MATHEMATICS

ON THE GENERALIZED SEMIGROUP RELATION IN THE STRONG OPERATOR TOPOLOGY *)

BY

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Abstract

Let $\mathfrak{X}$ be a Banach space, and let $\mathscr{E}(\mathfrak{X})$ denote the Banach algebra of endomorphisms of $\mathfrak{X}$. A one-parameter family of operator-valued functions $\{S(t),\ t \in R^+\}$, where $S(t): R^+ \to \mathscr{E}(\mathfrak{X})$, is said to be a generalized semigroup of operators on $\mathfrak{X}$ if (1) $S(s+t) - S(s)S(t) = F(s,t)$, $s, t \in R^+$, (2) $S(s)S(t) = S(t)S(s)$, (3) $S(O) = I$, where $F(s,t)$: $R^+ \times R^+ \to \mathscr{E}(\mathfrak{X})$. Solutions of Eq. (1) in the uniform operator topology were considered in an earlier paper of the authors (*Proc. Nat. Acad. Sci. U.S.A.* 60 (1968), 1170–1174). In this paper the authors investigate the analytical properties of Eq. (1) in the strong operator topology, when it is assumed that the perturbation family $F(s,t)$ is bounded relative to the family $S(t)$. A representation of $S(t)$ is given; and, as an example, a generalized semigroup of translations on $C[0, \infty]$ is considered.

1. *Introduction*

Let $\mathfrak{X}$ be a Banach space, and let $\mathscr{E}(\mathfrak{X})$ denote the Banach algebra of endomorphisms of $\mathfrak{X}$. A one-parameter family of operator-valued functions $\{S(t),\ t \in R^+\}$, where $S(t): R^+ \to \mathscr{E}(\mathfrak{X})$, is said to be a *generalized semigroup of operators* on $\mathfrak{X}$ if

$$(1) \qquad \begin{cases} S(s+t) - S(s)S(t) = F(s,t),\ s, t \in R^+, \\ S(s)S(t) = S(t)S(s),\ s, t \in R^+, \\ S(0) = I, \end{cases}$$

where $F(s,t): R^+ \times R^+ \to \mathscr{E}(\mathfrak{X})$. The generalized semigroup relation (1) was introduced by RICHARDSON [7]. The two-parameter family of operator-valued functions $F(s,t)$ can be regarded as a perturbation family which expresses the departure of $\{S(t),\ t \in R^+\}$ from a semigroup relation, which would hold in case $F(s,t) = \Theta$ (the null operator) for all $s, t \in R^+$. From (1) it follows immediately that (i) $F(0,t) = F(s,0) = \Theta$, (ii) $F(s,t) = F(t,s)$ (symmetry).

The study of operators $S(t)$ which arise as solutions in the uniform operator topology of functional equations of the form (1) was initiated by BUCHE and BHARUCHA-REID [1]. In this paper we investigate the analytical properties of the solution of Eq. (1) in the strong operator

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topology, when it is assumed that the perturbation family $F(s, t)$ is bounded relative to the family $S(t)$ (cf., KATO [4]). In Section 2 we establish a result on the rate of growth of $\|S(t)\|$, and obtain an extension of a result of PHILLIPS [6] on the connection between strong measurability and boundness of a semigroup. Section 3 is concerned with the properties of the (first) infinitesimal generator of the generalized semigroup; and in Section 4 we present results on the representation of $S(t)$ in the strong operator topology. Finally, in Section 5 we consider as a concrete example a generalized semigroup of translations on $C[0, \infty]$.

2. *Measurability and Boundedness of the Generalized Semigroup Operator*

The two-parameter family of operator-valued functions $\{F(s, t), s, t \in R^+\}$ will be said to be *relatively bounded* with respect to the one-parameter family $\{S(t), t \in R^+\}$ if

$$\|F(s, t)f\| \leqslant K\|S(s)S(t)f\|, \qquad f \in \mathfrak{X}. \tag{2}$$

We now prove the following lemma on the rate of growth of $\log \|S(t)\|$.

Lemma 1. *If* $p(t) = \log \|S(t)\|$ *is bounded on the interval* $(0, t_0)$ *for each* $t_0 > 0$, *and if* $F(s, t)$ *satisfies condition* (2), *then* $\limsup_{t\to\infty} t^{-1} \log \|S(t)\| < \infty$.

