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Well-posedness and weak rotation limit for the Ostrovsky equation

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A B S T R A C T

We consider the Cauchy problem of the Ostrovsky equation. We first prove the time local well-posedness in the anisotropic Sobolev space $H^{s,a}$ with $s > -a/2 - 3/4$ and $0 \leqslant a \leqslant -1$ by the Fourier restriction norm method. This result include the time local well-posedness in H^s with $s > -3/4$ for both positive and negative dissipation, namely for both $\beta\gamma > 0$ and $\beta\gamma < 0$. We next consider the weak rotation limit. We prove that the solution of the Ostrovsky equation converges to the solution of the KdV equation when the rotation parameter γ goes to 0 and the initial data of the KdV equation is in L^2. To show this result, we prove a bilinear estimate which is uniform with respect to γ.

1. Introduction

We consider the initial value problem of the Ostrovsky equation as follows:

$$\begin{cases} \partial_t u + c\partial_x u + \alpha u \partial_x u - \beta \partial_x^3 u = \gamma \partial_x^{-1} u, & (x,t) \in \mathbf{R} \times [0,\infty), \\ u(x,0) = \varphi(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $u(x,t)$ is a real valued function, c, α, β and γ are real valued constant parameter. The Ostrovsky equation has some physical models (see, e.g. [1,5,8,9,29]). For example, it describes the gravity

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waves propagating down a channel under the influence of Coriolis force. The parameter γ measures the effect of the Earth's rotation and is very small in real situations. When $\gamma = 0$, (1.1) is the Korteweg-de Vries equation, which is completely integrable. However, when $\gamma \neq 0$, (1.1) is known to be not integrable [9,30]. Our aim is to study the effect of the rotating term, namely, to understand differences and similarities between $\gamma = 0$ and $\gamma \neq 0$. Existence and stability of solitary waves and existence of traveling wave were studied in [16,21,22,24–27]. In the present paper, we consider more fundamental problems. One is the well-posedness, which means the existence of solutions and the uniqueness and the continuous dependence on initial data, the other is the convergence of the solutions when $\gamma \to 0$.

Before we proceed to our problems, for simpleness, we normalize parameters by the change of variables and the scaling as follows:

$$c = 0, \qquad \alpha = 1, \qquad \beta = +1 \text{ or } -1, \quad \gamma \in \mathbf{R}.$$

We first recall the known results for the KdV equation ($\gamma = 0$). In [3], Bourgain proved the time local well-posedness in L^2 by introducing the Fourier restriction norm method. In [19], Kenig, Ponce and Vega proved refined bilinear estimates to extend Bourgain's result to H^s with $s > -3/4$. Earlier results can be found in [2,10,17,18]. The lifetime of the solutions by the time local well-posedness results above, depends only on the size of the H^s norm of the initial data. Therefore, by combining the L^2 conservation law and time local results, we have the time global well-posedness in H^s with $s \geqslant 0$. Since the KdV equation on H^s for $s < 0$ has no conservation law, it seemed difficult to consider the long time behavior of solutions in H^s with $s < 0$. However, in [7], Colliander, Keel, Staffilani, Takaoka and Tao overcame this difficulty and proved the time global well-posedness with $s > -3/4$ by introducing a regularizing Fourier multiplier operator I and calculating a modified energy defined in H^s, which is called the "I-method". The value $s = -3/4$ seems to be critical. Nakanishi, Takaoka and Tsutsumi proved that the fundamental bilinear estimate used to prove the time local well-posedness fails when $s \leqslant -3/4$ in [28]. Christ, Colliander and Tao proved the time local well-posedness with $s \geqslant -3/4$ by studying the modified KdV equation and the Miura transform in [6]. They also proved the time local ill-posedness with $-1 \leqslant s < -3/4$ in the sense that the solution operator fails to be uniformly continuous with respect to the H^s norm (see also [4,20,31]).

