JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **84**, 270–281 (1981)

Equivalence, Reduction and Minimization of Finite Fuzzy-Automata

VLADIMIR V. TOPENCHAROV AND KETTY G. PEEVA

Center of Applied Mathematics, P. O. Box 384, 1000 Sofia, Bulgaria

Submitted by A. O. Esogbue

A new algebraic approach to the problem of equivalence, reduction and minimization of some kinds of fuzzy-automata is given. A system of necessary and sufficient conditions for equivalence of weakly initial fuzzy-automata is formulated. Some algorithms considering the equivalence for two fuzzy-automata are constructed.

One classical problem in the theory of automata is equivalence, reduction and minimization. The problem is completely solved [12] for the cases of deterministic automata; for stochastic automata it is studied in detail in [3, 5, 7]; an attempt for the case of fuzzy-automata is given in [9, 10].

The theoretical foundation [8] of the well known algorithm of Even [5] and the analogies between some aspects of the theory of rings (resp. modules) and the theory of semi-rings (resp. semi-modules) indicate the way for asking a general solution in the case of fuzzy-automata. Constructing the notion of noetherian semi-module, an algorithm for equivalence of some kinds of fuzzy-automata is exhibited.

In the following, all sets are supposed to be finite; if C is a set, we denote by $|C|$ its cardinality and by C^* the free semigroup of words on C with the empty word $\Lambda \in C^*$ as unity. The length of the word $w \in C^*$ is denoted by $l(w) \in \mathbb{N}$ and we express two words $u, v \in C^*$ having the same length $k \in \mathbb{N}$ writing $l(u/v) = k$. The terminology and the notations not especially indicated in the paper are according to [7].

1. Basic Notions

We recall the definitions of semi-ring and semi-module [1, 4, 6] and some notions of the theory of fuzzy-automata [9, 11, 14] in form appropriated for the following.

Let C be a set with two inner binary laws of composition

0022-247X/81/110270–12$02.00/0

$k_i: C \times C \to C$, $i = 1, 2$. We call the algebra (C, k_1, k_2) a *(commutative) semi-ring* if (C, k_1) and (C, k_2) are (commutative) semi-groups with unity and k_1 and k_2 are distributive one after the other.

Let E be a set, C be a semi-ring and let $k'_1: E \times E \to E$ and $k'_2: C \times E \to E$ be two laws of composition, the second being external. The algebra (E, k'_1, k'_2) is a *left semi-module* over C if for each $a, b \in C$ and $x, y \in E$ the following conditions hold:

(SM.1) (E, k'_1) is a commutative semi-group with unity:

(SM.2) $k'_2(a, k'_1(x, y)) = k'_1(k'_2(a, x), k'_2(a, y))$,

$k'_2(k_1(a, b), x) = k'_1(k'_2(a, x), k'_2(b, x))$;

(SM.3) $k'_2(a, k'_2(b, x)) = k'_2(k_2(a, b), x)$.

The mapping $h: (E, k'_1, k'_2) \to (E'', k''_1, k''_2)$ is called *morphism of semi-modules* if the following holds

$$h(k'_1(x, y)) = k''_1(h(x), h(y)) \qquad \text{and} \qquad h(k'_2(a, x)) = k''_2(a, h(x)).$$

By the same way the notion of *right semi-module* is defined. If C is a commutative semi-ring we talk about *semi-module*. The category of all C-semi-modules is denoted by C-SMod.

Let M be a C-semi-module. The set $X \subseteq M$ is a *system of generators* for M if X generates M. A *quasi-base* is the minimal system of generators for M. If the quasi-base is finite, the dimension of M (denote dim M) is the number of its vectors (elements of E).

Let X be a set, not necessarily finite and let C be a semi-ring. Putting

$$VX = \sum_{x \in X} a_x \cdot x, \qquad a_x \in C, x \in X,$$

where $a_x \neq 0$ only for a finite number of elements $x \in X$, it is easy to verify that VX is a semi-module according to the laws of composition of the semi-ring, called *free semi-module*. The set X is a minimal system of generators for VX.

DEFINITION 1. The C-semi-module M should be called *noetherian* if M is a noetherian object [6] in the category C-SMod.

The following two results are important for the theory and its applications; the proofs are omitted.

