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On book-wheel Ramsey number

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Abstract

In this paper we determine the following Ramsey numbers:

(1) $r(B_m, W_n) = 2n + 1$ for $m \geqslant 1$, $n \geqslant 5m + 3$, (2) $r(B_m, K_2 \vee C_n) = 2n + 3$ for $n \geqslant 9$ if $m = 1$ or $n \geqslant (m-1)(16m^3 + 16m^2 - 24m - 10) + 1$ if $m \geqslant 2$, where the book B_m is the join $K_2 \vee K_m^c$, W_n denotes a wheel with n spokes, and C_n denotes a cycle of length n.

Keywords: Book; Wheel; Ramsey number

If the edges of the complete graph are colored either red or blue, denote the spanning subgraph with all red edges and all blue edges, respectively, by R and B. Then, R and B is called a factorization of K_n and denoted by $K_n = R \oplus B$. For the graph G_1 and G_2 the Ramsey number $r(G_1, G_2)$ is the least positive integer n such that if $K_n = R \oplus B$ is an arbitrary factorization of the complete graph K_n, then either G_1 is a subgraph of R or G_2 is a subgraph of B. All graphs in this paper are both finite and simple. The edge set, vertex set, maximum degree and minimum degree of a graph G are denoted by $E(G), V(G), \Delta(G)$ and $\delta(G)$, respectively. Let C_n denote a cycle of length n. Let B_m and W_n denote the join $K_2 \vee k_M^C$ and wheel with n spokes, respectively. We refer the reader to [1] for any notation and terminology not explained here.

The following results are well known.

Theorem A (Burr and Erdős [3]). *If* $n \geqslant 5$, *then* $r(B_1, W_n) = 2n + 1$.

Theorem B (Zhou [6]). *If* $n \geqslant 9$, *then* $r(B_1, K_2 \vee C_n) = 2n + 3$.

Theorem C (Sheehan [5]). *If* $m \geqslant 2, n \geqslant (m-1)(16m^3 + 16m^2 - 24m - 10) + 1$, *then* $r(B_m, B_n) = 2n + 3$.

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We need the following simple lemmas to prove our main results. An end-block of a graph G is a maximal nonseparable subgraph of G with at most one cut vertex of G.

Lemma 1. *Suppose that H_1 and H_2 both are end-blocks of a graph G of connectivity at most one. If the complement of G does not contain B_m and $|V(G)| \geqslant 3m+1$, then $V(G) = V(H_1) \cup V(H_2)$.*

Proof. It is easy to see that if $V(G) \setminus (V(H_1) \cup V(H_2)) \neq \emptyset$, then $\max\{|V(H_1)|, |V(H_2)|, |V(G) \setminus (V(H_1) \cup V(H_2))|\} \geqslant m+1$, since $|V(G)| \geqslant 3m+1$. Hence, without loss of generality, we may assume that $|V(H_1)| \geqslant m+1$ when $|V(G) \setminus (V(H_1) \cup V(H_2))| \leqslant m$. Further, we can derive that there are at least m vertices in H_1 which are not adjacent in G to any vertex of $V(G) \setminus V(H_1)$. However, there exist two nonadjacent vertices in $G - V(H_1)$. This contradicts the hypothesis of $G^c \not\supseteq B_m$. When $|V(G) \setminus (V(H_1) \cup V(H_2))| \geqslant m+1$, and in $G[V(H_1) \cup V(H_2)]$ there exist two nonadjacent vertices which are not adjacent in G to any vertex of $V(G) \setminus (V(H_1) \cup V(H_2))$. This also contradicts the hypothesis of $G^c \not\supseteq B_m$. $\square$

Lemma 2. *If the graph G contains a cycle C of length m, but G does not contain any cycle of length $m+1$, and the complement of G has no K_r, then every vertex of $V(G) \setminus V(C)$ is adjacent to at most $r-2$ vertices of C.*

Proof. Suppose that $x_1 x_2 \ldots x_m$ is the sequence of the vertices of cycle C and $v \in V(G) \setminus V(C)$. Since $G \not\supseteq C_{m+1}$, it follows that $x_{i-1}x_{j-1} \notin E(G)$ and $x_{i-1}, x_{j-1} \notin N_G(v)$ for any vertices $x_i, x_j \in N_G(v) \cap V(C)$. This implies that $G[V(C)]$ contains an independent set with $|N_G(v) \cap V(C)|$ vertices which are not adjacent to v. Hence, since $G^c \not\supseteq K_r$, the lemma holds. $\square$

The closure $C(G)$ of a graph G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$ until no such pair remains.

Lemma 3 (Bondy and Murty [1, Theorem 4.4]). *A graph is hamiltonian if and only if its closure is hamiltonian.*

Lemma 4. *If G is a 2-connected graph with at least $n \geqslant 5m+3$ vertices and $G^c \not\supseteq B_m$, then $G \supset C_n$.*

Proof. When $m=1$, Bondy and Ingleton [2] have proved that G is pancyclic. That is to say G contains cycles of length s for all $s, 3 \leqslant s \leqslant |V(G)|$. So we may assume $m \geqslant 2$. We first show that G contains a cycle of length s, where $n-m-1 \leqslant s \leqslant n$. Defining $f(G) = \min\{d_G(u) + d_G(v) | u, v \in V(G), uv \notin E(G)\}$, we get $f(G) \geqslant |V(G)| - m - 1$ from $G^c \not\supseteq B_m$. By a result (if G is a 2-connected graph with $|V(G)| \geqslant k$ and $f(G) \geqslant k$, then G contains a cycle of length at least k) in [4], we get that G contains a cycle of length $l \geqslant |V(G)| - m - 1$. Let $C = v_1v_2 \ldots v_l$ denote this cycle. If $l \leqslant n$, we are done. If $l > n$, then, when $N_G(v_1) \cap \{v_{n-m-1}, v_{n-m}, \ldots, v_n\} \neq \emptyset$, clearly, $G \supseteq C_s, n-m-1 \leqslant s \leqslant n$. When $N_G(v_1) \cap \{v_{n-m-1}, v_{n-m}, \ldots, v_n\} = \emptyset$, let $q = \min\{i > n | v_i \in N_G(v_1) \cap V(C)\}$ and $D = \{v_{q-1}, v_{q-2}, \ldots, v_{q-m}\}$. From $v_1v_{n-m-1} \notin E(G)$ and $G^c[D \cup \{v_1, v_{n-m-1}\}] \not\supseteq B_m$, we can derive $N_G(v_{n-m-1}) \cap D \neq \emptyset$. Assuming $v_p \in N_G(v_{n-m-1}) \cap D$, we get a required cycle $v_1v_2 \ldots v_{n-m-1}v_pv_{p+1} \ldots v_q$. Therefore, if $G \not\supseteq C_n$, we may assume that $C = v_1v_2 \ldots v_s$ is the longest cycle in G such that its length is at most $n-1$. Of course, $n-m-1 \leqslant s \leqslant n-1$. Since $G^c \not\supseteq B_m$, so that $G^c \not\supseteq K_{m+2}$, from Lemma 2 we can obtain $|N_G(v) \cap V(C)| \leqslant m$ for each vertex v of $V(G) \setminus V(C)$. Further, we can conclude that $G[V(G) \setminus V(C)]$ is a complete graph since $G^c \not\supseteq B_m$. Furthermore, if $|V(G) \setminus V(C)| = 1$, say $x \in V(G) \setminus V(C)$, then $s = n-1$. Since G is 2-connected, one may assume $v_i, v_j \in N_G(x) \cap V(C)$, where $j > i$. It is easy to see that $d_G(v_{i-1}) \geqslant f(G) - d_G(x) \geqslant (|V(G)| - m - 1) - m = s - 2m$ by $v_{i-1} \notin N_G(x)$ and $|N_G(x) \cap V(C)| \leqslant m$. Similarly, one obtains $d_G(v_{j-1}) \geqslant s - 2m$. Hence if $v_{i-1}v_{j-1} \in E(G)$, $v_iv_{i+1} \ldots v_{j-1}v_{i-1}v_{i-2} \ldots v_jx$ is a cycle of length $s+1$. If $v_{i-1}v_{j-1} \notin E(G)$, then, clearly, $v_{i-1}v_{j-1} \in E(C(G))$ since $d_G(v_{i-1}) + d_G(v_{j-1}) \geqslant 2s - 4m = 2n - 4m - 2 > n$. So $C(G)$ is hamiltonian. Further, G is hamiltonian by Lemma 3. All the above contradict the assumption of $G \not\supseteq C_n$. Therefore, we may also assume $|V(G) \setminus V(C)| \geqslant 2$.

