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Cohomology of Semidirect Product Groups

Burt Totaro*

Department of Mathematics, University of Chicago, 5734 South University Avenue, Chicago, Illinois 60637

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We give examples of semidirect product groups $G \ltimes A$ such that the Hochschild–Serre spectral sequence $H^*(G, H^*A) \Rightarrow H^*(G \ltimes A)$ for $\mathbf{Z}/p$-cohomology has nonzero differentials. Until now, few such examples have been known, especially when the normal subgroup A is abelian. In particular, Benson and Feshbach [2] mentioned that in all known semidirect products with A abelian, the spectral sequence satisfies:

(1) All differentials after d_2 are 0.

(2) All differentials are 0 if $A = (S^1)^n$. (To be consistent with the notation for discrete groups A, H^*A here means the cohomology of the classifying space of A.)

(3) All differentials are 0 if $A = (\mathbf{Z}/2)^n$ and we consider cohomology with $\mathbf{Z}/2$ coefficients.

We give examples to show that all three statements can fail. In fact, there can be nonzero differentials at d_p or later in all of these cases. I expect that there can be nonzero differentials arbitrarily far along in the spectral sequence in all of these cases, but the problem remains open. (For semidirect products $G \ltimes A$ with A not abelian, Benson and Feshbach [2] gave examples of nonzero differentials arbitrarily far along in the spectral sequence for $\mathbf{Z}/2$-cohomology.)

It turns out that there is a very general reason why there will be nonzero differentials in some examples. If X is a G-space, then $H^*(G, C^*(X))$ admits Steenrod operations compatible with those on H^*G because it is the cohomology of the space $(X \times EG)/G$, whereas there is no reason for $H^*(G, M)$ to have Steenrod operations for a general G-module M. Thus Steenrod operations provide a fundamental obstruction for a G-module to

*E-mail address: totaro@math.uchicago.edu.

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be the representation of G on the cohomology of a G-space, as G. Carlsson found [4]; there is a useful exposition by Benson and Habegger [3]. If a semidirect product $G \ltimes A$ has the G-action on A given by the dual of such a G-module, we can show that there must be nonzero differentials in the Hochschild–Serre spectral sequence.

It is interesting to contrast these examples with Nakaoka's theorem that the Hochschild–Serre spectral sequence has no differentials for any wreath product $G \ltimes H^n$ [6, p. 50]. Here G and H are any finite groups and G acts on H^n through a permutation representation $G \hookrightarrow S_n$. It would be good to characterize algebraically the class of G-modules M over $\mathbf{Z}/p$, say for a p-group G, such that the semidirect product $G \ltimes M$ has no differentials in the spectral sequence: it seems to be fairly close to the class of permutation modules, but there are some other interesting examples.

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1. LEMMA ON STEENROD OPERATIONS

Let p be a prime number. Throughout this paper we write H^*X for $H^*(X, \mathbf{Z}/p)$. Also, let $H^{\cdot}X$ be $H^*(X, \mathbf{Z}/2)$ if $p = 2$, and let it be the even-dimensional subring $H^{ev}(X, \mathbf{Z}/p)$ is p is odd. Thus $H^{\cdot}X$ is always a commutative ring. Finally, for p odd, $P^i\colon H^nX \to H^{n+2(p-1)i}X$ are the usual Steenrod operations, and for $p = 2$ we use the same notation P^i to mean $P^i = Sq^i\colon H^nX \to H^{n+i}X$.

We need the following lemma, which is a variant of Proposition 3 in Landweber and Stong [7]. The proof is short enough to repeat here. Recall that the radical of an ideal I in a commutative ring R is the set of $a \in R$ such that $a^n \in I$ for some $n \geq 1$.

LEMMA 1. *Let p be a prime number, and let H^*X denote $H^*(X, \mathbf{Z}/p)$. Let $X \to Y$ be a map of spaces. Let $M \subset H^*X$ be any finitely generated graded $H^{\cdot}Y$-submodule. Then the radical of the annihilator of M is an ideal in $H^{\cdot}Y$ which is closed under the Steenrod operations P^i, $i \geq 0$.*

The interesting thing is that M is not assumed to be closed under the Steenrod operations, and as a result the annihilator of M is generally not closed under the Steenrod operations; but the radical of the annihilator of M behaves better.

