Flux Difference Splitting for Hyperbolic Systems of Conservation Laws with Source Terms

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Abstract—The second order accurate scalar scheme given in [1] is extended to hyperbolic systems of conservation laws with source terms using flux-difference splitting. Examples of the scheme are given for the shallow water equations and the Euler equations for duct flow.

1. INTRODUCTION

In a recent paper [1], a second order, upwind difference scheme was presented for scalar conservation laws with source terms and represented an extension of well-known schemes for homogeneous scalar laws. The scheme was completely determined, including the evaluation of the source terms. In this paper, we show how this scheme can be extended further to hyperbolic systems of conservation laws. The resulting upwind scheme is second order accurate, and examples of its application to the shallow water equations and the Euler equations of duct flow are provided. There remains the question of how to allow the creation of new extrema which are physical, whilst suppressing non-physical oscillations, and research on this is currently being undertaken. We note that current schemes in this area [2,3] are not second order accurate, but that part of the scheme arising from the flux terms is second order accurate. Moreover, both the flux and source terms are limited to maintain monotonicity, but it is not clear that this is the correct procedure that should be followed. We begin by recalling the first order and second order schemes for scalar equations given in [1].

2. SCALAR SCHEMES

Consider the hyperbolic problem

\[ u_t + f_x = h(x, u), \]  

for the function \( u = u(x, t) \) where \( f = f(u) \) is a convex flux function and the source term contains no derivatives of \( u \).

2.1. First Order Schemes

If we define a grid \( x_j = x_{j-1} + \Delta x \) in the \( x \)-direction with constant mesh spacing \( \Delta x \), and a grid in the \( t \)-direction \( t_n = t_{n-1} + \Delta t \) with mesh spacing \( \Delta t \), and denote by \( u^n_j \) an approximation to \( u(x_j, t_n) \), then the first order scheme in [1] is given by

\[ \]
\[ u_{j+1}^{n+1} = u_j^n - \nu^{-}_{j+\frac{1}{2}} (u_j^n - u_{j-1}^n) - \nu^{+}_{j+\frac{1}{2}} (u_{j+1}^n - u_j^n) \]
\[ + \Delta t \frac{\nu^{+}_{j+\frac{1}{2}}}{\nu^{-}_{j+\frac{1}{2}}} h(x_j, u_j^n) + \Delta t \frac{\nu^{+}_{j+\frac{1}{2}}}{\nu^{-}_{j+\frac{1}{2}}} h(x_{j+1}, u_{j+1}^n), \] (2.2)

where

\[ \nu_{j-\frac{1}{2}} = a_{j-\frac{1}{2}} \frac{\Delta t}{\Delta x}, \] (2.3)

\[ a_{j-\frac{1}{2}} = \begin{cases} f(u_{j-1}^n) - f(u_{j-1}^{n-1}) & u_{j-1}^n \neq u_{j-1}^{n-1}, \\ u_j^n - u_{j-1}^n & u_{j-1}^n = u_{j-1}^{n-1}, \end{cases} \] (2.4a,b)

is an approximation to the wavespeed \( a(u) = f'(u) \) in \([x_{j-1}, x_j]\), and

\[ \nu^{\pm}_{j-\frac{1}{2}} = a^{\pm}_{j-\frac{1}{2}} \frac{\Delta t}{\Delta x}, \] (2.5)

with

\[ p^{\pm} = \frac{1}{2} (p \pm |p|). \] (2.6)

This scheme can also be written cell-wise as

\[ u_{j-1}^{n+1} = u_{j-1}^n, \]
\[ u_j^{n+1} = u_j^n + \Delta t h(x_j, u_j^n) - \nu_{j-\frac{1}{2}} (u_j^n - u_{j-1}^n), \]
\[ v_{j-1}^{n+1} = v_{j-1}^n + \Delta t h(x_{j-1}, u_{j-1}^n) - \nu_{j-\frac{1}{2}} (u_{j-1}^n - u_{j-2}^n), \]
\[ u_{j+1}^{n+1} = u_{j+1}^n, \]

and a schematic representation of this is given in Figure 1.

\[ Figure 1. \text{Schematic representation of the scalar first order scheme.} \]
2.2. Second Order Scheme

The second order scheme in [1] is given by

$$u_{j}^{n+1} = u_{j}^{n} + \left( 1 + \frac{\Delta t}{2} (h_{u})_{j-\frac{1}{2}} \right) \left( \Delta t \frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}^{-}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-} (u_{j}^{n} - u_{j-1}^{n}) \right)$$

$$+ \left( 1 + \frac{\Delta t}{2} (h_{u})_{j+\frac{1}{2}} \right) \left( \Delta t \frac{\nu_{j+\frac{1}{2}}^{+}}{\nu_{j+\frac{1}{2}}^{-}} h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}^{-} (u_{j+1}^{n} - u_{j}^{n}) \right)$$

$$- \frac{1}{2} \left( 1 - |\nu_{j-\frac{1}{2}}| \right) \Delta t \left( \frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}^{-}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-} (u_{j}^{n} - u_{j-1}^{n}) \right)$$

$$+ \frac{1}{2} \left( 1 - |\nu_{j+\frac{1}{2}}| \right) \Delta t \left( \frac{\nu_{j+\frac{1}{2}}^{+}}{\nu_{j+\frac{1}{2}}^{-}} h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}^{-} (u_{j+1}^{n} - u_{j}^{n}) \right)$$

$$- \frac{1}{2} \left( 1 - |\nu_{j+\frac{1}{2}}| \right) \Delta t \left( \frac{\nu_{j+\frac{1}{2}}^{+}}{\nu_{j+\frac{1}{2}}^{-}} h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}^{-} (u_{j+1}^{n} - u_{j}^{n}) \right),$$

(2.8)

where

$$h_{j+\frac{1}{2}} = h \left( x_{j} \pm \frac{\Delta x}{2}, u_{j}^{n} \right), \quad \text{and} \quad h_{j}^{n} = h_{u}(x_{j}, u_{j}^{n}).$$

Written cell-wise, this scheme becomes

$$u_{j-1}^{n+1} = u_{j-1}^{n} + \left( 1 + \frac{\Delta t}{2} (h_{u})_{j-\frac{1}{2}} \right) \left( \Delta t \frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}^{-}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-} (u_{j}^{n} - u_{j-1}^{n}) \right)$$

$$- \frac{1}{2} \left( 1 - |\nu_{j-\frac{1}{2}}| \right) \Delta t \left( \frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}^{-}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-} (u_{j}^{n} - u_{j-1}^{n}) \right)$$

$$+ \frac{1}{2} \left( 1 - |\nu_{j+\frac{1}{2}}| \right) \Delta t \left( \frac{\nu_{j+\frac{1}{2}}^{+}}{\nu_{j+\frac{1}{2}}^{-}} h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}^{-} (u_{j+1}^{n} - u_{j}^{n}) \right),$$

(2.10a)

$$u_{j}^{n+1} = u_{j}^{n} + \left( 1 + \frac{\Delta t}{2} (h_{u})_{j-\frac{1}{2}} \right) \left( \Delta t \frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}^{-}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-} (u_{j}^{n} - u_{j-1}^{n}) \right)$$

$$- \frac{1}{2} \left( 1 - |\nu_{j-\frac{1}{2}}| \right) \Delta t \left( \frac{\nu_{j-\frac{1}{2}}^{+}}{\nu_{j-\frac{1}{2}}^{-}} h_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}}^{-} (u_{j}^{n} - u_{j-1}^{n}) \right)$$

$$+ \frac{1}{2} \left( 1 - |\nu_{j+\frac{1}{2}}| \right) \Delta t \left( \frac{\nu_{j+\frac{1}{2}}^{+}}{\nu_{j+\frac{1}{2}}^{-}} h_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}}^{-} (u_{j+1}^{n} - u_{j}^{n}) \right).$$

(2.10b)

Finally, as an increment stage $\psi$, together with a transfer stage $c$, the second order scheme is given by

$$u_{j-1}^{n+1} = u_{j-1}^{n} + c_{j-\frac{1}{2}}, \quad \nu_{j-\frac{1}{2}} > 0,$$

$$u_{j}^{n+1} = u_{j}^{n} + \psi_{j-\frac{1}{2}} - c_{j-\frac{1}{2}}, \quad \nu_{j-\frac{1}{2}} < 0,$$

$$u_{j+1}^{n+1} = u_{j+1}^{n} + \psi_{j+\frac{1}{2}} - c_{j+\frac{1}{2}},$$

$$u_{j}^{n+1} = u_{j}^{n} + c_{j-\frac{1}{2}},$$

(2.11b)
where
\[ \psi_{j+\frac{1}{2}} = \left(1 + \frac{\Delta t}{2} (h_u)_{j+\frac{1}{2}} \right) \left( \Delta t (c_{j+\frac{1}{2}} - \nu_{j+\frac{1}{2}} (u_j^n - u_{j-1}^n)) \right), \]
\[ c_{j-\frac{1}{2}} = -\frac{1}{2} \left(1 - |\nu_{j-\frac{1}{2}}| + \frac{\Delta t}{2} (h_u)_{j-\frac{1}{2}} \right) \left( \Delta t (c_{j-\frac{1}{2}} - \nu_{j-\frac{1}{2}} (u_j^n - u_{j-1}^n)) \right). \]

(See Figure 2 for a schematic representation.) It is the form given by (2.11a–d) that is used for computational purposes, and we extend this to systems in Section 6.