We next recall the known results for well-posedness of the Ostrovsky equation ($\gamma \neq 0$). Assume that φ is in $H^s \cap \dot{H}^{-1}$. Varlamov and Liu proved the time local well-posedness for $s > 3/2$ in [32]. Linares and Milanés extended this result to $s > 3/4$ in [23]. Huo and Jia extended this result to $s \geqslant -1/8$ by the Fourier restriction norm method in [13]. In [14], Isaza and Mejia considered the case $\varphi \in H^s$ without $\dot{H}^{-1}$ assumption and proved the time local well-posedness for $s > -3/4$ with $\beta\gamma < 0$ and for $s > -1/2$ with $\beta\gamma > 0$. In [12], Gui and Liu also considered the case $\varphi \in H^s$ without $\dot{H}^{-1}$ assumption and proved the time local well-posedness for $s > -7/12$ with $\beta\gamma > 0$. In [33], Wang and Cui considered the case $\varphi \in H^{s,a}_\gamma$, which is defined below. They proved the time local well-posedness for $s > -5/8$ and $0 \geqslant a > -1/2$ with $\beta\gamma < 0$. There is a gap between these lower bounds of s and the critical value of the KdV equation $-3/4$ when $\beta\gamma > 0$. In these papers, bilinear estimates plays an important role. We have refined the bilinear estimate (see, Proposition 3.2) to obtain the following theorem, which includes all results mentioned above.

Theorem 1.1. *Let $\gamma \neq 0$, $\beta = +1$ or -1 and $\varphi \in H^{s,a}_\gamma$. If $s > -a/2 - 3/4$ and $0 \geqslant a \geqslant -1$, then* (1.1) *is time locally well-posed. Moreover, we assume $|\gamma| < \Gamma$ and $-1/2 > a \geqslant -1$. Then, the lifetime* (*i.e. the size of the existence time of the solution*) *depends only on s, a, Γ and $\|\varphi\|_{H^{s,a}_\gamma}$.*

The definition of $H^{s,a}_\gamma$ for $s \in \mathbf{R}$ and $a \leqslant 0$ is as follows:

$$H^{s,a}_\gamma = \left\{ u \in \mathcal{S}' \;\middle|\; \|u\|_{H^{s,a}_\gamma} < +\infty \right\},$$

$$\|u\|_{H^{s,a}_\gamma} = \left\| \langle \xi \rangle^s \left\langle \gamma \xi^{-1} \right\rangle^{-a} \hat{u}(\xi) \right\|,$$

where $\langle\cdot\rangle = 1 + |\cdot|$ and $\hat{u} = \mathcal{F}_x u$ denote the Fourier transform with respect to x variable. Obviously, we have

$$H^{s,a}_\gamma = H^s \cap \dot{H}^a \quad \text{when } s \geqslant a,$$
$$H^{s,a}_\gamma \supset H^s \cap \dot{H}^a \quad \text{when } s < a,$$

and

$$H^{s,a}_\gamma = H^s \quad \text{when } a = 0.$$

Remark 1.1. When $a = -1$, the assumption on s in Theorem 1.1 is $s > -1/4$. When $a = 0$, the lower bound of s of the assumption in Theorem 1.1 is $-3/4$, which is equal to the critical value of the KdV equation. Since we cannot apply the Miura transform for the case $\gamma \neq 0$, it seems difficult to prove the time local well-posedness with $s = -3/4$.

The Ostrovsky equation has the L^2 conservation law and the following a priori estimates.