PROPOSITION 1. *For a semi-module $M \in C$-SMod the following conditions are equivalent*:

(a) *M is a noetherian C-semi-module*;

(b) *Each increasing sequence of sub-C-semi-modules of M, i.e.,* $M_1 \subset M_2 \subset \cdots \subset M_k \subset \cdots$, *such that* $M_i \neq M_{i-1}$, *is finite;*

(c) *For each sub-C-semi-module of M there exists a finite minimal system of generators;*

(d) *Each non-empty set G of sub-C-semi-modules of* $M \in C$-SMod *contains a maximal element.*

PROPOSITION 2. *If* $X \neq \varnothing$ *is a finite set, the free semi-module VX is noetherian.*

EXAMPLES. (1) Let be given the closed interval $I = [0, 1] \subset \mathbb{F}$; let us consider the binary operations $k_1 = \max$ and $k_2 = \min$ in I, according to the natural order in I; the algebra $(I, k_1, k_2) = ([0, 1], \max, \min)$ is a commutative semi-ring.

(2) Let L be a distributive lattice; the algebra $(L, \max \min)$, constructed and studied in [10] is a commutative semi-ring.

(3) Let X be a finite set and VX be the free semi-module generated by X over the semi-ring from the Example 1; the operations in the free semi-module are defined as follows:

$$k_1' = + : VX \times VX \to VX, \sum_{x \in X} a_x \cdot x + \sum_{x \in X} b_x \cdot x = \sum_{x \in X} \max(a_x, b_x) \cdot x,$$

$$k_1' = \cdot : [0, 1] \times VX \to VX, y \left(\sum_{x \in X} a_x \cdot x \right) = \sum_{x \in X} \min(y, a_x) \cdot x.$$

When X is finite, VX is a noetherian semi-module (see Proposition 2).

DEFINITION 2. The quadruple $A = (X, Q, Y, h)$, X, Q, Y being finite sets and $h: X \times Q \times Y \times Q \to [0, 1]$ being a map, should be called a *fuzzy-* (shortly F-) *automaton.*

As usual, X is the *input alphabet*, Y is the *output alphabet*, Q is the *set of states* for the F-automaton A; the map h is called *membership function* and we write $h(x_i, q_j, y_r, q_k) = a_{ij}^{rk} \in [0, 1]$.

It is easy to show that the classical definition of fuzzy-automaton [9, 13, 14] gives an automaton according to Definition 2 (see [1, 10]). For our purpose, however, this definition is preferable.

If the interval [0, 1] is replaced by the distributive lattice L (see Example 2) we obtain the more general notion of L-automaton, closely related to F-automaton.

Every F-automaton A defines the free semi-modules $V(X \times Q)$ and $V(Y \times Q)$ over the semi-ring [0, 1]. The membership function

$$h: V(X \times Q) \to V(Y \times Q)$$

is defined such that $h(x_i, q_j) = \sum_{r,k} a_{ij}^{rk}(y_r, q_k)$; its corresponding matrix $M_h = \|a_{ij}^{rk}\|$ characterize the work of the F-automaton.

Let us consider the words $u = x_1 \cdot x_2 \cdot \cdots \cdot x_p \in X^*$, $v = y_1 \cdot y_2 \cdot \cdots \cdot y_p \in Y^*$ and the matrices $M(x_i/y_r) = \|m_{jk}(x_i/y_r)\|$, $m_{jk}(x_i/y_r) = a_{ij}^{kr}$. With the maxi-min product of matrices denoted by $\circ$, we obtain the expression

$$M(u/v) = M(x_1/y_1) \circ M(x_2/y_2) \circ \cdots \circ M(x_p/y_p).$$

If $P = P(\Lambda/\Lambda)$ is a matrix-column of the type $|Q| \times 1$ which elements are equal to 1, the following composition is defined:

$$P(u/v) = M(u/v) \circ P.$$

Let A be an F-automaton with (Q, ε) as the F-set of initial states, sub-F-set of Q; $\varepsilon(q) \in [0, 1]$ defines the membership of $q \in Q$ as an initial state of A. We denote

$$\begin{aligned} \varepsilon_q^1(q') &= 1 \quad \text{if } q = q' \qquad & \varepsilon_q^0(q') &= 0 & \text{if } q = q' \\ &= 0 \quad \text{if } q \neq q' & &= a \in [0, 1] & \text{if } q \neq q' \end{aligned}$$

with the supplementary condition $\sum_{\bar{q} \in Q} \varepsilon_q^0(\bar{q}) \neq 0$ for ε_q^0. The F-automaton A, denoted in this case (A, ε_q^1) (resp. (A, ε)) is called *initial* (resp. *weakly initial*) if (Q, ε_q^1) (resp. (Q, ε)) is a sub-F-set of Q.