3. SYSTEMS OF CONSERVATION LAWS WITH SOURCE TERMS

In the remaining sections, we apply the first and second order schemes of Section 2 to systems of conservation laws with source terms.

Consider the system of hyperbolic conservation laws
\[ w_t + f_z = g(x, t), \]
for the function \( w = w(x, t) \), where \( f = f(w) \) and where the source term \( g = g(x, w) \) contains no derivatives of \( w \). We assume that the approximate solution of (3.1) is sought by solving the Riemann problem
\[ w(\psi_{j-\frac{1}{2}}, \psi_{j+\frac{1}{2}}) = g(x, w), \quad (x, t) \in (x_{j-1}, x_j) \times (t_n, t_{n+1}), \]
where \( \tilde{A}(w_{j-1}^n, w_j^n) \) is an approximation to the Jacobian matrix \( A(w) = \frac{\partial f}{\partial w}(w) \) and \( w_{j-1}^n, w_j^n \) represent piecewise constant states, i.e.,
\[ w(x, t_n) = \begin{cases} 
  w_{j-1}^n, & x \in (x_{j-1} - \frac{\Delta x}{2}, x_{j-1} + \frac{\Delta x}{2}) \\
  w_j^n, & x \in (x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2}). 
\end{cases} \]
4. FIRST ORDER SCHEME BY DIAGONALISATION

We begin by applying the first order scheme of Section 2.1 to (3.2) by diagonalising the matrix $A$.

Consider the cell $[x_{j-1}, x_j]$ and suppose that the approximate Jacobian matrix $\tilde{A}(w_{j-1}^n, w_j^n) = \tilde{A}_{j-\frac{1}{2}}$ has $m$ eigenvalues $\tilde{\lambda}_{j-\frac{1}{2}}, i = 1, \ldots, m$, with corresponding linearly independent eigenvectors $\tilde{e}_{j-\frac{1}{2}}, i = 1, \ldots, m$. If we write

$$\tilde{X}_{j-\frac{1}{2}} = [\tilde{e}_{j-\frac{1}{2}}, \ldots, m\tilde{e}_{j-\frac{1}{2}}]$$

as the modal matrix, then it is well-known that

$$\tilde{X}_{j-\frac{1}{2}}^{-1} \tilde{A}_{j-\frac{1}{2}} \tilde{X}_{j-\frac{1}{2}} = \tilde{\Lambda}_{j-\frac{1}{2}},$$

where

$$\tilde{\Lambda}_{j-\frac{1}{2}} = \text{diag}(\tilde{\lambda}_{j-\frac{1}{2}}, \ldots, m\tilde{\lambda}_{j-\frac{1}{2}})$$

is a diagonal matrix. Thus, if we define the independent variable $v$ by

$$v = \tilde{X}_{j-\frac{1}{2}}^{-1} w, \quad (x, t) \in [x_{j-1}, x_j] \times [t_n, t_{n+1}],$$

then equation (3.2) becomes

$$v_t + \tilde{\Lambda}_{j-\frac{1}{2}} v_x = h(z, v),$$

where

$$\tilde{X}_{j-\frac{1}{2}}^{-1} g(x, w) = \tilde{X}_{j-\frac{1}{2}}^{-1} g(x, \tilde{X}_{j-\frac{1}{2}} v) = h(x, v).$$

Equation (4.5) represents the set of scalar problems

$$\frac{\partial}{\partial t}(v_i) + \tilde{\Lambda}_{j-\frac{1}{2}} \frac{\partial}{\partial x}(v_i) = h_i(x, v_1, \ldots, v_m), \quad i = 1, \ldots, m,$$

where $v = (v_1, \ldots, v_m)^\top$. Equation (4.7) can now be solved using the first order upwind scheme given by (2.2), where we identify $\tilde{\lambda}_{j-\frac{1}{2}}$ with the approximation to $a_{j-\frac{1}{2}}$ for each $i$. Thus, the scheme for (4.7) cell-wise is

$$v_{j-1}^{n+1} = v_{j-1}^n - \Delta t \frac{\partial}{\partial x} \tilde{\lambda}_{j-\frac{1}{2}} (v_j^n - v_{j-1}^n) + \Delta t \frac{\partial}{\partial x} \tilde{\lambda}_{j-\frac{1}{2}} h_i(x_j, v_j^n, \ldots, v_m^n),$$

$$v_j^{n+1} = v_j^n - \Delta t \frac{\partial}{\partial x} \tilde{\lambda}_{j-\frac{1}{2}} (v_j^n - v_{j-1}^n) + \Delta t \frac{\partial}{\partial x} \tilde{\lambda}_{j-\frac{1}{2}} h_i(x_j, v_j^n, \ldots, v_m^n),$$

for $i = 1, \ldots, m$, where $\tilde{\lambda}_{j-\frac{1}{2}}^\pm$ are defined by (2.6). Equations (4.8a,b) can now be written in system form as

$$v_{j-1}^{n+1} = v_{j-1}^n - \Delta t \tilde{\lambda}_{j-\frac{1}{2}} (v^n_j - v_{j-1}^n) + \Delta t \tilde{\lambda}_{j-\frac{1}{2}} h(x_{j-1}, v^n_j),$$

$$v_j^{n+1} = v_j^n - \Delta t \tilde{\lambda}_{j-\frac{1}{2}}^+ (v^n_j - v_{j-1}^n) + \Delta t \tilde{\lambda}_{j-\frac{1}{2}}^- h(x_j, v^n_j),$$

where

$$\tilde{\lambda}_{j-\frac{1}{2}}^\pm = \text{diag}(\tilde{\lambda}_{j-\frac{1}{2}}^\pm, \ldots, m\tilde{\lambda}_{j-\frac{1}{2}}^\pm).$$
If we transform back using (4.4) and (4.6), equations (4.9a,b) become

\[
\begin{align*}
\varepsilon_n^{n+1} &= \varepsilon_n^n - \Delta t \frac{\partial}{\partial x} \tilde{A}_{j-\frac{1}{2}}^{-} (\varepsilon_j^n - \varepsilon_{j-1}^n) + \Delta t \tilde{A}_{j-\frac{1}{2}}^{-1} \tilde{A}_{j-\frac{1}{2}}^{-} g(x_{j-1}, \varepsilon_{j-1}^n), \\
\varepsilon_j^{n+1} &= \varepsilon_j^n - \Delta t \frac{\partial}{\partial x} \tilde{A}_{j-\frac{1}{2}}^{+} (\varepsilon_j^n - \varepsilon_{j-1}^n) + \Delta t \tilde{A}_{j-\frac{1}{2}}^{+1} \tilde{A}_{j-\frac{1}{2}}^{+} g(x_j, \varepsilon_j^n),
\end{align*}
\]

where \( \tilde{A}_{j-\frac{1}{2}}^{\pm} \) are given by

\[
\tilde{A}_{j-\frac{1}{2}}^{\pm} = \tilde{A}_{j-\frac{1}{2}}^{\pm}(\tilde{A}_{j-\frac{1}{2}}^{\pm} - 1)\tilde{A}_{j-\frac{1}{2}}^{\pm}
\]

as the positive and negative parts of \( \tilde{A}_{j-\frac{1}{2}} \), and have eigenvalues \( \tilde{\lambda}_{j-\frac{1}{2}}^{\pm} \), \( i = 1, \ldots, m \). The scheme given by (4.11a,b) can be written pointwise as

\[
\begin{align*}
\varepsilon_j^{n+1} &= \varepsilon_j^n - \Delta t \frac{\partial}{\partial x} \tilde{A}_{j-\frac{1}{2}}^{\pm} (\varepsilon_j^n - \varepsilon_{j-1}^n) - \Delta t \tilde{A}_{j+\frac{1}{2}}^{-} (\varepsilon_j^n - \varepsilon_{j+1}^n) \\
&\quad + \Delta t \tilde{A}_{j-\frac{1}{2}}^{-1} \tilde{A}_{j+\frac{1}{2}}^{+} g(x_j, \varepsilon_j^n) + \Delta t \tilde{A}_{j+\frac{1}{2}}^{-1} \tilde{A}_{j+\frac{1}{2}}^{-} g(x_j, \varepsilon_j^n),
\end{align*}
\]

which is an extension of the algorithm given by (2.2). To implement this scheme in an upwind manner, by looking at each of the \( m \)-waves with wavespeeds \( \tilde{\lambda}_{j-\frac{1}{2}}^{\pm} \), \( i = 1, \ldots, m \), we use the technique of flux-difference splitting as follows.

