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Orthogonality Preserving Transformations on Indefinite Inner Product Spaces: Generalization of Uhlhorn's Version of Wigner's Theorem¹

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We present an analogue of Uhlhorn's version of Wigner's theorem on symmetry transformations for the case of indefinite inner product spaces. This significantly generalizes a result of Van den Broek. The proof is based on our main theorem, which describes the form of all bijective transformations on the set of all rank-one idempotents of a Banach space which preserve zero products in both directions. © 2002 Elsevier Science (USA)

Key Words: symmetry transformations; indefinite inner product spaces; idempotents on Banach spaces

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Wigner's theorem on symmetry transformations plays a fundamental role in quantum mechanics. It states that any quantum mechanical invariance transformation (symmetry transformation) can be represented by a unitary or antiunitary operator on a complex Hilbert space and that, conversely, any operator of that kind represents an invariance transformation. In mathematical language, the result can be reformulated in the following way. If H is a complex Hilbert space and T is a bijective transformation on the set of all one-dimensional linear subspaces of H which preserves the angle between every pair of such subspaces (in the terminology of quantum mechanics, this angle is called a transition probability), then T is induced by either a unitary or an antiunitary operator U on H . This means that for every one-dimensional subspace L of H we have $T(L) = U[L] = \{Ux : x \in$

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L }. In his famous paper [11], Uhlhorn generalized this result by requiring only that T preserves the orthogonality between the one-dimensional subspaces of H . This is a significant achievement since Uhlhorn's transformation preserves only the logical structure of the quantum mechanical system in question while Wigner's transformation preserves its complete probabilistic structure. However, in the case when the dimension of H is not less than 3, Uhlhorn was able to obtain the same conclusion as Wigner.

In the last decades it has become quite clear that indefinite inner product spaces are even more useful than definite ones in describing several physical problems (see, for example, the introduction in [1]). This has raised the need to study Wigner's theorem in the indefinite setting as well (see [1, 3]). Our paper [7] was devoted to a generalization of Wigner's original theorem for indefinite inner product spaces. In the present paper, we treat Uhlhorn's version in that setting. Our approach here is different from that followed in [7]. Namely, it is based on a beautiful result of Ovchinnikov [10] describing the automorphisms of the poset of all idempotents on a separable Hilbert space of dimension at least 3, which result can be regarded as a "skew version" of the fundamental theorem of projective geometry. This result enables us to use operator algebraic tools to attack the problem. We note that this kind of machinery already proved effective in our former works [6, 8] where we obtained the Wigner-type results for different structures. We emphasize that in the literature there does exist an Uhlhorn-type result on symmetry transformations on indefinite inner product spaces. In fact, this is due to Van den Broek [3] (an application of his result can be found in [2], also see [4]). In that paper he considered indefinite inner product spaces induced by nonsingular self-adjoint operators on finite-dimensional complex Hilbert spaces. Moreover, in the proof of the main result he basically followed the original idea of Uhlhorn. In the present paper, we apply a completely different approach and obtain a much more general result, namely, a result concerning indefinite inner product spaces induced by any invertible bounded linear operator on a real or complex Hilbert space of any dimension (not less than 3). Quantum logics on spaces with such a general indefinite metric have been investigated by, for example, Matvejchuk in [5]. Our result will follow from the main theorem of the paper, which describes the form of all bijective transformations of the set of all rank-one idempotents on a Banach space which preserve zero products in both directions.

If X is a (real or complex) Banach space, then $B(X)$ stands for the algebra of all bounded linear operators on X . An operator $P \in B(X)$ is called an idempotent if $P^2 = P$. The set of all idempotents in $B(X)$ is denoted by $I(X)$ and $I_1(X)$ stands for the set of all rank-one elements of $I(X)$.

Now, our main result reads as follows.

MAIN THEOREM. *Let X be a (real or complex) Banach space of dimension at least 3. Let $\phi: I_1(X) \rightarrow I_1(X)$ be a bijective transformation with the property that*

$$PQ = 0 \Leftrightarrow \phi(P)\phi(Q) = 0$$

for all $P, Q \in I_1(X)$.

If X is real, then there exists an invertible bounded linear operator $A: X \rightarrow X$ such that ϕ is of the form

$$\phi(P) = APA^{-1} \quad (P \in I_1(X)). \quad (1)$$

If X is complex and infinite dimensional, then there exists an invertible bounded linear or conjugate-linear operator $A: X \rightarrow X$ such that ϕ is of form (1).

