\begin{array}{ccc}
DJ(\text{link}(1)) & \longrightarrow & DJ(\text{rest}\{2, \ldots, n\}) \\
\downarrow & & \downarrow \\
DJ(\text{star}(1)) & \longrightarrow & DJ(K)
\end{array}
$$

where $DJ(\text{star}(1)) = BT \times DJ(\text{link}(1))$. Mapping the four corners into $\prod_{i=1}^{n} BT$ and taking homotopy fibres gives a cube as in Lemma 3.1, and in particular a homotopy pushout of fibres

$$
\begin{array}{ccc}
S^1 \times \mathcal{Z}_{\text{link}(1)} & \longrightarrow & S^1 \times \mathcal{Z}_{\text{rest}\{2,\ldots,n\}} \\
\downarrow & & \downarrow \\
\mathcal{Z}_{\text{star}(1)} & \longrightarrow & \mathcal{Z}_K .
\end{array}
\tag{14}
$$

We wish to show that each of $\mathcal{Z}_{\text{link}(1)}$, $\mathcal{Z}_{\text{rest}\{2,\ldots,n\}}$, and $\mathcal{Z}_{\text{star}(1)}$ is homotopy equivalent to a wedge of spheres, and then identify the maps in the homotopy pushout in order to show that $\mathcal{Z}_K$ is also homotopy equivalent to a wedge of spheres.

This topological problem can be reformulated more generally for coordinate subspaces. We still assume that K is a shifted complex on the index set $[n]$. Let $X_1, \ldots, X_n$ be path-connected spaces. Let A be the coordinate subspace of $X_1 \times \cdots \times X_n$ associated to K. Then there is a homotopy pushout

$$
\begin{array}{ccc}
\text{Link}(1) & \longrightarrow & \text{Rest}\{2, \ldots, n\} \\
\downarrow & & \downarrow \\
\text{Star}(1) & \longrightarrow & A
\end{array}
\tag{15}
$$

where $\text{Star}(1) \simeq X_1 \times \text{Link}(1)$. Now compose each of the four corners with the inclusion $A \longrightarrow X_1 \times \cdots \times X_n$ and take homotopy fibres. Let F_L, F_S, F_R, and F_A be the homotopy fibres of the respective inclusions of $\text{Link}(1)$, $\text{Star}(1)$, $\text{Rest}\{2, \ldots, n\}$, and A into $X_1 \times \cdots \times X_n$. Then Lemma 3.1 says there is a homotopy pushout of fibres

$$
\begin{array}{ccc}
F_L & \longrightarrow & F_R \\
\downarrow & & \downarrow \\
F_S & \longrightarrow & F_A .
\end{array}
\tag{16}
$$

The homotopy pushout in (16) can be refined. First, consider the map $F_L \longrightarrow F_S$. As $\text{link}(1)$ is a simplicial complex on the vertices $\{2, \ldots, n\}$, the space $\text{Link}(1)$ is a coordinate subspace of $X_2 \times \cdots \times X_n$. Thus $F_L \simeq \Omega X_1 \times \overline{F}_L$, where $\overline{F}_L$ is the homotopy fibre of the inclusion $F_L \longrightarrow X_2 \times \cdots \times X_n$. Continuing, as $\text{Star}(1) \simeq X_1 \times \text{Link}(1)$, there is a homotopy pullback

$$
\begin{array}{ccccc}
\Omega X_1 \times \overline{F}_L & \longrightarrow & \text{Link}(1) & \longrightarrow & X_1 \times \cdots \times X_n \\
\downarrow & & \downarrow & & \| \\
F_S & \longrightarrow & X_1 \times \text{Link}(1) & \longrightarrow & X_1 \times \cdots \times X_n .
\end{array}
$$

As the map $\text{Link}(1) \longrightarrow X_1 \times \text{Link}(1)$ is the inclusion of the second factor, the previous homotopy pullback shows that $\overline{F}_L \simeq F_S$ and the map $\Omega X_1 \times \overline{F}_L \longrightarrow F_S$ is the projection. Next, consider the map $F_L \longrightarrow F_R$. As $\text{Rest}\{2, \ldots, n\}$ is a coordinate subspace of $X_2 \times \cdots \times X_n$, we have $F_R \simeq \Omega X_1 \times \overline{F}_R$, where $\overline{F}_R$ is the homotopy fibre of $\text{Rest}\{2, \ldots, n\} \longrightarrow X_2 \times \cdots \times X_n$. As $\text{Link}(1)$ is a subspace of $\text{Rest}\{2, \ldots, n\}$, the map $F_L \longrightarrow F_R$ becomes $\Omega X_1 \times F_S \xrightarrow{\;\Omega X_1 \times \gamma\;} \Omega X_1 \times \overline{F}_R$ for some map γ. Collecting all this information on the homotopy fibres, the homotopy pushout in diagram (16) becomes

a homotopy pushout

$$
\begin{array}{ccc}
\Omega X_1 \times F_S & \xrightarrow{\ \Omega X_1 \times \gamma\ } & \Omega X_1 \times \overline{F}_R \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
F_S & \xrightarrow{\hspace{2cm}} & F_A.
\end{array}
\tag{17}
$$

The goal is to identify the homotopy type of F_A. We do this in Theorem 9.4. It may be helpful to recall the definition of $\mathcal{G}_1^n$ in 7.3.