For the F-automaton A we define $S_\varepsilon(u/v) = \varepsilon \circ P(u/v)$, an entry indicating the maximal degree of membership for the input word u and the output word v, (Q, ε) being fixed.

Let $A = (X, Q, Y, h)$ and $A' = (X, Q', Y, h')$ be F-automata; (Q, ε) and (Q', ε') are sub-F-sets of Q and Q', respectively.

DEFINITION 3. Two initial automata (A, ε) and (A', ε') are *equivalent* (notation $(A, \varepsilon) \sim (A', \varepsilon')$) if $S_\varepsilon(u/v)_A = S_{\varepsilon'}(u/v)_{A'}$ for all $u \in X^*$ and $v \in Y^*$. In particular:

— let $A = A' = (X, Q, Y, h)$; if $(A, \varepsilon) \sim (A', \varepsilon')$, then ε *and* ε' *are equivalent on* Q (notation $\varepsilon \sim \varepsilon'$);

— if $(A, \varepsilon_q^1) \sim (A', \varepsilon_{q'}'^1)$, then the *states* $q \in Q$ *and* $q' \in Q'$ *are* equivalent (notation $q \sim q'$);

— $A = (X, Q, Y, h)$ is *equivalently embedded into* $A' = (X, Q', Y, h')$ if for each $q \in Q$ there exists an equivalent state $q' \in Q'$ of A' (notation $A \subsetsim A'$);

— A is *weakly equivalently embedded into* A' (notation $A \subsetapprox A'$) if for each $\varepsilon: Q \to [0, 1]$ there exists $\varepsilon': Q' \to [0, 1]$ such that $(A, \varepsilon) \sim (A', \varepsilon')$;

— A and A' are *equivalent* (notation $A \sim A'$) if $A \subsetsim A'$ and $A \supsetsim A'$;

—A and A' are *wekly equivalent* if $A \subsetapprox A'$ and $A' \subsetapprox A$ (notation $A \approx A'$).

DEFINITION 4. Let A be an F-automaton;

— A is in *reduced form* if for each $q, q' \in Q$ the relation $q \sim q'$ implies $q = q'$;

— A' is called a *reduct* of A if A' is in reduced form and equivalent to A;

— A is in *minimal form* if for each $\varepsilon^1_{q_i}$ $(i \leqslant |Q|)$ there does not exist $\varepsilon^0_{q_i}$ $(i \leqslant |Q|)$ such that $(A, \varepsilon^1_{q_i}) \sim (A, \varepsilon^0_{q_i})$;

— A' is called a *minimal* of A if it is in minimal form and if $A \approx A'$.

The above defined notions are in concordance with the classical theory of automata [7, 12] and coincide with the usual terminology in the cases of deterministic, nondeterministic and stochastic automata. For the F-automata [9] this is an attempt to unify the terminology.

2. EQUIVALENCE OF F-AUTOMATA—AN ALGEBRAIC APPROACH

For each F-automaton $A = (X, Q, Y, h)$ we define a map t: $V(X^* \times Y^*) \to VQ$ as follows:

$$t(\Lambda, \Lambda) = \sum_{q \in Q} q, \qquad t(u, v) = \sum_{q_j \in Q} p_j(u/v)\, q_j \qquad \text{if } l(u) = l(v)$$
$$= 0 \qquad \text{if } l(u) \neq l(v).$$

It is easy to verify that t is a morphism of semi-modules. Let us denote its corresponding matrix by M_t.

We construct the sequence $E_0 \subset E_1 \subset \cdots \subset E$ of subsets of $E = X^* \times Y^*$ obtained as follows:

$$E_0 = \{(\Lambda, \Lambda)\};...,$$
$$E_i = E_{i-1} \cup \{(u, v);\, u \in X^*, v \in Y^*, l(u) = l(v) = i\}.$$

Let $n = |\{p_j(x/y)|\, x \in X, y \in Y, j \leqslant |Q|\}|^{|Q|}$.