Since $|V(G) \setminus V(C)| \geqslant 2$ and G is 2-connected, we may choose two vertices v_i, v_j on C such that v_i, v_j are, respectively, adjacent to distinct vertices of $V(G) \setminus V(C)$ and $d_C(v_i, v_j)$ is as small as possible, where $d_C(v_i, v_j)$ is the length of the shortest (v_i, v_j)-segment of C. Say $x, y \in V(G) \setminus V(C)$ and $xv_i, yv_j \in E(G)$. Since $G^c \not\supseteq B_m$, and so that $G^c \not\supseteq K_{m+2}$, we have $|N_G(x) \cap V(C)| \leqslant m$ and $|N_G(y) \cap V(C)| \leqslant m$ from Lemma 2. Then $d_G(x) \leqslant |V(G) \setminus V(C)| + m - 1$ and $d_G(y) \leqslant |V(G) \setminus V(C)| + m - 1$. Now we consider two cases.

Case 1: $d_C(v_i, v_j) = 1$. Say $j = i + 1$. By the choice of cycle C we conclude that $s = n-1, v_{j+1} \notin N_G(V(G) \setminus (V(C) \cup \{x\}))$ and $v_{j+2} \notin N_G(V(G) \setminus (V(C) \cup \{y\}))$ (if $|V(G) \setminus V(C)| \geqslant 3$, then $yv_{j+2} \notin E(G)$ also) since $G[V(G) \setminus V(C)]$ is a complete graph. So we get

$$\begin{aligned} d_{G[V(C)]}(v_{j+1}) &\geqslant d_G(v_{j+1}) - 1 \geqslant f(G) - d_G(y) - 1 \\ &\geqslant |V(G)| - m - 1 - (|V(G) \setminus V(C)| + m - 1) - 1 \\ &= s - 2m - 1. \end{aligned} \tag{1}$$

Similarly, it follows that $d_{G[V(C)]}(v_{j+2}) \geqslant d_G(v_{j+2}) - 1 \geqslant f(G) - d_G(x) - 1 \geqslant s - 2m - 1$. Furthermore, if $v_{j+2}v_j \in E(G)$ or $v_{j+2}v_{j+4} \in E(G)$, then $xyv_jv_{j+2}v_{j+3} \ldots v_i$ or, respectively, $xyv_jv_{j+1}v_{j+2}v_{j+4}v_{j+5} \ldots v_i$ is a cycle longer than C. Hence $v_{j+2}v_j, v_{j+2}v_{j+4} \notin E(G)$. If $v_{j+2}v_{j+r} \in E(G)$ and $v_{j+r-2}v_{j+1} \in E(G)$ for $r \neq 1, 3$, the cycle $xyv_jv_{j+1}v_{j+r-2}v_{j+r-3} \ldots v_{j+2}v_{j+r}v_{j+r+1} \ldots v_i$ is longer than C, so that $v_{j+r-2}v_{j+1} \notin E(G)$ if $v_{j+r}v_{j+2} \in E(G)$ for each vertex v_{j+r} on $C, r \neq 1, 3$. So we have

$$\begin{aligned} d_{G[V(C)]}(v_{j+1}) &\leqslant s - 1 - (d_{G[V(C)]}(v_{j+2}) - 2) \\ &\leqslant s - 1 - (s - 2m - 1 - 2) = 2m + 2. \end{aligned} \tag{2}$$

However, $s-2m-1=n-1-2m-1>2m+2$ since $n\geqslant 5m+3$ and $m\geqslant 2$. That is to say inequality (1) contradicts (2).

Case 2: $d_C(v_i,v_j)\geqslant 2$. Since $G[V(G)\setminus V(C)]$ is a complete graph and $G\not\supseteq C_{s+1}$, we get $d_C(v_i,v_j)\geqslant |V(G)\setminus V(C)|+1$. By the choice of v_i,v_j and $G\not\supseteq C_{s+1}$, we also have $v_{j-2},v_{i-1}\notin N_G(V(G)\setminus V(C))$. Let $G'=G[\{x,y\}\cup V(C)\setminus\{v_{j-1}\}]$. It is easy to see $d_{G[V(C)]}(v_{i-1})=d_G(v_{i-1})\geqslant f(G)-d_G(x)\geqslant |V(G)|-m-1-(|V(G)\setminus V(C)|+m-1)=s-2m$. This implies $d_{G'}(v_{i-1})\geqslant s-2m-1$. Similarly, we have $d_{G'}(v_{j-2})\geqslant s-2m-1$. Furthermore, if $v_{i-1}v_{j-2}\in E(G)$, the cycle $xyv_jv_{j+1}\ldots v_{i-1}v_{j-2}v_{j-3}\ldots v_i$ contradicts the choice of C. Hence, $v_{i-1}v_{j-2}\notin E(G)$. So, we obtain $|V(G)\setminus V(C)|\leqslant m-1$ from $v_{i-1},v_{j-2}\notin N_G(V(G)\setminus V(C))$ and $G^c[\{v_{i-1},v_{j-2}\}\cup V(G)\setminus V(C)]\not\supseteq B_m$. This implies $s\geqslant n-m+1$. Further, we obtain $d_{G'}(v_{i-1})+d_{G'}(v_{j-2})\geqslant 2s-4m-2>s+1=|V(G')|$ since $s\geqslant n-m+1$ and $n\geqslant 5m+3$. Then, $v_{i-1}v_{j-2}\in E(C(G'))$, and $v_{i-1}v_{j-2}v_{j-3}\ldots v_ixyv_jv_{j+1}\ldots v_{i-1}$ is a hamiltonian cycle in $C(G')$. This implies that $G'\supset C_{s+1}$ by Lemma 3. This completes the proof of Lemma 4. $\square$

Lemma 5. *Suppose that G is a graph with $G^c\not\supseteq B_m$, where $m\geqslant 2$. Let $S\subset V(G)$ and $D=\{v\in S\mid d_{G[S]}(v)=|S|-1\}$. If $\delta(G[S])\geqslant|S|-m$ and $|S\setminus D|-(m-1)^2-m>0$, then there exists an edge xy in $G[S]$ such that $|N_G(x)\cap N_G(y)|\geqslant|S|+|V(G)\setminus S|/4-11m/4+3/4$.*

Proof. We will first show that $G[N_G(x)\cap S]$ is not a complete graph for any vertex x of S. If $x\in D$, we are done since $|S\setminus D|-(m-1)^2-m>0$. If $x\in S\setminus D$, since $\delta(G[S])\geqslant|S|-m$, there are at most $(|S\setminus N_G(x)|-1)(m-1)\leqslant(m-1)^2$ vertices of $N_G(x)\cap S$ which are adjacent in G^c to some vertices of $S\setminus(N_G(x)\cup\{x\})$. So there exist at least $|N_G(x)\cap S|-|D|-(m-1)^2\geqslant|S|-m-|D|-(m-1)^2>0$ vertices of $N_G(x)\cap S$ which are not adjacent in G^c to any vertex of $S\setminus(N_G(x)\cup\{x\})$ and whose degree in $G[S]$ is at most $|S|-2$. This implies that $G[N_G(x)\cap S]$ is not a complete graph.