5. IMPLEMENTATION OF THE FIRST ORDER SCHEME VIA FLUX-DIFFERENCE SPLITTING

We split the Jacobian matrix \( A = \frac{\partial f}{\partial \varepsilon} \), with eigenvalues \( \lambda \) and corresponding linearly independent eigenvectors \( e_i, i = 1, \ldots, m \), into

\[
A = A^+ + A^-,
\]

and \( g \), similarly, as

\[
g = g^+ + g^-,
\]

where

\[
g^{\pm} = A^{-1} A^{\pm} g.
\]

The matrices \( A^{\pm} \) are defined by

\[
A^{\pm} = X \Lambda^{\pm} X^{-1},
\]

where

\[
\Lambda^{\pm} = \text{diag}(\lambda^{\pm}, \ldots, m^{\pm}),
\]

\[
X = \{e_1, \ldots, m\},
\]

and \( \lambda^{\pm} \) are defined by (2.6). Expanding \( w(x_j, t_n + \Delta t) \) about \( (x_j, t_n) \) as a Taylor series and using (3.1) and (5.1)-(5.3), we obtain

\[
w(x_j, t_n + \Delta t) \approx w(x_j, t_n) + \Delta t \left[ g(x_j, t_n) - A w(x_j, t_n) \right] \\
= w(x_j, t_n) + \Delta t \left[ g^+(x_j, t_n) - A^+ w(x_j, t_n) \right] \\
+ \Delta t \left[ g^-(x_j, t_n) - A^- w(x_j, t_n) \right].
\]
Thus, defining $\tilde{A}_{j-\frac{1}{2}}, \tilde{g}_{j-\frac{1}{2}}$ as approximations to $A, g$ at $x_{j-\frac{1}{2}} = \frac{1}{2}(x_j + x_{j-1})$ and time level $n$, and splitting $\tilde{A}_{j-\frac{1}{2}}, \tilde{g}_{j-\frac{1}{2}}$ into

\begin{align*}
\tilde{A}_{j-\frac{1}{2}} &= \tilde{A}_{j-\frac{1}{2}}^+ + \tilde{A}_{j-\frac{1}{2}}^-,
\tilde{g}_{j-\frac{1}{2}} &= \tilde{g}_{j-\frac{1}{2}}^+ + \tilde{g}_{j-\frac{1}{2}}^-,
\end{align*}

where

\begin{equation}
\tilde{g}_{j-\frac{1}{2}}^\pm = \tilde{\tilde{g}}_{j-\frac{1}{2}}^\pm \tilde{g}_{j-\frac{1}{2}}.
\end{equation}

we obtain the following first order upwind scheme for (3.1) from (5.7):

\begin{equation}
\begin{align*}
\mathbf{w}_j^{n+1} = \mathbf{w}_j^n + \Delta t \left[ \tilde{g}_{j-\frac{1}{2}}^+ - \tilde{\gamma}_{j-\frac{1}{2}} \right] \frac{(\mathbf{w}_j^n - \mathbf{w}_{j-1}^n)}{\Delta x} + \Delta t \left[ \tilde{g}_{j+\frac{1}{2}}^- - \tilde{\gamma}_{j+\frac{1}{2}} \right] \frac{(\mathbf{w}_{j+1}^n - \mathbf{w}_j^n)}{\Delta x}.
\end{align*}
\end{equation}

(N.B. $\tilde{A}_{j-\frac{1}{2}}^\pm$ are associated with right (+) and left (−) travelling waves.)

Comparing equations (5.9), (4.13) and (5.8c), we see that we can take the approximations $\tilde{g}_{j \pm \frac{1}{2}}$ used for updating $\mathbf{w}_j^n$ to be $\tilde{g}_{j \pm \frac{1}{2}} = \mathbf{g}(x_j, \mathbf{w}_j^n)$.

To implement the algorithm given by (4.13) written cell-wise in (4.11a,b) we project:

\begin{equation}
\begin{align*}
\mathbf{w}_j^n - \mathbf{w}_{j-1}^n &= \sum_{i=1}^{m} i \tilde{\alpha}_{j-\frac{1}{2}} i \tilde{\beta}_{j-\frac{1}{2}},
\mathbf{g}(x_{j-1}, \mathbf{w}_{j-1}^n) &= -\frac{1}{\Delta x} \sum_{i=1}^{m} i \tilde{\alpha}_{j-\frac{1}{2}} i \tilde{\beta}_{j-\frac{1}{2}} i \tilde{e}_{j-\frac{1}{2}}, \quad \text{and}
\mathbf{g}(x_j, \mathbf{w}_j^n) &= \frac{1}{\Delta x} \sum_{i=1}^{m} i \tilde{\alpha}_{j-\frac{1}{2}} i \tilde{\beta}_{j-\frac{1}{2}} i \tilde{e}_{j-\frac{1}{2}},
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\mathbf{w}_j^n &= \mathbf{w}_{j-1}^n - \frac{\Delta t}{\Delta x} \sum_{i=1}^{m} i \tilde{\alpha}_{j-\frac{1}{2}} i \tilde{\beta}_{j-\frac{1}{2}} i \tilde{e}_{j-\frac{1}{2}}, \quad \text{and}
\mathbf{w}_j^n - \mathbf{w}_j^{n+1} &= \frac{\Delta t}{\Delta x} \sum_{i=1}^{m} i \tilde{\alpha}_{j-\frac{1}{2}} i \tilde{\beta}_{j-\frac{1}{2}} i \tilde{e}_{j-\frac{1}{2}},
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
i \tilde{\alpha}_{j-\frac{1}{2}} &= i \tilde{\alpha}_{j-\frac{1}{2}} + i \tilde{\beta}_{j-\frac{1}{2}}, \quad \text{and}
i \tilde{\beta}_{j-\frac{1}{2}} &= i \tilde{\alpha}_{j-\frac{1}{2}} + \tilde{\gamma}_{j-\frac{1}{2}}.
\end{align*}
\end{equation}

(N.B. $\tilde{A}_{j-\frac{1}{2}}, \tilde{A}_{j-\frac{1}{2}}^\pm$ have eigenvalues $i \tilde{\lambda}_{j-\frac{1}{2}}, i \tilde{\lambda}_{j-\frac{1}{2}}^\pm$, respectively.) The schematic representation of the scheme given by (5.13a)-(5.15) can be seen in Figure 3.
Written pointwise, equations (5.13a,b) become
\[ w_{j+1} = w_j - \sum_{i=1}^{m} \Delta t \delta_{j-i} \delta_{j-i} - \sum_{i=1}^{m} \Delta t \delta_{j+i} \delta_{j+i}, \tag{5.16} \]
where \( \delta_{j-\frac{1}{2}} = \frac{\Delta t}{\Delta x} \delta_{j-\frac{1}{2}} \). Equation (5.16) could have been derived from (4.13) by projecting \( w_j - w_{j-1} \), \( w_{j+1} - w_j \) onto the local eigenvectors \( \delta_{j-\frac{1}{2}}, \delta_{j+\frac{1}{2}} \), respectively, as given by (5.10), and projecting the term \( g(x_j, w^n_j) \) occurring as \( \delta_{j-\frac{1}{2}} \delta_{j+\frac{1}{2}} g \) onto the local eigenvectors \( \delta_{j-\frac{1}{2}}, \delta_{j+\frac{1}{2}} \), respectively, as given by (5.11) and (5.12).

We can use equations (4.4) and (5.10) to represent \( \beta_{j-\frac{1}{2}} \) as
\[ \beta_{j-\frac{1}{2}} = [\delta_{j-\frac{1}{2}} (w^n_j - w_{j-1}^n)]_{i}. \tag{5.17} \]
In addition, we can use equations (4.6), (5.11) and (5.12) to represent the 'additional wave-strengths' \( \beta_{j-\frac{1}{2}} \) as
\[ \beta_{j-\frac{1}{2}} = \frac{\Delta x}{\delta_{j-\frac{1}{2}}} h_i(x_j, v^n_j), \tag{5.18} \]
\[ \beta_{j-\frac{1}{2}} = \frac{\Delta x}{\delta_{j-\frac{1}{2}}} h_i(x_j, v^n_j). \tag{5.19} \]

Finally, we note that the scheme given by (5.16) is the scalar scheme given by (2.2) when applied to each of the \( m \)-waves.

6. SECOND ORDER SCHEME

We now derive a second order scheme for the solution of (3.1) using the scalar scheme of Section 2.2.