Theorem 9.4. *Let K be a shifted complex on the index set $[n]$. Let $X_1, \ldots, X_n$ be path-connected spaces and let A be the coordinate subspace of $X_1 \times \cdots \times X_n$ which corresponds to K. Use the notation and setup established in diagrams (15) and (17). Then the following hold:*

(a) $F_S \in \mathcal{G}_1^n$ *and* $\overline{F}_R \in \mathcal{G}_2^n$;

(b) *there are homotopy decompositions $F_S \simeq F_S^1 \vee F_S^2$ and $\overline{F}_R \simeq \overline{F}_R^1 \vee F_S^2$ such that $F_S^1, F_S^2 \in \mathcal{G}_1^n$, $\overline{F}_R^1 \in \mathcal{G}_2^n$ and there is a homotopy commutative diagram*

$$
\begin{array}{ccc}
F_S & \xrightarrow{\ \gamma\ } & \overline{F}_R \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
F_S^1 \vee F_S^2 & \xrightarrow{\ *\vee F_S^2\ } & \overline{F}_R^1 \vee F_S^2;
\end{array}
$$

(c) $F_A \in \mathcal{G}_1^n$.

Proof. We induct on n, the number of vertices. When $n = 1$, we have $A = X_1$, $\mathrm{Star}(1) = X_1$, $\mathrm{Rest}\{2, \ldots, n\} = *$, and $\mathrm{Link}(1) = *$. Composing into (the product space) X_1 and taking homotopy fibres, we immediately see that $F_S \simeq *$, $\overline{F}_R \simeq *$, γ is homotopic to the map from the basepoint to itself so part (b) trivially holds, and $F_A \simeq *$.

Assume the proposition holds for $n - 1$ vertices. First, applying Proposition 7.5(a) to the regular sequence from $X_1 \vee \cdots \vee X_n$ to $\mathrm{Star}(1)$ in Example 8.2 shows that $F_S \in \mathcal{G}_1^n$. Next, since $\mathrm{Rest}\{2, \ldots, n\}$ is a shifted complex on the vertices $\{2, \ldots, n\}$, the inductive hypothesis implies that $\overline{F}_R \in \mathcal{G}_2^n$. This proves part (a).

Assume part (b) for the moment. Lemma 3.5 applies to show there is a homotopy equivalence

$$
F_A \simeq F_S^2 \vee (\Omega X_1 * F_S^1) \vee (\Omega X_1 \ltimes \overline{F}_R^1).
$$

As $\overline{F}_R^1 \in \mathcal{G}_2^n$, it is a suspension and so $\Omega X_1 \ltimes \overline{F}_R^1 \simeq (\Omega X_1 \wedge \overline{F}_R^1) \vee \overline{F}_R^1$. Thus as $F_S^1, F_S^2 \in \mathcal{G}_1^n$ and $\overline{F}_R^1 \in \mathcal{G}_2^n$, we have $F_A \in \mathcal{G}_1^n$, proving part (c).

To prove part (b), we need to closely examine the map $F_S \xrightarrow{\gamma} \overline{F}_R$. This was defined in the setup for diagram (17) by a homotopy pullback

$$
\begin{array}{ccccc}
F_S & \longrightarrow & \mathrm{Link}(1) & \longrightarrow & X_2 \times \cdots \times X_n \\
\downarrow{\scriptstyle \gamma} & & \downarrow & & \| \\
\overline{F}_R & \longrightarrow & \mathrm{Rest}\{2, \ldots, n\} & \longrightarrow & X_2 \times \cdots \times X_n.
\end{array}
$$

By definition, $\text{Star}_R(2)$ is a coordinate subspace of $\text{Rest}\{2, \ldots, n\}$. In Example 8.3 we showed that $\text{Link}(1)$ is a coordinate subspace of $\text{Star}_R(2)$. Thus there is a diagram of iterated homotopy pullbacks

$$
\begin{array}{ccccc}
F_S & \longrightarrow & \text{Link}(1) & \longrightarrow & X_2 \times \cdots \times X_n \\
\downarrow{\scriptstyle \delta} & & \downarrow & & \| \\
F_{\overline{S}} & \longrightarrow & \text{Star}_R(2) & \longrightarrow & X_2 \times \cdots \times X_n \\
\downarrow{\scriptstyle \epsilon} & & \downarrow & & \| \\
\overline{F}_R & \longrightarrow & \text{Rest}\{2, \ldots, n\} & \longrightarrow & X_2 \times \cdots \times X_n
\end{array}
$$

where the pullbacks define the space $F_{\overline{S}}$ and the maps δ and ϵ. Hence $\gamma \simeq \epsilon \circ \delta$. We deal with each of δ and ϵ one at a time.

Applying Proposition 7.5(b) to the regular sequence from $\text{Link}(1)$ to $\text{Star}_R(2)$ in Example 8.3 shows that $F_S \simeq E_1 \vee E_2$, where $E_1, E_2 \in \mathcal{G}_1^n$, the restriction of δ to E_1 is null homotopic, and the restriction of δ to E_2 has a left homotopy inverse.