By $|S\setminus D|-(m-1)^2-m>0$ we may assume that $x_1,x_2\in S$ and $x_1x_2\notin E(G)$. From $G^c\not\supseteq B_m$, we obtain $|(N_G(x_1)\cup N_G(x_2))\cap(V(G)\setminus S)|\geqslant|V(G)\setminus S|-m+1$. Without loss of generality, we may assume that $|N_G(x_1)\cap V(G)\setminus S|\geqslant|N_G(x_2)\cap V(G)\setminus S|$. Then $|N_G(x_1)\cap V(G)\setminus S|=|N_G(x_1)\setminus S|\geqslant(|V(G)\setminus S|-m+1)/2$. Since $G[N_G(x_1)\cap S]$ is not a complete graph, we may also assume that $y_1,y_2\in N_G(x_1)\cap S$ and $y_1y_2\notin E(G)$. From $G^c\not\supseteq B_m$ we also get $|(N_G(y_1)\cup N_G(y_2))\cap(N_G(x_1)\setminus S)|\geqslant|N_G(x_1)\setminus S|-m+1$. Say $|N_G(y_1)\cap N_G(x_1)\setminus S|\geqslant|N_G(y_2)\cap N_G(x_1)\setminus S|$. So we have $|N_G(y_1)\cap N_G(x_1)\setminus S|\geqslant(|N_G(x_1)\setminus S|-m+1)/2\geqslant|V(G)\setminus S|/4-3m/4+3/4$. Morever, observing $\delta(G[S])\geqslant|S|-m$, one can get $|N_G(y_1)\cap N_G(x_1)\cap S|\geqslant|S|-2m$. Hence, $|N_G(y_1)\cap N_G(x_1)|\geqslant|S|-2m+|V(G)\setminus S|/4-3m/4+3/4=|S|+|V(G)\setminus S|/4-11m/4+3/4$. The proof of Lemma 5 is complete. $\square$

Theorem 1. $r(B_m,W_n)=2n+1$ *for* $m\geqslant 1$, $n\geqslant 5m+3$.

Proof. For $m=1$, the theorem is true by Theorem A, and so we need only to prove the theorem for $m\geqslant 2$. Consider the graph $H=K_n+K_n$. Since $H^c\not\supseteq B_m$ and $H\not\supseteq W_n$, it follows that $r(B_m, W_n) \geqslant 2n+1$. To prove the reverse inequality, let $K_{2n+1} = R \oplus B$ be an arbitrary factorization of the complete graph K_{2n+1}. We will first prove the following two propositions.

Proposition 1.1. *If $B \not\supseteq W_n$ and $\delta(R) \geqslant 2n/3 + m - 1/3$, then $R \supseteq B_m$.*

Proof. Since $B \not\supseteq W_n$, we have $R \supseteq B_1$ by Theorem A. So suppose that $x_1x_2x_3$ is a triangle in R. If $R \not\supseteq B_m$, so that $|N_R(x_i) \cap N_R(x_j)| \leqslant m-1$ for $i \neq j, i, j = 1, 2, 3$, then $|V(R)| \geqslant d_R(x_1) + d_R(x_2) + d_R(x_3) - |N_R(x_1) \cap N_R(x_2)| - |N_R(x_1) \cap N_R(x_3)| - |N_R(x_2) \cap N_R(x_3)| \geqslant 3\delta(R) - 3(m-1) \geqslant 3(2n/3 + m - 1/3) - 3(m-1) = 2n+2$, a contradiction.

Proposition 1.2. *If $R \not\supseteq B_m$ and $B \not\supseteq W_n$, then there exists a vertex v in B such that $B[N_B(v)]$ contains a block with at least n vertices.*

Proof. Since $R \not\supseteq B_m$ and $B \not\supseteq W_n$, we have that $\delta(R) < 2n/3 + m - 1/3$ by Proposition 1.1. Hence, $\triangle(B) > 2n - (2n/3 + m - 1/3) = 4n/3 - m + 1/3$. Choose a vertex v in B such that $d_B(v) = \triangle(B)$. If $B[N_B(v)]$ is 2-connected, we are done. Conversely, if the connectivity of $B[N_B(v)]$ is at most one, we may assume that H_1 and H_2 are both end-blocks in $B[N_B(v)]$ with $V(H_1) \cup V(H_2) = N_B(v)$ by Lemma 1. Without loss of generality, suppose that $|V(H_1)| \leqslant |V(H_2)|$ and let $V_1 = V(H_1) \setminus V(H_2)$, $V_2 = V(H_2) \setminus V(H_1)$ and $V_3 = V(B) \setminus (\{v\} \cup N_B(v))$. If $|V(H_2)| \geqslant n$, then the vertex v is required. So, we may also assume that $|V(H_2)| < n$. Then $|V_1| = |N_B(v)| - |V(H_2)| > 4n/3 - m + 1/3 - (n-1) = n/3 - m + 4/3$, and the $B[V_1]$ is a complete graph, since $R \not\supseteq B_m$ and $|N_B(V_1) \cap V_2| \leqslant 1$. Let $T_1 = \{x \in V_3 | N_B(x) \supseteq V_1\}$ and $T_2 = \{x \in V_3 | N_B(x) \supseteq V_2\}$. Now we consider two cases.

Case 1: $V_3 = T_1 \cup T_2$. In this case we must have $|V_1 \cup T_1| \geqslant n$ or $|V_2 \cup T_2| \geqslant n$. If $|V_1 \cup T_1| \geqslant n$, every vertex of V_1 is required since $B[V_1]$ is a complete graph. So we may assume that $|V_1 \cup T_1| < n$, and hence that $|V_2 \cup T_2| \geqslant n$. Further, if $B[V_2]$ is a complete graph, then every vertex of V_2 is also required. If $B[V_2]$ is not a complete graph, then $|V_1| \leqslant m$ since $R \not\supseteq B_m$ and $|N_B(V_2) \cap V_1| \leqslant 1$. Furthermore, noting that $|V(H_2)| < n$ and $|V_2 \cup T_2| \geqslant n$, we may assume $y \in T_2$. From $yv \notin E(B)$ and $R \not\supseteq B_m$, we can obtain $|N_B(y) \cap V_3| \geqslant |V_3| - m$. So we get $d_B(y) \geqslant |V_3| - m + |V_2| \geqslant |V(B)| - |V_1| - |\{v\}| - m - 1 \geqslant 2n - 2m - 1 > n + m > d_B(v)$, a contradiction.

