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Orthogonal spline collocation methods for partial differential equations ☆

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Abstract

This paper provides an overview of the formulation, analysis and implementation of orthogonal spline collocation (OSC), also known as spline collocation at Gauss points, for the numerical solution of partial differential equations in two space variables. Advances in the OSC theory for elliptic boundary value problems are discussed, and direct and iterative methods for the solution of the OSC equations examined. The use of OSC methods in the solution of initial–boundary value problems for parabolic, hyperbolic and Schrödinger-type systems is described, with emphasis on alternating direction implicit methods. The OSC solution of parabolic and hyperbolic partial integro-differential equations is also mentioned. Finally, recent applications of a second spline collocation method, modified spline collocation, are outlined.

1. Introduction

1.1. Preliminaries

In [70], Fairweather and Meade provided a comprehensive survey of spline collocation methods for the numerical solution of differential equations through early 1989. The emphasis in that paper is on various collocation methods, primarily smoothest spline collocation, modified spline collocation and orthogonal spline collocation (OSC) methods, for boundary value problems (BVPs) for ordinary differential equations (ODEs). Over the past decade, considerable advances have been made in the formulation, analysis and application of spline collocation methods, especially OSC for partial differential equations (PDEs). In this paper, we review applications of OSC (also called spline collocation at Gauss points) to elliptic, parabolic, hyperbolic and Schrödinger-type PDEs, as well

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as to parabolic and hyperbolic partial integro-differential equations. The emphasis throughout is on problems in two space variables.

A brief outline of the paper is as follows. In Section 2, OSC for linear two-point BVPs for ODEs is described. OSC for such problems was first analyzed in the seminal paper of deBoor and Swartz [32], which laid the foundation for the formulation and analysis of OSC methods for a wide variety of problems and the development of software packages for their solution; see [70]. In Section 2, we also describe the continuous-time OSC method and the discrete-time Crank–Nicolson OSC method for linear parabolic initial–boundary value problems (IBVPs) in one space variable and outline applications of OSC to other equations in one space variable such as Schrödinger-type equations. OSC methods for linear and nonlinear parabolic problems in one space variable were first formulated and analyzed in [59,61–63], and Cerutti and Parter [43] tied together the results of [32] and those of Douglas and Dupont. Following the approach of Douglas and Dupont, Houstis [83] considered OSC for nonlinear second-order hyperbolic problems.

Section 3 is devoted to elliptic BVPs. The development of the convergence theory of OSC for various types of elliptic problems is examined, followed by an overview of direct and iterative methods for solving the OSC equations. Several of these methods reduce to the OSC solution of linear BVPs for ODEs. The OSC solution of biharmonic problems is also described and the section closes with a discussion of recent work on domain decomposition OSC methods.

Section 4 concerns IBVPs problems for parabolic and hyperbolic equations and Schrödinger-type systems with emphasis on the formulation and analysis of alternating direction implicit (ADI) methods. The ADI methods considered are based on the two space variable Crank–Nicolson OSC method, and reduce to independent sets of one space variable problems of the type considered in Section 2. Spline collocation methods for certain types of partial integro-differential equations are also discussed. In Section 5, we give a brief synopsis of modified spline collocation methods for elliptic BVPs.

In the remainder of this section, we introduce notation that is used throughout the paper.

1.2. Notation

We denote the unit interval $(0,1)$ by I and the unit square $I\times I$ by Ω. Let π be a partition given by

$$\pi\colon\ 0=x^{(0)}<x^{(1)}<\cdots<x^{(N)}=1.$$

Let

$$\mathscr{M}_r(\pi)=\{v\in C^1(\bar I)\colon\ v|_{[x^{(i-1)},x^{(i)}]}\in P_r,\ i=1,2,\ldots,N\},$$

where P_r denotes the set of all polynomials of degree $\leqslant r$. Also let

$$\mathscr{M}_r^0(\pi)=\{v\in\mathscr{M}_r(\pi)\colon\ v(0)=v(1)=0\}.$$

Note that

$$\dim\mathscr{M}_r^0(\pi)\equiv M=N(r-1),\quad \dim\mathscr{M}_r(\pi)=M+2.$$

When $r=3$, the spaces $\mathscr{M}_r(\pi)$ $(\mathscr{M}_r^0(\pi))$ and $\mathscr{M}_r(\pi)\otimes\mathscr{M}_r(\pi)$ $(\mathscr{M}_r^0(\pi)\otimes\mathscr{M}_r^0(\pi))$ are commonly known as piecewise Hermite cubics and piecewise Hermite bicubics, respectively.

Let $\{\sigma_k\}_{k=1}^{r-1}$ be the nodes of the $(r-1)$-point Gauss–Legendre quadrature rule on I, and let the Gauss points in I be defined by

$$\xi_{(i-1)(r-1)+k} = x^{(i-1)} + h_i\sigma_k, \quad k = 1, 2, \ldots, r-1, \ i = 1, \ldots, N, \tag{1}$$

where $h_i = x^{(i)} - x^{(i-1)}$. We set $h = \max_i h_i$.

By optimal order estimates, we mean bounds on the error which are $O(h^{r+1-j})$ in the H^j norm, $j = 0, 1, 2$, and $O(h^{r+1})$ in the L^∞ norm.

2. Problems in one space variable

2.1. Boundary value problems for ordinary differential equations

In this section, we briefly discuss OSC for the two-point BVP

$$\mathscr{L}u \equiv -a(x)u'' + b(x)u' + c(x)u = f(x), \quad x \in I, \tag{2}$$

$$\mathscr{B}_0 u(0) \equiv \alpha_0 u(0) + \beta_0 u'(0) = g_0, \qquad \mathscr{B}_1 u(1) \equiv \alpha_1 u(1) + \beta_1 u'(1) = g_1, \tag{3}$$

where α_i, β_i, and g_i, $i = 0, 1$, are given constants. The OSC method for solving (2)–(3) consists in finding $u_h \in \mathscr{M}_r(\pi)$, $r \geqslant 3$, such that

$$\mathscr{L}u_h(\xi_j) = f(\xi_j), \quad j = 1, 2, \ldots, M, \qquad \mathscr{B}_0 u_h(0) = g_0, \quad \mathscr{B}_1 u_h(1) = g_1, \tag{4}$$

where $\{\xi_j\}_{j=1}^{M}$ are the *collocation points* given by (1). If $\{\phi_j\}_{j=1}^{M+2}$ is a basis for $\mathscr{M}_r(\pi)$, we may write

$$u_h(x) = \sum_{j=1}^{M+2} u_j \phi_j(x)$$

and hence the collocation equations (4) reduce to a system of linear algebraic equations for the coefficients $\{u_j\}_{j=1}^{M+2}$. If the basis is of Hermite type, B-splines or monomial basis functions, the coefficient matrices are *almost block diagonal* (ABD) [31]. For example, if $\boldsymbol{u} = [u_1, u_2, \ldots, u_{M+2}]^{\mathrm{T}}$, and $\boldsymbol{f} = [g_0, f(\xi_1), f(\xi_2), \ldots, f(\xi_{M-1}), f(\xi_M), g_1]^{\mathrm{T}}$, then, for Hermite-type or B-spline bases with standard orderings, the collocation equations have the form $A\boldsymbol{u} = \boldsymbol{f}$, where A has the ABD structure

$$\begin{bmatrix} D_0 & & & & & & \\ W_{11} & W_{12} & W_{13} & & & & \\ & & W_{21} & W_{22} & W_{23} & & \\ & & & & \ddots & & \\ & & & & W_{N1} & W_{N2} & W_{N3} \\ & & & & & & D_1 \end{bmatrix}. \tag{5}$$

The 1×2 matrices $D_0 = [\alpha_0 \ \ \beta_0]$, $D_1 = [\alpha_1 \ \ \beta_1]$ arise from the boundary conditions, and the matrices $W_{i1} \in \mathbb{R}^{(r-1)\times 2}$, $W_{i2} \in \mathbb{R}^{(r-1)\times(r-3)}$, $W_{i3} \in \mathbb{R}^{(r-1)\times 2}$ come from the collocation equations on the ith subinterval. Such systems are commonly solved using the package *colrow* [56,57]. The packages *abdpack* and *abbpack* [108–110] are designed to solve the systems arising when monomial bases are employed, in which case the ABD structure is quite different from that in (5). The solution procedures implemented in these packages are all variants of Gaussian elimination with partial pivoting. A review of methods for solving ABD systems is given in [3].