Suppose we split the Jacobian matrix \( A = \frac{\partial f}{\partial w} \) and the source term \( g \) as in (5.1)-(5.3), and expand \( w(x_j, t_n + \Delta t) \) about \( (x_j, t_n) \) as the truncated Taylor series
\[ w(x_j, t_n + \Delta t) \simeq w(x_j, t_n) + \Delta t w_t(x_j, t_n) + \frac{\Delta t^2}{2} w_{tt}(x_j, t_n). \tag{6.1} \]
Using equation (3.1), we have
\[ w_t = g - f_x = g - A w_x, \quad \text{and} \]
\[ w_{tt} = g_{tt} - g_x w_t - (f_x)_x = g_{w} w_t - (A w_t)_x \]
\[ = g_w (g - A w_x) - (A (g - A w_x))_x, \tag{6.3} \]
so that equation (6.1) becomes
\[ w(x_j, t_n + \Delta t) \simeq w + \Delta t (g - A w_x) + \frac{\Delta t^2}{2} g_{w} (g - A w_x) - \frac{\Delta t^2}{2} (A (g - A w_x))_x, \tag{6.4} \]
where the terms on the right-hand side are evaluated at \( (x_j, t_n) \). We approximate
\[ (g - A w_x) \simeq \frac{1}{2} \left[ g_{j-\frac{1}{2}} - \delta_{j-\frac{1}{2}} \frac{(w^n_j - w_{j-1}^n)}{\Delta x} + g_{j+\frac{1}{2}} - \delta_{j+\frac{1}{2}} \frac{(w^n_{j+1} - w^n_j)}{\Delta x} \right], \tag{6.5a} \]
\[ g_w(g - A w_x)|_{(x_j,t_n)} = \frac{1}{2} \left[ (g_w)_{j-\frac{1}{2}} \left( g_{j-\frac{1}{2}} - \tilde{A}_{j-\frac{1}{2}} \frac{(w^n_j - w^n_{j-1})}{\Delta x} \right) 
+ (g_w)_{j+\frac{1}{2}} \left( g_{j+\frac{1}{2}} - \tilde{A}_{j+\frac{1}{2}} \frac{(w^n_{j+1} - w^n_j)}{\Delta x} \right) \right], \quad (6.5b) \]

and
\[
(\tilde{A} (g - A w_x))_x \approx \frac{1}{\Delta x} \left[ \tilde{A}_{j+\frac{1}{2}} \left( g_{j+\frac{1}{2}} - \tilde{A}_{j+\frac{1}{2}} \frac{(w^n_{j+1} - w^n_j)}{\Delta x} \right) 
- \tilde{A}_{j-\frac{1}{2}} \left( g_{j-\frac{1}{2}} - \tilde{A}_{j-\frac{1}{2}} \frac{(w^n_j - w^n_{j-1})}{\Delta x} \right) \right], \quad (6.6) \]

where the approximations \( g_{j-\frac{1}{2}}, \ (g_w)_{j-\frac{1}{2}} \) to \( g, \ g_w \) at \( (x, t_n) \), \( x \in [x_{j-1}, x_j] \), will be determined by the scalar algorithm in Section 2.2. Substituting the expressions given by (6.5a)–(6.6) into (6.4) and using equations (5.8a–c) gives the following scheme centred on \( x_j \)

\[
w^{n+1}_j = w^n_j + \frac{\Delta t}{2} \left[ g^+_j \left( \frac{w^n_j - w^n_{j-1}}{\Delta x} \right) + g^-_j \left( \frac{w^n_j - w^n_{j-1}}{\Delta x} \right) \right] 
+ \frac{\Delta t^2}{2} \left[ \left( g_w^+_j \left( \frac{w^n_j - w^n_{j-1}}{\Delta x} \right) + g^-_j \left( \frac{w^n_j - w^n_{j-1}}{\Delta x} \right) \right) \right] 
- \frac{\Delta t^2}{2\Delta x} \left[ \tilde{A}^+_j \left( \frac{w^n_j - w^n_{j-1}}{\Delta x} \right) + \tilde{A}^-_j \left( \frac{w^n_j - w^n_{j-1}}{\Delta x} \right) \right]. \quad (6.7) \]

To express this scheme as an increment stage and a transfer stage as in Section 2.2, we rearrange equation (6.7) as

\[
w^{n+1}_j = w^n_j + \left( I + \frac{\Delta t}{2} (g_w)_{j-\frac{1}{2}} \right) \left( \Delta t g^+_j - \frac{\Delta t}{\Delta x} \tilde{A}^+_j \left( w^n_j - w^n_{j-1} \right) \right) 
+ \left( I + \frac{\Delta t}{2} (g_w)_{j+\frac{1}{2}} \right) \left( \Delta t g^-_j - \frac{\Delta t}{\Delta x} \tilde{A}^-_j \left( w^n_{j+1} - w^n_j \right) \right) 
- \frac{1}{2} \left( I - \frac{\Delta t}{\Delta x} \tilde{A}^-_j + \frac{\Delta t}{2} (g_w)_{j-\frac{1}{2}} \right) \left( \Delta t g^+_j - \frac{\Delta t}{\Delta x} \tilde{A}^+_j \left( w^n_j - w^n_{j-1} \right) \right) 
- \frac{1}{2} \left( I + \frac{\Delta t}{\Delta x} \tilde{A}^-_j + \frac{\Delta t}{2} (g_w)_{j+\frac{1}{2}} \right) \left( \Delta t g^-_j - \frac{\Delta t}{\Delta x} \tilde{A}^-_j \left( w^n_{j+1} - w^n_j \right) \right) 
+ \frac{1}{2} \left( I + \frac{\Delta t}{\Delta x} \tilde{A}^-_j + \frac{\Delta t}{2} (g_w)_{j-\frac{1}{2}} \right) \left( \Delta t g^+_j - \frac{\Delta t}{\Delta x} \tilde{A}^+_j \left( w^n_{j+1} - w^n_j \right) \right) 
+ \frac{1}{2} \left( I - \frac{\Delta t}{\Delta x} \tilde{A}^-_j + \frac{\Delta t}{2} (g_w)_{j+\frac{1}{2}} \right) \left( \Delta t g^-_j - \frac{\Delta t}{\Delta x} \tilde{A}^-_j \left( w^n_{j+1} - w^n_j \right) \right). \quad (6.8) \]

We project

\[
w^n_j - w^n_{j-1} = \sum_{i=1}^m i \tilde{\alpha}_j^i \hat{e}_j^i, \quad (6.9)\]

\[
\hat{e}_j^i = - \frac{1}{\Delta x} \sum_{i=1}^m i \tilde{\alpha}_j^i \hat{e}_j^i, \quad (6.10)\]

\[
\hat{\xi}_j^i = - \frac{1}{\Delta x} \sum_{i=1}^m i \tilde{\alpha}_j^i \hat{\xi}_j^i, \quad (6.11)\]
so that we can apply the scalar algorithm of Section 2.2 to each of the $m$-waves. Also,

$$\frac{\Delta t}{2} \left( g_w \right)_{j-\frac{1}{2}} \hat{e}_{j-\frac{1}{2}} = \sum_{i=1}^{m} \kappa_i \hat{\omega}_{j-\frac{1}{2}} \hat{e}_{j-\frac{1}{2}}, \quad \text{and} \quad (6.12)$$

$$\frac{\Delta t}{2} \left( g_w \right)_{j+\frac{1}{2}} \hat{e}_{j+\frac{1}{2}} = \sum_{i=1}^{m} \kappa_i \hat{\theta}_{j+\frac{1}{2}} \hat{e}_{j+\frac{1}{2}}, \quad (6.13)$$

Thus comparing equations (6.8)-(6.12) with the scheme of Section 2, we see that $\hat{\beta}_{j+\frac{1}{2}}^i$ and $\hat{\gamma}_{j-\frac{1}{2}}^i$ should be evaluated at $x_j + \Delta x/2, w^n_j$ and $x_j - \Delta x/2, w^n_j$. Similarly, $\hat{\kappa}_i \hat{\psi}_{j+\frac{1}{2}}^i$ should be evaluated at $x_j, w^n_j$. This means that $g_{j-\frac{1}{2}} = g(x_j - \Delta x/2, w^n_j)$, $g_{j+\frac{1}{2}} = g(x_j + \Delta x/2, w^n_j)$, $(g_w)_{j-\frac{1}{2}} = g_w(w^n_j, w^n_j)$. Thus, we project

$$g(x_j + \Delta x/2, w^n_j) = -\frac{1}{\Delta x} \sum_{i=1}^{m} \bar{\lambda}_{j+\frac{1}{2}}^i \bar{\beta}_{j+\frac{1}{2}}^i \delta_{j+\frac{1}{2}}, \quad (6.14)$$

so that

$$\bar{\beta}_{j+\frac{1}{2}} = -\left[ \frac{1}{\bar{\lambda}_{j+\frac{1}{2}}} \bar{\chi}_{j+\frac{1}{2}} \left( g(x_j + \Delta x/2, w^n_j) \right) \right] \frac{\Delta x}{\bar{\lambda}_{j+\frac{1}{2}}} h_i \left( x_j + \frac{\Delta x}{2}, v^n_j \right), \quad (6.15)$$

and

$$g \left( x_j - \Delta x/2, w^n_j \right) = -\frac{1}{\Delta x} \sum_{i=1}^{m} \bar{\lambda}_{j-\frac{1}{2}}^i \bar{\gamma}_{j-\frac{1}{2}}^i \bar{\theta}_{j-\frac{1}{2}}, \quad (6.16)$$

so that

$$\bar{\gamma}_{j-\frac{1}{2}} = -\left[ \frac{1}{\bar{\lambda}_{j-\frac{1}{2}}} \bar{\chi}_{j-\frac{1}{2}} \left( g(x_j - \Delta x/2, w^n_j) \right) \right] \frac{\Delta x}{\bar{\lambda}_{j-\frac{1}{2}}} h_i \left( x_j - \frac{\Delta x}{2}, v^n_j \right). \quad (6.17)$$