If X is complex and finite dimensional, then we can suppose that our transformation ϕ acts on the space of $n \times n$ complex matrices ($n = \dim X$). In this case there is a nonsingular matrix $A \in M_n(\mathbb{C})$ and a ring automorphism h of \mathbb{C} such that ϕ is of the form

$$\phi(P) = Ah(P)A^{-1} \quad (P \in I_1(\mathbb{C}^n)). \quad (2)$$

Here $h(P)$ denotes the matrix obtained from P by applying h to every entry of it.

Our main theorem can be summarized by saying that every bijective transformation on $I_1(X)$ which preserves zero products in both directions comes from a linear or conjugate-linear algebra automorphism of $B(X)$ if X is real or complex and infinite dimensional, and it comes from a semilinear algebra automorphism of $B(X)$ if X is complex and finite dimensional. Replying to a remark of the referee, we note that our result probably has no serious physical meaning. This is because the poset of all idempotents on a Banach space (the partial order among idempotents is defined in Section 2) does not form a lattice in general and hence it is not a geometry or a logic in the sense of quantum mechanics (see [12]). In fact, the poset of idempotents is not to be confused with the lattice of subspaces of a linear space as the idempotents are determined not by one but two complementary subspaces. However, our main theorem will easily imply our result Corollary 2 generalizing Uhlhorn's version of Wigner's theorem for indefinite inner product spaces which statement we believe has serious physical meaning. On the other hand, it will be clear from the proof presented that one can readily get a very similar result as in our theorem for the form of zero product preserving transformations on the set of rank-one idempotents on different Banach spaces (also see the remark after Corollary 2) which has an

interesting mathematical consequence. Namely, it implies that the real Banach spaces as topological vector spaces are completely determined by the set of their rank-one idempotents with the relation of zero product.

In our paper [7], we presented a Wigner-type result for pairs of ray transformations [7, Theorem 1] which enabled us to generalize the result of Bracci et al. [1] for indefinite inner product spaces generated by any invertible bounded linear (not necessarily self-adjoint) operator on a Hilbert space. Now, our main result above can be applied to obtain the following corollary, which is a Banach space analogue and hence a remarkable generalization (in the mathematical sense) of the main result in [7] that was formulated for (complex) Hilbert spaces.

For the formulation of our corollary, we need some concepts and notation. Following the terminology of Uhlhorn, for any vector $x \in X$, the set \underline{x} of all nonzero scalar multiples of x is called the ray generated by x . The set of all rays in X is denoted by \underline{X} . The dual space of X (that is the set of all bounded linear functionals on X) is denoted by X' . For any $x \in X, f \in X'$ we use the common and convenient notation $\langle x, f \rangle$ for $f(x)$. We say that the rays $\underline{x} \in \underline{X}$ and $\underline{f} \in \underline{X}'$ are orthogonal to each other, in notation $\underline{x} \cdot \underline{f} = 0$, if we have $\langle y, g \rangle = 0$ for all $y \in \underline{x}$ and $g \in \underline{f}$. The Banach space adjoint of an operator $A \in B(X)$ is denoted by A' . We extend the concept of adjoints also for conjugate-linear operators. If A is a bounded conjugate-linear operator on the complex Banach space X , then its adjoint $A' : X' \rightarrow X'$ (which is also a bounded conjugate-linear operator) is defined by $A'f = \overline{f \circ A}$ ($f \in X'$). If X is a linear space over \mathbb{K} (\mathbb{K} denotes the real or complex field) and h is a ring automorphism of \mathbb{K} , then the function $A : X \rightarrow X$ is called h -semilinear if it is additive and $A(\lambda x) = h(\lambda)Ax$ holds for every $x \in X$ and $\lambda \in \mathbb{K}$. If X is a finite-dimensional complex linear space and h is a ring automorphism of \mathbb{C} , then for any h -semilinear operator A , the adjoint A' of A is defined by $A'f = h^{-1} \circ f \circ A$ ($f \in X'$). Clearly, $A' : X' \rightarrow X'$ is an h^{-1} -semilinear operator.

After this preparation we can formulate our first corollary as follows.

COROLLARY 1. *Let X be a (real or complex) Banach space of dimension not less than 3. Let $T : \underline{X} \rightarrow \underline{X}$ and $S : \underline{X}' \rightarrow \underline{X}'$ be bijective transformations with the property that*

$$T\underline{x} \cdot S\underline{f} = 0 \quad \text{if and only if} \quad \underline{x} \cdot \underline{f} = 0$$

for every $\underline{x} \in \underline{X}$ and $\underline{f} \in \underline{X}'$.