2.2. Parabolic problems in one space variable

Consider the parabolic IBVP

$$u_t + \mathscr{L}u = f(x,t), \quad (x,t) \in I \times (0,T],$$
$$\mathscr{B}_0 u(0,t) \equiv \alpha_0 u(0,t) + \beta_0 u_x(0,t) = g_0(t), \quad t \in (0,T],$$
$$\mathscr{B}_1 u(1,t) \equiv \alpha_1 u(1,t) + \beta_1 u_x(1,t) = g_1(t), \quad t \in (0,T],$$
$$u(x,0) = u_0(x), \quad x \in \bar{I},$$

where

$$\mathscr{L}u = -a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u. \tag{6}$$

The continuous-time OSC approximation is a differentiable map $u_h : [0,T] \to \mathscr{M}_r(\pi)$ such that

$$[(u_h)_t + \mathscr{L}u_h](\xi_i, t) = f(\xi_i, t), \quad i = 1,2,\ldots,M, \ \ t \in (0,T],$$
$$\mathscr{B}_0 u_h(0,t) = g_0(t), \qquad \mathscr{B}_1 u_h(1,t) = g_1(t), \quad t \in (0,T], \tag{7}$$

where $u_h(\cdot,0) \in \mathscr{M}_r(\pi)$ is determined by approximating the initial condition using either Hermite or Gauss interpolation. With $u_h(x,t) = \sum_{j=1}^{M+2} u_j(t)\phi_j(x)$, where $\{\phi_j\}_{j=1}^{M+2}$ is a basis for $\mathscr{M}_r(\pi)$, (7) is an initial value problem for a first-order system of ODEs. This system can be written as

$$B\boldsymbol{u}_h'(t) + A(t)\boldsymbol{u}_h(t) = \boldsymbol{F}(t), \quad t \in (0,T], \quad \boldsymbol{u}(0) \text{ prescribed}, \tag{8}$$

where B and $A(t)$ are both ABD matrices of the same structure.

A commonly used discrete-time OSC method for solving (7) is the Crank–Nicolson OSC method [63]. This method consists in finding $u_h^k \in \mathscr{M}_r(\pi)$, $k=1,\ldots,K$, which satisfies the boundary conditions and, for $k = 0,\ldots,K-1$,

$$\left[\frac{u_h^{k+1} - u_h^k}{\Delta t} + \mathscr{L}^{k+1/2} u_h^{k+1/2}\right](\xi_m) = f(\xi_m, t_{k+1/2}), \quad m = 1,2,\ldots,M,$$

where

$$K\Delta t = T, \quad t_{k+1/2} = (k+1/2)\Delta t, \quad u_h^{k+1/2} = (u_h^k + u_h^{k+1})/2$$

and $\mathscr{L}^{k+1/2}$ is the operator $\mathscr{L}$ of (6) with $t = t_{k+1/2}$. In matrix–vector form, this method can be written as

$$[B + \tfrac{1}{2}\Delta t A(t_{k+1/2})]\boldsymbol{u}_h^{k+1} = [B - \tfrac{1}{2}\Delta t A(t_{k+1/2})]\boldsymbol{u}_h^k + \Delta t \boldsymbol{F}(t_{k+1/2}),$$

which is essentially the trapezoidal method for (8). Thus, with a standard basis, an ABD system must be solved at each time step.

Several time-dependent problems in one space variable (with homogeneous Dirichlet boundary conditions) have been solved using a method of lines (MOL) approach using OSC with monomial bases for the spatial discretization, which results in an initial value problem for a system of differential algebraic equations (DAEs). In a series of papers, Robinson and Fairweather [124–126] adopted this approach in the solution of the cubic Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + q|u|^2 u = 0, \quad (x,t) \in I \times (0,T],$$

where $i^2 = -1$ and q is a given positive constant, and in the so-called two-dimensional parabolic equation of Tappert

$$u_t = \frac{i}{2}k_0[n^2(x,t) - 1 + iv(x,t)]u + \frac{i}{2k_0}u_{xx}, \quad (x,t) \in I \times (0,T],$$

where $n(x,t)$ and $v(x,t)$ are given functions. In each case, the DAEs are solved using the package D02NNF from the NAG Library. In [126], an optimal order L^2 estimate of the error in the semidiscrete (continuous-time) approximation at each time level is derived. In [124,125], it is shown that the use of monomial bases is particularly convenient in problems in layered media. In [122,123], this work is extended to the Schrödinger equation with general power nonlinearity

$$iu_t + u_{xx} + q|u|^{p-1}u = 0, \quad (x,t) \in I \times (0,T]$$

and to the generalized nonlinear Schrödinger equation

$$iu_t + u_{xx} + q_c|u|^2u + q_q|u|^4u + iq_m(|u|^2)_xu + iq_u|u|^2u_x = 0, \quad (x,t) \in I \times (0,T],$$

where q_c, q_q, q_m and q_u are real constants.

In [112], the MOL OSC approach is used in the solution of the Rosenau equation

$$u_t + u_{xxxxt} = f(u)_x, \quad (x,t) \in I \times (0,T],$$

where

$$f(u) = \sum_{i=1}^{n} \frac{c_i u^{p_i+1}}{p_i + 1}$$

with $c_i \in \mathbb{R}$ and p_i a nonnegative integer. This equation is first reformulated as a system by introducing the function

$$v = -u_{xx}$$

to obtain

$$u_t - v_{xxt} = f(u)_x, \quad v + u_{xx} = 0, \quad (x,t) \in I \times (0,T].$$

Here the DAEs are solved using an implicit Runge–Kutta method. Optimal order L^2 and L^∞ error estimates are obtained. The same approach is adopted and similar error estimates obtained in [111] for the Kuramoto–Sivashinsky equation

$$u_t + vu_{xxxx} + u_{xx} + uu_x = 0, \quad (x,t) \in I \times (0,T],$$

where v is a positive constant.

Several software packages have been developed for solving systems of nonlinear IBVPs in one space variable using an MOL OSC approach. The first, *pdecol* [107], uses B-spline bases and solves the resulting linear systems using the code *solveblok* [31]. The code *epdcol* [92] is a variant of *pdecol* in which *solveblok* is replaced by *colrow* [57]. In the MOL code based on OSC with monomial bases described in [116], the linear algebraic systems are solved using *abdpack* [110].

3. Elliptic boundary value problems

3.1. Introduction

In this section, we discuss OSC for the Dirichlet BVP

$$\begin{aligned} Lu &= f(x), \quad x = (x_1, x_2) \in \Omega, \\ u(x) &= g(x), \quad x \in \partial\Omega, \end{aligned} \tag{9}$$

where the linear second-order partial differential operator L is in the divergence form

$$Lu = -\sum_{i=1}^{2} (a_i(x) u_{x_i})_{x_i} + \sum_{i=1}^{2} b_i(x) u_{x_i} + c(x) u \tag{10}$$

with a_1, a_2 satisfying

$$0 < a_{\min} \leqslant a_i(x), \quad x \in \Omega, \; i = 1, 2, \tag{11}$$

or in the nondivergence form

$$Lu = -\sum_{i,j=1}^{2} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{2} b_i(x) u_{x_i} + c(x) u, \quad a_{12} = a_{21} \tag{12}$$

with a_{ij} satisfying

$$a_{\min} \sum_{i=1}^{2} \eta_i^2 \leqslant \sum_{i,j=1}^{2} a_{ij}(x) \eta_i \eta_j, \quad x \in \Omega, \; \eta_1, \eta_2 \in \mathbb{R}, \; a_{\min} > 0. \tag{13}$$

Although the principal part of L is elliptic in the sense of (11) or (13), the operator L itself could be indefinite. It should be noted that, while the divergence form (10) is more natural for finite element Galerkin methods, the nondivergence form (12) is more appropriate for spline collocation methods (cf., for example, [90], where modified spline collocation is considered). The BVP most frequently considered in the literature is Poisson's equation with homogeneous Dirichlet boundary conditions, viz.,

$$\begin{aligned} -\Delta u &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{14}$$

where Δ denotes the Laplacian.

The OSC problem for (9) consists in finding $u_h \in \mathscr{M}_r(\pi) \otimes \mathscr{M}_r(\pi)$, $r \geqslant 3$, such that

$$\begin{aligned} Lu_h(\xi_m, \xi_n) &= f(\xi_m, \xi_n), \quad m, n = 1, \ldots, M, \\ u_h(x) &= \tilde{g}(x), \quad x \in \partial\Omega, \end{aligned} \tag{15}$$

where, on each side of $\partial\Omega$, $\tilde{g}$ is an approximation to g in $\mathscr{M}_r(\pi)$.

3.2. Convergence theory

Prenter and Russell [120] considered (15) for $r = 3$, $g = 0$, and L of (10) with $b_1 = b_2 = 0$ and $c \geqslant 0$. Assuming the existence of the OSC solution and uniform boundedness of partial derivatives of certain divided difference quotients, they derived optimal order H^1 and L^2 error estimates. For the same r and L, but nonhomogeneous boundary conditions, Bialecki and Cai [15] chose $\tilde{g}$ to be the piecewise Hermite cubic interpolant of g or the piecewise cubic interpolant of g at the boundary collocation points, that is, the Gauss interpolant. In both cases, they used energy inequalities to prove existence and uniqueness of the OSC solution for sufficiently small meshsize h, and superconvergence properties of the piecewise Hermite bicubic interpolant of u to derive an optimal order H^1 error estimate.

Percell and Wheeler [117] studied (15) for $r \geqslant 3$, $g = 0$, and L of (10) with $a_1 = a_2$. They proved existence and uniqueness of the OSC solution for sufficiently small h and derived optimal order H^1 and L^2 error estimates. The assumption $a_1 = a_2$ is essential in their approach, which reduces the analysis to that for the Laplacian Δ. Dillery [58] extended the results of Percell and Wheeler to $g \neq 0$ and $a_1 \neq a_2$.