Case 2: $V_3 \setminus (T_1 \cup T_2) \neq \emptyset$. Let $T_3 = V_3 \setminus (T_1 \cup T_2)$ and let $x \in T_3$. Since $R \not\supseteq B_m$ and $|N_B(u) \cap V_1| \leqslant 1$ for any vertex u of $V_2 \setminus N_B(x)$, we get $|N_B(x) \cap V_1| \geqslant |V_1| - m$. Similarly, we have $|N_B(x) \cap V_2| \geqslant |V_2| - m$. From $R \not\supseteq B_m$ and $xv \notin E(B)$ we can also get $|N_B(x) \cap V_3| \geqslant |V_3| - m$. Hence, $d_B(x) \geqslant |V_1| - m + |V_2| - m + |V_3| - m \geqslant |V(B)| - 3m - 2 = 2n - 3m - 1$. Further, observing that $|V(H_2)| < n$ and $n \geqslant 5m+3$, we have $|V_1| = d_B(v) - |V(H_2)| \geqslant d_B(x) - |V(H_2)| \geqslant 2n - 3m - 1 - (n-1) \geqslant 2m+3$ by the choice of vertex v. Hence, $B[V_2]$ also is a complete graph, since $R \not\supseteq B_m$. Without loss of generality, we may assume that $|V_1 \cup T_1| \geqslant |V_2 \cup T_2|$. Then $|V_1 \cup T_1| \geqslant (|V(B) \setminus T_3| - 2)/2 = n - (|T_3| + 1)/2$. On the other hand, since every vertex of T_3 is adjacent to at least $|V_1| - m$ vertices of V_1, as x is, $|[T_3, V_1]| \geqslant |T_3|(|V_1| - m)$ in B. Observing that $|V_1| \geqslant 2m+2$, we obtain $|T_3|(|V_1|-m)/|V_1|=|T_3|-m|T_3|/|V_1|>|T_3|-m|T_3|/2m=|T_3|/2$. This implies that there must exist a vertex y in V_1 satisfying $|N_B(y)\cap T_3|>|T_3|/2$. Further, we can see that $d_B(y)\geqslant|(N_B(y)\cap T_3)\cup(V_1\cup T_1)\cup\{v\}|-1>|T_3|/2+n-(|T_3|+1)/2+1-1=n-1/2$, and the subgraph of B induced by $(N_B(y)\cap T_3)\cup(V_1\cup T_1)\cup\{v\}\setminus\{y\}\subseteq N_B(y)$ is 2-connected. Thus, the proof of the proposition is complete. □

Proof of Theorem 1. If $R\not\supseteq B_m$ and $B\not\supseteq W_n$, by Proposition 1.2 we may choose a vertex v in B such that $B[N_B(v)]$ contains a block G with at least n vertices. By Lemma 4 we have that G contains a C_n, that is to say $B\supseteq W_n$, a contradiction.

Theorem 2. *$r(B_m,K_2\vee C_n)=2n+3$ for $n\geqslant 9$ if $m=1$ or $n\geqslant(m-1)(16m^3+16m^2-24m-10)+1$ if $m\geqslant 2$.*

Proof. When $m=1$, the theorem is true by Theorem B. Now, we shall prove the theorem for $m\geqslant 2$. Consider the graph $F=K_{n+1}+K_{n+1}$. It is obvious that $F\not\supseteq K_2\vee C_n$ and $F^{\mathrm{c}}\not\supseteq B_m$. Hence $r(B_m,K_2\vee C_n)\geqslant 2n+3$. Let $K_{2n+3}=R\oplus B$ be an arbitrary factorization of the complete graph K_{2n+3}. We shall show that either R contains a B_m or B contains a $K_2\vee C_n$, that is to say $r(B_m,K_2\vee C_n)\leqslant 2n+3$, we first prove the following.

Proposition 2.1. *If $R\not\supseteq B_m$* and *$B\not\supseteq K_2\vee C_n$, then there exists an edge uv in B such that $B[N_B(u)\cap N_B(v)]$ contains a block with at least n vertices.*

Proof. Choose an edge uv in B such that $|N_B(u)\cap N_B(v)|$ is as large as possible. By Theorem C and $R\not\supseteq B_m$, we obtain $|N_B(u)\cap N_B(v)|\geqslant n$. If $B[N_B(u)\cap N_B(v)]$ is 2-connected, we are done. Therefore, we may assume that the connectivity of $B[N_B(u)\cap N_B(v)]$ is at most one. Further, by Lemma 1, we may assume that H_1 and H_2 both are end-blocks of $B[N_B(u)\cap N_B(v)]$ with $V(H_1)\cup V(H_2)=N_B(u)\cap N_B(v)$. Suppose that $|V(H_1)|\leqslant|V(H_2)|$. Let $V_1=V(H_1)\setminus V(H_2)$, $V_2=V(H_2)\setminus V(H_1)$ and $V_3=V(B)\setminus(N_B(u)\cap N_B(v)\cup\{u,v\})$, and let $T_1=\{x\in V_3|N_B(x)\supseteq V_1\}$ and $T_2=\{x\in V_3|N_B(x)\supseteq V_2\}$. If $V(H_2)|\geqslant n$, then uv is the required edge, so we may also assume that $|V(H_2)|<n$. Now, we consider two cases.

Case 1: $V_3=T_1\cup T_2$. In this case it is easy to see that $|V_1\cup V_2\cup T_1\cup T_2|=2n$ when $V(H_1)\cap V(H_2)\neq\emptyset$ or $|V_1\cup V_2\cup T_1\cup T_2|=2n+1$ when $V(H_1)\cap V(H_2)=\emptyset$. So, we get either $|V_1\cup T_1|\geqslant n$ or $|V_2\cup T_2|\geqslant n$.

Subcase 1.1: $|V_2\cup T_2|\geqslant n$. If $B[V_2]$ has at least 4 vertices of degree $|V_2|-1$, then, clearly, every edge of the subgraph of B induced by these vertices is required. If $B[V_2]$ has at most 3 vertices of degree $|V_2|-1$, we have that $|V_1|<m$ and $\delta(B[V_2])\geqslant|V_2|-m$ since $R\not\supseteq B_m$ and $|N_B(V_2)\cap V_1|\leqslant 1$. Thus, $|V_2|-3-(m-1)^2-m\geqslant|N_B(u)\cap N_B(v)|-|V_1|-1-3-(m-1)^2-m>n-m-4-(m-1)^2-m>0$. Applying Lemma 5 to the graph $B[V_2\cup V_3\setminus T_2]$, we may assume that xy is an edge of $B[V_2]$ satisfying $|N_B(x)\cap N_B(y)\cap(V_2\cup V_3\setminus T_2)|\geqslant|V_2|+|V_3\setminus T_2|/4-11m/4+3/4$. Observing that $|V_3\setminus T_2|=|V(B)|-|V_2\cup T_2|-|V(H_1)|-|\{u,v\}|\geqslant|V(B)|-|V_2\cup T_2|-m-2$, we get $|N_B(x)\cap N_B(y)|\geqslant|V_2|+|V_3\setminus T_2|/4-11m/4+3/4+|T_2|+|\{u,v\}|\geqslant 3|V_2\cup T_2|/4+|V(B)|/$

$4 - 3m + 9/4 \geqslant 5n/4 - 3m + 3 > n + m > |N_B(u) \cap N_B(v)|$. This contradicts the choice of the edge uv.

Subcase 1.2: $|V_2 \cup T_2| < n$. In this case we obtain $|V_1 \cup T_1| \geqslant |V(B) \setminus (V_2 \cup T_2 \cup \{u, v\})| - 1 \geqslant 2n + 3 - (n - 1 + 2) - 1 = n + 1$ and we have the $B[V_1]$ is a complete graph, since $|N_B(V_1) \cap V_2| \leqslant 1$ and $R \not\supseteq B_m$. Hence, if $|V_1| \geqslant 4$, every edge of $B[V_1]$ is required.