If X is real, then there exists an invertible bounded linear operator $A : X \rightarrow X$ such that T, S are of the forms

$$T\underline{x} = \underline{Ax} \quad \text{and} \quad S\underline{f} = \underline{A^{-1}f} \quad (0 \neq x \in X, 0 \neq f \in X'). \tag{3}$$

If X is complex and infinite dimensional, then there exists an invertible bounded linear or conjugate-linear operator $A : X \rightarrow X$ such that T, S are of forms (3).

If X is complex and finite dimensional, then there exist a ring automorphism h of \mathbb{C} and an invertible h -semilinear operator $A : X \rightarrow X$ such that T, S are of forms (3).

The operator A above is unique up to multiplication by a scalar.

Finally, as a consequence of Corollary 1, we shall present our Uhlhorn-type version of Wigner's theorem for indefinite inner product spaces that was promised in the abstract. As mentioned above, our result is a far-reaching generalization of the main result in [3], where a similar assertion in the particular case when H is finite dimensional and the generating invertible operator η is self-adjoint was presented.

Let η be an invertible bounded linear operator on a Hilbert space H . Denote by $(x, y)_\eta$ the quantity $\langle \eta x, y \rangle$ ($x, y \in H$). We write $\underline{x} \cdot_\eta \underline{y} = 0$ if $\langle \eta x_0, y_0 \rangle = 0$ holds for every $x_0 \in \underline{x}$ and $y_0 \in \underline{y}$. The ray transformation $T : \underline{H} \rightarrow \underline{H}$ is called a symmetry transformation on the indefinite inner product space \underline{H} generated by η if

$$T\underline{x} \cdot_\eta T\underline{y} = 0 \quad \Leftrightarrow \quad \underline{x} \cdot_\eta \underline{y} = 0$$

for all $\underline{x}, \underline{y} \in \underline{H}$. We say that the transformation $T : \underline{H} \rightarrow \underline{H}$ is induced by the invertible linear or conjugate-linear operator $U : H \rightarrow H$ if $T\underline{x} = \underline{Ux}$ for every $0 \neq x \in H$.

COROLLARY 2. *Let H be a (real or complex) Hilbert space of dimension not less than 3 and let $\eta \in \mathcal{B}(H)$ be invertible. Suppose that $T : \underline{H} \rightarrow \underline{H}$ is a bijective transformation with the property that*

$$T\underline{x} \cdot_\eta T\underline{y} = 0 \quad \text{if and only if} \quad \underline{x} \cdot_\eta \underline{y} = 0$$

holds for every $\underline{x}, \underline{y} \in \underline{H}$.

If H is real, then T is induced by an invertible bounded linear operator U on H . Similarly, if H is complex, then T is induced by an invertible bounded linear or conjugate-linear operator U on H .

The operator U inducing T is unique up to multiplication by a scalar.

If H is real, then the invertible bounded linear operator $U : H \rightarrow H$ induces a symmetry transformation on \underline{H} if and only if

$$(Ux, Uy)_\eta = c(x, y)_\eta \quad (x, y \in H)$$

holds for some constant $c \in \mathbb{R}$.

If H is complex, then the invertible bounded linear operator $U : H \rightarrow H$ induces a symmetry transformation on \underline{H} if and only if

$$(Ux, Uy)_\eta = c(x, y)_\eta \quad (x, y \in H)$$

holds for some constant $c \in \mathbb{C}$. Similarly, the invertible bounded conjugate-linear operator $U : H \rightarrow H$ induces a symmetry transformation on \underline{H} if and only if

$$(Ux, Uy)_\eta = d(y, x)_{\eta^*} \quad (x, y \in H)$$

holds for some constant $d \in \mathbb{C}$. Here, η^* denotes the Hilbert space adjoint of η .

Remark. Observe that in contrast with the Main Theorem and Corollary 1, in Corollary 2 above general semilinear operators do not appear.

In Uhlhorn’s paper [11] it was mentioned that, for physical reasons, one should consider ray transformations between different spaces. It will be clear from the proofs below that one can generalize our result in that direction easily.

We should point out that, as will be clear from their proofs, in Corollaries 1 and 2 there is in fact no need to assume the injectivity of the transformations T, S . We have posed this condition only for the sake of “symmetry”.

Finally, we note that we are convinced that our result could somehow be extended for the case of quaternionic Hilbert spaces, which have also been proved to be important in the applications of mathematics in certain physical problems. The first step in this direction could be an extension of Ovchinnikov’s result for that case. However, we leave the whole (we believe challenging) problem open.