Bialecki [11] analyzed (15) for $g = 0$, and L as in (12) in the case $r \geqslant 3$. He established existence and uniqueness of the OSC solution for sufficiently small h and derived the optimal order H^2, H^1, and L^2 error estimates by proving H^2 stability of the OSC problem using a transformation of Bernstein and Nitsche's trick, and by bounding the truncation error using superconvergence properties of an interpolant of u in $\mathcal{M}_r^0(\pi) \otimes \mathcal{M}_r^0(\pi)$. The results of [11] extend and generalize all previous theoretical OSC results for two-dimensional BVPs, which do not consider mixed partial derivatives and provided no H^2 convergence analysis.

Houstis [84] considered (15) for $r = 3$, $g = 0$, and two cases of L. In the first case, L is given by (12), and in the second case L is that of (10) with $b_1 = b_2 = 0$ and $c \geqslant 0$. Using a Green's function approach (cf. [32] for two-point BVPs), he derived an optimal order L^∞ error estimate in the first case, and an optimal order L^2 error estimate in the second. However, it appears that his analysis is based on the unrealistic assumption that a partial derivative of the corresponding Green's function be uniformly bounded.

It was proved in [32] that OSC possesses superconvergence properties for two-point BVPs. Specifically, if the exact solution of the problem is sufficiently smooth and if the OSC solution $u_h \in \mathcal{M}_r(\pi)$, $r \geqslant 3$, then, at the partition nodes, the values of the OSC solution and the values of its first derivative approximate the corresponding values of the exact solution with error $O(h^{2r-2})$. For BVPs on rectangles, the same rate of convergence in the values of the OSC solution and the values of its first partial derivatives at the partition nodes was first observed numerically in [22] for $r = 3$, and in [21] for $r > 3$ and has since been observed in all applications of OSC to elliptic BVPs and IBVPs which we have examined. For (14) with $r = 3$ and a uniform partition, Bialecki [12] proved the fourth-order convergence rate in the first-order partial derivatives at the partition nodes. The approach used in [12] is a combination of a discrete Fourier method and a discrete maximum principle applied in the two different coordinate directions.

Grabarski [74] considered OSC with $r = 3$ for the solution of a nonlinear problem comprising (14) with $f(x)$ replaced with $f(x, u)$. He proved the existence and uniqueness of the OSC solution using Browder's theorem and derived an optimal order H^1 error estimate using a superconvergence property of the piecewise Hermite bicubic interpolant of u.

Aitbayev and Bialecki [1] analyzed OSC with $r = 3$ for the nonlinear problem (9) with $g = 0$ and

$$Lu = -\sum_{i,j=1}^{2} a_{ij}(x,u)u_{x_i x_j} + \sum_{i=1}^{2} b_i(x,u)u_{x_i} + c(x,u)u,$$

where a_{ij} satisfy the ellipticity condition

$$a_{\min}\sum_{i=1}^{2}\eta_i^2 \leqslant \sum_{i,j=1}^{2} a_{ij}(x,s)\eta_i\eta_j, \quad x\in\Omega,\ s\in\mathbb{R},\ \eta_1,\eta_2\in\mathbb{R},\ a_{\min}>0.$$

For sufficiently small h, they proved existence and uniqueness of the OSC solution, and established optimal order H^2 and H^1 error bounds. The approach of [1] is based on showing consistency and stability of the nonlinear OSC problem. Consistency is proved using superconvergence properties of the piecewise Hermite bicubic interpolant of u and stability is established using Banach's lemma. Then existence, uniqueness and error bounds are obtained using general results similar to those of [94]. Newton's method is analyzed for the solution of the resulting nonlinear OSC problem.

For the solution of second-order linear elliptic boundary value problems with differential operators of the form (12), Houstis et al. [86] described three OSC algorithms which use piecewise Hermite bicubics. The first, GENCOL, is for problems on general two-dimensional domains, the second for problems on rectangular domains with general linear boundary conditions, and the third, INTCOL, for problems on rectangular domains with uncoupled boundary conditions. FORTRAN implementations of these algorithms are given in [87,88]; see also [121]. In [100], the algorithm INTCOL is extended to regions whose sides are parallel to the axes. No convergence analysis of the algorithms is provided. However, in [115], Mu and Rice present an experimental study of the algorithm GENCOL on a large sample of problems in nonrectangular regions which demonstrates that the rate of convergence of the OSC solution at the nodes is fourth order.

3.3. Direct methods for linear OSC problems

Consider the special case of (9) with $g=0$ and $Lu=-u_{x_1x_1}+L_2u$, where

$$L_2u=-a_2(x_2)u_{x_2x_2}+b_2(x_2)u_{x_2}+c_2(x_2)u, \tag{16}$$

which includes (14) in polar, cylindrical, and spherical coordinate systems. In recent years, several matrix decomposition algorithms have been developed for the fast solution of the linear algebraic systems arising when finite difference, finite element Galerkin and spectral methods are applied to problems of this type. These methods, which are described in [20], depend on knowledge of the eigensystem of the second derivative operator subject to certain boundary conditions. In this section, we first describe an OSC matrix decomposition algorithm developed in [6,22,23,68] for the case in which $r=3$ and the partition of Ω is uniform at least in the x_1 direction. Then we formulate a matrix decomposition algorithm for $r\geqslant 3$ for the solution of more general elliptic problems [21].

Let $\{\phi_n\}_{n=1}^{M}$ be a basis for $\mathcal{M}_3^0(\pi)$, and write the piecewise Hermite bicubic approximation in the form

$$u_h(x_1,x_2)=\sum_{m=1}^{M}\sum_{n=1}^{M}u_{m,n}\phi_m(x_1)\phi_n(x_2).$$

If $\boldsymbol{u}=[u_{1,1},\dots,u_{1,M},\dots,u_{M,1},\dots,u_{M,M}]^{\mathrm{T}}$, then the collocation equations (15) can be written as

$$(A_1\otimes B+B\otimes A_2)\boldsymbol{u}=\boldsymbol{f}, \tag{17}$$

where $\boldsymbol{f}=[f_{1,1},\dots,f_{1,M},\dots,f_{M,1},\dots,f_{M,M}]^{\mathrm{T}}$, $f_{m,n}=f(\xi_m,\xi_n)$,

$$A_1=(-\phi_n''(\xi_m))_{m,n=1}^{M},\quad A_2=(L_2\phi_n(\xi_m))_{m,n=1}^{M},\quad B=(\phi_n(\xi_m))_{m,n=1}^{M}. \tag{18}$$

In [22], real nonsingular matrices $\Lambda = \mathrm{diag}(\lambda_j)_{j=1}^{M}$ and Z, a matrix of sines and cosines, are determined such that

$$Z^{\mathrm{T}}B^{\mathrm{T}}A_1Z = \Lambda, \quad Z^{\mathrm{T}}B^{\mathrm{T}}BZ = I. \tag{19}$$

System (17) can then be written in the form

$$(Z^{\mathrm{T}}B^{\mathrm{T}} \otimes I)(A_1 \otimes B + B \otimes A_2)(Z \otimes I)(Z^{-1} \otimes I)\boldsymbol{u} = (Z^{\mathrm{T}}B^{\mathrm{T}} \otimes I)\boldsymbol{f},$$

which becomes, on using (19),

$$(\Lambda \otimes B + I \otimes A_2)(Z^{-1} \otimes I)\boldsymbol{u} = (Z^{\mathrm{T}}B^{\mathrm{T}} \otimes I)\boldsymbol{f}.$$

Hence the matrix decomposition algorithm for solving (17) takes the following form:

Algorithm A

1. Compute $\boldsymbol{g} = (Z^{\mathrm{T}}B^{\mathrm{T}} \otimes I)\boldsymbol{f}$.
2. Solve $(\Lambda \otimes B + I \otimes A_2)\boldsymbol{v} = \boldsymbol{g}$.
3. Compute $\boldsymbol{u} = (Z \otimes I)\boldsymbol{v}$.

In step 1, matrix–vector multiplications involving the matrix B^{T} require a total of $\mathrm{O}(N^2)$ arithmetic operations. FFT routines can be used to perform multiplications by the matrix Z^{T} in step 1 and by the matrix Z in step 3. The corresponding cost of each of these steps is $\mathrm{O}(N^2 \log N)$ operations. Step 2 consists of solving M almost block diagonal linear systems of order M, the coefficient matrix of the jth system being of the form $A_2 + \lambda_j B$. The use of *colrow* [57] to solve these systems requires $\mathrm{O}(N^2)$ operations. Thus the total cost of the algorithm is $\mathrm{O}(N^2 \log N)$ operations. Note that step 2 is equivalent to solving the OSC problem for a two-point BVP of the form

$$-v'' + \lambda_j v = f(x), \quad x \in I, \quad v(0) = v(1) = 0.$$

Algorithm A is Algorithm II of [22]. Algorithm I of that paper describes an OSC matrix decomposition procedure for (14) in which the linear system (17) is diagonalized by applying FFTs in both variables. This algorithm costs twice as much as Algorithm A for (14).

Sun and Zamani [137] also developed a matrix decomposition algorithm for solving the OSC equations (15) for (14). Their algorithm is based on the fact that the eigenvalues of the matrix $B^{-1}A_1$ are real and distinct [64] and hence there exists a real nonsingular matrix Q such that $B^{-1}A_1 = Q\Lambda Q^{-1}$. Sun and Zamani's algorithm, which also requires $\mathrm{O}(N^2 \log N)$ operations, appears to be more complicated and less efficient than Algorithm A, which hinges on the existence of a real nonsingular matrix Z satisfying (19). In particular, the utilization of the second equation in (19) distinguishes Algorithm A and makes it more straightforward.