Moreover,

$$\frac{\Delta t}{2} \left( g_w \right)_{j-\frac{1}{2}} \bar{\chi}_{j-\frac{1}{2}} = X_{j-\frac{1}{2}} \Omega_{j-\frac{1}{2}}, \quad (6.18)$$

$$\frac{\Delta t}{2} \left( g_w \right)_{j+\frac{1}{2}} \bar{\chi}_{j+\frac{1}{2}} = \bar{X}_{j+\frac{1}{2}} \bar{\Omega}_{j+\frac{1}{2}}, \quad (6.19)$$

where $\Omega_{j-\frac{1}{2}} = \left\{ \kappa \hat{\omega}_{j-\frac{1}{2}}, \right\}$, $\bar{\Omega}_{j+\frac{1}{2}} = \left\{ \kappa \bar{\theta}_{j+\frac{1}{2}} \right\}$, and hence

$$\bar{\Omega}_{j-\frac{1}{2}} = \frac{\Delta t}{2} \bar{X}_{j-\frac{1}{2}} \left( g_w \right)_{j-\frac{1}{2}} \bar{X}_{j-\frac{1}{2}} = \frac{\Delta t}{2} \bar{X}_{j-\frac{1}{2}} \bar{g}_w(x_j, w^n_j) \bar{X}_{j-\frac{1}{2}} \quad (6.20)$$

$$\frac{\Delta t}{2} \bar{X}_{j+\frac{1}{2}} \left( g_w \right)_{j+\frac{1}{2}} \bar{X}_{j+\frac{1}{2}} = \frac{\Delta t}{2} \bar{X}_{j+\frac{1}{2}} \bar{g}_w(x_j, w^n_j) \bar{X}_{j+\frac{1}{2}} \quad (6.21)$$

We note that the expressions given for $\bar{\beta}_{j+\frac{1}{2}}^i$, $\bar{\gamma}_{j-\frac{1}{2}}^i$, $\kappa \hat{\omega}_{j-\frac{1}{2}}$, $\kappa \bar{\theta}_{j+\frac{1}{2}}$ by (6.14)-(6.21) are consistent with the algorithm of Section 2. Using equations (5.8b,c) and (6.9)-(6.11), we can write

$$\Delta t g_{j-\frac{1}{2}} \mp \Delta x \bar{X}_{j-\frac{1}{2}} \left( w^n_j \cdot w^n_{j-1} \right) = -\frac{\Delta t}{\Delta x} \sum_{i=1}^{m} \bar{\lambda}_{j-\frac{1}{2}}^i \bar{\gamma}_{j-\frac{1}{2}}^i \bar{\theta}_{j-\frac{1}{2}}, \quad (6.22)$$

$$\Delta t g_{j+\frac{1}{2}} \mp \Delta x \bar{X}_{j+\frac{1}{2}} \left( w^n_{j+1} - w^n_j \right) = -\frac{\Delta t}{\Delta x} \sum_{i=1}^{m} \bar{\lambda}_{j+\frac{1}{2}}^i \bar{\gamma}_{j+\frac{1}{2}}^i \bar{\theta}_{j+\frac{1}{2}}, \quad (6.23)$$
where
\[ i\delta_{j+\frac{1}{2}} = i\alpha_{j+\frac{1}{2}} + i\beta_{j+\frac{1}{2}}, \quad \text{and} \]
\[ i\epsilon_{j-\frac{1}{2}} = i\alpha_{j-\frac{1}{2}} + i\gamma_{j-\frac{1}{2}}. \]

Finally, we need to consider expressions like
\[ D = \left( I + \frac{\Delta t}{\Delta x} \tilde{A}_{j+\frac{1}{2}} + \frac{\Delta t}{2} (gw)_{j+\frac{1}{2}} \right) \left( \Delta t g^-_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x} (w^n_{j+1} - w^n_j) \right). \]

Now, using equation (6.23), we can write
\[ D = -\frac{\Delta t}{\Delta x} \left( I + \frac{\Delta t}{\Delta x} \tilde{A}_{j+\frac{1}{2}} + \frac{\Delta t}{2} (gw)_{j+\frac{1}{2}} \right) \left( \sum_{i=1}^{m} i\tilde{\lambda}_{j+\frac{1}{2}}^- i\delta_{j+\frac{1}{2}} + i\epsilon_{j+\frac{1}{2}} \right) \]
\[ = -\frac{\Delta t}{\Delta x} \sum_{i=1}^{m} \left( I + \frac{\Delta t}{\Delta x} \tilde{A}_{j+\frac{1}{2}} + \frac{\Delta t}{2} (gw)_{j+\frac{1}{2}} \right) i\tilde{\lambda}_{j+\frac{1}{2}}^- i\delta_{j+\frac{1}{2}}. \]

Also, since $\tilde{A}_{j+\frac{1}{2}}$ has eigenvalues $i\tilde{\lambda}_{j+\frac{1}{2}}$ with eigenvectors $i\tilde{\alpha}_{j+\frac{1}{2}}$, we can use (6.13) to rewrite equation (6.26) as
\[ D = -\frac{\Delta t}{\Delta x} \sum_{i=1}^{m} i\tilde{\lambda}_{j+\frac{1}{2}}^- i\delta_{j+\frac{1}{2}} \left( i\epsilon_{j+\frac{1}{2}} + \frac{\Delta t}{\Delta x} i\tilde{\lambda}_{j+\frac{1}{2}} + \sum_{k=1}^{m} i\tilde{\lambda}_{j+\frac{1}{2}}^- i\tilde{\alpha}_{j+\frac{1}{2}} \right) \]
\[ = -\sum_{i=1}^{m} i\tilde{\nu}_{j+\frac{1}{2}}^- i\delta_{j+\frac{1}{2}} \left( (I + i\nu_{i+\frac{1}{2}}) i\tilde{\alpha}_{j+\frac{1}{2}} + i\tilde{\xi}_{j+\frac{1}{2}} \right), \]
\[ \text{where} \]
\[ i\tilde{\xi}_{j+\frac{1}{2}} = \sum_{k=1}^{m} i\tilde{\lambda}_{j+\frac{1}{2}}^- i\tilde{\alpha}_{j+\frac{1}{2}}, \quad \text{and} \]
\[ i\tilde{\nu}_{j+\frac{1}{2}} - \frac{\Delta t}{\Delta x} i\tilde{\lambda}_{j+\frac{1}{2}} = \frac{\Delta t}{\Delta x} i\tilde{\lambda}_{j+\frac{1}{2}}^+. \]

Similarly, defining
\[ i\tilde{\zeta}_{j-\frac{1}{2}} = \sum_{k=1}^{m} i\tilde{\lambda}_{j-\frac{1}{2}}^- i\tilde{\alpha}_{j-\frac{1}{2}}, \]
we can write equation (6.8) as
\[ w^{n+1}_j = w^n_j - \sum_{i=1}^{m} i\tilde{\nu}_{j-\frac{1}{2}}^- i\tilde{\epsilon}_{j-\frac{1}{2}} \left( i\tilde{\epsilon}_{j-\frac{1}{2}} + i\tilde{\zeta}_{j-\frac{1}{2}} \right) - \sum_{i=1}^{m} i\tilde{\nu}_{j+\frac{1}{2}}^- i\tilde{\zeta}_{j+\frac{1}{2}} \left( i\tilde{\epsilon}_{j+\frac{1}{2}} + i\tilde{\xi}_{j+\frac{1}{2}} \right) \]
\[ + \sum_{i=1}^{m} \frac{i}{2} i\tilde{\nu}_{j-\frac{1}{2}}^- i\tilde{\epsilon}_{j-\frac{1}{2}} \left( (1 - i\tilde{\nu}_{j-\frac{1}{2}}^-) i\tilde{\epsilon}_{j-\frac{1}{2}} + i\tilde{\zeta}_{j-\frac{1}{2}} \right) \]
\[ + \sum_{i=1}^{m} \frac{i}{2} i\tilde{\nu}_{j+\frac{1}{2}}^- i\tilde{\epsilon}_{j+\frac{1}{2}} \left( (1 - i\tilde{\nu}_{j+\frac{1}{2}}^-) i\tilde{\epsilon}_{j+\frac{1}{2}} + i\tilde{\xi}_{j+\frac{1}{2}} \right) \]
\[ - \sum_{i=1}^{m} \frac{i}{2} i\tilde{\nu}_{j-\frac{1}{2}}^- i\tilde{\zeta}_{j-\frac{1}{2}} \left( (1 - i\tilde{\nu}_{j-\frac{1}{2}}^-) i\tilde{\epsilon}_{j-\frac{1}{2}} + i\tilde{\zeta}_{j-\frac{1}{2}} \right) \]
\[ - \sum_{i=1}^{m} \frac{i}{2} i\tilde{\nu}_{j+\frac{1}{2}}^- i\tilde{\zeta}_{j+\frac{1}{2}} \left( (1 - i\tilde{\nu}_{j+\frac{1}{2}}^-) i\tilde{\epsilon}_{j+\frac{1}{2}} + i\tilde{\xi}_{j+\frac{1}{2}} \right). \]
Finally, we can summarise this scheme by writing it in the cell-wise fashion