2. PROOFS

In the proofs we need some additional notation and definitions.

Let X be a (real or complex) Banach space. The ideal of all finite rank operators in $B(X)$ is denoted by $F(X)$. Two idempotents P, Q in $B(X)$ are said to be (algebraically) orthogonal if $PQ = QP = 0$. There is a natural partial order on $I(X)$. Namely, for any $P, Q \in I(X)$ we write $P \leq Q$ if $PQ = QP = P$. Clearly, $P \leq Q$ holds if and only if the range $\text{rng } P$ of P is a subset of the range of Q and the kernel $\ker P$ of P contains the kernel of Q . The symbol $I_f(X)$ stands for the collection of all finite rank idempotents in $B(X)$. The natural embedding of X into its second dual X'' is denoted by κ . If $x \in X$ and $f \in X'$, then $x \otimes f$ stands for the operator (of rank at most 1)

defined by

$$(x \otimes f)(z) = \langle z, f \rangle x \quad (z \in X).$$

Clearly, $x \otimes f$ is a rank-one idempotent if and only if $\langle x, f \rangle = 1$. It is easy to see that the elements of $F(X)$ are exactly the operators $A \in B(X)$ which can be written as finite sums of the form

$$A = \sum_i x_i \otimes f_i \tag{4}$$

with $x_1, \dots, x_n \in X$ and $f_1, \dots, f_n \in X'$. Using this representation, the trace of A is defined by

$$\text{tr } A = \sum_i \langle x_i, f_i \rangle.$$

It is known that $\text{tr } A$ is well defined, that is, it does not depend on the particular representation (4) of A . Denote by $M_n(\mathbb{K})$ the algebra of all $n \times n$ matrices with entries in \mathbb{K} .

In the proof of our main result we shall need the following lemma.

LEMMA. *For any $P_1, P_2 \in I_f(X)$ there exists a $P \in I_f(X)$ such that $P_1, P_2 \leq P$.*

Proof. The assertion will follow from the following observation. Let $M, N \subset X$ be closed subspaces. Suppose that M is of finite codimension and N is of finite dimension. Then there exists an idempotent $P \in I_f(X)$ such that $\ker P \subset M$ and $\text{rng } P \supset N$. Indeed, since every finite-dimensional subspace of a Banach space is complemented, we can find a closed subspace K in X such that $K \oplus (M \cap N) = M$. Since the sum of a closed and a finite dimensional subspace is closed, it follows that $M + N$ is closed and has finite codimension. So, there is a finite-dimensional subspace L in X such that $(M + N) \oplus L = X$. We clearly have

$$K \oplus (N \oplus L) = X.$$

Now, there exists an idempotent $P \in I_f(X)$ such that $\ker P = K$ and $\text{rng } P = N \oplus L$. This verifies our observation.

If $P_1, P_2 \in I_f(X)$, then $\ker P_1 \cap \ker P_2$ is of finite corank and $\text{rng } P_1 + \text{rng } P_2$ is of finite rank. Now, the idempotent $P \in I_f(X)$ obtained according to the observation above clearly has the property that $P_1, P_2 \leq P$. This completes the proof.

Proof of the Main Theorem. We first extend ϕ to the set $I_f(X)$ of all finite rank idempotents in $B(X)$. If $0 \neq P \in I_f(X)$, then there are mutually (algebraically) orthogonal rank-one idempotents $P_1, \dots, P_n \in B(X)$ such that $P = \sum_i P_i$. Clearly, $\phi(P_1), \dots, \phi(P_n)$ are also mutually orthogonal rank-one idempotents. Let us define

$$\tilde{\phi}(P) = \sum_i \phi(P_i).$$

We have to show that $\tilde{\phi}$ is well defined. In order to do this, let $Q_1, \dots, Q_n \in B(X)$ be mutually orthogonal rank-one idempotents with sum P . Pick any $R \in I_1(X)$. We have

$$\begin{aligned} \left(\sum_i \phi(P_i)\right)\phi(R) = 0 &\Leftrightarrow \phi(P_i)\phi(R) = 0 \quad (i = 1, \dots, n) \Leftrightarrow \\ P_i R = 0 \quad (i = 1, \dots, n) &\Leftrightarrow \left(\sum_i P_i\right)R = 0. \end{aligned}$$

Similarly, we obtain

$$\left(\sum_i \phi(Q_i)\right)\phi(R) = 0 \Leftrightarrow \left(\sum_i Q_i\right)R = 0.$$