Algorithm A can be generalized to problems in which, on the sides $x_2 = 0, 1$ of $\partial\Omega$, u satisfies either the Robin boundary conditions

$$\alpha_0 u(x_1, 0) + \beta_0 u_{x_2}(x_1, 0) = g_0(x_1), \quad \alpha_1 u(x_1, 1) + \beta_1 u_{x_2}(x_1, 1) = g_1(x_1), \quad x_1 \in \bar{I},$$

where α_i, β_i, $i = 0, 1$, are constants, or the periodic boundary conditions

$$u(x_1, 0) = u(x_1, 1), \quad u_{x_2}(x_1, 0) = u_{x_2}(x_1, 1), \quad x_1 \in \bar{I}.$$

On the sides $x_1 = 0, 1$ of $\partial\Omega$, u may be subject to either Dirichlet conditions, Neumann conditions, mixed Dirichlet–Neumann boundary conditions or periodic boundary conditions. Details are given in [6,23,68]. Other extensions have been considered, to problems in three dimensions [119] and to OSC with higher degree piecewise polynomials [134].

For the case of Poisson's equation with pure Neumann or pure periodic boundary conditions, Bialecki and Remington [28] formulated a matrix decomposition approach for determining the least-squares solution of the singular OSC equations when $r = 3$.

In [127], an eigenvalue analysis is presented for spline collocation differentiation matrices corresponding to periodic boundary conditions. In particular, the circulant structure of piecewise Hermite cubic matrices is used to develop a matrix decomposition FFT algorithm for the OSC solution of a general second-order PDE with constant coefficients. The proposed algorithm, whose cost is $O(N^2 \log N)$, requires the use of complex arithmetic. An eigenvalue analysis for Dirichlet and Neumann boundary conditions and arbitrary collocation points is presented in [135].

In [13], Algorithm A was extended to the OSC solution of the polar form of the Poisson's equation on a disk

$$r^{-1}(ru_r)_r + r^{-2}u_{\theta\theta} = f(r,\theta), \quad (r,\theta) \in (0,1) \times (0,2\pi), \tag{20}$$

subject to Dirichlet or Neumann boundary conditions. The new algorithm is remarkably simple, due, in part, to a new treatment of the boundary condition on the side of the rectangle corresponding to the center of the disk. For Dirichlet boundary conditions, the OSC solution is obtained as the superposition of two OSC solutions on the rectangle. (The superposition approach was also used in [138] for a finite difference scheme.) The first OSC solution is obtained using the FFT matrix decomposition method of [6] which is based on the knowledge of the eigensystem for the OSC discretization of the second derivative operator subject to periodic boundary conditions. A simple analytical formula is derived for the second OSC solution (this is not the case for the finite difference scheme of [138]). For Neumann boundary conditions, the corresponding OSC problem is singular. In this case, the matrix decomposition method is modified to obtain an OSC approximation corresponding to the particular continuous solution with a specified value at the center of the disk. Each algorithm requires $O(N^2 \log N)$ operations. While the numerical results demonstrate fourth-order accuracy of the OSC solution and third-order accuracy of its partial derivatives at the nodes, the analysis of the OSC scheme is an open question.

Sun [132] considered the piecewise Hermite bicubic OSC solution of (20) on an annulus and on a disk. In the case of the annulus, Sun's FFT matrix decomposition algorithm is based on that of [137]. For the disk, the approach of [138] is used to derive an additional equation corresponding to the center of the disk. In contrast to [13], this equation is not solved independently of the rest of the problem. A convergence analysis for piecewise Hermite bicubic OSC solution of (20) on an annulus or a disk has yet to be derived.

For the case $r \geqslant 3$, we now describe an algorithm for determining OSC approximations to solutions of BVPs of the form

$$(L_1 + L_2)u = f(x), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega,$$

where

$$L_1 u = -a_1(x_1)u_{x_1x_1} + c_1(x_1)u$$

and L_2 is given by (16), with $a_i \geqslant a_{\min} > 0$, $c_i \geqslant 0$, $i = 1,2$. The collocation equations can again be written in the form (17) with A_2 and B as in (18) and $A_1 = (L_1\phi_n(\xi_m))_{m,n=1}^M$. As before, these matrices are ABD for the usual choices of bases of $\mathscr{M}_r^0(\pi)$.

Now, let $W = \operatorname{diag}(h_1w_1, h_1w_2, \ldots, h_1w_{r-1}, \ldots, h_Nw_1, h_Nw_2, \ldots, h_Nw_{r-1})$, where $\{w_i\}_{i=1}^{r-1}$ are the weights of the $(r-1)$-point Gauss–Legendre quadrature rule on I. For v defined on I, let $D(v) = \operatorname{diag}(v(\xi_1), v(\xi_2), \ldots, v(\xi_M))$. If

$$F = B^{\mathrm{T}}WD(1/a_1)B, \quad G = B^{\mathrm{T}}WD(1/a_1)A_1, \tag{21}$$

then F is symmetric and positive definite, and G is symmetric. Hence, there exist a real $\Lambda = \operatorname{diag}(\lambda_j)_{j=1}^M$ and a real nonsingular Z such that

$$Z^{\mathrm{T}}GZ = \Lambda, \quad Z^{\mathrm{T}}FZ = I, \tag{22}$$

[73]. By (21), the matrices Λ and Z can be computed by using the decomposition $F = HH^{\mathrm{T}}$, where $H = B^{\mathrm{T}}[WD(1/a_1)]^{1/2}$, and solving the symmetric eigenproblem for

$$C = H^{-1}GH^{-\mathrm{T}} = [WD(1/a_1)]^{1/2}A_1B^{-1}[WD(1/a_1)]^{-1/2}$$

to obtain

$$Q^{\mathrm{T}}CQ = \Lambda \tag{23}$$

with Q orthogonal. If $Z = B^{-1}[WD(1/a_1)]^{-1/2}Q$, then Λ and Z satisfy (22). Thus,

$$[Z^{\mathrm{T}}B^{\mathrm{T}}WD(1/a_1) \otimes I](A_1 \otimes B + B \otimes A_2)(Z \otimes I) = \Lambda \otimes B + I \otimes A_2,$$

which leads to the following matrix decomposition algorithm, Algorithm II of [21]:

Algorithm B

1. Determine Λ and Q satisfying (23).
2. Compute $\boldsymbol{g} = (Q^{\mathrm{T}}[WD(1/a_1)]^{1/2} \otimes I)\boldsymbol{f}$.
3. Solve $(\Lambda \otimes B + I \otimes A_2)\boldsymbol{v} = \boldsymbol{g}$.
4. Compute $\boldsymbol{u} = (B^{-1}[W_1D_1(1/a_1)]^{-1/2}Q \otimes I)\boldsymbol{v}$.

Steps 1, 3, and 4 each involve solving M independent ABD systems which are all of order M. In Step 1, C can be determined efficiently from $B^{\mathrm{T}}[WD(1/a_1)]^{1/2}C = A_1^{\mathrm{T}}[WD(1/a_1)]^{1/2}$. Computing the columns of C requires solving ABD systems with the same coefficient matrix, the transpose of the ABD matrix $[WD(1/a_1)]^{1/2}B$. This ABD matrix is factored once and the columns of C determined. This factored form is also used in Step 4. In Step 3, the ABD matrices have the form $A_2 + \lambda_jB$, $j = 1,2,\ldots,M$. Assuming that on average only two steps of the QR algorithm are required per eigenvalue when solving the symmetric, tridiagonal eigenvalue problem corresponding to (23), the total cost of this algorithm is $\mathrm{O}(r^3N^3 + r^4N^2)$.

Matrix decomposition algorithms similar to those of [21] are described in [118] for the OSC solution of Poisson's equation in three dimensions. The authors claim that these algorithms are competitive with FFT-based methods since the cost of solving one-dimensional collocation eigenvalue problems is low compared to the total cost.

Bialecki [9] developed cyclic reduction and Fourier analysis-cyclic reduction methods for the solution of the linear systems which arise when OSC with piecewise Hermite bicubics is applied to (14). On a uniform partition, the cyclic reduction and Fourier analysis-cyclic reduction methods require $O(N^2 \log N)$ and $O(N^2 \log(\log N))$ arithmetic operations, respectively.