Proof. Since $\|F(s, t)\| \leqslant K\|S(s)S(t)\|$, it follows that

$$\|S(s+t)\| \leqslant (K+1)\|S(s)S(t)\|,$$

and hence

$$\|S(2t)\| \leqslant (K+1)\|S(t)\|^2,$$

and

$$\|S(3t)\| \leqslant (K+1)\|S(2t)S(t)\| \leqslant (K+1)^2\|S(t)\|^3.$$

By induction, $\|S(nt)\| \leqslant (K+1)^{n-1}\|S(t)\|^n$ for any positive integer n and $t > 0$. Consider now a fixed $t_0 > 0$. Let n be a positive integer such that $nt_0 \leqslant t < (n+1)t_0$. Then

$$\begin{aligned} S(t) &= S(nt_0 + t - nt_0) \\ &= S(nt_0)S(t - nt_0) + F(nt_0, t - nt_0), \end{aligned}$$

so that

$$\begin{aligned} \|S(t)\| &\leqslant (K+1)(K+1)^{n-1}\|S(t_0)\|^n\|S(t - nt_0)\| \\ &\leqslant (K+1)^n\|S(t_0)\|^n p_0 \\ &\leqslant (K+1)^n p_0^{n+1}, \end{aligned}$$

where $p_0 = \sup_{0 \leqslant t \leqslant t_0} \|S(t)\|$. Hence

$$t^{-1} \log \|S(t)\| \leqslant nt^{-1} \log (K+1) + (n+1)t^{-1} \log p_0,$$

from which it follows that

$$\limsup_{t\to\infty} t^{-1} \log \|S(t)\| \leqslant t_0^{-1}[\log (K+1) + \log p_0]. \tag{3}$$

Corollary 1. *There exist positive constants M and β such that* $\|S(t)\| \leqslant Me^{\beta t}$.

We now state and prove a result on the relationship between the strong measurability and boundedness of $S(t)$.

Lemma 2. *If $S(t)$ is strongly measurable, and $F(s, t)$ satisfies condition* (2), *then $\|S(t)\|$ is bounded on every finite interval $[t_1, t_2]$, $t_1, t_2 \in R^+$, $t_1 < t_2$.*

Proof. The proof is based on arguments of Miyadera [5]. In view of the uniform boundedness theorem it would suffice to prove that $\|S(t)f\|$ is bounded in $[t_1, t_2]$ for each $f \in \mathfrak{X}$. Suppose that this is not true for some $f \in \mathfrak{X}$. Then there exists a $\tau \in (t_1, t_2)$ and a sequence $\{\tau_n\} \subset [t_1, t_2]$ such that $\tau_n \to \tau$ and $\|S(\tau_n)f\| \geqslant n$ for all n. On the other hand $\|S(t)f\|$ being measurable, there exist a constant N and a measurable set $G \subset [0, \tau]$, with measure $\mu(G) > \tau/2$, such that $\sup_{t \in G} \|S(t)f\| \leqslant N$.

Let $E_n = \{(\tau_n - \delta) : \delta \in G \cap [0, \tau_n]\}$. Then E_n is measurable, and for n sufficiently large $\mu(E_n) \geqslant \tau/2$. Now for $\delta \in G \cap [0, \tau_n]$ we have

$$\begin{aligned} n &\leqslant \|S(\tau_n)f\| \\ &\leqslant \|(S(\tau_n - \delta)S(\delta) + F(\tau_n - \delta, \delta))f\| \\ &\leqslant (K+1)\|S(\tau_n - \delta)\| \, \|S(\delta)f\| \\ &\leqslant (K+1)N\|S(\tau_n - \delta)\|, \end{aligned}$$

and hence $\|S(t)\| \geqslant n/(K+1)N$ for all $t \in E_n$. Denoting $\limsup_n E_n$ by E we see that $\|S(t)\| = \infty$ for all $t \in E$, and $\mu(E) \geqslant \tau/2 > 0$, which is a contradiction.

The above lemma is an extension of a result of Phillips [6] for semigroups of operators.