For ϵ, we appeal to the inductive hypothesis. Let $\text{link}_R(2)$ be $\text{link}(2)$ within $\text{rest}\{2, \ldots, n\}$. Since $\text{rest}\{2, \ldots, n\}$ is a shifted complex, it is the pushout of $\text{star}_R(2)$ and $\text{rest}\{3, \ldots, n\}$ over $\text{link}_R(2)$. This results in a homotopy pushout of the corresponding coordinate subspaces (in $X_2 \times \cdots \times X_n$)

$$
\begin{array}{ccc}
\text{Link}_R(2) & \longrightarrow & \text{Rest}\{3, \ldots, n\} \\
\downarrow & & \downarrow \\
\text{Star}_R(2) & \longrightarrow & \text{Rest}\{2, \ldots, n\}.
\end{array}
$$

Let $F_{\overline{L}}$ and $F_{\overline{R}}$ respectively be the homotopy fibres of the inclusions of $\text{Link}_R(2)$ and $\text{Rest}\{3, \ldots, n\}$ into $X_2 \times \cdots \times X_n$. Recall that $F_{\overline{S}}$ and $\overline{F}_R$ respectively have been defined as the homotopy fibres of the inclusions of $\text{Star}_R(2)$ and $\text{Rest}\{2, \ldots, n\}$ into $X_2 \times \cdots \times X_n$. As in diagram (16), when all four corners of the pushout above are mapped into $X_2 \times \cdots \times X_n$, we obtain a homotopy pushout of fibres

$$
\begin{array}{ccc}
F_{\overline{L}} & \longrightarrow & F_{\overline{R}} \\
\downarrow & & \downarrow \\
F_{\overline{S}} & \stackrel{\epsilon}{\longrightarrow} & \overline{F}_R.
\end{array}
$$

Arguing as for diagram (17), this homotopy pushout of fibres refines to a homotopy pushout

$$
\begin{array}{ccc}
\Omega X_2 \times F_{\overline{S}} & \stackrel{1 \times \overline{\gamma}}{\longrightarrow} & \Omega X_2 \times \overline{F}_{\overline{R}} \\
\downarrow & & \downarrow \\
F_{\overline{S}} & \stackrel{\epsilon}{\longrightarrow} & \overline{F}_R
\end{array}
$$

where $\overline{F}_{\overline{R}}$ is the homotopy fibre of the inclusion $\text{Rest}\{3, \ldots, n\} \longrightarrow X_3 \times \cdots \times X_n$. Since the underlying shifted complex $\text{rest}\{2, \ldots, n\}$ is on $n - 1$ vertices, by inductive hypothesis we can assume that there are homotopy decompositions $F_{\overline{S}} \simeq D_1 \vee D_2$ and $\overline{F}_{\overline{R}} \simeq D_3 \vee D_2$, where $D_1, D_2 \in \mathcal{G}_2^n$, $D_3 \in \mathcal{G}_3^n$, and under these decompositions $\overline{\gamma}$ becomes the map $D_1 \vee D_2 \xrightarrow{\ast \vee D_2} D_3 \vee D_2$. Applying

Lemma 3.5 we obtain a homotopy equivalence $\overline{F}_R \simeq D_2 \vee (\Omega X_2 * D_1) \vee (\Omega X_2 \ltimes D_3)$. Further, letting $M = (\Omega X_2 * D_1) \vee (\Omega X_2 \ltimes D_3)$, the map ϵ becomes $D_1 \vee D_2 \xrightarrow{*\vee D_2} M \vee D_2$.

Now consider the composite $\gamma : F_S \xrightarrow{\delta} F_{\overline{S}} \xrightarrow{\epsilon} \overline{F}_R$. The decomposition of $F_{\overline{S}}$ can be used to refine the decomposition $F_S \simeq E_1 \vee E_2$, as follows. Since the restriction of δ to E_2 has a left homotopy inverse, the decomposition $F_{\overline{S}} \simeq D_1 \vee D_2$ results in a decomposition $E_2 \simeq E_2^1 \vee E_2^2$, where E_2^i retracts off D_i. Let $F_S^1 = E^1 \vee E_2^1$, and let $F_S^2 = E_2^2$. Then $F_S \simeq F_S^1 \vee F_S^2$. Now combining the decompositions of F_S, $F_{\overline{S}}$, and $\overline{F}_R$ with their effects on δ and ϵ we obtain a homotopy commutative diagram

$$
\begin{array}{ccccc}
F_S & \xrightarrow{\delta} & F_{\overline{S}} & \xrightarrow{\epsilon} & \overline{F}_R \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
F_S^1 \vee F_S^2 & \xrightarrow{*\vee i} & D_1 \vee D_2 & \xrightarrow{*\vee D_2} & M \vee D_2
\end{array}
$$

where i has a left homotopy inverse. Let C be the homotopy cofibre of i. Then as $D_2 \in \mathcal{G}_2^n$ it is a suspension and so the fact that i has a left homotopy inverse implies that there is a homotopy decomposition $D_2 \simeq F_S^2 \vee C$. Let $\overline{F}_R^1 = M \vee C$. Then $\overline{F}_R \simeq \overline{F}_R^1 \vee F_S^2$ and the previous diagram shows that under this altered decomposition of $\overline{F}_R$ the map $\gamma \simeq \epsilon \circ \delta$ becomes $F_S^1 \vee F_S^2 \xrightarrow{*\vee F_S^2} \overline{F}_R^1 \vee F_S^2$, which proves part (b). $\quad\square$

With Theorem 9.4 in hand, we can prove Theorem 9.1 as a special case.