Since $\sum_i P_i = \sum_i Q_i$, these imply that

$$\left(\sum_i \phi(P_i)\right)\phi(R) = 0 \Leftrightarrow \left(\sum_i \phi(Q_i)\right)\phi(R) = 0.$$

As $\phi(R)$ runs through the set $I_1(X)$, we deduce that the kernels of the idempotents $\sum_i \phi(P_i)$ and $\sum_i \phi(Q_i)$ are the same. A similar argument shows that the ranges of these two idempotents are also equal. Therefore, we have

$$\sum_i \phi(P_i) = \sum_i \phi(Q_i).$$

This shows that the transformation $\tilde{\phi}$ is well defined. It is now easy to verify that $\tilde{\phi}: I_f(X) \rightarrow I_f(X)$ is a bijection which preserves the order, the orthogonality and the rank in both directions. In fact, only the injectivity is not trivial but it follows from an argument quite similar to the one proving $\tilde{\phi}$ is well defined.

Pick a finite rank idempotent $P_0 \in B(X)$ whose rank is at least 3. Consider the set $I_{P_0}(X)$ of all idempotents $P \in B(X)$ for which $P \leq P_0$. Let $M = \ker P_0$

and $N = \text{rng } P_0$. We have $M \oplus N = X$. Denote by $B(X, M, N)$ the set of all operators A in $B(X)$ for which $A(N) \subset N$ and $A(M) = \{0\}$. Clearly, we have $I_{P_0}(X) \subset B(X, M, N)$. Considering the transformation $A \mapsto A|_N$ we get an algebra isomorphism from $B(X, M, N)$ onto $B(N)$. Moreover, $B(N)$ is obviously isomorphic to $M_n(\mathbb{K})$. Denote the so-obtained algebra isomorphism from $B(X, M, N)$ onto $M_n(\mathbb{K})$ by ψ . Similarly, we have an algebra isomorphism ψ' from $B(X, \ker \phi(P_0), \text{rng } \phi(P_0))$ onto $M_n(\mathbb{K})$. Therefore, the transformation $P \mapsto \Psi(P) = \psi'(\tilde{\phi}(\psi^{-1}(P)))$ is a bijection of the set of all idempotents in $M_n(\mathbb{K})$ which preserves the order \leq in both directions. The form of all such transformations is described on p. 186 in [10]. In particular, it follows from that form that there is a ring-automorphism h_{P_0} of \mathbb{K} such that

$$\text{tr } \Psi(P)\Psi(Q) = h_{P_0}(\text{tr } PQ)$$

holds for all idempotents P, Q in $M_n(\mathbb{K})$. Since ψ, ψ' are algebra isomorphisms, it follows that they preserve rank-one idempotents. This implies that ψ, ψ' preserve the traces of rank-one operators, from which we conclude that they are generally trace preserving. It follows that

$$\text{tr } \tilde{\phi}(P)\tilde{\phi}(Q) = h_{P_0}(\text{tr } PQ) \quad (P, Q \in I_{P_0}(X)). \quad (5)$$

We claim that in fact h_{P_0} does not depend on P_0 . Indeed, let $P_1 \in I_f(X)$ be such that $P_0 \leq P_1$. Considering the corresponding ring automorphism h_{P_1} of \mathbb{K} , by (5) we get that

$$h_{P_0}(\text{tr } PQ) = h_{P_1}(\text{tr } PQ)$$

holds for every $P, Q \in I_{P_0}(X)$. Clearly, $\text{tr } PQ$ runs through \mathbb{K} as P, Q run through $I_{P_0}(X)$. This shows that $h_{P_0} = h_{P_1}$. Since for any two finite rank idempotents there is a finite rank idempotent majorizing both of them (this is just the content of our lemma), we have the independence of h_{P_0} from P_0 . Therefore, there exists a ring automorphism h of \mathbb{K} such that

$$\text{tr } \tilde{\phi}(P)\tilde{\phi}(Q) = h(\text{tr } PQ) \quad (P, Q \in I_f(X)). \quad (6)$$

We now extend $\tilde{\phi}$ from $I_f(X)$ onto $F(X)$. For any $P_1, \dots, P_n \in I_f(X)$ and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ we define

$$\Phi \left(\sum_i \lambda_i P_i \right) = \sum_i h(\lambda_i) \tilde{\phi}(P_i).$$