3. *The infinitesimal generator*

In what follows we assume that $S(t)$ is strongly continuous for $t \in R^+$. This implies boundedness on every finite interval of R^+. Let $A_h = (S(h) - I)/h$, $h > 0$. The *(first) infinitesimal generator* A of the generalized semigroup $\{S(t), t \in R^+\}$ is defined by

$$Af = \lim_{h \to 0} A_h f, \; f \in \mathscr{D}(A), \tag{4}$$

where $\mathscr{D}(A) \subset \mathfrak{X}$, and $\mathscr{D}(A)$ is the set of elements $f \in \mathfrak{X}$ for which the limit exists. Clearly $\mathscr{D}(A)$ is a linear subspace of $\mathfrak{X}$, and A is a linear operator. Let

$$\begin{cases} Q(t)f = \lim_{s \to 0} \dfrac{F(s, t)}{s} f \\ \quad\;\;\; = \lim_{s \to 0} \dfrac{F(s, t-s)}{s} f, \end{cases} \tag{5}$$

and let $\mathscr{D}(Q(t))$ be the set of all $f \in \mathfrak{X}$ for which the limit exists. In what follows we assume that $\mathscr{D}(Q(t))$ is independent of t; and, writing $\mathscr{D}(Q(t)) = \mathscr{D}(Q)$, we assume that $\mathscr{D}(Q)$ is dense in $\mathfrak{X}$. Following DUNFORD and SCHWARTZ ([2], pp. 619–621), the following result is easily obtained from the above assumptions and the properties of $F(s, t)$.

Lemma 3. (a) *If* $\mathscr{D}(A) = \mathscr{D}(Q)$, *then* $\mathscr{D}(A)$ *is invariant under the transformation* $S(t)$ *for any* $t \in R^+$; (b) *For each* $f \in \mathscr{D}(A) \cap \mathscr{D}(Q)$,

$$\begin{cases} \dfrac{d}{dt}(S(t)f) = S(t)Af + Q(t)f \\ \qquad\qquad\quad = AS(t)f + Q(t)f, \end{cases} \tag{6}$$

and for all $s, t \in R^+$, $s < t$

$$[S(t) - S(s)]f = \int_0^t S(\tau)Afd\tau + \int_0^t Q(\tau)fd\tau \tag{7}$$

(c) *If* $\mathscr{D}(A) = \mathscr{D}(Q)$, *then* $\mathscr{D}(A)$ *is dense in* $\mathfrak{X}$, *and* A *is a closed operator on* $\mathscr{D}(A)$.

4. *Representation of the Generalized Semigroup*

In this section we shall obtain a representation of $S(t)$ as a strong limit of a sequence of operators on $\mathfrak{X}$, convergence being uniform on every finite interval of R^+. The theorem stated below is a generalization of the corresponding theorem for semigroups of operators and for the generalized semigroup of operators in the uniform operator topology.

Theorem 1. *Under the general assumptions of Section* 3, *and the hypothesis that* $\mathscr{D}(A) = \mathscr{D}(Q)$,

$$S(t) = \lim_{h \to 0} \left[\exp\{tA_h\} f + \int_0^t \exp\{(t-\tau) A_h\} \frac{F(h, \tau)}{h} f d\tau \right] \tag{8}$$

for each $f \in \mathfrak{X}$, *and uniformly for each* t *in any finite interval.*

Proof. Let $f \in \mathscr{D}(A)$. Then by (6) and (7) the following holds:

$$\begin{aligned} S(t)f - \exp\{tA_h\}f &= \\ &= \int_0^t \frac{d}{d\tau} [\exp\{(t-\tau) A_h\} S(\tau)f]\, d\tau = \\ &= \int_0^t \exp\{(t-\tau) A_h\} (-A_h) S(\tau) f d\tau + \\ &+ \int_0^t \exp\{(t-\tau) A_h\} [S(\tau) Af + Q(\tau)f]\, d\tau = \\ &= -\int_0^t \exp\{(t-\tau) A_h\}(A_h - A) S(\tau) f d\tau + \\ &+ \int_0^t \exp\{(t-\tau) A_h\} Q(\tau) f d\tau. \end{aligned}$$

Hence

$$S(t)f - \exp\{tA_h\} f - \int_0^t \exp\{(t-\tau) A_h\} \frac{F(h,\tau)}{h} f d\tau =$$
$$= - \int_0^t \exp\{(t-\tau) A_h\} S(\tau) [A_h - A] f d\tau -$$
$$- \int_0^t \exp\{(t-\tau) A_h\} \left[\frac{F(h,\tau)}{h} - Q(\tau)\right] f d\tau.$$