PROPOSITION 3. *The following statements hold*:

(a) VE_i *is a sub-semi-module of* VE_{i+1}, *for each* $i = 0, 1,...$;

(b) *If* $tVE_i = tVE_{i+1}$, *then* $tVE_i = tVE_{i+p}$ *for each* $p = 0, 1,...$;

(c) *The quasi-base of* tVE *contains at most* n *elements*;

(d) $tVE_{n-1} = tVE_n = \cdots = tVE$.

Proof. (a) According to the construction of $E_0, E_1,...,$ which are sets of generators (quasi-bases) for the semi-modules VE_i, $i = 0, 1,...,$ we have $VE_0 \subset VE_1 \subset \cdots \subset VE$. (b) The morphism of semi-modules t being a linear operator, the image of the sequence $VE_0 \subset VE_1 \subset \cdots \subset VE$ is the following sequence $tVE_0 \subset tVE_1 \subset \cdots \subset tVE$; hence (see Proposition 2) for $i \in \mathbb{N}$ and $p = 0, 1, \ldots,$ we have $tVE_i = tVE_{i+p}$. (c) The semimodule VQ being noetherian and since $tVE \subseteq VQ$, holds $\dim tVE \leqslant n$. If $tVE = VQ$, the equality holds. (d) Let us consider the sub-semimodule tVE_i; it contains a certain number of vectors of the quasi-base. If $tVE_i \subset tVE_{i+1}$, the sub-semimodule contains at least one supplementary vector of the quasi-base. Hence if $tVE = VQ$ and since each quasi-base tE_i contains exactly a new (supplementary) vector of the quasi-base of tVE, we obtain:

$$tVE_0 \subset tVE_1 \subset \cdots \subset tVE_{n-1} = tVE_n = \cdots = tVE,$$

having obviously $\dim tVE_{n-1} = n$.

This result reinforces some statements and algorithms of [10].

THEOREM 1. *Let (A, ε) and (A', ε') be two weakly initial F-automata. $(A, \varepsilon) \sim (A', \varepsilon')$ iff $\varepsilon \circ t = \varepsilon' \circ t'$.*

Proof. If $l(u) = l(v)$ for $(u, v) \in X^* \times Y^*$, according to Definition 6:

$$S_\varepsilon(u/v)_A = S_{\varepsilon'}(u/v)_{A'}$$

and since $S_\varepsilon(u/v) = \varepsilon \circ (M(u/v) \circ P)$, we obtain

$$\varepsilon \circ (M(u/v) \circ P) = \varepsilon' \circ (M'(u/v) \circ P);$$

this expression is equivalent to $\varepsilon \circ t(u, v) = \varepsilon' \circ t'(u, v)$ for each couple $(u, v) \in X^* \times Y^*$ such that $l(u) = l(v)$; if $l(u) \neq l(v)$, according to the definition of t, holds $t(u, v) = t'(u, v) = 0$, i.e., $\varepsilon \circ t = \varepsilon' \circ t'$.

Conversely, let $\varepsilon \circ t = \varepsilon' \circ t'$; obviously $\varepsilon \circ t(u, v) = \varepsilon' \circ t'(u, v)$ for each couple $(u, v) \in X^* \times Y^*$ and hence $\varepsilon \circ t(u/v) = \varepsilon' \circ t'(u/v)$; but

$$\begin{aligned} M_t(u/v) &= M(u/v) \circ P \qquad &\text{if} \quad l(u) = l(v) \\ &= 0 &\text{if} \quad l(u) \neq l(v); \end{aligned}$$

it follows $\varepsilon \circ (M(u/v) \circ P) = \varepsilon' \circ (M'(u/v) \circ P)$, i.e., $S_\varepsilon(u/v)_A = S_{\varepsilon'}(u/v)_{A'}$ for each $(u, v) \in X^* \times Y^*$ such that $l(u) = l(v)$.

A similar result is given in [9, 10].