Proof. It suffices to prove that the radical of the annihilator in $H^{\cdot}Y$ of a single element $x \in H^*X$ is closed under all P^i's. For $2i > \dim x$ (or $i > \dim x$, in case $p = 2$) we have $P^ix = 0$. So there is a positive integer r

large enough that $P^i x = 0$ for $i \geq p^r$. Then, for any $a \in H^{\cdot}Y$ and $i \geq 0$, we have

$$P^{ip^r}(a^{p^r}x) = (P^i a)^{p^r} x.$$

This follows from the Cartan identity for Steenrod operations, which says that the total Steenrod operation $P = 1 + P^1 + P^2 + \cdots$ is a ring homomorphism from H^*X to itself, so that

$$\begin{aligned} P(a^{p^r}x) &= P(a)^{p^r} P(x) \\ &= \left(\sum_{k \geq 0} (P^k a)^{p^r} \right) \left(\sum_{0 \leq l < p^r} P^l x \right). \end{aligned}$$

The earlier identity follows by equating terms in the appropriate dimension. Now if a belongs to the radical of the annihilator of x, we may assume that $a^{p^r}x = 0$. The identity then shows that $P^i a$ is in the radical of the annihilator of x for all $i \geq 0$. Q.E.D.

2. COHOMOLOGY OF SEMIDIRECT PRODUCT GROUPS

Let $G \ltimes A$ be a semidirect product of groups, where we need not assume that A is abelian, although that is where I have applications for the theorem. Let $H^*G = H^*(G, \mathbf{Z}/p)$ for a fixed prime number p, and let $H^{\cdot}G$ be H^*G for $p = 2$, $H^{ev}G$ for p odd.

A group G is defined to be of type VFP for $\mathbf{Z}/p$-coefficients if it has a subgroup of finite index whose total cohomology with coefficients in any module of finite dimension over $\mathbf{Z}/p$ is finite-dimensional. Finite groups as well as arithmetic groups, such as $GL_n\mathbf{Z}$, are examples of groups of type VFP.

THEOREM 1. *Suppose that G has type VFP for $\mathbf{Z}/p$ and that H^iA is finite-dimensional for each i. Let r be the smallest number ≥ 1 such that $H^rA \neq 0$. If the Hochschild–Serre spectral sequence for computing $H^*(G \ltimes A)$ has all differentials into $H^*(G, H^rA)$ equal to 0, then the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, H^rA)$ is closed under the Steenrod operations P^i.*

Proof. Venkov and Evens proved that for finite groups G, $H^{\cdot}G$ is a noetherian ring and $H^*(G, M)$ is a finitely generated $H^{\cdot}G$-module for all $\mathbf{Z}/p$ G-modules M of finite dimension over $\mathbf{Z}/p$ [1, p. 130]. The Hochschild–Serre spectral sequence shows that these properties generalize to groups G of type VFP for $\mathbf{Z}/p$.

Since we have a semidirect product, the 0th row of the spectral sequence, H^*G, splits off from $H^*(G \ltimes A)$ as an H^*G-module in a natural way. The remaining piece of $H^*(G \ltimes A)$ has a filtration by H^*G-submodules, with the bottom piece of the filtration isomorphic to $H^*(G, H^rA)/$(all differentials). If, as we assume, there are no differentials mapping into the rth row, then we have exhibited $H^*(G, H^rA)$ as an H^*G-submodule of $H^*(G \ltimes A)$.

By Lemma 1, even though the Steenrod operations need not map $H^*(G, H^rA)$ into itself, the radical of the annihilator of $H^*(G, H^rA)$ in $H^{\cdot}G$ is closed under the Steenrod operations. Q.E.D.

Corollary 1. *For each prime number p, there are semidirect products $(\mathbf{Z}/p)^2 \ltimes (\mathbf{Z}/p)^n$, $(\mathbf{Z}/p)^2 \ltimes \mathbf{Z}^n$, and $(\mathbf{Z}/p)^2 \ltimes (S^1)^n$ such that the Hochschild–Serre spectral sequence with $\mathbf{Z}/p$ coefficients does not degenerate. More precisely there will be nonzero differentials mapping into $H^*(G, H^1A)$ in the first two cases and into $H^*(G, H^2A)$ in the last case. We can take $n = 2p^2$.*

Proof. Let $G = (\mathbf{Z}/p)^2$. Following Benson [1, pp. 190–195], we will exhibit a $\mathbf{Z}G$-module L_ζ which is free as a $\mathbf{Z}$-module such that the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ is not closed under the Steenrod operations P^i. (One can define a module L_ζ with this property for any finite group G of p-rank ≥ 2, but we will just prove what we need for $G = (\mathbf{Z}/p)^2$.) Then, if we define an abelian group A with G-action by $A = \mathrm{Hom}(L_\zeta, \mathbf{Z}/p)$, $A = \mathrm{Hom}(L_\zeta, \mathbf{Z})$, or $A = B\,\mathrm{Hom}(L_\zeta, \mathbf{Z})$ (in the last case $A \cong (S^1)^n$), then the lowest-dimensional cohomology of A (H^rA where $r = 1, 1, 2$, respectively) is isomorphic to $L_\zeta \otimes \mathbf{Z}/p$ as a G-module. By Theorem 1, there are nonzero differentials in the spectral sequence of the extension $G \ltimes A$ with $\mathbf{Z}/p$ coefficients in these three cases. In fact there are nonzero differentials mapping into $H^*(G, H^rA)$.