\[
w_{j-1}^{n+1} = w_j^n - \sum_{i=1}^{m} \tilde{v}_{j-\frac{1}{2}}^i \tilde{\delta}_{j-\frac{1}{2}}^i \left( i\tilde{\varepsilon}_{j-\frac{1}{2}}^i + i\tilde{\xi}_{j-\frac{1}{2}}^i \right)
\]

\[
+ \sum_{i=1}^{m} \frac{1}{2} \tilde{v}_{j-\frac{1}{2}}^i \tilde{\delta}_{j-\frac{1}{2}}^i \left( (1 - |i\tilde{v}_{j-\frac{1}{2}}^i|) \tilde{\varepsilon}_{j-\frac{1}{2}}^i + i\tilde{\xi}_{j-\frac{1}{2}}^i \right)
\]

\[
- \sum_{i=1}^{m} \frac{1}{2} \tilde{v}_{j+\frac{1}{2}}^i \tilde{\delta}_{j-\frac{1}{2}}^i \left( (1 - |i\tilde{v}_{j-\frac{1}{2}}^i|) \tilde{\varepsilon}_{j-\frac{1}{2}}^i + i\tilde{\xi}_{j-\frac{1}{2}}^i \right),
\] (6.32a)

\[
w_j^{n+1} = w_j^n - \sum_{i=1}^{m} \tilde{v}_{j+\frac{1}{2}}^i \tilde{\varepsilon}_{j+\frac{1}{2}}^i \left( i\tilde{\varepsilon}_{j+\frac{1}{2}}^i + i\tilde{\xi}_{j+\frac{1}{2}}^i \right)
\]

\[
+ \sum_{i=1}^{m} \frac{1}{2} \tilde{v}_{j+\frac{1}{2}}^i \tilde{\varepsilon}_{j+\frac{1}{2}}^i \left( (1 - |i\tilde{v}_{j-\frac{1}{2}}^i|) \tilde{\varepsilon}_{j+\frac{1}{2}}^i + i\tilde{\xi}_{j+\frac{1}{2}}^i \right)
\]

\[
- \sum_{i=1}^{m} \frac{1}{2} \tilde{v}_{j-\frac{1}{2}}^i \tilde{\varepsilon}_{j+\frac{1}{2}}^i \left( (1 - |i\tilde{v}_{j-\frac{1}{2}}^i|) \tilde{\varepsilon}_{j+\frac{1}{2}}^i + i\tilde{\xi}_{j+\frac{1}{2}}^i \right),
\] (6.32b)

where \(\tilde{\varepsilon}_{j+\frac{1}{2}}, \tilde{\varepsilon}_{j-\frac{1}{2}}, \tilde{\varepsilon}_{j+\frac{1}{2}}, \tilde{\varepsilon}_{j-\frac{1}{2}}\) are as before,

\[
i\tilde{\varepsilon}_{j+\frac{1}{2}} = i\tilde{\varepsilon}_{j+\frac{1}{2}} + i\tilde{\delta}_{j+\frac{1}{2}}^i,
\] (6.33)

\[
i\tilde{\varepsilon}_{j-\frac{1}{2}} = i\tilde{\varepsilon}_{j-\frac{1}{2}} + i\tilde{\delta}_{j-\frac{1}{2}}^i,
\] (6.34)

and \(\tilde{\delta}_{j+\frac{1}{2}}, \tilde{\delta}_{j-\frac{1}{2}}\) represent projections onto the local eigenvectors \(\tilde{\varepsilon}_{j+\frac{1}{2}}, \tilde{\varepsilon}_{j-\frac{1}{2}}\) of \(g(x_j - \Delta x/2, w_{j-1}^n), g(x_j - \Delta x/2, w_{j}^n)\), respectively.

In addition, following equations (6.12), (6.13), (6.20), (6.21), (6.28) and (6.30), \(i\tilde{\xi}_{j+\frac{1}{2}}, i\tilde{\xi}_{j-\frac{1}{2}}\) represent \((\Delta t/2) \frac{\partial g}{\partial w}(x_{j-1}, w_{j-1}^n), (\Delta t/2) \frac{\partial g}{\partial w}(x_j, w_{j}^n)\) applied to \(i\tilde{\varepsilon}_{j+\frac{1}{2}}\). Finally, we can write equations (6.32a,b) in a similar form to the scheme of Section 2.2, i.e., in increment \((\phi, \psi)\) and transfer \((b, c)\) form, as

\[
w_{j-1}^{n+1} = w_j^n + i\phi_{j-\frac{1}{2}}^i + i\psi_{j+\frac{1}{2}}^i, \quad i\lambda_{j+\frac{1}{2}}^i > 0,
\] (6.35a)

\[
w_j^{n+1} = w_j^n + i\phi_{j-\frac{1}{2}}^i - i\psi_{j-\frac{1}{2}}^i, \quad i\lambda_{j-\frac{1}{2}}^i > 0,
\] (6.35b)

for each \(i = 1, \ldots, m\), where

\[
i\phi_{j-\frac{1}{2}} = -i\tilde{v}_{j-\frac{1}{2}}^i \tilde{\varepsilon}_{j-\frac{1}{2}}^i \left( i\tilde{\varepsilon}_{j+\frac{1}{2}}^i + i\tilde{\xi}_{j+\frac{1}{2}}^i \right)
\] (6.36)

\[
i\psi_{j-\frac{1}{2}} = -i\tilde{v}_{j-\frac{1}{2}}^i \tilde{\delta}_{j-\frac{1}{2}}^i \left( i\tilde{\varepsilon}_{j+\frac{1}{2}}^i + i\tilde{\xi}_{j+\frac{1}{2}}^i \right)
\] (6.37)

\[
i\psi_{j-\frac{1}{2}} = -\frac{1}{2} \tilde{v}_{j-\frac{1}{2}}^i \tilde{\varepsilon}_{j-\frac{1}{2}}^i \left( (1 - |i\tilde{v}_{j-\frac{1}{2}}^i|) \tilde{\varepsilon}_{j-\frac{1}{2}}^i + i\tilde{\xi}_{j+\frac{1}{2}}^i \right)
\] (6.38)

\[
i\psi_{j-\frac{1}{2}} = -\frac{1}{2} \tilde{v}_{j-\frac{1}{2}}^i \tilde{\varepsilon}_{j-\frac{1}{2}}^i \left( (1 - |i\tilde{v}_{j-\frac{1}{2}}^i|) \tilde{\varepsilon}_{j-\frac{1}{2}}^i + i\tilde{\xi}_{j+\frac{1}{2}}^i \right).
\] (6.39)

This scheme is represented schematically in Figure 4, and it is this form that would be used for computational purposes. There remains the question of how to suppress non-physical oscillations and we mention this in Section 8.

In the next section, we discuss the special cases of compressible flow in a duct of variable cross section, and incompressible flow in a channel.
7. EXAMPLES

In this section, we discuss the application of the algorithm of Section 6 to particular systems of conservation laws. First, we consider the Euler equations of gas dynamics with source terms arising from flow in a narrow duct of smoothly varying cross-section. Second, we consider the non-linear shallow water equations with source terms arising from flow in a channel whose lower surface is smoothly varying.

7.1. Euler Equations

The Euler equations for the compressible flow of an ideal gas in a duct of cross-section $S(x)$ can be written as

$$w_t + f_w = g,$$  \hspace{1cm} (7.1)

where

$$w = S(x) (\rho, \rho u, e)^T$$  \hspace{1cm} (7.2)

$$f(w) = S(x) (\rho u, \rho u + \rho u^2, u (e + p))^T$$  \hspace{1cm} (7.3)

$$g(x, w) = (0, p S'(x), 0)^T,$$  \hspace{1cm} and

$$e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2.$$  \hspace{1cm} (7.4)

The quantities $\rho = \rho(x,t)$, $u = u(x,t)$, $p = p(x,t)$, $e = e(x,t)$ and $\gamma$ represent the density, velocity, pressure, total energy, and the ratio of specific heat capacities of the fluid, respectively, at a general point $x$ and at time $t$. The special cases $S(x) = 1$, $x$, $x^2$ refer to flows with slab, cylindrical or spherical symmetry, respectively. Following the approach in [2], we define new variables $R = S(x) \rho$, $U = u$, $P = S(x) p$, $E = S(x) e$, so that

$$w = (R, RU, E)^T$$  \hspace{1cm} (7.6)

$$f = (RU, P + RU^2, U(E + P))^T$$  \hspace{1cm} (7.7)

$$g = \left(0, \frac{P S'(x)}{S(x)}, 0\right)^T$$  \hspace{1cm} and

$$E = \frac{P}{\gamma - 1} + \frac{1}{2} R U^2.$$  \hspace{1cm} (7.8)

For the algorithm of Section 6 we devise a linearised Riemann problem as given by (3.2). In the specific example of this section, we use the linearised Riemann problem proposed in [2].