COROLLARY 1. *Let A be an F-automaton. $\varepsilon \sim \varepsilon'$ iff $S_\varepsilon(u/v)_A = S_{\varepsilon'}(u/v)_A$ for each $(u, v) \in X^* \times Y^*$ such that $l(u/v) \leqslant n - 1$.*

Proof. If $(A, \varepsilon) \sim (A, \varepsilon')$, then $S_\varepsilon(u/v)_A = S_{\varepsilon'}(u/v)_A$ for $l(u/v) = 0, 1, \ldots$; hence $l(u/v) \leqslant n$. If $S_\varepsilon(u/v)_A = S_{\varepsilon'}(u/v)_A$ for each $(u, v) \in X^* \times Y^*$ such that $l(u/v) \leqslant n - 1$, according to Proposition 3(d) it follows $\varepsilon \circ (M(u/v) \circ P) = \varepsilon' \circ (M(u/v) \circ P)$; hence $(A, \varepsilon) \sim (A, \varepsilon')$.

This is the fuzzy-interpretation of the well-known Carlyle theorem [3] for equivalence of stochastic automata.

COROLLARY 2. *For a given F-automaton A the following statements are equivalent*:

(a) $\varepsilon \sim \varepsilon'$;

(b) $\varepsilon \circ M_t = \varepsilon' \circ M_t$.

COROLLARY 3 [10]. *For a given F-automaton A the following statements are equivalent*:

(a) $\varepsilon^1_{q_i} \sim \varepsilon^1_{q_j}$;

(b) *The ith and jth rows in the matrix M_t are identical.*

Proof. Let $\varepsilon^1_{q_i} \sim \varepsilon^1_{q_j}$; according to the Corollary 2 of Theorem 1, $\varepsilon^1_{q_i} \circ M_t = \varepsilon^1_{q_j} \circ M_t$: but by the construction of ε^1_q this means that the ith and jth rows in M_t are identical, hence $(a) \Rightarrow$ (b). The inverse implication (b) $\Rightarrow$ (a) is directly verified.

The following auxiliary result is an important criterion to ascertain the equivalence of two F-automata.

LEMMA. *If* $(A, \varepsilon) \sim (A', \varepsilon')$ *then* $\dim(\operatorname{Im} t) = \dim(\operatorname{Im} t')$.

Proof. According to Theorem 1, $\varepsilon \circ t = \varepsilon' \circ t' \Leftrightarrow \varepsilon \circ M_{t'}$. This matrix equality leads to $\dim(\operatorname{Im} t) = \dim(\operatorname{Im} t')$.

Let A and A' be two F-automata.

THEOREM 2. *If ε is given, the problem of finding ε', if it exists, such that $(A, \varepsilon) \sim (A', \varepsilon')$ is algorithmically decidable.*

As a proof we give the algorithm (see Fig. 1).

The computing program is not easy to realize, useful standard programs are missing.

3. REDUCTION AND MINIMIZATION OF FUZZY-AUTOMATA

The problem of reduction and minimization of F-automata is a consequence of the theory of equivalence of F-automata, but they have a high importance in applications. This part is a completion of very rich ideas of [10].

Closely connected with the problem of reduction of F-automata is the following statement:

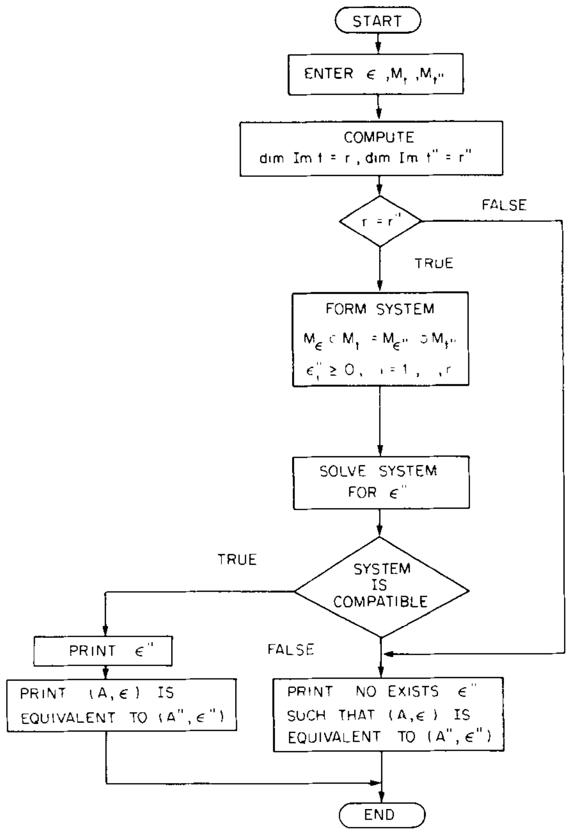

FIGURE 1

THEOREM 3. *Let M_t be the matrix associated to the F-automaton A. If M_t contains two identical rows, there exist two F-automata A' and A'', with $|Q|-1$ states each, such that $A \sim A'$ and $A \sim A''$.*