Proposition 1.2. *Let u be a solution of* (1.1)*. Then, for $-1 \leqslant a \leqslant 0$, we have*

$$\|u(\cdot,t)\|^2_{L^2} = \|\varphi\|^2_{L^2}, \tag{1.2}$$

$$\sup_{0\leqslant t\leqslant T} \|u(\cdot,t)\|^2_{\dot{H}^{-1}} \leqslant C\big(\|\varphi\|^2_{\dot{H}^{-1}} + T^{4/3}|\gamma|^{4/3}\|\varphi\|^{10/3}_{L^2}\big), \tag{1.3}$$

$$\sup_{0\leqslant t\leqslant T} \|u(\cdot,t)\|^2_{H^{0,a}_\gamma} \leqslant C\big(\|\varphi\|^2_{H^{0,a}_\gamma} + T^{4/3}|\gamma|^2\|\varphi\|^{10/3}_{L^2}\big). \tag{1.4}$$

By (1.4), we can extend the time local solutions in Theorem 1.1 to time global ones.

Theorem 1.3. *Let $\gamma \neq 0$, $\beta = +1$ or -1 and $\varphi \in H^{s,a}_\gamma$. If $s \geqslant 0$ and $-1 \leqslant a \leqslant 0$, then* (1.1) *is time globally well-posed.*

In [15], Isaza and Mejia proved a priori estimate by I-method and obtained the time global well-posedness of (1.1) in H^s with $s > -3/10$ for both $\beta\gamma > 0$ and $\beta\gamma < 0$. However, the time global well-posedness in $H^{s,a}_\gamma$ with $s < 0$ and $a < 0$ is still open.

We next consider the convergence of the solutions when $\gamma \to 0$. Let u_n and v be the solutions of the following equations:

$$\begin{cases} \partial_t u_n - \beta\partial_x^3 u_n + u_n\partial_x u_n = \gamma_n \partial_x^{-1} u_n, & (x,t) \in \mathbf{R}\times[0,\infty), \\ u_n(x,0) = \varphi_n(x), & x \in \mathbf{R}, \end{cases} \tag{1.5}$$

$$\begin{cases} \partial_t v - \beta\partial_x^3 v + v\partial_x v = 0, & (x,t) \in \mathbf{R}\times[0,\infty), \\ v(x,0) = \psi(x), & x \in \mathbf{R}. \end{cases} \tag{1.6}$$

In [25], Liu and Varlamov proved the following proposition.

Proposition 1.4. *Let $s > 3/2$, $\psi = \varphi_n \in H^s \cap \dot{H}^{-1}$ and $T > 0$. Then,*

$$\sup_{0\leqslant t\leqslant T} \|v(t) - u_n(t)\|_{L^2} \to 0,$$

when $\gamma_n \to 0$.

In this proposition, there is a gap of regularity between the convergence of the solutions and the assumption on the initial data. Our aim is to obtain a result similar to this proposition under more natural assumption, namely, the initial data are in L^2. The main tool of the proof of Proposition 1.4 is the L^2 inequality for $w_n := v - u_n$ as follows:

$$\|w_n(T)\|_{L^2}^2 - \|w_n(0)\|_{L^2}^2 \leqslant C\left(\left|\int_0^T\int w_n^2 \partial_x v\,dx\,dt\right| + \left|\gamma_n \int_0^T\int w_n \partial_x^{-1} u_n\,dx\,dt\right|\right).$$

$|\int w_n^2 \partial_x v\,dx|$ is bounded by $\|w_n\|_{L^2}^2 \|v\|_{H^{3/2+\epsilon}}$ by the Sobolev embedding. This is the reason why we need the assumption $s > 3/2$ in Proposition 1.4. To overcome this difficulty, we combine a time local bilinear estimate and a priori estimates and we obtain the following theorem.

Theorem 1.5. *Let $\beta = +1$ or -1, $\psi \in L^2$, $\varphi_n \in L^2 \cap \dot{H}^{-1}$, $\Gamma > 0$, $M > 0$ and $T > 0$. Assume that $|\gamma_n| \leqslant \Gamma$, $\|\psi\|_{L^2} < M$ and $\|\varphi_n\|_{H^{0,-1}_{\gamma_n}} < M$. Then, we have*