3.4. Iterative methods for linear OSC problems

The OSC problem (15) with $r \geqslant 3$, $g = 0$ and separable $L = L_1 + L_2$, where

$$L_i u = -(a_i(x_i) u_{x_i})_{x_i} + c_i(x_i) u, \quad i = 1, 2 \tag{24}$$

or

$$L_i u = -a_i(x_i) u_{x_i x_i} + c_i(x_i) u, \quad i = 1, 2, \tag{25}$$

$a_i \geqslant a_{\min} > 0$, $c_i \geqslant 0$, can be solved by an alternating direction implicit (ADI) method. The matrix–vector form of this method is defined as follows: given $\boldsymbol{u}^{(0)}$, for $k = 0, 1, \ldots,$ compute $\boldsymbol{u}^{(k+1)}$ from

$$\begin{aligned}
&[(A_1 + \gamma_{k+1}^{(1)} B) \otimes B]\boldsymbol{u}^{(k+1/2)} = \boldsymbol{f} - [B \otimes (A_2 - \gamma_{k+1}^{(1)} B)]\boldsymbol{u}^{(k)}, \\
&[B \otimes (A_2 + \gamma_{k+1}^{(2)} B)]\boldsymbol{u}^{(k+1)} = \boldsymbol{f} - [(A_1 - \gamma_{k+1}^{(2)} B) \otimes B]\boldsymbol{u}^{(k+1/2)},
\end{aligned}$$

where, for $i=1,2$, $A_i = (L_i \phi_n(\xi_m))_{m,n=1}^M$, and $B = (\phi_n(\xi_m))_{m,n=1}^M$ as before, and $\gamma_{k+1}^{(1)}$, $\gamma_{k+1}^{(2)}$ are acceleration parameters. Introducing

$$\boldsymbol{v}^{(k)} = (B \otimes I)\boldsymbol{u}^{(k)}, \quad \boldsymbol{v}^{(k+1/2)} = (I \otimes B)\boldsymbol{u}^{(k+1/2)},$$

we obtain

$$\begin{aligned}
&[(A_1 + \gamma_{k+1}^{(1)} B) \otimes I]\boldsymbol{v}^{(k+1/2)} = \boldsymbol{f} - [I \otimes (A_2 - \gamma_{k+1}^{(1)} B)]\boldsymbol{v}^{(k)}, \\
&[I \otimes (A_2 + \gamma_{k+1}^{(2)} B)]\boldsymbol{v}^{(k+1)} = \boldsymbol{f} - [(A_1 - \gamma_{k+1}^{(2)} B) \otimes I]\boldsymbol{v}^{(k+1/2)},
\end{aligned}$$

which, at each iteration step, avoids unnecessary multiplications by B, and the solution of systems with coefficient matrix B. Note that each step of the ADI method requires the solution of ABD linear systems similar to those arising in step 3 of Algorithm B.

Dyksen [64] used the ADI method for (14) with $r = 3$, and a uniform partition. The eigenvalues (taken in increasing order) of the OSC eigenvalue problem corresponding to $-v'' = \lambda v$, $v(0) = v(1) = 0$, are used as the acceleration parameters. The cost of the resulting algorithm with $2N$ ADI iterations is $O(N^3)$. Based on numerical results, Dyksen claims that reasonable accuracy is achieved with fewer than $2N$ ADI iterations. Numerical results are included to demonstrate that the same acceleration parameters work well for more general operators L_i. In [65], Dyksen considered the solution of certain elliptic problems in three space variables in a rectangular parallelepiped using Hermite bicubic OSC to discretize in x_1 and x_2 and symmetric finite differences in x_3. The resulting equations were solved using an ADI approach.

Cooper and Prenter [53] analyzed the ADI method for L_i of (24) with $r = 3$, and a nonuniform partition. They proved convergence of the ADI method with arbitrary positive acceleration parameters but did not determine an optimal sequence of such parameters. Generalizations of this approach are discussed in [50,51].

For L_i of (25), $r \geqslant 3$, and a nonuniform partition, Bialecki [7] showed that, with Jordan's selection of the acceleration parameters, the cost of the ADI method is $O(N^2 \log^2 N)$ for obtaining an approximation to the OSC solution within the accuracy of the truncation error.

For non-separable, positive-definite, self-adjoint and nonself-adjoint L of the form (10), Richardson and minimal residual preconditioned iterative methods were presented in [10] for solving (15) with $r = 3$. In these methods, the OSC discretization of $-\Delta$, $-\Delta_h$, is used as a preconditioner since it is shown that the OSC discretization of L, L_h, and $-\Delta_h$ are spectrally equivalent on the space $\mathscr{M}_3^0(\pi) \otimes \mathscr{M}_3^0(\pi)$ with respect to the collocation inner product, that is, the discrete inner product defined by the collocation points. At each iteration step, OSC problems with $-\Delta_h$ are solved using Algorithm A. As an alternative, for self-adjoint, positive definite L of (10), in [18] the general theory of additive and multiplicative Schwarz methods is used to develop multilevel methods for (15). In these methods, variable coefficient additive and multiplicative preconditioners for L_h are, respectively, sums and products of "one-dimensional" operators which are defined using the energy collocation inner product generated by the operator $(L_h^* + L_h)/2$, where L_h^* is the operator adjoint to L_h with respect to the collocation inner product. This work includes implementations of additive and multiplicative preconditioners. Numerical tests show that multiplicative preconditioners are faster than additive preconditioners.

For L of the form (10) with $a_1 = a_2 = 1$, and u subject to a combination of Dirichlet, Neumann and Robin's boundary conditions, an application of a bilinear finite element preconditioner is considered in [98] for the solution of the corresponding OSC problem on a uniform partition. The finite element matrix $\tilde{L}_N$ associated with the self-adjoint and positive-definite operator $\tilde{L}u = -\Delta u + \tilde{c}(x)u$ is used to precondition the OSC matrix L_N corresponding to $r = 3$. Bounds and clustering results are obtained for the singular values of the preconditioned matrix $\tilde{L}_N^{-1} L_N$. This approach is used in [102] to solve the systems arising in the OSC solution of problems of linear elasticity, and is extended to quasiuniform partitions in [97].

The application of finite difference preconditioners to the solution of (15) is discussed in [136] and [95]. The theory of [136] appears to be applicable only for the case $g = 0$, and $Lu = -\Delta u + cu$, with c a positive constant, for $r = 3$ and a uniform partition. It should be noted that, in this special case, as well as in the case when c is a function of either x_1 or x_2, (15) can be solved by Algorithm A. Even if c is a function of both x_1 and x_2 and nonnegative, (15) can be solved very efficiently by the preconditioned conjugate gradient (PCG) method with $-\Delta_h$ as a preconditioner. In [95], the same operator L as in [98] (but with u subject to homogeneous Dirichlet boundary conditions) is preconditioned by a finite difference operator. Again bounds and clustering results are obtained for the singular values of the preconditioned matrix.

Some researchers [79,101,133] have investigated applications of classical iterative methods, such as Jacobi, Gauss-Seidel or SOR, to the solution of (15) for (14) for $r = 3$ and a uniform partition. In [131], an application of these methods was considered for $r = 3$, a nonuniform partition, and $L = L_1 + L_2$ with L_i of (25) and $g = 0$. As in the case of the finite difference discretization, classical iterative methods applied to these specialized problems are not as efficient as Algorithm A or the ADI method; see [129].

In [2], iterative methods are developed and implemented for the solution of (15) with L of the form (12). The PCG method is applied, with Δ_h^2 as a preconditioner, to the solution of the normal problem

$$L_h^* L_h u_h = L_h^* f_h.$$

Using an H^2 stability result of [11], it is proved that the convergence rate of this method is independent of h. On a uniform partition, the preconditioned OSC problem is solved with cost $O(N^2 \log N)$ by a modification of Algorithm A.

Lai et al. [100] considered the application of iterative methods (SOR and CG types, and GMRES) for the solution of piecewise Hermite bicubic OSC problems on regions with sides parallel to the axes.

3.5. Biharmonic problems

A common approach to solving the biharmonic equation

$$\Delta^2 u = f(x), \quad x \in \Omega$$

is to use the splitting principle in which an auxiliary function $v = \Delta u$ is introduced and the biharmonic equation rewritten in the form

$$-\Delta u + v = 0, \quad -\Delta v = -f(x), \quad x \in \Omega. \tag{26}$$

In the context of the finite element Galerkin method, this approach is known as the mixed method of Ciarlet and Raviart [48].

Using this approach, Lou et al. [106] derived existence, uniqueness and convergence results for piecewise Hermite bicubic OSC methods and developed implementations of these methods for the solution of three biharmonic problems. The boundary conditions for the first problem comprise $u = g_1$ and $\Delta u = g_2$ on $\partial\Omega$, and the problem becomes one of solving two nonhomogeneous Dirichlet problems for Poisson's equation. The resulting linear systems can be solved effectively with cost $O(N^2 \log N)$ on a uniform partition using Algorithm A. In this case, optimal order H^j error estimates, $j = 0, 1, 2$, are derived. In the second problem, the boundary condition in the first problem on the horizontal sides of $\partial\Omega$, $\Delta u = g_2$, is replaced by the condition $u_{x_2} = g_3$. Optimal order H^1 and H^2 error estimates are derived and a variant of Algorithm A is formulated for the solution of the corresponding algebraic problem. This algorithm also has cost $O(N^2 \log N)$ on a uniform partition. The third problem is the biharmonic Dirichlet problem,

$$\begin{aligned} &-\Delta u + v = 0, \quad -\Delta v = -f(x), \quad x \in \Omega, \\ &u = g_1(x), \quad u_n = g_2(x), \quad x \in \partial\Omega, \end{aligned} \tag{27}$$

where the subscript n denotes the outward normal derivative. Again optimal order H^1 and H^2 error estimates are derived. In this case, the OSC linear system is rather complicated and is solved by a direct method which is based on the capacitance matrix technique with the second biharmonic problem as the auxiliary problem. On a uniform partition, the total cost of the capacitance matrix method for computing the OSC solution is $O(N^3)$ since the capacitance system is first formed explicitly and then solved by Gauss elimination. Results of some numerical experiments are presented which, in particular, demonstrate the fourth-order accuracy of the approximations and the superconvergence of the derivative approximations at the mesh points.