We have to show that Φ is well defined. Let $Q_1, \dots, Q_m \in I_f(X)$ and $\mu_1, \dots, \mu_m \in \mathbb{K}$ be such that

$$\sum_i \lambda_i P_i = \sum_j \mu_j Q_j.$$

It follows that

$$\sum_i \lambda_i P_i R = \sum_j \mu_j Q_j R$$

holds for every $R \in I_f(X)$. Taking traces we obtain

$$\sum_i \lambda_i \operatorname{tr} P_i R = \sum_j \mu_j \operatorname{tr} Q_j R.$$

By (6) it follows that

$$\sum_i \lambda_i h^{-1}(\operatorname{tr} \tilde{\phi}(P_i) \tilde{\phi}(R)) = \sum_j \mu_j h^{-1}(\operatorname{tr} \tilde{\phi}(Q_j) \tilde{\phi}(R)).$$

This implies that

$$h^{-1} \left(\sum_i h(\lambda_i) \operatorname{tr} \tilde{\phi}(P_i) \tilde{\phi}(R) \right) = h^{-1} \left(\sum_j h(\mu_j) \operatorname{tr} \tilde{\phi}(Q_j) \tilde{\phi}(R) \right),$$

that is,

$$h^{-1} \left(\operatorname{tr} \left(\sum_i h(\lambda_i) \tilde{\phi}(P_i) \tilde{\phi}(R) \right) \right) = h^{-1} \left(\operatorname{tr} \left(\sum_j h(\mu_j) \tilde{\phi}(Q_j) \tilde{\phi}(R) \right) \right).$$

This gives

$$\operatorname{tr} \left(\sum_i h(\lambda_i) \tilde{\phi}(P_i) \right) \tilde{\phi}(R) = \operatorname{tr} \left(\sum_j h(\mu_j) \tilde{\phi}(Q_j) \right) \tilde{\phi}(R).$$

Since $\tilde{\phi}(R)$ runs through the set $I_f(X)$, we obtain

$$\sum_i h(\lambda_i) \tilde{\phi}(P_i) = \sum_j h(\mu_j) \tilde{\phi}(Q_j).$$

Therefore, Φ is well defined. Since the finite rank idempotents linearly generate $F(X)$, it follows that Φ is a surjective h -semilinear transformation on $F(X)$ which preserves the rank-one idempotents and their linear spans. We can now apply a result of Omladič and Šemrl describing the form of all such transformations. In fact, if, for example, X is real, then by [9, Main Result] either there exists an invertible bounded linear operator $A : X \rightarrow X$ such that

$$\phi(P) = APA^{-1} \quad (P \in I_1(X)) \quad (7)$$

or there exists an invertible bounded linear operator $B: X' \rightarrow X$ such that

$$\phi(P) = BP'B^{-1} \quad (P \in I_1(X)).$$

If we had this second possibility, then we would get that

$$\phi(P)\phi(Q) = 0 \Leftrightarrow BP'Q'B^{-1} = 0 \Leftrightarrow P'Q' = 0 \Leftrightarrow QP = 0$$

for every $P, Q \in I_1(X)$. On the other hand, we know that

$$\phi(P)\phi(Q) = 0 \Leftrightarrow PQ = 0.$$

So, we would have

$$PQ = 0 \Leftrightarrow QP = 0$$

for every $P, Q \in I_1(X)$, which is an obvious contradiction. Therefore, ϕ is of form (7).

If X is complex, then one can argue in a very similar way referring to [9, Main Result] again (in the infinite dimensional case) or to [9, Theorem 4.5] (in the finite-dimensional case). The proof is complete. ■

Proof of Corollary 1. We define a bijective transformation $\phi: I_1(X) \rightarrow I_1(X)$ which preserves zero products in both directions.

First, for every $0 \neq x \in X$ pick a vector from the ray $T\underline{x}$. In that way we get a transformation, which will be denoted by the same symbol T , from $X \setminus \{0\}$ into itself with the property that for every vector $0 \neq y \in X$, there exists a vector $0 \neq x \in X$ such that $y = \lambda Tx$ for some nonzero scalar $\lambda \in \mathbb{K}$. We do the same with the other transformation S . Clearly, we have

$$\langle Tx, Sf \rangle = 0 \quad \text{if and only if} \quad \langle x, f \rangle = 0 \quad (8)$$

for every nonzero $x \in X$ and nonzero $f \in X'$.