Proof of Theorem 9.1. In this case, each space X_i equals BT, the classifying space of the torus, the coordinate subspace A equals $DJ(K)$, and the homotopy fibre F_A equals $\mathcal{Z}_K$. Theorem 9.4(c) says that $\mathcal{Z}_K \in \mathcal{G}_1^n$, meaning that $\mathcal{Z}_K$ is homotopy equivalent to a wedge of summands of the form $\Omega BT_{i_1} * \cdots * \Omega BT_{i_m}$. Such a summand is homotopy equivalent to S^{m+1} since $\Omega BT \simeq S^1$. Thus $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, and so $K \in \mathcal{F}_0$. $\quad\square$

A special case of a shifted complex is the full i-skeleton $\Delta^i(n)$ of the standard simplex $\Delta(n)$ on n vertices. For path-connected spaces $X_1, \ldots, X_n$, let T_k^n be the coordinate subspace associated to $\Delta^{n-k}(n)$. Specifically,

$$T_k^n = \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n \mid \text{at least } k \text{ of the } x_i\text{'s are } *\}.$$

In particular, $T_0^n = X_1 \times \cdots \times X_n$, T_1^n is the fat wedge, $T_{n-1}^n = X_1 \vee \cdots \vee X_n$, and $T_n^n = *$. Let F_k^n be the homotopy fibre of the inclusion $T_k^n \longrightarrow T_0^n$. By Theorem 9.4, F_k^n is homotopy equivalent to a wedge of summands of the form $\Omega X_{i_1} * \cdots * \Omega X_{i_m}$, where $1 \leq i_1 < \cdots < i_m \leq n$. This wedge can be calculated explicitly using the iteration in Proposition 8.1 to reproduce a result first obtained in a different context by Porter [12]. For a space X and a positive integer j, let $j \cdot X$ be the wedge sum of j copies of X. Let $X^{(j)}$ be the j-fold smash of X with itself.

Theorem 9.5 (*Porter*). *For $n \geq 1$, let $X_1, \ldots, X_n$ be path-connected spaces. Let k be such that $1 \leq k \leq n-1$. Then there is a homotopy equivalence*

$$F_k^n \simeq \bigvee_{j=n-k+1}^{n} \left(\bigvee_{1 \leq i_1 < \cdots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k} \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_j} \right). \quad \square$$

Corollary 9.6. *As in Theorem 9.5, if $X_i = X$ for each $1 \leq i \leq n$ then there is a homotopy equivalence*

$$F_k^n \simeq \bigvee_{j=n-k+1}^{n} \binom{n}{j}\binom{j-1}{n-k} \Sigma^{n-k}(\Omega X)^{(j)}. \quad \square$$

The case of relevance to us is Corollary 9.6 applied when $X = \mathbb{C}P^{\infty}$. Then T_k^n corresponds to $DJ(K)$ for $K = \Delta^{n-k}(n)$. The homotopy fibre F_k^n corresponds to $\mathcal{Z}_K$. Since $\Omega X \simeq S^1$, we obtain:

Corollary 9.7. *If $K = \Delta^{n-k}(n)$, then*

$$\mathcal{Z}_K \simeq \bigvee_{j=n-k+1}^{n} \binom{n}{j}\binom{j-1}{n-k} S^{n-k+j}. \quad \square$$

We now return to Examples 9.2 and 9.3 to identify the homotopy type of $\mathcal{Z}_K$ for two shifted complexes K which are not skeletons of a standard simplex.

Example 9.8. Let K be the simplicial complex in Example 9.2. Observe that

  (i) star(1) consists of vertices $\{1, 2, 3, 4\}$ and edges $\{12, 13, 14\}$;
 (ii) rest$\{2, 3, 4\}$ consists of vertices $\{2, 3, 4\}$ and edges $\{23, 24\}$;
(iii) link(2) consists of vertices $\{2, 3, 4\}$ and no higher-dimensional simplices;
 (iv) star(2) coincides with rest$\{2, 3, 4\}$;
  (v) rest$\{3, 4\}$ consists of vertices $\{3, 4\}$ and no higher-dimensional simplices;
 (vi) link(2) coincides with rest$\{3, 4\}$.

The homotopy pushout for $\mathcal{Z}_K$ in (14) refines as in (17) to a homotopy pushout

$$
\begin{array}{ccc}
\Omega BT_1 \times \mathcal{Z}_{\text{link}(1)} & \xrightarrow{1 \times \gamma} & \Omega BT_1 \times \mathcal{Z}_{\text{rest}\{2,3,4\}} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
\mathcal{Z}_{\text{link}(1)} & \longrightarrow & \mathcal{Z}_K.
\end{array}
$$

As link(1) consists only of three vertices, Corollary 9.7 applied to $\Delta^0(3)$ shows that $\mathcal{Z}_{\text{link}(1)} \simeq 3S^3 \vee 2S^4$. Similarly, as rest$\{2, 3, 4\}$ consists only of two vertices, Corollary 9.7 applied to $\Delta^0(3)$ shows that $\mathcal{Z}_{\text{rest}\{2,3,4\}} \simeq S^3$. Keeping track of coordinates in Theorem 9.4(b) shows that γ sends two of the S^3 summands and both S^4 summands of $\mathcal{Z}_{\text{link}(1)}$ to the basepoint and it sends the remaining S^3 summand identically onto itself. Substituting, the preceding homotopy pushout becomes

$$
\begin{array}{ccc}
S^1 \times (S^3 \vee 2S^3 \vee 2S^4) & \xrightarrow{1 \times (1 \vee * \vee *)} & S^1 \times S^3 \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
3S^3 \vee 2S^4 & \longrightarrow & \mathcal{Z}_K.
\end{array}
$$

Lemma 3.4 then says that $\mathcal{Z}_K \simeq (S^1 * (2S^3 \vee 2S^4)) \vee S^3 \simeq S^3 \vee 2S^5 \vee 2S^6$.