The approximate Jacobian matrix $\tilde{A}(w_j, w_j^n)$, to $A = \frac{\partial f}{\partial w}$ in the cell $[x_{j-1}, x_j]$ at time level $n$ is

$$\tilde{A} = \begin{bmatrix}
0 & 1 & 0 \\
\frac{1}{2} (\gamma - 3) \tilde{U}^2 & (3 - \gamma) \tilde{U} & \gamma - 1 \\
\frac{1}{2} (\gamma - 1) \tilde{U}^3 - \tilde{H} \tilde{U} & \tilde{H} - (\gamma - 1) \tilde{U}^2 & \gamma \tilde{U}
\end{bmatrix},$$  \hspace{1cm} (7.10)
where the averages of $U$ and the enthalpy $H = (E + P)/R$ are given by

$$\bar{U} = \frac{\sqrt{R_{j-1}} U_{j-1} + \sqrt{R_j} U_j}{\sqrt{R_{j-1}} + \sqrt{R_j}},$$

$$\bar{H} = \frac{\sqrt{R_{j-1}} H_{j-1} + \sqrt{R_j} H_j}{\sqrt{R_{j-1}} + \sqrt{R_j}}.$$  

(7.11, 7.12)

The eigenvalues of $\hat{A}$ are

$$1, 2, 3 \hat{\lambda}_{j-\frac{1}{2}} = \hat{U} + \bar{a}, \hat{U} - \bar{a}, \hat{U},$$

(7.13a–c)

where

$$\bar{a} = \sqrt{(\gamma - 1)(\bar{H} - \frac{1}{2} \bar{U}^2)},$$

(7.14)

with corresponding eigenvectors

$$1, 2, 3 \bar{e}_{j-\frac{1}{2}} = \begin{pmatrix} 1 \\ \hat{U} + \bar{a} \\ \bar{H} + \bar{U} \bar{a} \end{pmatrix}, \begin{pmatrix} 1 \\ \hat{U} - \bar{a} \\ \bar{H} - \bar{U} \bar{a} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \bar{U}^2 \end{pmatrix}.$$  

(7.15a–c)

Thus, the modal matrix $X_{j-\frac{1}{2}} = \left[1 \bar{e}_{j-\frac{1}{2}}, 2 \bar{e}_{j-\frac{1}{2}}, 3 \bar{e}_{j-\frac{1}{2}}\right]$ has the inverse

$$X^{-1}_{j-\frac{1}{2}} = \frac{1}{2a^2} \begin{bmatrix} \frac{1}{2} (\gamma - 1) \bar{U}^2 - \bar{U} \bar{a} & \bar{a} - (\gamma - 1) \bar{U} & \gamma - 1 \\ \frac{1}{2} (\gamma - 1) \bar{U}^2 + \bar{U} \bar{a} & -\bar{a} - (\gamma - 1) \bar{U} & \gamma - 1 \\ 2a^2 - (\gamma - 1) \bar{U}^2 & 2\bar{U} (\gamma - 1) & -2(\gamma - 1) \end{bmatrix}.$$  

(7.16)

To apply the scheme of Section 6 given by (6.32a,b) we need to calculate the quantities $i\bar{\bar{a}}_{j-\frac{1}{2}}$, $i\bar{\bar{b}}_{j-\frac{1}{2}}$, $i\bar{\bar{c}}_{j-\frac{1}{2}}$, $i\bar{\bar{d}}_{j-\frac{1}{2}}$, $i\bar{\bar{e}}_{j-\frac{1}{2}}$ and $i\bar{\bar{f}}_{j-\frac{1}{2}}$ for $i = 1, 2, 3$. Denoting

$$\Delta Y = Y_j - Y_{j-1},$$

(7.17)

equations (5.17), (7.6) and (7.16) yield

$$i\bar{\bar{a}}_{j-\frac{1}{2}} = \frac{1}{2a^2} \left( \Delta P + \bar{a} (\Delta (RU) - \bar{U} \Delta R) \right),$$

$$i\bar{\bar{b}}_{j-\frac{1}{2}} = \frac{1}{2a^2} \left( \Delta P - \bar{a} (\Delta (RU) - \bar{U} \Delta R) \right),$$

$$i\bar{\bar{c}}_{j-\frac{1}{2}} = \Delta R - \frac{\Delta P}{a^2},$$

(7.18a,b,c)

where we have used equation (7.9) and the property of $\bar{U}$ that

$$\Delta (RU^2) - 2\bar{U} \Delta (RU) + \bar{U}^2 \Delta R = 0.$$  

(7.19)

If we define the average of $R_{j-1}$, $R_j$

$$\bar{R} = \frac{1}{2a} \frac{\sqrt{R_{j-1} R_j}}{1},$$

(7.20)

then

$$\Delta (RU) - \bar{U} \Delta R = \bar{R} \Delta U,$$  

(7.21)

so that the expressions given in (7.18a,b) simplify to

$$i\bar{\bar{a}}_{j-\frac{1}{2}} = \frac{1}{2a^2} (\Delta P \pm \bar{R} \bar{a} \Delta U).$$  

(7.22a,b)
In addition, using equations (6.15), (6.17), (7.8), (7.13a-c) and (7.16) we obtain

\[ \tilde{\beta}_j = \kappa \frac{(\gamma - 1) \tilde{U} - \tilde{a}}{\tilde{U} + \tilde{a}} P_{j-1}, \]  

(7.23a)

\[ \tilde{\gamma} = \kappa \frac{(\gamma - 1) \tilde{U} - \tilde{a}}{\tilde{U} - \tilde{a}} P_{j}, \]  

(7.23b)

\[ \tilde{\kappa} = -2\kappa (\gamma - 1) P_{j-1}, \]  

(7.23c)

and

\[ \tilde{\gamma}_j = \kappa \frac{(\gamma - 1) \tilde{U} - \tilde{a}}{\tilde{U} + \tilde{a}} P_{j}, \]  

(7.24a)

\[ \tilde{\gamma}_{j-1} = \kappa \frac{(\gamma - 1) \tilde{U} + \tilde{a}}{\tilde{U} - \tilde{a}} P_{j}, \]  

(7.24b)

\[ \tilde{\gamma}_{j-1} = -2\kappa (\gamma - 1) P_{j}, \]  

(7.24c)

where

\[ \kappa = \frac{S'(x_j - \Delta x) - \Delta x}{2\tilde{a}^2 \Delta x S(x_j - \Delta x / 2)}. \]  

(7.25)

Finally, using equations (7.8) and (7.9)

\[ \mathbf{g} = \left(0, \frac{S'(x)}{S(x)} (\gamma - 1) \left(E - \frac{1}{2} \frac{M^2}{\kappa} \right), 0 \right)^T, \]  

(7.26)

where \( M = \mathcal{R} \mathcal{U}, \) so that, from (7.6)

\[ \mathbf{w} = (\mathcal{R}, M, E)^T \]  

(7.27)

and hence,

\[ \mathbf{g} \mathbf{w} = (\gamma - 1) \frac{S'(x)}{S(x)} \left( \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} U^2 & -U & 1 \\ 0 & 0 & 0 \end{array} \right). \]  

(7.28)

Therefore, the expressions \( \tilde{\xi}_{j-\frac{1}{2}}, \tilde{\zeta}_{j-\frac{1}{2}} \) of Section 6 become

\[ \tilde{\xi}_{j-\frac{1}{2}} = \left(0, \tau_{j-1} \left( \frac{\tilde{a}^2}{\gamma - 1} + \frac{1}{2} (\tilde{U} - U_{j-1})^2 + \tilde{a} (\tilde{U} - U_{j-1}) \right), 0 \right)^T, \]  

(7.29a)

\[ \tilde{\xi}_{j-\frac{1}{2}} = \left(0, \tau_{j-1} \left( \frac{\tilde{a}^2}{\gamma - 1} + \frac{1}{2} (\tilde{U} - U_{j-1})^2 - \tilde{a} (\tilde{U} - U_{j-1}) \right), 0 \right)^T, \]  

(7.29b)

\[ \tilde{\xi}_{j-\frac{1}{2}} = \left(0, \tau_{j-1} \left( \frac{1}{2} (\tilde{U} - U_{j-1})^2, 0 \right) \right)^T. \]  

(7.29c)

\[ \tilde{\zeta}_{j-\frac{1}{2}} = \left(0, \tau_{j} \left( \frac{\tilde{a}^2}{\gamma - 1} + \frac{1}{2} (\tilde{U} - U_{j})^2 + \tilde{a} (\tilde{U} - U_{j}) \right), 0 \right)^T, \]  

(7.30a)

\[ \tilde{\zeta}_{j-\frac{1}{2}} = \left(0, \tau_{j} \left( \frac{\tilde{a}^2}{\gamma - 1} + \frac{1}{2} (\tilde{U} - U_{j})^2 - \tilde{a} (\tilde{U} - U_{j}) \right), 0 \right)^T, \]  

(7.30b)

\[ \tilde{\zeta}_{j-\frac{1}{2}} = \left(0, \tau_{j} \left( \frac{1}{2} (\tilde{U} - U_{j})^2, 0 \right) \right)^T. \]  