Suppose that $|V_1| = 3$. If $|V_2 \cup T_2| \leqslant n - 2$, then, when $V(H_1) \cap V(H_2) \neq \emptyset$, denote by xy the edge of $B[V_1]$ whose end-vertices are both adjacent to vertex of $V(H_1) \cap V(H_2)$. Thus, the edge xy satisfies $|N_B(x) \cap N_B(y)| \geqslant |V(B) \setminus (V_2 \cup T_2 \cup \{x, y\})| \geqslant 2n + 3 - (n - 2 + 2) = n + 3$. When $V(H_1) \cap V(H_2) = \emptyset$, every edge xy of $B[V_1]$ also satisfies $|N_B(x) \cap N_B(y)| \geqslant n + 3$. So we deduce $|V(H_2)| \geqslant n$ from the choice of the edge uv. This contradicts the assumption of $|V(H_2)| < n$. Therefore, $|V_2 \cup T_2| = n - 1$. Hence, we get $|V(H_1) \cup T_1 \setminus T_2| = |V(B) \setminus (V_2 \cup T_2 \cup \{u, v\})| = 2n + 3 - (n - 1 + 2) = n + 2$ and may assume $x_1, x_2 \in V(H_1) \cup T_1 \setminus T_2$ and $x_1 x_2 \notin E(B)$ since $B[V(H_1) \cup T_1 \setminus T_2] \not\supseteq K_2 \vee C_n$. Let $D = \{x \in V_2 | d_{B[V_2]}(x) = |V_2| - 1\}$. If $|D| \geqslant 3m + 3$, we obtain $|(N_B(x_1) \cup N_B(x_2)) \cap D| \geqslant |D| - m + 1$ since $R[\{x_1, x_2\} \cup D] \not\supseteq B_m$. Without loss of generality, suppose that $|N_B(x_1) \cap D| \geqslant |N_B(x_2) \cap D|$. Then $|N_B(x_1) \cap D| \geqslant (|D| - m + 1)/2 \geqslant (3m + 3 - m + 1)/2 \geqslant 4$, so that there exists an edge xy in $B[N_B(x_1) \cap D]$ such that $N_B(x) \cap N_B(y) \supseteq T_2 \cup (V_2 \setminus \{x, y\}) \cup \{x_1, u, v\}$ and the subgraph of B induced by $T_2 \cup (V_2 \setminus \{x, y\}) \cup \{x_1, u, v\}$ is 2-connected. Now, one can see that xy is the required edge. If $|D| \leqslant 3m + 2$, then $|V_2 \setminus D| - (m - 1)^2 - m \geqslant |N_B(u) \cap N_B(v)| - |V_1| - 1 - |D| - (m - 1)^2 - m \geqslant n - 3 - 1 - (3m + 2) - (m - 1)^2 - m > 0$ and we have that $\delta(B[V_2]) \geqslant |V_2| - m, |V_1| < m$ since $|N_B(V_1) \cap V_2| \leqslant 1$ and $R \not\supseteq B_m$. Applying Lemma 5 to the graph $B[V_2 \cup V_3 \setminus T_2]$ we may assume that xy is an edge of $B[V_2]$ satisfying $|N_B(x) \cap N_B(y) \cap (V_2 \cup V_3 \setminus T_2)| \geqslant |V_2| + |V_3 \setminus T_2|/4 - 11m/4 + 3/4$. Observing that $|V_3 \setminus T_2| = |V(B)| - |V_2 \cup T_2| - |V(H_1)| - |\{u, v\}| \geqslant 2n + 3 - (n - 1) - 4 - 2 = n - 2$, we get $|N_B(x) \cap N_B(y)| \geqslant |V_2| + |V_3 \setminus T_2|/4 - 11m/4 + 3/4 + |T_2| + |\{u, v\}| = |V_2 \cup T_2| + |V_3 \setminus T_2|/4 - 11m/4 + 11/4 \geqslant n - 1 + (n - 2)/4 - 11m/4 + 11/4 > n + m > |N_B(u) \cap N_B(v)|$, a contradiction.

Suppose that $|V_1| = 2$ and denote the unique edge in $B(V_1)$ by xy. Then $|N_B(x) \cap N_B(y)| \geqslant |V(B)| - |V_2 \cup T_2| - 2 \geqslant 2n + 3 - (n - 1) - 2 = n + 2$. Further, we have $|V(H_2)| \geqslant n + 2 - |V_1| = n$ by the choice of edge uv. This contradicts the assumption that $|V(H_2)| < n$.

Suppose that $|V_1| = 1$ and let $D = \{x \in V_2 | d_{B[V_2]}(x) = |V_2| - 1\}$. Then $|V_2| \geqslant n - 2$, so that $|T_2| \leqslant 1$ since $|V_2 \cup T_2| < n$. Further, we get $|T_1 \setminus T_2| \geqslant |V(B)| - |V_2 \cup T_2| - |V_1| - 1 - |\{u, v\}| \geqslant 2n + 3 - (n - 1) - 1 - 1 - 2 = n$. Hence, if $|D| \leqslant 3m + 12$, we have $|V_2 \setminus D| - (m - 1)^2 - m \geqslant n - 2 - (3m + 12) - (m - 1)^2 - m > 0$ and can derive $\delta(B[V_2]) \geqslant |V_2| - m$ from $N_B(V_1) \cap V_2 = \emptyset$ and $R \not\supseteq B_m$. Using Lemma 5 to the graph $B[V_2 \cup T_1 \setminus T_2]$, we can find an edge xy in $B[V_2]$ such that $|N_B(x) \cap N_B(y) \cap (V_2 \cup T_1 \setminus T_2)| \geqslant |V_2| + |T_1 \setminus T_2|/4 - 11m/4 + 3/4 \geqslant n - 2 + n/4 - 11m/4 + 3/4 > n + m > |N_B(u) \cap N_B(v)|$. This is a contradiction. If $|D| \geqslant 3m + 13$, then since $n > |V_2 \cup T_2| \geqslant |V_2| \geqslant n - 2$ there are two cases to consider: $|V_2 \cup T_2| = n - 1$ or $|V_2 \cup T_2| = n - 2$. If $|V_2 \cup T_2| = n - 1$, then $|V(H_1) \cup T_1 \setminus T_2| = |V(B) \setminus (V_2 \cup T_2 \cup \{u, v\})| = 2n + 3 - (n - 1 + 2) = n + 2$. Since $B[V(H_1) \cup T_1 \setminus T_2] \not\supseteq K_2 \vee C_n$, we may assume that $x_1, x_2 \in V(H_1) \cup T_1 \setminus T_2$ and $x_1 x_2 \notin E(B)$. Hence, we get $|(N_B(x_1) \cup N_B(x_2)) \cap D| \geqslant |D| - m + 1$ since $R[\{x_1, x_2\} \cup D] \not\supseteq B_m$.