Let $x \in X$ and $f \in X'$ be such that $\langle x, f \rangle \neq 0$. Define

$$\phi\left(\frac{1}{\langle x, f \rangle} x \otimes f\right) = \frac{1}{\langle Tx, Sf \rangle} Tx \otimes Sf.$$

We show that ϕ is well defined. Let $x_0 \in X$ and $f_0 \in X'$ be such that $\langle x_0, f_0 \rangle \neq 0$ and suppose that

$$\frac{1}{\langle x, f \rangle} x \otimes f = \frac{1}{\langle x_0, f_0 \rangle} x_0 \otimes f_0.$$

This implies that x, x_0 belong to the same ray in X and the same holds true for f, f_0 in X' . Consequently, Tx, Tx_0 and Sf, Sf_0 generate equal rays in X and X' , respectively. Therefore, the ranges and the kernels of the idempotents $\frac{1}{\langle Tx, Sf \rangle} Tx \otimes Sf$ and $\frac{1}{\langle Tx_0, Sf_0 \rangle} Tx_0 \otimes Sf_0$ are equal, which implies the equality of these two idempotents. Hence, we obtain that ϕ is well defined.

By the ‘‘almost surjectivity’’ property of the vector–vector transformations T, S we obtain the surjectivity of ϕ . The injectivity of ϕ can be proved by an argument like the one we used to prove ϕ is well defined. The transformation ϕ preserves zero products in both directions, which is a consequence of (8).

Now, we can apply our main theorem. Suppose first that X is real. Then our transformation ϕ is of form (1) with some invertible bounded linear operator A on X . If $x \in X$ and $f \in X'$ are such that $\langle x, f \rangle \neq 0$, then from the equality

$$\begin{aligned} \frac{1}{\langle Tx, Sf \rangle} Tx \otimes Sf &= \phi \left(\frac{1}{\langle x, f \rangle} x \otimes f \right) \\ &= A \frac{1}{\langle x, f \rangle} x \otimes f A^{-1} = \left(\frac{1}{\langle x, f \rangle} Ax \right) \otimes (A^{-1}f) \end{aligned} \quad (9)$$

we deduce that Tx is a scalar multiple of Ax and Sf is a scalar multiple of $A^{-1}f$. This gives us that $T\underline{x} = \underline{Ax}$ and $S\underline{f} = \underline{A^{-1}f}$.

If X is complex infinite dimensional, then one can argue in a very similar way.

Finally, let X be complex and finite dimensional. In that case there exist a ring automorphism h of \mathbb{C} and an invertible h -semilinear operator $A : X \rightarrow X$ such that ϕ is of the form

$$\phi(P) = APA^{-1} \quad (P \in I_1(X)).$$

This comes from a rewriting of form (2) appearing in the formulation of our main theorem. Now, one can easily verify that we have the following

equality very similar to (9):

$$\frac{1}{\langle Tx, Sf \rangle} Tx \otimes Sf = \left(\frac{1}{h(\langle x, f \rangle)} Ax \right) \otimes (A^{-1}f).$$

This yields $T\underline{x} = \underline{Ax}$ and $S\underline{f} = \overline{A^{-1}f}$ ($x \in \underline{X}, f \in \underline{X}'$).

The assertion concerning essential uniqueness is a consequence of the following easy fact whose proof requires only elementary linear algebra. If A, B are semilinear operators on a vector space Y over \mathbb{K} with ranks at least 2 such that Ay, By are linearly dependent for every $y \in Y$, then A, B are linearly dependent. This completes the proof of Corollary 1. ■

Proof of Corollary 2. Just as in the proof of Corollary 1, we can define an “almost surjective” transformation (that is, one that has values in every ray) on the underlying Hilbert space H , denoted by the same symbol T , such that

$$\langle \eta Tx, Ty \rangle = 0 \quad \text{if and only if} \quad \langle \eta x, y \rangle = 0 \quad (x, y \in H \setminus \{0\}).$$

We can rewrite this equivalence first as

$$\langle \eta T \eta^{-1} x, Ty \rangle = 0 \quad \text{if and only if} \quad \langle x, y \rangle = 0 \quad (x, y \in H \setminus \{0\})$$

and next as

$$\langle Tx, \eta T \eta^{-1} y \rangle = 0 \quad \text{if and only if} \quad \langle x, y \rangle = 0 \quad (x, y \in H \setminus \{0\}).$$

Now, we apply Corollary 1. To be honest, we should point out that although that result is formulated for Banach spaces and hence dual spaces and Banach space adjoints of operators appear there, the very same argument can be applied to conclude that our present transformation T is generated by some invertible operator U on H . We learn from Corollary 1 that U is linear if H is real, it is either linear or conjugate-linear if H is complex infinite dimensional and, finally, U is semilinear if H is complex finite-dimensional. From the proof of the remaining part of our corollary it will be clear that this general semilinear case in fact does not occur.