(7.30c)
where
\[ \tau_j = \frac{\Delta t (\gamma - 1) S'(x_j)}{2 S(x_j)}. \]

We observe, however, that the \( \tilde{\xi}_{j-\frac{1}{2}}, \tilde{\xi}_{j-\frac{1}{2}} \) occur in the second order terms of (6.7) and, since
\[
\tilde{U} - U_j = -\frac{\Delta U_j}{1 + \sqrt{R_j / R_{j-1}}}, \tag{7.32}
\]
\[
\tilde{U} - U_{j-1} = -\frac{\Delta U_j}{1 + \sqrt{R_{j-1} / R_j}}, \tag{7.33}
\]
we could approximate the expressions in (7.29a)-(7.30c) by
\[
\tilde{\xi}_{j-\frac{1}{2}} = 2\tilde{\xi}_{j-\frac{1}{2}} = \frac{\tilde{\alpha}^2 \Delta t S'(x_{j-1})}{2 S(x_{j-1})} (0, 1, 0)^T, \quad \tilde{\xi}_{j-\frac{1}{2}} = 0, \tag{7.34a-c}
\]
\[
\tilde{\xi}_{j-\frac{1}{2}} = 2\tilde{\xi}_{j-\frac{1}{2}} = \frac{\tilde{\alpha}^2 \Delta t S'(x_j)}{2 S(x_j)} (0, 1, 0)^T, \quad \tilde{\xi}_{j-\frac{1}{2}} = 0. \tag{7.35a-c}
\]

7.2. Non-linear Shallow Water Equations

The shallow water equations for the flow of an incompressible fluid in a channel of rectangular cross-section can be written as
\[
w_t + \mathbf{f}_w = \mathbf{g}, \tag{7.36}
\]
where
\[
w = (g(\eta + h), g(\eta + h)u)^T, \tag{7.37}
\]
\[
f(w) = (g(\eta + h)u, g(\eta + h) u^2 + \frac{1}{2} g^2 (\eta + h)^2)^T, \quad \text{and} \quad \tag{7.38}
\]
\[
g(x, w) = (0, g^2(\eta + h) h'(x))^T. \tag{7.39}
\]
The quantities \( \eta = \eta(x, t), u = u(x, t) \) and \( h(x) \) represent the free surface elevation, velocity and the undisturbed depth of the fluid, respectively, at a general point \( x \) and at time \( t \). The acceleration due to gravity is represented by \( g \). Following the approach in [3], we define \( \phi = g(\eta + h) \) so that
\[
w = (\phi, \phi u)^T \tag{7.40}
\]
\[
f = (\phi u, \phi u^2 + \frac{1}{2} \phi^2)^T, \quad \text{and} \quad \tag{7.41}
\]
\[
g = (0, g\phi h'(x))^T. \tag{7.42}
\]
For the algorithm of Section 6 we devise a Riemann problem as given by (3.2). In the specific example of this section, we use the Riemann problem proposed in [3].

The approximate Jacobian matrix \( \tilde{A}(w^*_j, w_j) \) in the cell \([x_{j-1}, x_j]\) at time level \( n \) is
\[
\tilde{A} = \begin{bmatrix} 0 & 1 \\ \phi - \bar{u}^2 & 2 \bar{u} \end{bmatrix}, \tag{7.43}
\]
where the averages of \( u \) and \( \phi \) are given by
\[
\bar{u} = \frac{\sqrt{\phi_{j-1}} u_{j-1} + \sqrt{\phi_j} u_j}{\sqrt{\phi_{j-1}} + \sqrt{\phi_j}}, \tag{7.44}
\]
\[
\bar{\phi} = \frac{1}{2} (\phi_{j-1} + \phi_j). \tag{7.45}
\]
The eigenvalues of \( \tilde{A} \) are
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\[ 1,2 \lambda_{j-\frac{1}{2}} = \bar{u} + \bar{\psi}, \quad \bar{u} - \bar{\psi}, \quad (7.46a,b) \]

with corresponding eigenvectors

\[ 1,2 \tilde{E}_{j-\frac{1}{2}} = \begin{pmatrix} 1 \\ \bar{u} + \bar{\psi} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \bar{u} - \bar{\psi} \end{pmatrix}, \quad (7.47a,b) \]

where

\[ \bar{\psi} = \sqrt{\phi}. \quad (7.48) \]

Thus, the modal matrix \( \tilde{X}_{j-\frac{1}{2}} = \begin{bmatrix} 1 \tilde{E}_{j-\frac{1}{2}}, & 2 \tilde{E}_{j-\frac{1}{2}} \end{bmatrix} \) has the inverse

\[ \tilde{X}^{-1}_{j-\frac{1}{2}} = \frac{1}{2\bar{\psi}} \begin{bmatrix} \bar{\psi} - \bar{u} & 1 \\ \bar{\psi} + \bar{u} & -1 \end{bmatrix}. \quad (7.49) \]

To apply the scheme of Section 6 given by (6.32a,b) we need to calculate the quantities \( \sigma_{j-\frac{1}{2}}, \sigma_{j-\frac{1}{2}}, i\tilde{\xi}_{j-\frac{1}{2}}, i\tilde{\xi}_{j-\frac{1}{2}} \) and \( i\tilde{\xi}_{j-\frac{1}{2}} \) for \( i = 1,2 \). Denoting

\[ \Delta Y = Y_j - Y_{j-1}, \quad (7.50) \]

equations (5.17), (7.40) and (7.49) yield

\[ 1\sigma_{j-\frac{1}{2}} = \frac{1}{2\bar{\psi}} \left( \bar{\psi} \Delta \phi + (\Delta (\phi u) - \bar{u} \Delta \phi) \right), \quad (7.51a) \]
\[ 2\sigma_{j-\frac{1}{2}} = \frac{1}{2\bar{\psi}} \left( \bar{\psi} \Delta \phi - (\Delta (\phi u) - \bar{u} \Delta \phi) \right). \quad (7.51b) \]

If we define the average of \( \phi_{j-1}, \phi_j \)

\[ \tilde{\phi} = \sqrt{\phi_{j-1} \phi_j}, \quad (7.52) \]

then

\[ \Delta (\phi u) - \bar{u} \Delta \phi = \tilde{\phi} \Delta u, \quad (7.53) \]

so that the expressions given in (7.51a,b) simplify to

\[ 1,2 \tilde{\sigma}_{j-\frac{1}{2}} = \frac{1}{2} \Delta \phi \pm \frac{1}{2} \frac{\tilde{\phi}}{\psi} \Delta u. \quad (7.54a-b) \]

In addition, using equations (6.15), (6.17), (7.42), (7.46a,b) and (7.49), we obtain

\[ 1\tilde{\beta}_{j-\frac{1}{2}} = -\frac{\kappa \phi_{j-1}}{\bar{u} + \bar{\psi}}, \quad (7.55a) \]
\[ 2\tilde{\beta}_{j-\frac{1}{2}} = \frac{\kappa \phi_{j-1}}{\bar{u} - \bar{\psi}}, \quad (7.55b) \]

and

\[ 1\tilde{\gamma}_{j-\frac{1}{2}} = -\frac{\kappa \phi_{j}}{\bar{u} + \bar{\psi}}, \quad (7.56a) \]
\[ 2\tilde{\gamma}_{j-\frac{1}{2}} = \frac{\kappa \phi_{j}}{\bar{u} - \bar{\psi}}, \quad (7.56b) \]

where

\[ \kappa = \frac{gh'}{2\bar{\psi}} \left( x_j - \frac{\Delta x}{2} \right). \quad (7.57) \]
Finally, using equations (7.40) and (7.42),

\[ \mathbf{g}_w - gh'(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]  

(7.58)

Therefore, the expressions of Section 6 become

\[ r_{j-\frac{1}{2}} = (0, r_{j-1}), \]  

(7.59a)

\[ r_{j+\frac{1}{2}} = (0, r_{j-1}), \]  

(7.59b)

\[ \tau_j = \frac{1}{2} \Delta t g h'(x_j). \]  

(7.60a)

\[ \tilde{\tau}_j = (0, \tau_j), \]  

(7.60b)

(7.61)

8. CONCLUSIONS

We have extended the second order scheme in [1] for scalar conservation laws with source terms to hyperbolic systems of conservation laws with source terms. In particular, we have seen that the upwinding of the source terms is consistent with a characteristic based method, and we have seen where to evaluate the source terms. We have written the scheme in a simple ‘increment’ and ‘transfer’ form. This scheme has also been applied to two specific examples of inhomogeneous equations. One question remains, however. Scalar, homogeneous equations do not allow new extrema to be formed, and that is why flux limiters are used to avoid non-physical oscillations arising from the use of classical second order methods [4]. However, scalar, inhomogeneous equations may allow new extrema to be created due to the presence of the source terms. Therefore, is it appropriate to apply flux limiters to the second order scheme of Section 6? If so, which terms should be ‘limited,’ bearing in mind that we should not suppress creation of new extrema that are physically correct?

As yet, this question is unresolved, and research in this area is currently being undertaken.

REFERENCES