Example 9.9. Let K be the shifted complex in Example 9.3. Observe that

(i) star(1) consists of vertices $\{1, 2, 3, 4, 5\}$ and edges $\{12, 13, 14, 15\}$;

(ii) rest$\{2, 3, 4, 5\}$ is the simplicial complex discussed in Examples 9.2 and 9.8;

(iii) link(1) consists of vertices $\{2, 3, 4, 5\}$ and no higher-dimensional simplices.

As in Example 9.8, there is a homotopy pushout

$$
\begin{array}{ccc}
\Omega BT_1 \times \mathcal{Z}_{\text{link}(1)} & \xrightarrow{\ 1 \times \gamma\ } & \Omega BT_1 \times \mathcal{Z}_{\text{rest}\{2,3,4,5\}} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
\mathcal{Z}_{\text{link}(1)} & \longrightarrow & \mathcal{Z}_K.
\end{array}
$$

As link(1) consists only of four vertices, Corollary 9.7 applied to $\Delta^0(4)$ shows that $\mathcal{Z}_{\text{link}(1)} \simeq 6S^3 \vee 8S^4 \vee 3S^5$. By Example 9.8, $\mathcal{Z}_{\text{rest}\{2,3,4,5\}} \simeq S^3 \vee 2S^5 \vee 2S^6$. Keeping track of the coordinates in Theorem 9.4(b) shows that γ sends five of the S^3 summands and all the S^4 and S^5 summands to the basepoint, and it sends the remaining S^3 by the identity map onto the S^3 summand of $\mathcal{Z}_{\text{rest}\{2,3,4,5\}}$. Substituting, the preceding homotopy pushout becomes

$$
\begin{array}{ccc}
S^1 \times (S^3 \vee 5S^3 \vee 8S^4 \vee 3S^5) & \xrightarrow{\ 1 \times (1 \vee * \vee * \vee *)\ } & S^1 \times (S^3 \vee 2S^5 \vee 2S^6) \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
6S^3 \vee 8S^4 \vee 3S^5 & \longrightarrow & \mathcal{Z}_K.
\end{array}
$$

Applying Lemma 3.5 then shows that

$$
\mathcal{Z}_K \simeq (S^1 * (5S^3 \vee 8S^4 \vee 3S^5)) \vee (S^1 \ltimes (2S^5 \vee 2S^6)) \vee S^3 \simeq S^3 \vee 7S^5 \vee 12S^6 \vee 5S^7.
$$

10. Topological extensions

At this point, we have shown that if a simplicial complex K is shifted, then its moment-angle complex $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. Next, we want to consider other non-shifted simplicial complexes K for which $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, or for which $\Sigma\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. Note again that torsion can occur in the cohomology ring of $\mathcal{Z}_K$ for certain simplicial complexes K, making it impossible for $\mathcal{Z}_K$ to be even stably homotopic to a wedge of spheres.

We consider how three combinatorial operations – the disjoint union of simplicial complexes, gluing along a common face and the join of simplicial complexes – alter the homotopy type of the moment-angle complex. Recall that for given simplicial complexes K_1 and K_2 on sets $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, the *join* $K_1 * K_2$ is the simplicial complex

$$
K_1 * K_2 := \{\sigma \subset \mathcal{S}_1 \cup \mathcal{S}_2 \mid \sigma = \sigma_1 \cup \sigma_2, \sigma_1 \in K_1, \sigma_2 \in K_1\}
$$

on the set $\mathcal{S}_1 \cup \mathcal{S}_2$.

Theorem 10.1. *Let K_1 and K_2 be simplicial complexes such that $\mathcal{Z}_{K_1}$ and $\mathcal{Z}_{K_2}$ are homotopy equivalent to wedges of spheres. Then the following hold:*

(1) *if $K = K_1 \bigcup_\sigma K_2$ is obtained by gluing along a common face, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres;*

(2) *if $K = K_1 \coprod K_2$ is the disjoint union of simplicial complexes, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres;*

(3) *if $K = K_1 * K_2$ is the join of simplicial complexes, then $\mathcal{Z}_K$ is not homotopy equivalent to a wedge of spheres but $\Sigma \mathcal{Z}_K$ is.*