Without loss of generality, we assume that $|N_B(x_1)\cap D|\geqslant|N_B(x_2)\cap D|$. Then $|N_B(x_1)\cap D|\geqslant(|D|-m+1)/2\geqslant(3m+13-m+1)/2=m+7$. So there is an edge xy in $B[N_B(x_1)\cap D]$ such that $N_B(x)\cap N_B(y)\supseteq(V_2\backslash\{x,y\})\cup T_2\cup\{x_1,u,v\}$ and the subgraph of B induced by $(V_2\setminus\{x,y\})\cup T_2\cup\{x_1,u,v\}$ is 2-connected. That is to say that xy is the required edge. If $|V_2\cup T_2|=n-2$, then $T_2=\emptyset$, so that $|V(H_1)\cup T_1|=|V(B)\setminus(V_2\cup\{u,v\})|=2n+3-(n-2+2)=n+3$. Since $B[V(H_1)\cup T_1]\not\supseteq K_2\vee C_n$, we may assume that $x_1,x_2\in V(H_1)\cup T_1$ and $x_1x_2\notin E(B)$. Further, we get $|(N_B(x_1)\cup N_B(x_2))\cap D|\geqslant|D|-m+1$ since $R[\{x_1,x_2\}\cup D]\not\supseteq B_m$. Without loss of generality, we may assume that $|N_B(x_1)\cap D|\geqslant|N_B(x_2)\cap D|$, so that $|N_B(x_1)\cap D|\geqslant(|D|-m+1)/2$. By $B[(V(H_1)\cup T_1)\setminus\{x_1\}]\not\supseteq K_2\vee C_n$, we may also assume that $y_1,y_2\in(V(H_1)\cup T_1)\setminus\{x_1\}$ and $y_1y_2\notin E(B)$. Similarly, we obtain $|(N_B(y_1)\cup N_B(y_2))\cap N_B(x_1)\cap D|\geqslant|N_B(x_1)\cap D|-m+1$ since $R[\{y_1,y_2\}\cup(N_B(x_1)\cap D)]\not\supseteq B_m$. Suppose, without loss of generality, that $|N_B(y_1)\cap N_B(x_1)\cap D|\geqslant|N_B(y_2)\cap N_B(x_1)\cap D|$. Then $|N_B(y_1)\cap N_B(x_1)\cap D|\geqslant(|N_B(x_1)\cap D|-m+1)/2\geqslant(|D|-3m+3)/4\geqslant(3m+13-3m+3)/4=4$. Obviously, there is an edge xy in $B[N_B(y_1)\cap N_B(x_1)\cap D]$ such that $N_B(x)\cap N_B(y)\supseteq(V_2\setminus\{x,y\})\cup\{x_1,y_1,u,v\}$ and the subgraph of B induced by $(V_2\setminus\{x,y\})\cup\{x_1,y_1,u,v\}$ is 2-connected. So xy is the required edge.

Case 2: $V_3\setminus(T_1\cup T_2)\neq\emptyset$. In this case the proof proceeds by showing the following facts.

Fact 1. *If* $N_B(u)\cap V_3\setminus(T_1\cup T_2)\neq\emptyset$, *then* $|N_B(u)\cap V_3|\leqslant 3m+1$; *similarly, if* $N_B(v)\cap V_3\setminus(T_1\cup T_2)\neq\emptyset$, *then* $|N_B(v)\cap V_3|\leqslant 3m+1$.

Proof. We only prove the first part of the fact; the proof of the second is the same. Let $x\in N_B(u)\cap V_3\setminus(T_1\cup T_2)$. Then $V_1\setminus N_B(x)\neq\emptyset$ and $V_2\setminus N_B(x)\neq\emptyset$, so we get $N_B(x)\cap V_i|\geqslant|V_i|-m$, $i=1,2$, since $R\not\supseteq B_m$ and $|[V_1,V_2]|\leqslant 1$ in B. Morever, we have $|N_B(x)\cap N_B(u)\cap V_3|\geqslant|N_B(u)\cap V_3|-m$ since $xv\notin E(B), R[\{x,v\}\cup(N_B(u)\cap V_3)]\not\supseteq B_m$ and $N_B(v)\cap N_B(u)\cap V_3=\emptyset$. Hence, if $|N_B(u)\cap V_3|\geqslant 3m+2$, then $|N_B(x)\cap N_B(u)|\geqslant|N_B(x)\cap V_1|+|N_B(x)\cap V_2|+|N_B(x)\cap N_B(u)\cap V_3|\geqslant|V_1|-m+|V_2|-m+3m+2-m>|N_B(u)\cap N_B(v)|$. This contradicts the choice of edge uv.

Fact 2. *If* $|N_B(u)\cap N_B(v)|\geqslant 2n-8m-1$, *then Proposition* 2.1 *is true.*

Proof. Since $|N_B(u)\cap N_B(v)|\geqslant 2n-8m-1$ and $|V(H_1)|\leqslant|V(H_2)|<n$, we get $|V_1|\geqslant n-8m$ and $|V_3|=|V(B)\setminus(N_B(u)\cap N_B(v)\cup\{u,v\})|\leqslant 8m+2$. Further, we have that $B[V_1]$ and $B[V_2]$ both are complete graphs, since $R\not\supseteq B_m$. Without loss of generality, we may assume that $|V_1\cup T_1|\geqslant|V_2\cup T_2|$. Then $|V_1\cup T_1|\geqslant(|V(B)|-|\{u,v\}\cup V_3\setminus(T_1\cup T_2)|-1)/2=n-|V_3\setminus(T_1\cup T_2)|/2$. On the other hand, for each vertex x of $V_3\setminus(T_1\cup T_2)$, we have $|N_B(x)\cap V_1|\geqslant|V_1|-m$ since $V_2\setminus N_B(x)\neq\emptyset$, $|N_B(V_2)\cap V_1|\leqslant 1$ and $R\not\supseteq B_m$. This implies that there exist at most $m|V_3\setminus(T_1\cup T_2)|\leqslant m(8m+2)$ vertices of V_1 which are adjacent in R to some vertices of $V_3\setminus(T_1\cup T_2)$, and so there exist at least $|V_1|-m|V_3\setminus(T_1\cup T_2)\geqslant n-8m-m(8m+2)>4$ vertices of V_1 which are adjacent in B to each vertex of $V_3\setminus(T_1\cup T_2)$. Let $D=\{x\in V_1|N_B(x)\supseteq V_3\setminus(T_1\cup T_2)\}$ and let $xy\in E(B[D])$.

Then $N_B(x)\cap N_B(y)\supseteq(V_1\setminus\{x,y\})\cup T_1\cup\{u,v\}\cup V_3\setminus(T_1\cup T_2)$ and the subgraph of B induced by $(V_1\setminus\{x,y\})\cup T_1\cup\{u,v\}\cup V_3\setminus(T_1\cup T_2)$ is 2-connected, so the edge xy is required.

Fact 3. *If* $|V_3\setminus(N_B(u)\cup N_B(v)\cup T_1\cup T_2)|\geqslant m+1$, *then Proposition* 2.1 *is true.*

Proof. Since $R\not\supseteq B_m$ and u,v both are not adjacent in B to any vertex of $V_3\setminus(N_B(u)\cup N_B(v)\cup T_1\cup T_2)$, $\delta(B[V_3\setminus(N_B(u)\cup N_B(v)\cup T_1\cup T_2)])\geqslant|V_3\setminus(N_B(u)\cup N_B(v)\cup T_1\cup T_2)| - m\geqslant 1$. So we may assume that $x_1x_2\in E(B[V_3\setminus(N_B(u)\cup N_B(v)\cup T_1\cup T_2)])$ and can obtain $|N_B(x_1)\cap V_3\setminus N_B(u)|\geqslant|V_3\setminus N_B(u)|-m$, since $x_1u\notin E(B)$ and $R[\{x_1,u\}\cup V_3\setminus N_B(u)]\not\supseteq B_m$. Analogously, we have $|N_B(x_1)\cap V_3\setminus N_B(v)|\geqslant|V_3\setminus N_B(v)|-m$, and so we get $|N_B(x_1)\cap V_3|\geqslant|(V_3\setminus N_B(u))\cup(V_3\setminus N_B(v))|-2m=|V_3|-2m$ since $(V_3\cap N_B(u))\cap(V_3\cap N_B(v))=\emptyset$. Similarly, we have $|N_B(x_2)\cap V_3|\geqslant|V_3|-2m$. Morever, we can also get $|N_B(x_i)\cap V_j|\geqslant|V_j|-m$, $i,j=1,2$, since $R\not\supseteq B_m, V_j\setminus N_B(x_i)\neq\emptyset$, $i,j=1,2$, and $|[V_1,V_2]|\leqslant 1$ in B. So $|N_B(u)\cap N_B(v)|\geqslant|N_B(x_1)\cap N_B(x_2)|\geqslant|V_1|-2m+|V_2|-2m+|V_3|-4m\geqslant|V(B)\setminus\{u,v\}|-1-8m=2n-8m$, so that Proposition 2.1 holds by Fact 2.