Proof. Let in M_t the rows corresponding to the states q_i and q_j be identical and let $Q' = Q - \{q_i\}$, $Q'' = Q - \{q_j\}$; the corresponding matrix $M_{t'}$ (resp. $M_{t''}$) for the F-automaton A' (resp. A'') is obtained by M_t eliminating the ith (resp. the jth) row. We shall prove that $A \sim A'$ (resp. $A \sim A''$). The equivalent state to $q \in Q$, $q_i \neq q \neq q_j$ is $q \in Q'$ (resp. $q \in Q''$) and vice versa, because $\varepsilon_q'^1 \circ M_t = \varepsilon_q'^1 \circ M_{t'}$ (resp. $\varepsilon_q^1 \circ M_t = \varepsilon_q''^1 \circ M_{t''}$). The equivalent state to $q = q_i$, $q_j \in Q$ respectively is the state $q_i \in Q'$ (resp. $q_j \in Q''$). The state equivalent to $q_i \in Q'$ (resp. $q_j \in Q''$) is $q_i \in Q$ (resp. $q_j \in Q$). The proof in these conditions is a consequence of the definition of ε_q^1, of the construction of M_t and a direct verification holds.

COROLLARY. *For every F-automaton there exists a reduced F-automaton. All reduced F-automata associated to a given F-automaton have sets of states with the same cardinality.*

THEOREM 4. *For finite F-automata the relation of equivalence is decidable.*

The block-scheme (Fig. 2) of the algorithm proving the equivalence of two F-automata A and A' is in fact the proof of the Theorem 4.

The following result is connected with the existence and the explicit construction of a minimal F-automaton to a given F-automaton.

THEOREM 5. *Let $A = (X, Q, Y, h)$ be an F-automaton. If $\varepsilon^1_{q_b} \sim \varepsilon^0_{q_b}$ and*