Therefore

$$\left\| S(t) f - \exp\{tA_h\} f - \exp\{(t-\tau) A_h\} \frac{F(h,\tau)}{h} f d\tau \right\| \leqslant \int_0^t \|\exp\{(t-\tau) A_h\| d\tau \cdot$$
$$\cdot \sup_{0 \leqslant \tau \leqslant t} \left\{ \|S(\tau)\| \, \|(A_h - A) f\| + \left\| \left[\frac{F(h,\tau)}{h} - Q(\tau)\right] f \right\| \right\}.$$

The right hand side tends to zero as $h \to 0$ in view of our hypothesis, and the definitions of A and $Q(t)$; and since A is a strong infinitesimal generator of a one-parameter semigroup of operators. By Lemma 3 (c), $\mathscr{D}(A)$ is dense in $\mathfrak{X}$; thus the conclusion of the theorem follows by the Banach-Steinhaus theorem.

It is of interest, especially in applications, to have estimates of $\|S(t) - - T(t)\|$; that is, bounds for the departure of $S(t)$ from a semigroup operator. We state the following result, which can be verified by straightforward calculations:

Theorem 2. *Let* $T(t) = s - \lim_{h \to 0} e^{A_h t}$, *and let* $\|e^{tA_h} f\| \leqslant Ce^{\gamma t}\|f\|$ *for* $t \in [0, T]$ *(T finite). Then for each* $f \in \mathfrak{X}$ *and uniformly for* $t \in [0, T]$

(9) $$\|[S(t) - T(t)] f\| \leqslant C[(e^{\gamma t} - 1)/\gamma] \limsup_{\substack{h \to 0 \\ 0 \leqslant \tau \leqslant t}} \left\| \frac{F(h,\tau)}{h} f \right\|.$$

5. *A Generalized Semigroup of Translations on* $C[0, \infty]$.

We define a family of operators $\{S(t), t \in R^+\}$, $S(t)\colon R^+ \to \mathscr{E}(C[0, \infty])$ as follows:

(10) $$[S(t) f](u) = f(u+t) + \frac{\alpha t}{1+t} f(u), \; f \in C[0, \infty],$$

where $|\alpha| < 1$. Clearly $\lim_{t \to 0} (S(t)f)(u) = f(u)$, and $\|S(t)\| = 1 + \frac{|\alpha| t}{1+t}$. Since

$$(F(s,t) f)(u) =$$
$$= -\alpha \left[\frac{t}{1+t} f(u+s) + \frac{s}{1+s} f(u+t) - \alpha \left(\frac{s+t}{1+s+t} - \frac{\alpha st}{(1+s)(1+t)}\right) f(u)\right],$$

we have

$$\|F(s,t) f\| \leqslant \|S(s) S(t) f\|, \; f \in C[0, \infty], \; s, t \geqslant 0.$$

Also

$$(Q(t)\,f)(u) = -\alpha\left[\frac{t}{1+t}f'(u) + f(u+t) - \alpha\left(\frac{1}{(1+t)^2} - \frac{\alpha t}{1+t}\right) f(u)\right].$$

Thus $\mathscr{D}(Q(t))$ is independent of t, and is defined whenever $D = \dfrac{d}{du}$ is defined. The (first) infinitesimal generator is given by

$$Af(u) = \lim_{h\to 0}\frac{1}{h}\left[f(u+h) + \frac{\alpha h}{1+h}f(u) - f(u)\right] = (D+\alpha)\,f(u).$$

Hence $\mathscr{D}(A) = \mathscr{D}(Q(t))$, and $\mathscr{D}(A)$ is dense in $\mathfrak{X}$. Let $\Delta_h f(u) = f(u+h) - f(u)$, $h \geqslant 0$. Then, by the representation theorem

$$(S(t)\,f)\,(u) = \lim_{h\to 0}\left[\left(\exp\left\{t\frac{\Delta h}{h} + \frac{\alpha t}{1+h}\right\}\right) f(u) - \right.$$

$$-\alpha\int_0^t\left(\exp\left\{(t-\tau)\frac{\Delta h}{h} + \frac{\alpha(t-\tau)}{1+h}\right\}\right)\cdot$$

$$\left.\cdot\left(\frac{\tau}{1+\tau}\frac{\Delta_h}{h} + \frac{1}{1+h}\Delta_\tau + \frac{\tau(1+(1+\alpha)\,(1+h+\tau))}{(1+h)\,(1+\tau)\,(1+h+\tau)}\right)\cdot f(u)\,dt\right].$$

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