Piecewise Hermite bicubic OSC methods for the biharmonic Dirichlet problem (27) were considered by Cooper and Prenter [52] who proposed an ADI OSC method for the discretization of the BVP and the solution of the discrete problem, and by Sun [130] who presented an algorithm, the cost of which is $O(N^3 \log N)$. Numerical results show that these methods produce approximations which are fourth-order accurate at the nodes, but no rigorous proof of this result is provided in [52] or [130].

In [14], Bialecki developed a very efficient Schur complement method for obtaining the piecewise Hermite bicubic OSC solution to the biharmonic Dirichlet problem (27). In this approach, which is similar to that of [29] for finite differences, the OSC biharmonic Dirichlet problem is reduced to a Schur complement system involving the approximation to v on the vertical sides of $\partial\Omega$ and to an auxiliary OSC problem for a related biharmonic problem with v, instead of u_n, specified on the two vertical sides of $\partial\Omega$. The Schur complement system with a symmetric and positive definite matrix is solved by the PCG method with a preconditioner obtained from the OSC problem for a related biharmonic problem with v, instead of u_n, specified on the two horizontal sides of $\partial\Omega$. On a uniform partition, the cost of solving the preconditioned system and the cost of multiplying the Schur complement matrix by a vector are $O(N^2)$ each. With the number of PCG iterations proportional to $\log N$, the cost of solving the Schur complement system is $O(N^2 \log N)$. The solution of the auxiliary OSC problem is obtained using a variant of Algorithm A at a cost of $O(N^2 \log N)$. Thus the total cost of solving the OSC biharmonic Dirichlet problem is $O(N^2 \log N)$, which is essentially the same as that of Bjørstad's algorithm [29] for solving the second-order finite difference biharmonic Dirichlet problem. Numerical results indicate that the L^2, H^1, and H^2 errors in the OSC approximations to u and v are of optimal order. Convergence at the nodes is fourth order for the approximations to u, v, and their first-order derivatives.

3.6. Domain decomposition

Bialecki [8] used a domain decomposition approach to develop a fast solver for the piecewise Hermite bicubic OSC solution of (14). The square Ω is divided into parallel strips and the OSC solution is first obtained on the interfaces by solving a collection of independent tridiagonal linear systems. Algorithm A is then used to compute the OSC solution on each strip. Assuming that the strips have the same width and that their number is proportional to $N/\log N$, the cost of the domain decomposition solver is $O(N^2 \log(\log N))$.

Mateescu et al. [113] considered linear systems arising from piecewise Hermite bicubic collocation applied to general linear two-dimensional second-order elliptic PDEs on Ω with mixed boundary conditions. They constructed an efficient, parallel preconditioner for the GMRES(k) method. The focus in [113] is on rectangular domains decomposed in one dimension.

For the same problem as in [8], Bialecki and Dillery [17] analyzed the convergence rates of two Schwarz alternating methods. In the first method, Ω is divided into two overlapping subrectangles, while three overlapping subrectangles are used in the second method. Fourier analysis is used to obtain explicit formulas for the convergence factors by which the H^1 norm of the error is reduced in one iteration of the Schwarz methods. It is shown numerically that while these factors depend on the size of the overlap, they are independent of h. Using a convex function argument, Kim and Kim [96] bounded theoretically the convergence factors of [17] by quantities that depend only on the way that Ω is divided into the overlapping subrectangles.

In [16], an overlapping domain decomposition method was considered for the solution of the piecewise Hermite bicubic OSC problem corresponding to (14). The square is divided into overlapping squares and the additive Schwarz, conjugate gradient method involves solving independent OSC problems using Algorithm A.

Lai et al. [99] considered a generalized Schwarz splitting method for solving elliptic BVPs with interface conditions that depend on a parameter that might differ in each overlapping region. The method is coupled with the piecewise Hermite bicubic collocation discretization to solve the corresponding BVP in each subdomain. The main objective of [99] is the mathematical analysis of the iterative solution of the so-called enhanced generalized Schwarz splitting collocation equation corresponding to (14).

Bialecki and Dryja [19] considered the piecewise Hermite bicubic OSC solution of

$$\Delta u = f \text{ in } \hat{\Omega}, \quad u = 0 \text{ on } \partial\hat{\Omega}, \tag{28}$$

where $\hat{\Omega}$ is the L-shaped region given by

$$\hat{\Omega} = (0,2) \times (0,1) \cup (0,1) \times (1,2). \tag{29}$$

The region $\hat{\Omega}$ is partitioned into three nonoverlapping squares with two interfaces. On each square, the approximate solution is a piecewise Hermite bicubic that satisfies Poisson's equation at the collocation points in the subregion. The approximate solution is continuous throughout the region and its normal derivatives are equal at the collocation points on the interfaces, but continuity of the normal derivatives across the interfaces is not guaranteed. The solution of the collocation problem is first reduced to finding the approximate solution on the interfaces. The discrete Steklov–Poincaré operator corresponding to the interfaces is self-adjoint and positive definite with respect to the discrete inner product associated with the collocation points on the interfaces. The approximate solution on the interfaces is computed using the PCG method with the preconditioner obtained from two discrete Steklov–Poincaré operators corresponding to two pairs of the adjacent squares. Once the solution of the discrete Steklov–Poincaré equation is obtained, the collocation solutions on subregions are computed using Algorithm A. On a uniform partition, the total cost of the algorithm is $O(N^2 \log N)$, where the number of unknowns is proportional to N^2.

4. Time-dependent problems

4.1. Parabolic and hyperbolic problems

In this section, we discuss OSC for the IBVP

$$\begin{aligned} &u_t + (L_1 + L_2)u = f(x,t), \quad (x,t) \in \Omega_T \equiv \Omega \times (0,T], \\ &u(x,0) = g_1(x), \quad x \in \bar{\Omega}, \\ &u(x,t) = g_2(x,t), \quad (x,t) \in \partial\Omega \times (0,T] \end{aligned} \tag{30}$$

and the second-order hyperbolic IBVP

$$\begin{aligned} &u_{tt} + (L_1 + L_2)u = f(x,t), \quad (x,t) \in \Omega_T, \\ &u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \quad x \in \bar{\Omega}, \\ &u(x,t) = g_3(x,t), \quad (x,t) \in \partial\Omega \times (0,T], \end{aligned} \tag{31}$$

where the second-order differential operators L_1 and L_2 are of the form

$$L_1 u = -(a_1(x,t)u_{x_1})_{x_1} + b_1(x,t)u_{x_1} + c(x,t)u, \quad L_2 u = -(a_2(x,t)u_{x_2})_{x_2} + b_2(x,t)u_{x_2} \tag{32}$$

or

$$L_1u = -a_1(x,t)u_{x_1x_1} + b_1(x,t)u_{x_1} + c(x,t)u, \quad L_2u = -a_2(x,t)u_{x_2x_2} + b_2(x,t)u_{x_2} \tag{33}$$

with a_1, a_2 satisfying

$$0 < a_{\min} \leqslant a_i(x,t) \leqslant a_{\max}, \quad (x,t) \in \Omega_T, \quad i = 1,2. \tag{34}$$

For $r \geqslant 3$, Greenwell-Yanik and Fairweather [76] analyzed continuous-time OSC methods and Crank–Nicolson schemes for nonlinear problems of the form (30) and (31) with $f(x,t,u)$ in place of $f(x,t)$, $g_2 = 0$, L_1 and L_2 of (33) with $a_1 = a_2$, and $c = 0$. Using the OSC solution of the corresponding elliptic problem as a comparison function, they derived optimal order L^2 error estimates.

For $r = 3$, Grabarski [75] considered the continuous-time OSC method for a nonlinear problem, (30) with $f(x,t,u)$ in place of $f(x,t)$, $g_2 = 0$, L_1 and L_2 of (33) with $a_1 = a_2$, $b_1 = b_2 = c = 0$. He derived an optimal order H^1 error estimate using a superconvergence property of the piecewise Hermite bicubic interpolant of u.