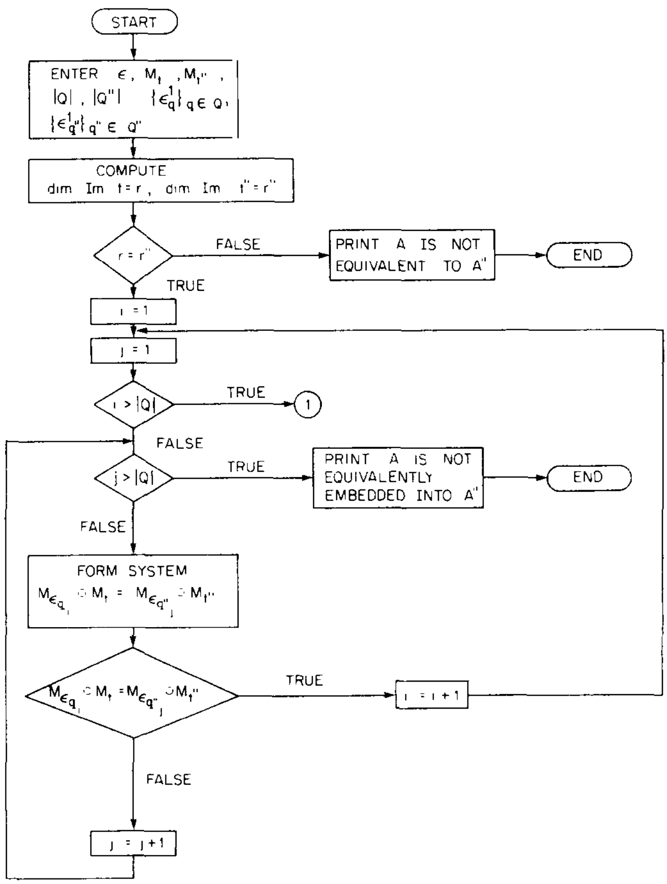

FIGURE 2

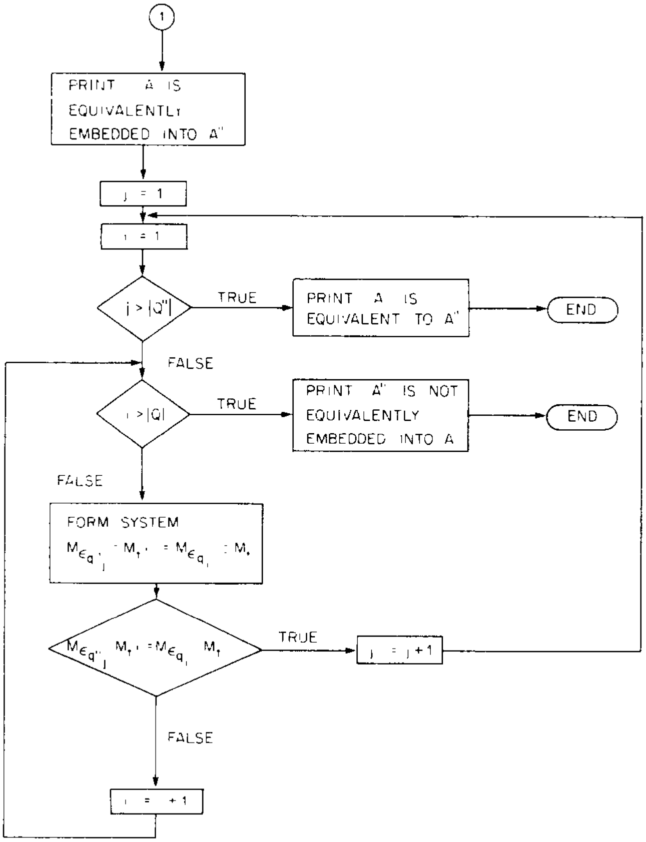

FIG. 2—*Continued.*

$\varepsilon^0_{q_b}$ *contains* $1 \in [0, 1]$ *as a component, there exists an F-automaton* $\bar{A} = (X, \bar{Q}, Y, \bar{h})$, *with* $|Q| - 1$ *states, such that* $A \approx \bar{A}$.

Proof. Let $(Q, \varepsilon^1_{q_b})$ and $(Q, \varepsilon^0_{q_b})$ be a sub-F-set of Q; let

$$\varepsilon^0_{q_b} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b-1}, 0, \varepsilon_{b+1}, \ldots, \varepsilon_n)$$

verify the condition of the theorem and

$$\varepsilon^1_{q_b} = (0, \ldots, 0, 1, 0, \ldots, 0)$$

be equivalent to $\varepsilon^0_{q_b}$. We construct the F-automaton $\bar{A} = (X, \bar{Q}, Y, \bar{h})$ as follows:

$$\bar{Q} = Q - \{q_b\}, \qquad \bar{h}: V(X \times \bar{Q}) \to V(Y \times \bar{Q}), \qquad \bar{h}(x_i, q_j) = \sum_{r,k} \bar{a}^{rk}_{ij}(y_r, q_k),$$

where $\bar{a}_{ij}^{rk} = \max(a_{ij}^{rk}, \min(\varepsilon_k^0, a_{ij}^{rb}))$. We shall show the states q_j, $j \neq b$, with the same indices in A and $\bar{A}$ are equivalent. For the words with length $l = 1$ we have

$$\bar{a}_{ij}^r = \max_{k \neq b}(\bar{a}_{ij}^{rk}) = \max(\max_{k \neq b}(a_{ij}^{rk}), \max_{k \neq b}(\min(\varepsilon_k^0, a_{ij}^{rb}))) = \max_k(a_{ij}^{rk}) = a_{ij}^r.$$

Writing the last equality we have in mind that $\varepsilon_{q_b}^0$ contains $1 \in [0, 1]$ as a component, i.e., $\max_{k \neq b}(\min(\varepsilon_k^0, a_{ij}^{rb})) = a_{ij}^{rb}$, because

$$\max_{k \neq b}(\min(\varepsilon_k^0, a_{ij}^{rb})) = \min(\max_{k \neq b}(\varepsilon_k^0), a_{ij}^{rb}) = a_{ij}^{rb}.$$