Fact 1 implies that if $|N_B(u)\cap V_3\setminus(T_1\cup T_2)|\geqslant 3m+2$ or $|N_B(v)\cap V_3\setminus(T_1\cup T_2)|\geqslant 3m+2$, then Proposition 2.1 holds. Combining Fact 1 with Fact 3 we conclude that if $|V_3\setminus(T_1\cup T_2)|\geqslant 7m+3$, then Proposition 2.1 holds. For the case $|V_3\setminus(T_1\cup T_2)|\leqslant 7m+2$ we have following.

Fact 4. *If* $|V_3\setminus(T_1\cup T_2)|\leqslant 7m+2$, *then Proposition* 2.1 *is also true.*

Proof. The proof is broken into two cases.

Subcase 2.1: $|V_2\cup T_2\cup V_3\setminus(T_1\cup T_2)|\geqslant n$. Let $D=\{x\in V_2|d_{B[V_2]}(x)=|V_2|-1\}$. For each vertex v of $V_3\setminus(T_1\cup T_2)$ we have $|N_B(v)\cap V_2|\geqslant|V_2|-m$ since $|N_B(V_1)\cap V_2|\leqslant 1, V_1\setminus N_B(v)\neq\emptyset$ and $R[\{v\}\cup V_1\cup V_2]\not\supseteq B_m$. This implies that there are at most $|v_3\setminus(T_1\cup T_2)|m\leqslant(7m+2)m$ vertices of V_2 which are adjacent in R to some vertices of $V_3\setminus(T_1\cup T_2)$. So, if $|D|\geqslant(7m+2)m+4$, let $S=\{x\in D|N_B(x)\supseteq V_3\setminus(T_1\cup T_2)\}$. Clearly, $|S|\geqslant|D|-(7m+2)m\geqslant 4$. Let xy be an edge of $B[S]$. Clearly, $N_B(x)\cap N_B(y)\supseteq(V_2\setminus\{x,y\})\cup T_2\cup\{u,v\}\cup V_3\setminus(T_1\cup T_2)$ and the subgraph of B induced by $(V_2\setminus\{x,y\})\cup T_2\cup\{u,v\}\cup V_3\setminus(T_1\cup T_2)$ is 2-connected, so xy is the required edge. If $|D|\leqslant(7m+2)m+3$, so that $|V_2\setminus D|-(m-1)^2-m\geqslant(n-1)/2-(7m+2)m-3-(m-1)^2-m>0$, then $|V(H_1)|\leqslant m$ and $\delta(B[V_2])\geqslant|V_2|-m$, since $R[V_1\cup V_2]\not\supseteq B_m$ and $|N_B(V_2)\cap V_1|\leqslant 1$. Applying Lemma 5 to the graph $B[V_2\cup T_1\setminus T_2]$, we may assume that xy is an edge of $B[V_2]$ such that $|N_B(x)\cap N_B(y)\cap(V_2\cup T_1\setminus T_2)|\geqslant|V_2|+|T_1\setminus T_2|/4-11m/4+3/4$. Observing that $|V_2\cup T_2|\geqslant n-|V_3\setminus(T_1\cup T_2)|\geqslant n-7m-2$ and $|T_1\setminus T_2|=|V(B)\setminus(V_2\cup T_2\cup V(H_1)\cup\{u,v\}|-|V_3\setminus(T_1\cup T_2)|\geqslant 2n+3-|V_2\cup T_2|-m-2-(7m+2)=2n-|V_2\cup T_2|-8m-1$, we obtain $|N_B(x)\cap N_B(y)|\geqslant|V_2|+|T_1\setminus T_2|/4-11m/4+3/4+|T_2|+|\{u,v\}|\geqslant|V_2\cup T_2|+(2n-|V_2\cup T_2|-8m-1)/4-11m/4+11/4=3|V_2\cup T_2|/4+n/2-19m/4+5/2\geqslant 3(n-7m-2)/4+n/2-19m/4+5/2>n+m>|N_B(u)\cap N_B(v)|$, contradicting the choice of the edge uv.

Subcase 2.2: $|V_2\cup T_2\cup V_3\setminus(T_1\cup T_2)|<n$. In this case we have $|V_1\cup T_1\setminus T_2|\geqslant n+1$ and also have that $B[V_1]$ is a complete graph since $|N_B(V_1)\cap V_2|\leqslant 1$, $|V_2|\geqslant(n-1)/2$

and $R[V_1 \cup V_2] \not\supseteq B_m$. So we conclude that if $|V_1| \geqslant 4$, then Proposition 2.1 is true, since $N_B(u) \cap N_B(v) \supseteq V_1$.