The essential uniqueness of U can be verified as in the proof of Corollary 1. As for the third part of the statement, we present the proof only in the complex finite-dimensional case. In all other cases one can argue in a quite similar way. So, let h be a ring automorphism of \mathbb{C} . Suppose that the invertible h -semilinear operator $U: H \rightarrow H$ induces a symmetry transformation. Then, we have

$$\langle \eta Ux, Uy \rangle = 0 \Leftrightarrow \langle \eta x, y \rangle = 0$$

for every $x, y \in H$. This implies that

$$h^{-1}(\langle \eta Ux, Uy \rangle) = 0 \iff \langle \eta x, y \rangle = 0 \quad (x, y \in H).$$

If we fix $y \in H$, then the functions $x \mapsto h^{-1}(\langle \eta Ux, Uy \rangle)$ and $x \mapsto \langle \eta x, y \rangle$ are linear functionals with the same kernel. We deduce that these functionals differ only by a scalar multiple. Hence, there exists a $c(y) \in \mathbb{C}$ such that

$$h^{-1}(\langle \eta Ux, Uy \rangle) = c(y)\langle \eta x, y \rangle \tag{10}$$

for every $x, y \in H$. Similarly, for every $x \in H$ there exists a scalar $d(x) \in \mathbb{C}$ such that

$$h^{-1}(\langle Uy, \eta Ux \rangle) = d(x)\langle y, \eta x \rangle \quad (x, y \in H).$$

Defining $g : \mathbb{C} \rightarrow \mathbb{C}$ by $g(\lambda) = \overline{h(\lambda)}$ ($\lambda \in \mathbb{C}$), we can write this last equality as

$$g^{-1}(\langle \eta Ux, Uy \rangle) = \overline{d(x)}\langle \eta x, y \rangle \quad (x, y \in H). \tag{11}$$

It follows from (10) and (11) that

$$\langle \eta Ux, Uy \rangle = C(y)h(\langle \eta x, y \rangle) \quad \text{and} \quad \langle \eta Ux, Uy \rangle = D(x)g(\langle \eta x, y \rangle)$$

for every $x, y \in H$, where C, D are complex-valued functions on H . We then have

$$C(y)h(\langle \eta x, y \rangle) = D(x)g(\langle \eta x, y \rangle)$$

for every $x, y \in H$. It is easy to see that C, D are in fact constant functions. Indeed, pick any $y_1, y_2 \in H$ which are linearly independent. Then we have $x, z \in H$ such that $y_1 = \eta x$, $z \perp \eta x$ and $y_2 = \eta x + z$. Since $\langle \eta x, y_1 \rangle = \langle \eta x, y_2 \rangle$, it follows from the equality above that $C(y_1) = C(y_2)$. In case $y_1, y_2 \in H \setminus \{0\}$ are linearly dependent, we can choose $y_3 \in H$ such that y_1, y_3 and y_2, y_3 are both linearly independent and we get $C(y_1) = C(y_2)$. Since $C(0)$ does not count, we obtain that C is really constant. A similar argument applies to D . It follows that we have constants $C, D \in \mathbb{C}$ such that

$$\langle \eta Ux, Uy \rangle = Ch(\langle \eta x, y \rangle)$$

and

$$\langle \eta Ux, Uy \rangle = D\overline{h(\langle \eta x, y \rangle)}.$$

Since these hold for every $x, y \in H$ and we have $h(1) = 1$, it follows that $C = D$. This implies that h is self-adjoint in the sense that $h(\lambda) = \overline{h(\lambda)}$ ($\lambda \in \mathbb{C}$). It is well known that the only ring automorphisms of \mathbb{C} with this

property are the identity and the conjugation. In fact, this is an easy consequence of the fact that the only ring automorphism of \mathbb{R} is the identity. It now follows that either U is linear and we have

$$(Ux, Uy)_\eta = C(x, y)_\eta \quad (x, y \in H) \quad (12)$$

or U is conjugate-linear and we have

$$(Ux, Uy)_\eta = C(y, x)_{\eta^*} \quad (x, y \in H). \quad (13)$$

It is obvious that if $U : H \rightarrow H$ is either an invertible linear operator on H such that (12) holds or an invertible conjugate-linear operator such that (13) holds, then U induces a symmetry transformation.

The remaining part of the proof can be carried out in a similar, but simpler, way. ■

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