Proof. (1) Let $DJ(K_i), i = 1, 2, DJ(\sigma)$ and $DJ(K)$ be the corresponding Davis–Januszkiewicz spaces. Each vertex in K_i, σ or K corresponds to a coordinate in $DJ(K_i), DJ(\sigma)$ or $DJ(K)$ respectively. List the vertices of K_1 as $\{1, \ldots, l, \ldots, m\}$, where the vertices of σ are $\{l + 1, \ldots, m\}$. List the vertices of K_2 as $\{l + 1, \ldots, m, \ldots, n\}$. Regard $DJ(K_1)$ as a subspace of $\prod_{i=1}^m \mathbb{C}P^\infty$. Let D_1 be the image of $DJ(K_1)$ under the map $\prod_{i=1}^m \mathbb{C}P^\infty \longrightarrow \prod_{i=1}^n \mathbb{C}P^\infty$ given by the inclusion of the first m coordinates. Similarly, regard $DJ(K_2)$ as a subspace of $\prod_{i=l+1}^n \mathbb{C}P^\infty$, and let D_2 be its image under the map $\prod_{i=l+1}^n \mathbb{C}P^\infty \longrightarrow \prod_{i=1}^n \mathbb{C}P^\infty$ given by the inclusion of the last $n - l$ coordinates. Since σ is a simplex, $DJ(\sigma)$ is a product of $m - l$ copies of $\mathbb{C}P^\infty$. Let D_3 be the image of $DJ(\sigma)$ in $\prod_{i=1}^n \mathbb{C}P^\infty$ under the map $\prod_{i=l+1}^m \mathbb{C}P^\infty \longrightarrow \prod_{i=1}^n \mathbb{C}P^\infty$ given by the inclusion of the middle $m - l$ coordinates. Let D be the topological pushout

$$
\begin{array}{ccc}
D_3 & \longrightarrow & D_1 \\
\downarrow & & \downarrow \\
D_2 & \longrightarrow & D.
\end{array}
\tag{18}
$$

Then $D = DJ(K)$ and is a subspace of $\prod_{i=1}^n \mathbb{C}P^\infty$.

For notational convenience, let $BT^n = \prod_{i=1}^n \mathbb{C}P^\infty$. Map each of the four corners of pushout (18) into BT^n and take homotopy fibres. This gives homotopy fibrations

$$
\begin{aligned}
F &\longrightarrow D \longrightarrow BT^n \\
F_1 \times N &\longrightarrow D_1 \longrightarrow BT^n \\
M \times F_2 &\longrightarrow D_2 \longrightarrow BT^n \\
M \times N &\longrightarrow D_3 \longrightarrow BT^n
\end{aligned}
$$

where the first homotopy fibration defines F, F_1 is the homotopy fibre of $D_1 \longrightarrow \prod_{i=1}^m \mathbb{C}P^\infty$, F_2 is the homotopy fibre of $D_2 \longrightarrow \prod_{i=l+1}^n \mathbb{C}P^\infty$, $M = \prod_{i=1}^l S^1$, and $N = \prod_{i=m+1}^n S^1$. Including D_3 into D_1 gives a homotopy pullback diagram

$$
\begin{array}{ccccccc}
\Omega BT^n & \longrightarrow & M \times N & \longrightarrow & D_3 & \longrightarrow & BT^n \\
\Big\| & & \downarrow{\scriptstyle \theta} & & \downarrow & & \Big\| \\
\Omega BT^n & \longrightarrow & F_1 \times N & \longrightarrow & D_1 & \longrightarrow & BT^n
\end{array}
$$

for some map θ of fibres. We now identify θ. With $BT^m = \prod_{i=1}^m \mathbb{C}P^\infty$, the pullback just described is the product of the homotopy pullback

$$\begin{array}{ccccccc}
\Omega BT^m & \longrightarrow & M & \longrightarrow & D_3 & \longrightarrow & BT^m \\
\| & & \downarrow{\scriptstyle\theta'} & & \downarrow & & \| \\
\Omega BT^m & \longrightarrow & F_1 & \longrightarrow & D_1 & \longrightarrow & BT^m
\end{array}$$

and the path-loop fibration $N \longrightarrow * \longrightarrow \prod_{i=m+1}^{n} \mathbb{C}P^\infty$. So $\theta = \theta' \times N$. Further, $M = \prod_{i=1}^{l} S^1$ is a retract of $\Omega BT^m \simeq \prod_{i=1}^{m} S^1$ and $\Omega BT^m \longrightarrow F_1$ is null homotopic since ΩBT^m is a retract of $\Omega D_1 = \Omega DJ(K_1)$. Hence $\theta' \simeq *$ and so $\theta \simeq * \times N$. A similar argument for the inclusion of D_3 into D_2 shows that the map of fibres $M \times N \longrightarrow M \times F_2$ is homotopic to $M \times *$.

Collecting all this information about homotopy fibres, Lemma 3.1 shows that there is a homotopy pushout

$$\begin{array}{ccc}
M \times N & \xrightarrow{\ *\times N\ } & F_1 \times N \\
\downarrow{\scriptstyle M\times*} & & \downarrow \\
M \times F_2 & \longrightarrow & F.
\end{array}$$