We define the $\mathbf{Z}G$-module L_ζ as follows. Let $x, y \in H^2G$ span the space of Bocksteins of elements of H^1, so that x and y generate a polynomial subring of H^*G, and let ζ be a homogenous irreducible polynomial in x, y over $\mathbf{Z}/p$ of degree $d > 1$. Then ζ gives an element of $H^{2d}G$, which even lifts to $H^{2d}(G, \mathbf{Z})$ since x and y are integral classes. Fix such a lift. Let

$$\cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

be a projective resolution of $\mathbf{Z}$ as a $\mathbf{Z}G$-module, and let $\Omega^i\mathbf{Z}$ be the image of P_i in P_{i-1}; it may depend on the resolution, although that is irrelevant to us. Then the lift of ζ in $H^{2d}(G, \mathbf{Z})$ can be represented by a map $\Omega^{2d}\mathbf{Z} \to \mathbf{Z}$ of $\mathbf{Z}G$-modules. Let L_ζ be the kernel, so that we have a short exact sequence of $\mathbf{Z}G$-modules,

$$0 \to L_\zeta \to \Omega^{2d}\mathbf{Z} \to \mathbf{Z} \to 0.$$

These are torsion-free abelian groups, because $\Omega^{2d}\mathbf{Z}$ is a submodule of P_{2d-1}. The cohomology of G with coefficients in $(\Omega^{2d}\mathbf{Z}) \otimes \mathbf{Z}/p$ is just H^*G shifted up by $2d$, at least in dimensions $\geq 2d+1$, and the map

$$\zeta\colon (\Omega^{2d}\mathbf{Z}) \otimes \mathbf{Z}/p \to \mathbf{Z}/p$$

gives a map

$$H^iG = H^{i+2d}\big(G, (\Omega^{2d}\mathbf{Z}) \otimes \mathbf{Z}/p\big) \to H^{i+2d}(G, \mathbf{Z}/p)$$

which is multiplication by $\zeta \in H^{2d}G$. Multiplication by ζ is an injective map on H^*G (for $p=2$, H^*G is a polynomial ring; for p odd, H^*G is the tensor product of a polynomial ring and an exterior algebra, with ζ in the polynomial subring). So the short exact sequence above, which remains exact on tensoring with $\mathbf{Z}/p$, determines $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ in high dimensions: for $i \geq 2d+2$,

$$H^k\big(G, L_\zeta \otimes \mathbf{Z}/p\big) = H^{i-1}G/(\zeta).$$

Knowing $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ in dimensions $\geq 2d+2$ is enough if we only want to know the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$; namely, this radical is the ideal $(\sqrt{\zeta})$ in $H^{\cdot}G = H^*G$ for $p=2$, or (ζ) in $H^{\cdot}G = H^{ev}G$ for p odd. But Serre [11] showed that if an ideal in $H^{\cdot}G$ is closed under the Steenrod operations, then the corresponding algebraic subset of Spec $H^{\cdot}G = A^2_{\mathbf{Z}/p}$ is a finite union of $\mathbf{Z}/p$-linear subspaces. Since the polynomial ζ is irreducible of degree >1 over $\mathbf{Z}/p$, Serre's theorem shows that the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ is not closed under the Steenrod operations, which is the property of L_ζ we want.

Specifically, let ζ be an irreducible quadratic polynomial over $\mathbf{Z}/p$, so that $d=2$ above. There is a resolution of $\mathbf{Z}$ over $\mathbf{Z}G$, where $G = (\mathbf{Z}/p)^2$, of the form

$$\cdots \to (\mathbf{Z}G)^3 \to (\mathbf{Z}G)^2 \to (\mathbf{Z}G)^1 \to \mathbf{Z} \to 0,$$

and one computes that $\Omega^4\mathbf{Z}$ is a $\mathbf{Z}G$-module of $\mathbf{Z}$-rank $2p^2+1$ for this resolution. So L_ζ is a $\mathbf{Z}G$-module of $\mathbf{Z}$-rank $2p^2$, and we can take $A = (\mathbf{Z}/p)^{2p^2}$, $A = \mathbf{Z}^{2p^2}$, or $A = (S^1)^{2p^2}$ for our example. Q.E.D.