ADI OSC methods, without convergence analysis, have been used over the last 20 years to solve certain parabolic problems (see, for example, [5,33,36–42,54,80–82]). In general, the ADI Crank–Nicolson scheme for (30) consists in finding $u_h^k \in \mathcal{M}_r(\pi) \otimes \mathcal{M}_r(\pi)$, $k = 1,\dots,K$, such that, for $k = 0,\dots,K-1$,

$$\left[\frac{u_h^{k+1/2} - u_h^k}{\Delta t/2} + L_1^{k+1/2}u_h^{k+1/2} + L_2^{k+1/2}u_h^k\right](\xi_m,\xi_n) = f(\xi_m,\xi_n,t_{k+1/2}), \quad m,n = 1,\dots,M,$$

$$\left[\frac{u_h^{k+1} - u_h^{k+1/2}}{\Delta t/2} + L_1^{k+1/2}u_h^{k+1/2} + L_2^{k+1/2}u_h^{k+1}\right](\xi_m,\xi_n) = f(\xi_m,\xi_n,t_{k+1/2}), \quad m,n = 1,\dots,M, \tag{35}$$

where $L_1^{k+1/2}$ and $L_2^{k+1/2}$ are the differential operators L_1 and L_2 with $t = t_{k+1/2}$, respectively, $u_h^0 \in \mathcal{M}_r(\pi) \otimes \mathcal{M}_r(\pi)$, $u_h^k|_{\partial\Omega}$, $k = 1,\dots,K$, are prescribed by approximating the initial and boundary conditions of (30) using either piecewise Hermite or Gauss interpolants, and for each $n = 1,\dots,M$, $u_h^{k+1/2}(\cdot,\xi_n) \in \mathcal{M}_r(\pi)$ satisfies

$$u_h^{k+1/2}(\alpha,\xi_n) = [(1/2)(u_h^{k+1} + u_h^k) + (\Delta t/4)L_2^{k+1/2}(u_h^{k+1} - u_h^k)](\alpha,\xi_n), \quad \alpha = 0,1.$$

For $r \geqslant 3$, Fernandes and Fairweather [72] analyzed (35) for the heat equation with homogeneous Dirichlet boundary conditions, which is the special case of (30) with $L_iu = -u_{x_ix_i}$, $i = 1,2$, and $g_2 = 0$. They proved second-order accuracy in time and optimal order accuracy in space in the L^2 and H^1 norms.

For $r = 3$, Bialecki and Fernandes [24] considered two- and three-level Laplace-modified (LM) and ADI LM schemes for the solution of (30) with L_1 and L_2 of (32). Also in [24], a Crank–Nicolson ADI OSC scheme (35), with $r = 3$, $L_2^{k+1/2}$ replaced by L_2^k in the first equation and by L_2^{k+1} in the second equation, was considered for L_1 and L_2 of (32) with $b_1 = b_2 = c = 0$. The stability proof of the scheme hinges on the fact that the operators L_1 and L_2 are nonnegative definite with respect to the collocation inner product. The derived error estimate shows that the scheme is second-order accurate in time and third-order accurate in space in a norm which is stronger than the L^2 norm but weaker than the H^1 norm.

In [25], scheme (35) was considered for $r = 3$, and L_1 and L_2 of (33). It was shown that the scheme is second-order accurate in time and of optimal accuracy in space in the H^1 norm. The analysis in [25] can be easily extended to the case $r \geqslant 3$. A new efficient implementation of the scheme is presented and tested on a sample problem for accuracy and convergence rates in various norms. Earlier implementations of ADI OSC schemes are based on determining, at each time level, a two-dimensional approximation defined on Ω. In the new implementation, at each time level, one-dimensional approximations are determined along horizontal and vertical lines passing through Gauss points and the two-dimensional approximation on Ω is determined only when desired. Note that the two-level, parameter-free ADI OSC scheme does not have a finite element Galerkin counterpart. The method of [60] of comparable accuracy is the three-level ADI LM scheme requiring the selection of a stability parameter.

A nonlinear parabolic IBVP on a rectangular polygon with variable coefficient Robin boundary conditions is considered in [27]. An approximation to the solution at a desired time value is obtained using an ADI extrapolated Crank–Nicolson scheme in which OSC with $r \geqslant 3$ is used for spatial discretization. For rectangular and L-shaped regions, an efficient B-spline implementation of the scheme is described and numerical results are presented which demonstrate the accuracy and convergence rates in various norms.

Fernandes and Fairweather [72] considered the wave equation with homogeneous Dirichlet boundary conditions, which is the special case of the hyperbolic problem (31) with $L_i u = -u_{x_i x_i}$, $i = 1, 2$, and $g_1 = 0$. First, the wave equation is rewritten as a system of two equations in two unknown functions. Then an ADI OSC scheme with $r \geqslant 3$ is developed and its second-order accuracy in time and optimal order accuracy in space in the L^2 and H^1 norms are derived.

In [71], two schemes are formulated and analyzed for the approximate solution of (31) with L_1 and L_2 of (32). OSC with $r = 3$ is used for the spatial discretization, and the resulting system of ODEs in the time variable is discretized using perturbations of standard finite difference procedures to produce LM and ADI LM schemes. It is shown that these schemes are unconditionally stable, and of optimal order accuracy in time and of optimal order accuracy in space in the H^1 norm, provided that, in each scheme, the LM stability parameter is chosen appropriately. The algebraic problems to which these schemes lead are also described and numerical results are presented for an implementation of the ADI LM scheme to demonstrate the accuracy and rate of convergence of the method.

In [26], the approximate solution of (31) with L_1 and L_2 of (33) is considered. The new ADI OSC scheme consists in finding $u_h^k \in \mathscr{M}_3(\pi) \otimes \mathscr{M}_3(\pi)$, $k = 2, \ldots, K$, such that, for $k = 1, \ldots, K-1$,

$$\begin{aligned} &[I + \tfrac{1}{2}(\Delta t)^2 L_1^k]\tilde{u}_h^k(\xi_m, \xi_n) = f(\xi_m, \xi_n, t_k) + 2(\Delta t)^{-2} u_h^k(\xi_m, \xi_n), \quad m, n = 1, \ldots, M, \\ &[I + \tfrac{1}{2}(\Delta t)^2 L_2^k](u_h^{k+1} + u_h^{k-1})(\xi_m, \xi_n) = (\Delta t)^2 \tilde{u}_h^k(\xi_m, \xi_n), \quad m, n = 1, \ldots, M, \end{aligned} \tag{36}$$

where L_1^k and L_2^k are the differential operators of (33) with $t = t_k$, and $u_h^0, u_h^1 \in \mathscr{M}_3(\pi) \otimes \mathscr{M}_3(\pi)$, $u_h^k|_{\partial\Omega}$, $k = 2, \ldots, M$, are prescribed by approximating the initial and boundary conditions of (31) using either piecewise Hermite or Gauss interpolants. In the first equation of (36), for each $n = 1, \ldots, M$, $\tilde{u}_h^k(\cdot, \xi_n) \in \mathscr{M}_3(\pi)$ satisfies

$$\tilde{u}_h^k(\alpha, \xi_n) = (\Delta t)^{-2}[I + \tfrac{1}{2}(\Delta t)^2 L_2^k](u_h^{k+1} + u_h^{k-1})(\alpha, \xi_n), \quad \alpha = 0, 1.$$

It is shown in [26] that scheme (36) is second-order accurate in time and of optimal order accuracy in space in the H^1 norm. An efficient implementation of the scheme is similar to that for the scheme of [25] and involves representing u_h^k in terms of basis functions with respect to x_2 alone while $\tilde{u}_h^k$

is represented in terms of basis functions with respect to x_1 only. It is interesting to note that, for variable coefficient hyperbolic problems, the parameter-free ADI OSC scheme (36) does not have a finite element Galerkin counterpart.

In [66], an OSC method is considered for the solution of PDEs that arise in investigating invariant tori for dynamical systems.

4.2. Schrödinger-type problems

In [103], Crank–Nicolson and ADI OSC schemes based on the Crank–Nicolson approach are formulated and analyzed for the approximate solution of the linear Schrödinger problem

$$\begin{aligned}
&\psi_t - \mathrm{i}\Delta\psi + \mathrm{i}\sigma(x,t)\psi = f(x,t), \quad (x,t) \in \Omega_T,\\
&\psi(x,0) = \psi^0(x), \quad x \in \Omega,\\
&\psi(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T],
\end{aligned} \tag{37}$$

where σ is a real function, while ψ, ψ^0 and f are complex valued. A problem of this type of current interest is the so-called parabolic wave equation which arises in wave propagation problems in underwater acoustics; see, for example, [128]. The functions ψ, f and ψ^0 are written as $\psi_1 + \mathrm{i}\psi_2$, $f_1 + \mathrm{i}f_2$ and $\psi_1^0 + \mathrm{i}\psi_2^0$, respectively. Taking real and imaginary parts of (37) then yields

$$\begin{aligned}
&\boldsymbol{u}_t + S(-\Delta + \sigma(x,t))\boldsymbol{u} = \boldsymbol{F}(x,t), \quad (x,t) \in \Omega_T,\\
&\boldsymbol{u}(x,0) = \boldsymbol{u}^0(x), \quad x \in \Omega,\\
&\boldsymbol{u}(x,t) = \boldsymbol{0}, \quad (x,t) \in \partial\Omega \times (0,T],
\end{aligned} \tag{38}$$

where $\boldsymbol{u} = [\psi_1\ \psi_2]^{\mathrm{T}}$, $\boldsymbol{F} = [f_1\ f_2]^{\mathrm{T}}$, and $\boldsymbol{u}^0 = [\psi_1^0\ \psi_2^0]^{\mathrm{T}}$ are real-valued vector functions, and

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Hence, (38), and thus (37), is not parabolic but a Schrödinger-type system of partial differential equations. OSC with $r \geqslant 3$ is used for the spatial discretization of (38). The resulting system of ODEs in the time variable is discretized using the trapezoidal rule to produce the Crank–Nicolson OSC scheme, which is then perturbed to obtain an ADI OSC scheme of the form (35). The stability of these schemes is examined, and it is shown that they are second-order accurate in time and of optimal order accuracy in space in the H^1 and L^2 norms. Numerical results are presented which confirm the analysis.