$$\sup_{0\leqslant t\leqslant T} \|v(t) - u_n(t)\|_{L^2} \leqslant C_0\left(\|\psi - \varphi_n\|_{L^2} + |\gamma_n|\|\varphi_n\|_{\dot{H}^{-1}} + |\gamma_n|^{5/3}\|\varphi_n\|_{L^2}^{5/3}\right) \tag{1.7}$$

for the solutions of (1.5)–(1.6), *where C_0 depends only on Γ, T and M.*

Corollary 1.6. *Let $\beta = +1$ or -1, $\psi \in L^2$, $\varphi_n \in L^2 \cap \dot{H}^{-1}$ and $T > 0$. Then, we have*

$$\sup_{0\leqslant t\leqslant T} \|v(t) - u_n(t)\|_{L^2} \to 0$$

for the solutions of (1.5)–(1.6), *when*

$$\|\psi - \varphi_n\|_{L^2} \to 0, \qquad \gamma_n\|\varphi_n\|_{\dot{H}^{-1}} \to 0, \quad \gamma_n \to 0.$$

Remark 1.2. In Theorem 1.5 and Corollary 1.6, ψ is not necessary to be in $\dot{H}^{-1}$ and $\|\varphi_n\|_{\dot{H}^{-1}}$ is not necessary to be bounded.

In general, it seems difficult to apply the Fourier restriction norm method to this kind of limit problem because the Fourier restriction norm depends on the linear part of the equation which include the parameter γ. In our problem, the Fourier restriction norm $X^{s,a,b}_{\beta,\gamma}$ defined below depends on γ. However, we prove a bilinear estimate which is uniform with respect to γ when $-1/2 > a \geqslant -1$ (see, Proposition 3.2). Moreover, from Lemma 2.2, we have uniform estimate as follows:

$$\|u\|_{Y^{s,b}_{\beta}} \leqslant \|u\|_{X^{s,a,b}_{\beta,\gamma}}$$

for $-a \geqslant b \geqslant 0$. These are the crucial points of our proof.

In Section 2, we give some notations and preliminary lemmas. In Section 3, we prove Theorem 1.1. In Section 4, we prove Proposition 1.2 and Theorem 1.5.

2. Notations and preliminary lemmas

Throughout this paper $C > 0$ denotes various constants depending only on s, a, b, b' and Γ, not depending on γ. $C_0, C_1, C_2, \dots$ are constants which may depend on γ and other parameters. The notation $P \lesssim Q$ denotes the estimate $P \leqslant CQ$. We use $P \sim Q$ to denote $P \lesssim Q \lesssim P$.

We define the Fourier restriction norms for the Ostrovsky equation and the KdV equation as follows:

$$\begin{aligned}\left\|u(x,t)\right\|_{X^{s,a,b}_{\beta,\gamma}} &= \left\|V_{\beta,\gamma}(-t)u(x,t)\right\|_{H^b_t H^{s,a}_\gamma} \\ &= \left\|\langle\xi\rangle^s \left\langle\gamma\xi^{-1}\right\rangle^{-a} \left\langle\tau+\beta\xi^3+\gamma\xi^{-1}\right\rangle^b \tilde{u}(\xi,\tau)\right\|_{L^2_{\xi,\tau}},\end{aligned}$$

$$\begin{aligned}\left\|u(x,t)\right\|_{Y^{s,b}_{\beta}} &= \left\|U_{\beta}(-t)u(x,t)\right\|_{H^b_t H^{s}_x} \\ &= \left\|\langle\xi\rangle^s \left\langle\tau+\beta\xi^3\right\rangle^b \tilde{u}(\xi,\tau)\right\|_{L^2_{\xi,\tau}},\end{aligned}$$

where $V_{\beta,\gamma}(t) = \exp\{t\beta\partial_x^3 + t\gamma\partial_x^{-1}\}$ and $U_\beta(t) = \exp\{t\beta\partial_x^3\}$. $\tilde{u} = \mathcal{F}_{x,t}u$ denotes the Fourier transform with respect to t and x variables. We define the $X^{s,a,b}_{\beta,\gamma}$ space and the $Y^{s,b}_{\beta}$ space as follows:

$$\begin{aligned}X^{s,a,b}_{\beta,\gamma} &= \left\{u \in \mathcal{S}'\left(\mathbf{R}^2\right) \;\middle|\; \|u\|_{X^{s,a,b}_{\beta,\gamma}} < +\infty\right\}, \\ Y^{s,b}_{\beta} &= \left\{u \in \mathcal{S}'\left(\mathbf{R}^2\right) \;\middle|\; \|u\|_{Y^{s,b}_{\gamma}} < +\infty\right\}.\end{aligned}$$

We can easily check that the $X^{s,a,b}_{\beta,\gamma}$ space is continuously embedded in $C_t(\mathbf{R} : H^{s,a}_\gamma)$ and the $Y^{s,b}_{\gamma}$ space is continuously embedded in $C_t(\mathbf{R} : H^s)$ when $b > 1/2$. Put

$$\begin{aligned}\hat{u}_{low} &= \hat{u}|_{|\xi|\leqslant 10|\gamma|}, & \hat{u}_{high} &= \hat{u}|_{|\xi|\geqslant |\gamma|/10}, \\ \hat{u}_{l} &= \hat{u}|_{|\xi|\leqslant 11|\gamma|}, & \hat{u}_{h} &= \hat{u}|_{|\xi|\geqslant 9|\gamma|}.\end{aligned}$$

Since $\langle\gamma\xi^{-1}\rangle \sim 1$ and $\langle\tau+\beta\xi^3+\gamma\xi^{-1}\rangle \sim \langle\tau+\beta\xi^3\rangle$ for $|\xi| \geqslant |\gamma|/10$, we have the following lemma.

Lemma 2.1. *For any $s, a, b \in \mathbf{R}$ and $u \in Y^{s,b}_{\beta}$, we have*

$$\|u_{high}\|_{Y^{s,b}_{\beta}} \lesssim \|u_{high}\|_{X^{s,a,b}_{\beta,\gamma}} \lesssim \|u_{high}\|_{Y^{s,b}_{\beta}} \leqslant \|u\|_{Y^{s,b}_{\beta}}.$$

We can easily prove the following lemma by the triangle inequality.

Lemma 2.2. *Let $s \in \mathbf{R}$ and $-a \geqslant b \geqslant 0$. Then, for any function $u \in X^{s,a,b}_{\beta,\gamma}$, we have*

$$\|u\|_{Y^{s,b}_{\beta}} \leqslant \|u\|_{X^{s,a,b}_{\beta,\gamma}}.$$

The following lemma is a variant of the bilinear estimate proved by Isaza and Mejia in [14].

Lemma 2.3. *Let $\beta = +1$ or -1, $\min\{2b - 1/2, 5/8\} \geqslant b' \geqslant b > 1/2$. Then, for any $u, v \in X^{0,0,b}_{\beta,\gamma}$, we have*

$$\left\| \partial_x(uv) \right\|_{X^{0,0,b'-1}_{\beta,\gamma}} \leqslant C_1 \|u\|_{X^{0,0,b}_{\beta,\gamma}} \|v\|_{X^{0,0,b}_{\beta,\gamma}},$$

where $C_1 > 0$ may depend on b, b' and γ.

Proof. The case $\gamma = 1$ follows from Lemma 1.1 and Lemma 1.2 with $s = 0$ in [14]. The case $\gamma \neq 1$ follows by rescaling $\tau' = |\gamma|^{3/4}\tau$, $\xi' = |\gamma|^{1/4}\xi$ because $\langle \tau + \beta\xi^3 + \gamma\xi^{-1} \rangle = \langle |\gamma|^{3/4} |\tau' + \beta\xi'^3 + \xi'^{-1}| \rangle$. □

We define smooth cut-off functions $\rho(t)$, $\rho_{t_0}(t) \in C^\infty$ such that

$$\rho(t) = \begin{cases} 1, & \text{for } |t| \leqslant 1, \\ 0, & \text{for } |t| > 2, \end{cases} \qquad \rho_{t_0} = \rho(t/t_0).$$

The following lemmas are basic tools of the Fourier restriction norm method. For the proofs, see e.g. [3,11,19].