Lemma 3.3 then gives a homotopy decomposition

$$F \simeq (M * N) \vee (M \ltimes F_2) \vee (F_1 \rtimes N).$$

To show that F is homotopy equivalent to a wedge of spheres, we show that each of $M * N$, $M \ltimes F_2$, and $F_1 \rtimes N$ are homotopy equivalent to wedges of spheres. First, observe that the suspension of a product of spheres is homotopy equivalent to a wedge of spheres. Since M and N are products of copies of S^1, $M * N$ is therefore homotopy equivalent to a wedge of spheres. Second, as F_2 is homotopy equivalent to a wedge of (connected) spheres we can write $F_2 \simeq \Sigma F_2'$, where F_2' is a wedge of spheres which possibly includes copies of S^0. We then have $M \ltimes F_2 \simeq M \ltimes (\Sigma F_2') \simeq \Sigma M \vee (\Sigma M \wedge F_2')$. Now ΣM is homotopy equivalent to a wedge of spheres. This also implies that $\Sigma M \wedge F_2'$ is homotopy equivalent to a wedge of suspensions of F_2'. That is, $M \wedge F_2 \simeq \Sigma M \wedge F_2'$ is homotopy equivalent to a wedge of copies of F_2 and suspensions of F_2. Therefore, as F_2 is homotopy equivalent to a wedge of spheres so is $M \wedge F_2$. Hence $M \ltimes F_2$ is homotopy equivalent to a wedge of spheres. The decomposition of the summand $F_1 \rtimes N$ into a wedge of spheres is exactly as for $M \ltimes F_2$.

(2) Let $K = K_1 \coprod K_2$ be the disjoint union of two simplicial complexes K_1 and K_2 on the index sets $[m]$ and $[n]$ respectively. Then their disjoint union $K = K_1 \coprod K_2$ is a simplicial complex on the index set $[m+n]$ obtained as the result of gluing K_1 to K_2 along the empty face. Applying part (1) then shows that $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. Moreover, the homotopy type of $\mathcal{Z}_K$ is given by

$$\mathcal{Z}_K \simeq \left(\prod_{i=1}^{m} S^1 * \prod_{j=1}^{n} S^1 \right) \vee \left(\mathcal{Z}_{K_1} \rtimes \prod_{i=1}^{n} S^1 \right) \vee \left(\prod_{i=1}^{m} S^1 \ltimes \mathcal{Z}_{K_2} \right).$$

(3) The Davis–Januszkiewicz space of the join $K = K_1 * K_2$ of two simplicial complexes K_1 and K_2 on the index sets $[m]$ and $[n]$ has the following form:

$$\begin{aligned}
DJ(K) = \bigcup_{\sigma \in K} BT^\sigma &= \bigcup_{\sigma_1 \cup \sigma_2 \in K} BT^{\sigma_1} \times BT^{\sigma_2} = \left(\bigcup_{\sigma_1 \in K_1} BT^{\sigma_1} \right) \times \left(\bigcup_{\sigma_2 \in K_2} BT^{\sigma_2} \right) \\
&= DJ(K_1) \times DJ(K_2).
\end{aligned}$$

Therefore the fibration

$$DJ(K_1 * K_2) \longrightarrow BT^{m+n}$$

associated to the join of K_1 and K_2 is the product fibration

$$DJ(K_1) \times DJ(K_2) \longrightarrow BT^m \times BT^n.$$

Hence $\mathcal{Z}_{K_1 * K_2} \simeq \mathcal{Z}_{K_1} \times \mathcal{Z}_{K_2}$, immediately implying part (3). (In fact, the homotopy equivalence between $\mathcal{Z}_{K_1 * K_2}$ and $\mathcal{Z}_{K_1} \times \mathcal{Z}_{K_2}$ can be improved to a homeomorphism. See [4, Lemma 2.1.4].)   □

Theorem 10.1 can be applied to two shifted complexes K_1 and K_2. However, the simplicial complex K in parts (1) and (2) need not be shifted. So the Theorem substantially extends the family of simplicial complexes for which the moment-angle complex is homotopy equivalent to a wedge of spheres.

Theorem 10.1 can also be useful for calculating the homotopy types of moment-angle complexes $\mathcal{Z}_K$. To illustrate, we give an alternative calculation of that in Example 9.8.

Example 10.2. Let K be the shifted complex in Example 9.2. Then K is obtained by gluing two copies of the 1-skeleton of the standard simplex $\Delta(3)$ (on three vertices) along a common edge. Specifically, $K = K_1 \cup_\sigma K_2$, where K_1 consists of vertices $\{1, 2, 3\}$ and edges $\{12, 13, 23\}$, K_2 consists of vertices $\{2, 3, 4\}$ and edges $\{23, 24, 34\}$, and σ is the common edge $\{23\}$. Using the notation in the proof of Theorem 10.1(a), the formula $F \simeq (M * N) \vee (M \ltimes F_2) \vee (F_1 \rtimes N)$ corresponds to $\mathcal{Z}_K \simeq (S^1 * S^1) \vee (S^1 \ltimes \mathcal{Z}_{K_2}) \vee (\mathcal{Z}_{K_1} \ltimes S^1)$. Both K_1 and K_2 are copies of $\Delta^1(3)$ so Corollary 9.7 says that $\mathcal{Z}_{K_1} \simeq \mathcal{Z}_{K_2} \simeq S^5$. Hence $\mathcal{Z}_K \simeq S^3 \vee 2S^5 \vee 2S^6$.