Steve Siegel pointed out to me that in the special case of semidirect products $(\mathbf{Z}/2)^2 \ltimes (\mathbf{Z}/2)^n$, we can take the cohomology classes x and y in the above construction to be in H^1 rather than H^2, with the result that there is a semidirect product of this type with nonzero Hochschild–Serre differentials for $n = 4$, rather than $n = 2p^2 = 8$.

3. COMMENTS

At this point we have answered negatively two of the three questions raised in [2]: the spectral sequence of a semidirect product $G \ltimes A$ does not always degenerate for $A = (S^1)^n$, nor for $A = (\mathbf{Z}/2)^n$ with $\mathbf{Z}/2$ coefficients. (Examples where the differential d_2 was nonzero were known before for $A = \mathbf{Z}^n$ [5; 9, pp. 28–29] and for $A = (\mathbf{Z}/p)^n$, at least when $p \geq 5$ [10].)

We can also answer no to the remaining question asked in [2], whether the d_2 differential is the only one which can be nonzero in the spectral sequence for semidirect products by an abelian group. The point is that for cohomology with $\mathbf{Z}/p$ coefficients, the only differentials which can be nonzero in the spectral sequence for a semidirect product $G \ltimes \mathbf{Z}^n$ are the d_i's with $i \equiv 1 \pmod{p-1}$, starting with d_p. This is an easy consequence of Lieberman's trick, that is, of the action of the multiplicative monoid of the positive integers on $G \ltimes \mathbf{Z}^n$ by fixing G and acting on the obvious way on $\mathbf{Z}^n$ [8, p. 262]. Thus, for the semidirect products $(\mathbf{Z}/p)^2 \ltimes \mathbf{Z}^n$ produced in Corollary 1, there is a nonzero differential at d_p or later. The same argument shows that for $(\mathbf{Z}/p)^2 \ltimes (S^1)^n$ as in Corollary 1, there is a nonzero differential at d_{2p-1} or later. (In this case the possibly nonzero differentials are d_i for $i \equiv 1 \pmod{2(p-1)}$.)

Also, for semidirect products $G \ltimes (\mathbf{Z}/p)^n$ as constructed in Corollary 1, there will be a nonzero differential at d_p or later. The point is that, in the Corollary, the G-action on $(\mathbf{Z}/p)^n$ lifts to an action on $A = \mathbf{Z}^n$. The resulting homomorphism $G \ltimes A \to G \ltimes (A/p)$ gives a map of spectral sequences which is an isomorphism on row 1 of the E_2 term:

$$H^i\big(G, H^1(A/p)\big) \xrightarrow{\cong} H^i(G, H^1 A).$$

Since there are no differentials into $H^i(G, H^1 A)$ until d_p or later, there are no differentials into $H^i(G, H^1(A/p))$ until d_p or later. Moreover Corollary 1 says that there will be a nonzero differential into $H^i(G, H^1(A/p))$ sometime, thus necessarily at d_p or later.

The question remains whether the $\mathbf{Z}/p$-cohomology spectral sequence for semidirect products $G \ltimes (\mathbf{Z}/p)^n$ or $G \ltimes \mathbf{Z}^n$ can have nonzero differentials after d_p. I expect that there can be nonzero differentials arbitrarily far along in the spectral sequence.

REFERENCES

1. D. Benson, ‘‘Representations and Cohomology. II. Cohomology of Groups and Modules,’’ Cambridge Univ. Press, Cambridge, 1991.
2. D. Benson and M. Feshbach, On the cohomology of split extensions, *Proc. Amer. Math. Soc.* **121** (1994), 687–690.
3. D. Benson and N. Habegger, Varieties for modules and a problem of Steenrod, *J. Pure Appl. Algebra* **44** (1987), 13–34.
4. G. Carlsson, A counterexample to a conjecture of Steenrod, *Invent. Math.* **64** (1981), 171–174.
5. L. Charlap and A. Vasquez, Characteristic classes for modules over groups I, *Trans. Amer. Math. Soc.* **137** (1969), 533–549.
6. L. Evens, ‘‘The Cohomology of Groups,’’ Oxford Univ. Press, New York, 1991.
7. P. Landweber and R. Stong, The depth of rings of invariants, *in* ‘‘Number Theory (New York, 1984–1985),’’ pp. 259–274, Lecture Notes in Mathematics, Vol. 1240, Springer-Verlag, Berlin, 1987.
8. C.-H. Sah, Cohomology of split group extensions, *J. Algebra* **29** (1974), 255–302.
9. C.-H. Sah, Cohomology of split group extensions, II, *J. Algebra* **45** (1977), 17–68.
10. S. Siegel, ‘‘Cohomology and Group Extensions,’’ Thesis, University of Chicago, 1993.
11. J.-P. Serre, Sur la dimension cohomologique des groupes profinis, *Topology* **3** (1965), 413–420.