Suppose that $|V_1|=3$. If $|V_2 \cup T_2 \cup V_3 \setminus (T_1 \cup T_2)| \leqslant n-2$, then, when $V(H_1) \cap V(H_2) \neq \emptyset$, let xy denote the edge of H_1 whose end vertices both are adjacent to the vertex of $V(H_1) \cap V(H_2)$, and so the edge xy satisfies $|N_B(x) \cap N_B(y)| \geqslant n+3$. When $V(H_1) \cap V(H_2) = \emptyset$, we have $|N_B(x) \cap N_B(y)| \geqslant n+3$ for each edge xy of H_1. So, by the choice of edge uv, we get $|N_B(u) \cap N_B(v)| \geqslant n+3$ when either $V(H_1) \cap V(H_2) \neq \emptyset$ or $V(H_1) \cap V(H_2) = \emptyset$. Then $|V(H_2)| \geqslant n$. This contradicts the assumption of $|V(H_2)| < n$. Therefore, $|V_2 \cup T_2 \cup V_3 \setminus (T_1 \cup T_2)| \geqslant n-1$. On the other hand, since $|V_1 \cup T_1 \setminus T_2| \geqslant n+1$, $|N_B(x) \cap N_B(y)| \geqslant n+1$ for each edge xy of $B[V_1]$, $|V_2| \geqslant |N_B(u) \cap N_B(v)| - |V_1| - 1 \geqslant n+1-3-1 = n-3$ by the choice of the edge uv, and so that $|T_2 \cup V_3 \setminus T_1 \cup T_2)| \leqslant 2$. Hence, if $B[T_1 \setminus T_2]$ is a complete graph, let $x \in V_3 \setminus (T_1 \cup T_2)$ and $y \in V_2 \setminus N_B(x)$. Thenk $|(N_B(x) \cup N_B(y)) \cap T_1 \setminus T_2| \geqslant |T_1 \setminus T_2| - m+1$, since $R[\{x,y\} \cup T_1 \setminus T_2] \not\supseteq B_m$. Say, without loss of generality, $|N_B(x) \cap T_1 \setminus T_2| \geqslant |N_B(y) \cap T_1 \setminus T_2|$. Clearly, $B[N_B(x) \cap T_1 \setminus T_2]$ contains a required edge since $|V_1 \cup T_1 \setminus T_2| \geqslant n+1$ and $B[V_1 \cup T_1 \setminus T_2]$ is a complete graph. If $B[T_1 \setminus T_2]$ is not a complete graph, we assume that $x_1, x_2 \in T_1 \setminus T_2$ and $x_1x_2 \notin E(B)$. Let $D = \{x \in V_2 | d_{B[V_2]}(x) = |V_2| - 1\}$. We have $|(N_B(x_1) \cup N_B(x_2)) \cap D| \geqslant |D| - m+1$ since $R[\{x_1,x_2\} \cup D] \not\supseteq B_m$ and get $|N_B(x) \cap D| \geqslant |D| - m$ for each vertex x of $V_3 \setminus (T_1 \cup T_2)$ since $V_1 \setminus N_B(x) \neq \emptyset$, $|N_B(V_1) \cap V_2| \leqslant 1$, and $R[\{x\} \cup V_1 \cup V_2] \not\supseteq B_m$. So there exist at most $|V_3 \setminus (T_1 \cup T_2)| m \leqslant 2m$ vertices of D which are adjacent in R to some vertices of $V_3 \setminus (T_1 \cup T_2)$. Further, if $|D| \geqslant 5m+7$, we, without loss of generality, assume that $|N_B(x_1) \cap D| \geqslant |N_B(x_2) \cap D|$. Then $|N_B(x_1) \cap D| \geqslant (|D| - m+1)/2$. Let $S = \{x \in N_B(x_1) \cap D N_B(x) \supseteq V_3 \setminus (T_1 \cup T_2)\}$. So $|S| \geqslant (|D| - m+1)/2 - 2m \geqslant 4$. Let $xy \in E(B[S])$. It is easy to see that $N_B(x) \cap N_B(y) \supseteq (V_2 \setminus \{x,y\}) \cup T_2 \cup \{x_1, u, v\} \cup V_3 \setminus (T_1 \cup T_2)$ and the subgraph of B induced by $(V_2 \setminus \{x,y\}) \cup T_2 \cup \{x_1,u,v\} \cup V_3 \setminus (T_1 \cup T_2)$ is 2-connected. So the edge xy is required. If $|D| \leqslant 5m+6$, so that $|V_2 \setminus D| - (m-1)^2 - m \geqslant n-3-(5m+6)-(m-1)^2 - m > 0$, then $|V_1| < m$ and $\delta(B[V_2]) \geqslant |V_2| - m$ since $R[V_1 \cup V_2] \not\supseteq B_m$ and $|N_B(V_2) \cap V_1| \leqslant 1$. Applying Lemma 5 to the graph $B[V_2 \cup T_1]$ and observing that $|V_2| \geqslant n-3$ and $|T_1| \geqslant n-2$ (since $|V_1 \cup T_1| \geqslant n+1$), we can find an edge xy in $B[V_2]$ such that $|N_B(x) \cap N_B(y) \cap (V_2 \cup T_1)| \geqslant |V_2| + |T_1|/4 - 11m/4 + 3/4 \geqslant n-3+(n-2)/4 - 11m/4 + 3/4 > n+m > |N_B(u) \cap N_B(v)|$, a contradiction.

Suppose that $|V_1|=2$. Let xy be a unique edge in $B[V_1]$. Then $|N_B(x) \cap N_B(y)| \geqslant |V(B) \setminus (V_2 \cup T_2 \cup V_3 \setminus (T_1 \cup T_2)| - 2 \geqslant 2n+3-(n-1)-2 = n+2$. By the choice of the edge uv, we have $|V(H_2)| \geqslant n$. This contradicts the assumption $|V(H_2)| < n$.

Suppose that $|V_1| = 1$. Then $|V_2| \geqslant n-2$, and so that $|V_2 \cup T_2 \cup V_3 \setminus (T_1 \cup T_2)| = n-1, T_2 = \emptyset$ and $|V_3 \setminus (T_1 \cup T_2)| = 1$ by $V_3 \setminus (T_1 \cup T_2) \neq \emptyset$ and $|V_2 \cup T_2 \cup V_3 \setminus (T_1 \cup T_2)| < n$. Further we get $|V_1 \cup T_1| \geqslant n+1$. If $B[T_1]$ is a complete graph, we assume that $x_1 \in V_3 \setminus (T_1 \cup T_2)$, $x_2 \in V_2 \setminus N_B(x_1)$ and can derive $|(N_B(x_1) \cup N_B(x_2)) \cap T_1| \geqslant |T_1| - m+1$ by $R[\{x_1,x_2\} \cup T_1] \not\supseteq B_m$, so that $B[V_1 \cup T_1 \cup \{x_1,x_2\}] \supseteq K_2 \vee C_n$. This contradicts the hypothesis of Proposition 2.1, if $B[T_1]$ is not a complete graph. Let $D = \{x \in V_2 \mid d_{B[V_2]}(x) = |V_2| - 1\}$. Then, if $|D| \geqslant 3m+5$, assume that $y_1, y_2 \notin T_1$ and $y_1y_2 \notin E(B)$. Thus, we obtain $|(N_B(y_1) \cup N_B(y_2)) \cap D| \geqslant |D| - m+1$ since $R[\{y_1,y_2\} \cup D] \not\supseteq B_m$. Without loss of generality, we assume that $|N_B(y_1) \cap D| \geqslant |N_B(y_2) \cap D|$, so that $|N_B(y_1) \cap D| \geqslant (|D| -$

$m+1)/2$. Further, let $y_3 \in V_3 \setminus (T_1 \cup T_2)$. We can obtain $|N_B(y_3) \cap N_B(y_1) \cap D| \geqslant |N_B(y_1) \cap D| - m + 1 \geqslant (|D| - m + 1)/2 - m + 1 \geqslant (3m + 5 - m + 1)/2 - m + 1 = 4$ since $N_B(y_3) \not\supseteq V_1$ and $R[\{y_3\} \cup V_1 \cup (N_B(y_1) \cap D)] \not\supseteq B_m$. Suppose that xy is an edge in $B[N_B(y_3) \cap N_B(y_1) \cap D]$. It is easy to see that $N_B(x) \cap N_B(y) \supseteq \{y_1, y_3, u, v\} \cup (V_2 \setminus \{x, y\})$ and the subgraph of B induced by $\{y_1, y_3, u, v\} \cup (V_2 \setminus \{x, y\})$ is 2-connected. So the edge xy is required. If $|D| \leqslant 3m + 4$, so that $|V_2 \setminus D| - (m-1)^2 - m \geqslant n - 2 - (3m + 4) - (m-1)^2 - m > 0$, we have $\delta(B[V_2]) \geqslant |V_2| - m$ since $R[V_1 \cup V_2] \not\supseteq B_m$ and $N_B(V_1) \cap V_2 = \emptyset$. Applying Lemma 5 to the graph $B[V_2 \cup T_1]$ and observing that $|V_2| \geqslant n - 2$ and $|T_1| \geqslant n$, we can find an edge xy in $B[V_2]$ such that $|N_B(x) \cap N_B(y) \cap (V_2 \cup T_1)| \geqslant |V_2| + |T_1|/4 - 11m/4 + 3/4 \geqslant n - 2 + n/4 - 11m/4 + 3/4 > |N_B(u) \cap N_B(v)|$. Again a contradiction. This completes the proof of Proposition 2.1. □

Proof of Theorem 2. If $R \not\supseteq B_m$ and $B \not\supseteq K_2 \vee C_n$, then by Proposition 2.1, we may assume that uv is an edge in B such that $B[N_B(u) \cap N_B(v)]$ contains a block G with at least n vertices. From Lemma 4 we have $G \supseteq C_n$, so that $B \supseteq K_2 \vee C_n$, a contradiction. □

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