Suppose the states q_j, $j \neq b$, w-equivalent, i.e., for arbitrary words $u \in X^*$, $v \in Y^*$ such that $l(u) = l(v) = w$, the following holds: $\bar{p}_j(u/v)_{\bar{A}} = p_j(u/v)_A$. According to the hypothesis $\max_{k \neq b}(\min(\varepsilon_k^0, p_k(u/v)) = p_b(u/v)$. Then we obtain

$$\bar{p}_j(x_i u/y_r v) = \max_{k \neq b}(\min(\bar{a}_{ij}^{rk}, \bar{p}_k(u/v))) = \max(\max_{k \neq b}(\min(a_{ij}^{rk}, p_k(u/v))),$$

$$\min(a_{ij}^{rb}, \max_{k \neq b}(\min(\varepsilon_k^0, p_k(u/v)))))$$

$$= \max_k(\min(a_{ij}^{rk}, p_k(u/v))) = p_j(x_i u/y_r v),$$

i.e., the states with the same indices for automata A and $\bar{A}$ are $(w+1)$-equivalent and thus equivalent. For each $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b-1}, \varepsilon_{b+1}, \ldots, \varepsilon_n)$ for $\bar{A}$, there exist an equivalent $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b-1}, 0, \varepsilon_{b+1}, \ldots, \varepsilon_n)$ for the automaton A. For a given $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ for the automaton A, the corresponding equivalent $\bar{\varepsilon} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n)$ for $\bar{A}$ is defined by the correspondence $\bar{\varepsilon}_i = \max(\varepsilon_i, \min(\varepsilon_b, \varepsilon_{i'}^0))$, $i \neq b$, where $\varepsilon_{i'}^0$ is the ith component in the vector $\varepsilon_{q_b}^0$. Indeed, having in mind the definition, we obtain

$$S_\varepsilon(u/v)_A = \varepsilon \circ P(u/v) = \max_i(\min(\varepsilon_i, p(u/v))$$

$$= \max(\max_{i \neq b}(\min(\varepsilon_i, p_i(u/v)), \min(\varepsilon_b, \max(\varepsilon_i^0, p_i(u/v)))))$$

$$= \max_{i \neq b}(\min(\max(\varepsilon_i, \min(\varepsilon_b, \varepsilon_i^0))), \bar{p}_i(u/v))$$

$$= \max_{i \neq b}(\min(\bar{\varepsilon}_i, \bar{p}_i(u/v))) = S_{\bar{\varepsilon}}(u/v)_{\bar{A}}.$$

REFERENCES

1. J. BRUNNER, Zur Theorie der R-fuzzy Automaten, II, *Wiss. Z. Tech. Univ. Dresden* **27**, No. 3/4 (1978), 693–695.

2. J. BRUNNER AND W. WECHLER, Zur Theorie der R-fuzzy Automaten, I, *Wiss. Z. Tech. Univ. Dresden* **26**, No. 3/4 (1977), 647–652.
3. J. W. CARLYLE, Reduced forms for stochastic sequential machines, *J. Math. Anal. Appl.* **7** (1963), 167–175.
4. S. EILENBERG, "Automata, Languages and Machines," Vol. A, Academic Press, New York/London, 1974.
5. S. EVEN, Comments on the minimization of stochastic machines, *IEEE Trans. Electron. Comp.* **14**, No. 4 (1965), 634–637.
6. C. FAITH, "Algebra: Rings, Modules, Categories," Springer-Verlag, New York/Heidelberg/Berlin, 1973.
7. A. PAZ, "Introduction to Probabilistic Automata," Academic Press, New York/London, 1971.
8. K. PEEVA, "Categories of Stochastic Automata," Ph.D. dissertation, Center of Applied Mathematics, Sofia, 1977.
9. E. S. SANTOS, Max-product machines, *J. Math. Anal. Appl.* **37** (1972), 667–686.
10. E. S. SANTOS, On reduction of Maxi-min machines, *J. Math. Anal. Appl.* **40** (1972), 60–78.
11. E. S. SANTOS AND W. G. WEE, General formulation of sequential machines, *Inform. and Control* **12** (1968), 5–10.
12. P. H. STARKE, "Abstract Automata," VEB Deutscher Verlag der Wiss., Berlin, 1969.
13. W. G. WEE AND K. S. FU, A formulation of fuzzy automata and its application as a model of learning system, *IEEE Trans. Systems Sci. Cybernet.* **5**, No. 3 (1969).
14. L. A. ZADEH, The concept of a linguistic variable and its application to approximate reasoning, Memo. ERL-M 411, Univ. of California, Berkeley, Calif., 1973.