Li et al. [104] considered the OSC solution of the following problem governing the transverse vibrations of a thin square plate clamped at its edges:

$$\begin{aligned}
&u_{tt} + \Delta^2 u = f(x,t), \quad (x,t) \in \Omega_T,\\
&u(x,0) = g_0(x), \quad u_t(x,0) = g_1(x), \quad x \in \Omega,\\
&u(x,t) = 0, \quad u_n(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T].
\end{aligned} \tag{39}$$

With $u_1 = u_t$, and $u_2 = \Delta u$, and $\boldsymbol{U} = [u_1 \;\; u_2]^{\mathrm{T}}$, $\boldsymbol{F} = [f \;\; 0]^{\mathrm{T}}$, and $\boldsymbol{G} = [g_1 \;\; \Delta g_0]^{\mathrm{T}}$, this problem can be reformulated as the Schrödinger-type problem

$$\begin{aligned} &\boldsymbol{U}_t - S\Delta \boldsymbol{U} = \boldsymbol{F}, \quad (x,t) \in \Omega_T, \\ &\boldsymbol{U}(x,0) = \boldsymbol{G}(x), \quad x \in \Omega, \\ &u_1(x,t) = (u_1)_n(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T]. \end{aligned} \tag{40}$$

An approximate solution of (39) is computed by first determining an approximation to the solution U of (40) using the Crank–Nicolson OSC scheme with $r=3$. The existence, uniqueness and stability of this scheme are analyzed and it is shown to be second-order accurate in time and of optimal order accuracy in space in the H^1 and H^2 norms. An approximation to the solution u of (39) with these approximation properties is determined by integrating the differential equation $u_t = u_1$ using the trapezoidal method with u_1 replaced by its Crank–Nicolson OSC approximation.

Similar results also hold for vibration problems with other boundary conditions, specifically, "hinged" boundary conditions,

$$u(x,t) = 0, \quad \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T]$$

and conditions in which the vertical sides are hinged and the horizontal sides are clamped:

$$\begin{aligned} &u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ &\Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega_1 \times (0,T], \\ &u_n(x,t) = 0, \quad (x,t) \in \partial\Omega_2 \times (0,T], \end{aligned}$$

where $\partial\Omega_1 = \{(\alpha, x_2)\colon \alpha = 0,1,\; 0 \leqslant x_2 \leqslant 1\}$ and $\partial\Omega_2 = \{(x_1, \alpha)\colon 0 \leqslant x_1 \leqslant 1,\; \alpha = 0,1\}$. For these boundary conditions, it is also possible to formulate and analyze Crank–Nicolson ADI OSC methods which cost $\mathrm{O}(N^2)$ per time step. Details are given in [105] for piecewise polynomials of arbitrary degree. In the case of (39), no ADI method has been found and, for the case $r = 3$, to solve the linear systems arising at each time step of the Crank–Nicolson scheme, a capacitance matrix method, which was used effectively in [106] for the solution of Dirichlet biharmonic problems, is employed. The cost per time step of this method is $\mathrm{O}(N^2 \log N)$. Results of numerical experiments are presented which confirm the theoretical analysis and also indicate that the L^2 norm of the error is of optimal order for the first and third choices of boundary conditions, a result which has yet to be proved analytically.

4.3. Partial integro-differential equations

In various fields of engineering and physics, systems which are functions of space and time are often described by PDEs. In some situations, such a formulation may not accurately model the physical system because, while describing the system as a function at a given time, it fails to take into account the effect of past history. Particularly in such fields as heat transfer, nuclear reactor dynamics, and viscoelasticity, there is often a need to reflect the effects of the "memory" of the system. This has resulted in the inclusion of an integral term in the basic PDE yielding a partial integro-differential equation (PIDE). For example, consider PIDEs of the form

$$u_t = \mu\Delta u + \int_0^t a(t-s)\Delta u(x,s)\,\mathrm{d}s + f(x,t), \quad (x,t) \in \Omega_T, \tag{41}$$

which arise in several areas such as heat flow in materials with memory, and in linear viscoelastic problems. When $\mu > 0$, the equation is parabolic; in the case where $\mu = 0$, if the memory function a is differentiable and $a(0) > 0$, the equation is then hyperbolic because (41) can be differentiated with respect to t to give

$$u_{tt} = a(0)\Delta u + \int_0^t a'(t-s)\Delta u(x,s)\,\mathrm{d}s + f_t(x,t), \quad (x,t) \in \Omega_T.$$

However, if a has a "strong" singularity at the origin, that is, $\lim_{t\to 0} a(t) = \infty$, Eq. (41) can only be regarded as intermediate between parabolic and hyperbolic.

In recent years, considerable attention has been devoted to the development and analysis of finite element methods, particularly finite element Galerkin methods, for the solution of parabolic and hyperbolic PIDEs; see, for example, [114] and references cited therein. Much of this work has attempted to carry over to PIDEs standard results for PDEs. OSC methods for PIDEs were first considered by Yanik and Fairweather [143] who formulated and analyzed discrete-time OSC methods for parabolic and hyperbolic PIDEs in one space variable of the form

$$c(x,t,u)\frac{\partial^j u}{\partial t^j} - u_{xx} = \int_0^t f(x,t,s,u(x,s),u_x)\,\mathrm{d}s, \quad (x,t) \in (0,1)\times(0,T],$$

where $j = 1,2$, respectively. In the case of two space variables, techniques developed in [76] can be used to derive optimal order estimates for fully discrete approximations to equations of the form

$$c(x,t)u_t - \Delta u = \int_0^t f(x,t,s,u(x,s))\,\mathrm{d}s, \quad (x,t) \in \Omega_T.$$

However this analysis does not extend to spline collocation methods applied to the PIDE (41).

Fairweather [67] examined two types of spline collocation methods, OSC and modified cubic spline collocation (see Section 5) for (41) in the special case when $\mu = 0$. The error analyses presented in [67] are based on the assumption that the memory function a is "positive", that is,

$$\int_0^t v(\sigma)\int_0^\sigma a(\sigma - s)v(s)\,\mathrm{d}s\,\mathrm{d}\sigma \geqslant 0 \tag{42}$$

for every $v \in C[0,T]$ and for every $t \in [0,T]$. This condition is guaranteed by easily checked sign conditions on the function a and its derivatives, namely,

$$a \in C^2[0,T], \quad (-1)^k a^{(k)}(t) \geqslant 0, \quad k = 0,1,2, \quad a' \not\equiv 0.$$

In the hyperbolic case (μ=0), condition (42) excludes a large class of memory functions. By employing a different approach involving the Laplace transform with respect to time, Yan and Fairweather [142] gave a complete analysis of the stability and convergence of the continuous-time OSC method for PIDEs of the form (41) which is more general than that presented in [67]. Their convergence results hold under much weaker conditions on the function a than (42). For example, a can be a C^2 function with only the additional condition $a(0) > 0$. Moreover, a may be singular, for example, $a(t) = \mathrm{e}^{-\beta t}t^{-\alpha}$ with $0 < \alpha < 1$ and $\beta \in R$.

5. Modified spline collocation methods

The first spline collocation method proposed for the solution of two-point boundary value problems of the form (2)–(3) was the nodal cubic spline method. In this method, one seeks an approximate solution in the space of cubic splines which satisfies the differential equation at the nodes of the partition π and the boundary conditions. De Boor [30] proved that this procedure is second-order accurate and no better, which led researchers to seek cubic spline collocation methods of optimal accuracy, namely, fourth order. Such a method, defined only on a uniform partition, was developed independently by Archer [4] and Daniel and Swartz [55]. In this method, often called modified cubic spline collocation, either the nodal cubic spline collocation solution is improved using a deferred correction-type procedure (the two-step method) or a fourth-order approximation is determined directly by collocating a high-order perturbation of the differential equation (the one-step method). Similar approaches were adopted in [85] in the development of optimal quadratic spline collocation methods for problems of the form (2)–(3) and in [91] for quintic spline collocation methods for general linear fourth-order two-point boundary value problems.

Modified spline collocation methods have also been derived for the solution of elliptic boundary value problems on rectangles; Houstis et al. [89,90] considered both approaches using bicubic splines, and Christara [45] has examined the use of biquadratics. Various direct and iterative methods for the solution of the resulting linear systems were considered for the cubic case in two and three space variables in [77,78,139], and in the biquadratic case in [44,46,47]. In several cases, analysis is carried out only for Helmholtz problems with constant coefficients and homogeneous Dirichlet boundary conditions.

Recently, for Poisson and Helmholtz equations, matrix decomposition algorithms have been developed for both the biquadratic and bicubic cases. In [49], the two-step method was considered for biquadratic splines. In this matrix decomposition approach, the coefficient matrix is diagonalized using FFTs as in Algorithm I of [22]. In [69], the two-step bicubic method was used for various combinations of boundary conditions but a matrix decomposition procedure for the one-step method has been derived only for Dirichlet problems.

A practical advantage of modified spline collocation over OSC is that, for a given partition, there are fewer unknowns with the same degree of piecewise polynomials, thereby reducing the size of the linear systems. However, it does require a uniform partition over the whole of the domain and is only applicable for low degree splines.

It should be noted that modified spline collocation methods may have considerable potential in computational physics, for example, where traditional spline collocation seems to be gaining in popularity over finite element Galerkin methods; see, for example, [34,35,93,140,141].

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