Lemma 2.4. *Let $\beta = +1$ or -1, $\gamma \in \mathbf{R}$ and $s, a, b \in \mathbf{R}$. Then, for any function $f \in H^{s,a}_\gamma$, we have*

$$\left\| \rho(t) V_{\beta,\gamma}(t) f \right\|_{X^{s,a,b}_{\beta,\gamma}} \lesssim \|f\|_{H^{s,a}_\gamma}$$

and for any function $f \in H^s$, we have

$$\left\| \rho(t) U_\beta(t) f \right\|_{Y^{s,b}_\beta} \lesssim \|f\|_{H^s}.$$

Lemma 2.5. *Let $\beta = +1$ or -1, $\gamma \in \mathbf{R}$, $s, a \in \mathbf{R}$, $1/2 < b < b' \leqslant 1$ and $0 < t_0 < 1$. Then, for any function $F \in X^{s,a,b}_{\beta,\gamma}$, we have*

$$\left\| \rho_{t_0}(t) \int_0^t V_{\beta,\gamma}(t - t') F(t')\, dt' \right\|_{X^{s,a,b}_{\beta,\gamma}} \lesssim t_0^{b'-b} \|F\|_{X^{s,a,b'-1}_{\beta,\gamma}}$$

and for any function $F \in Y^{s,b}_\beta$, we have

$$\left\| \rho_{t_0}(t) \int_0^t U_\beta(t - t') F(t')\, dt' \right\|_{Y^{s,b}_\beta} \lesssim t_0^{b'-b} \|F\|_{Y^{s,b'-1}_\beta}.$$

Lemma 2.6.

(i) *Let $0 \leqslant r < p + q - 1$ and $r \leqslant \min\{p, q\}$. Then, for any $l, m \in \mathbf{R}$, we have*

$$\int_{-\infty}^{\infty} \frac{1}{\langle x - l \rangle^p \langle x - m \rangle^q}\, dx \leqslant \frac{C_2}{\langle l - m \rangle^r},$$

where C_2 depends only on p, q and r.

(ii) *Let $p > 1/2$. Then, for any $l, m \in \mathbf{R}$, we have*

$$\int_{-\infty}^{\infty} \frac{1}{\langle x-l\rangle^{p}|x-m|^{1/2}}\,dx \leqslant \frac{C_3}{\langle l-m\rangle^{1/2}},$$

where C_3 depends only on p.

The following argument was originally used in [19].

Lemma 2.7. *For a subset $\Omega \subset \mathbf{R}^4$, we define the characteristic function χ_Ω as follows:*

$$\chi_\Omega(\tau,\xi,\tau_1,\xi_1) = \begin{cases} 1, & \text{for } (\tau,\xi,\tau_1,\xi_1) \in \Omega, \\ 0, & \text{for } (\tau,\xi,\tau_1,\xi_1) \notin \Omega \end{cases}$$

and we put

$$\widetilde{F_\Omega(u,v)} = |\xi| \int_{\mathbf{R}^2} \chi_\Omega \tilde{u}(\tau-\tau_1,\xi-\xi_1)\tilde{v}(\tau_1,\xi_1)\,d\tau_1\,d\xi_1.$$