11. Algebra

Let A be a polynomial ring on n variables $k[x_1, \ldots, x_n]$ over a field k and let $R = A/I$, where I is homogeneous ideal. In this section we shall be interested in the nature of $\mathrm{Tor}_R(k, k)$: specifically, in identifying a class of rings R for which all Massey products in $\mathrm{Tor}_A(R, k)$ vanish and how this impacts upon the Poincaré series of R. Recall that the Poincaré series of R is the formal power series

$$P(R) = \sum_{i=0}^{\infty} b_i t^i$$

where $b_i = \dim_k \mathrm{Tor}_R^i(k, k)$ are the Betti numbers of R. It has been conjectured by Kaplansky and Serre that $P(R)$ is always a rational function. The regular local rings were the first rings for which $P(R)$ was explicitly computed. In this case Serre [14] showed that $P(R) = (1 + t)^n$. Tate [15] showed that if R is a complete intersection, then there exist non-negative integers m, n such that

$$P(R) = \frac{(1 + t)^n}{(1 - t^2)^m}.$$

Golod [8] made a far-reaching contribution to the problem by showing that if certain homology operations on the Koszul complex vanish, then there exist non-negative integers $n, c_1, \ldots, c_n$ such that

$$P(R) = \frac{(1 + t)^n}{1 - \sum_{i=1}^{n} c_i t^{i+1}}.$$

In general not much is known about the rationality of $P(R)$, although there is an inequality due to Golod [8] showing that $P(R)$ is always bounded (coefficient-wise) by a rational function.

In the past, describing various properties of $\text{Tor}_R(k, k)$ has been largely an algebraic problem. Further on, we translate the problem of rationality of the Poincaré series into topology by using recent results of toric topology. Then by using our results on the homotopy type of the complement of a coordinate subspace arrangement, we find a class of rings R for which $P(R)$ is a rational function determined by $P(\text{Tor}_A(R, k))$.

In what follows R will be the Stanley–Reisner ring $k[K]$ of an arbitrary simplicial complex K on n vertices. Recall from Definition 1.4 that the Stanley–Reisner ring $k[K]$ is *Golod* if all Massey products in $\text{Tor}_{k[v_1,...,v_n]}(k[K], k)$ vanish. Buchstaber and Panov [1] proved that

$$\text{Tor}^*_{k[K]}(k, k) \cong H^*(\Omega DJ(K); k).$$

This isomorphism now lets us exploit the topological properties of the loop space $\Omega DJ(K)$ to obtain further information about $\text{Tor}_R(k, k)$. Looking at the split fibration

$$\Omega \mathcal{Z}_K \longrightarrow \Omega DJ(K) \longrightarrow T^n$$

we have

$$\text{Tor}^*_R(k, k) \cong H^*(\Omega DJ(K); k) = H^*(T^n; k) \otimes H^*(\Omega \mathcal{Z}_K; k).$$

A calculation using the bar resolution shows that

$$P(H^*(\Omega \mathcal{Z}_K; k)) \leq P(T(\Sigma^{-1} H^*(\mathcal{Z}_K; k)))$$

where $\Sigma^{-1} H^*(\mathcal{Z}_K; k)$ is the desuspension of the module $H^*(\mathcal{Z}_K; k)$. Therefore

$$P(R) \leq (1 + t)^n P(T(\Sigma^{-1} H^*(\mathcal{Z}_K; k))) = \frac{t(1 + t)^n}{t - P(H^*(\mathcal{Z}_K; k))}.$$

Looking at the Eilenberg–More spectral sequence (the bar resolution) that computes the cohomology of the fibre in the path-loop fibration $\Omega \mathcal{Z}_K \longrightarrow * \longrightarrow \mathcal{Z}_K$, we conclude that the above equality is reached when the differentials are trivial. According to May, the differentials are determined by the Massey products and therefore they are trivial when all the Massey products in $H^*(\mathcal{Z}_K)$ vanish. As $H^*(\mathcal{Z}_K; k) \cong \text{Tor}_{k[v_1,...,v_n]}(k[K], k)$ [1], an equality for $P(R)$ is obtained when the Stanley–Reisner ring $k[K]$ is Golod. This proves the following theorem.

Theorem 11.1. *For a simplicial complex K,*

$$P(k[K]) \leq \frac{t(1 + t)^n}{t - P(H^*(\mathcal{Z}_K; k))}. \tag{19}$$

Equality is obtained when $k[K]$ is Golod.

We proceed by describing a new class of Golod rings using topological methods.

Theorem 11.2. *If $K \in \mathcal{F}_0$, then $k[K]$ is a Golod ring.*

Proof. By definition of the family $\mathcal{F}_0$, when $K \in \mathcal{F}_0$ then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. Therefore in the cohomology of $\mathcal{Z}_K$ all cup products and higher Massey products are trivial. On

the other hand, recall that Buchstaber and Panov [1] proved that

$$H^*(\mathcal{Z}_K; k) \cong \mathrm{Tor}_{k[v_1,\ldots,v_n]}(k[K], k).$$

Therefore in $\mathrm{Tor}_{k[v_1,\ldots,v_n]}(k[K], k)$ all Massey products are trivial. Now by definition, the ring $k[K]$ is Golod. $\quad\square$

We finish by proving that the Poincaré series of a ring belonging to the class defined in Theorem 11.2 is a rational function.

Corollary 11.3. *If $K \in \mathcal{F}_0$, then the Poincaré series of the ring $k[K]$ has the following form*

$$P(k[K]) = \frac{t(1 + t)^n}{t - P(H^*(\mathcal{Z}_K; k))}.$$

Proof. As $k[K]$ is a Golod ring, in (19) equality holds. $\quad\square$

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