If we have

$$\sup_{\tau,\xi} \frac{|\xi|^2\langle\xi\rangle^{2s}\langle\gamma\xi^{-1}\rangle^{-2a}}{\langle\tau+\beta\xi^3+\gamma\xi^{-1}\rangle^{2(1-b')}} \int_{\mathbf{R}^2} \chi_\Omega \frac{\langle\xi_1\rangle^{-2s}\langle\gamma\xi_1^{-1}\rangle^{2a}}{\langle\tau_1+\beta\xi_1^3+\gamma\xi_1^{-1}\rangle^{2b}} \times \frac{\langle\xi-\xi_1\rangle^{-2s}\langle\gamma(\xi-\xi_1)^{-1}\rangle^{2a}}{\langle\tau-\tau_1+\beta(\xi-\xi_1)^3+\gamma(\xi-\xi_1)^{-1}\rangle^{2b}}\,d\tau_1\,d\xi_1 \leqslant M^2 \tag{2.8}$$

or

$$\sup_{\tau_1,\xi_1} \frac{\langle\xi_1\rangle^{-2s}\langle\gamma\xi_1^{-1}\rangle^{2a}}{\langle\tau_1+\beta\xi_1^3+\gamma\xi_1^{-1}\rangle^{2b}} \int_{\mathbf{R}^2} \chi_\Omega \frac{|\xi|^2\langle\xi\rangle^{2s}\langle\gamma\xi^{-1}\rangle^{-2a}}{\langle\tau+\beta\xi^3+\gamma\xi^{-1}\rangle^{2(1-b')}} \times \frac{\langle\xi-\xi_1\rangle^{-2s}\langle\gamma(\xi-\xi_1)^{-1}\rangle^{2a}}{\langle\tau-\tau_1+\beta(\xi-\xi_1)^3+\gamma(\xi-\xi_1)^{-1}\rangle^{2b}}\,d\tau\,d\xi \leqslant M^2 \tag{2.9}$$

for a constant $M > 0$, then we have

$$\big\|F_\Omega(u,v)\big\|_{X^{s,a,b'-1}_{\beta,\gamma}} \leqslant M\|u\|_{X^{s,a,b}_{\beta,\gamma}}\|v\|_{X^{s,a,b}_{\beta,\gamma}}. \tag{2.10}$$

Proof. (i) We first consider the case that (2.8) holds. Put

$$G(\tau,\xi) = \langle\xi\rangle^s\big\langle\gamma\xi^{-1}\big\rangle^{-a}\big\langle\tau+\beta\xi^3+\gamma\xi^{-1}\big\rangle^b\tilde{u}(\tau,\xi),$$
$$H(\tau,\xi) = \langle\xi\rangle^s\big\langle\gamma\xi^{-1}\big\rangle^{-a}\big\langle\tau+\beta\xi^3+\gamma\xi^{-1}\big\rangle^b\tilde{v}(\tau,\xi).$$

By the Schwartz inequality, we have

$$\left|\widetilde{F_\Omega(u,v)}\right|^2 \leqslant |\xi|^2 \int_{\mathbf{R}^2} \chi_\Omega \frac{\langle\xi_1\rangle^{-2s}\langle\gamma\xi_1^{-1}\rangle^{2a}}{\langle\tau_1+\beta\xi_1^3+\gamma\xi_1^{-1}\rangle^{2b}} \frac{\langle\xi-\xi_1\rangle^{-2s}\langle\gamma(\xi-\xi_1)^{-1}\rangle^{2a}}{\langle\tau-\tau_1+\beta(\xi-\xi_1)^3+\gamma(\xi-\xi_1)^{-1}\rangle^{2b}}\,d\tau_1\,d\xi_1$$
$$\times \int_{\mathbf{R}^2} \left|G(\tau-\tau_1,\xi-\xi_1)\right|^2 \left|H(\tau_1,\xi_1)\right|^2 d\tau_1\xi_1.$$

Therefore, we obtain

$$\begin{aligned}\left\|F_\Omega(u,v)\right\|^2_{X^{s,a,b'-1}_{\beta,\gamma}} &\leqslant M^2 \int_{\mathbf{R}^4} \left|G(\tau-\tau_1,\xi-\xi_1)\right|^2 \left|H(\tau_1,\xi_1)\right|^2 d\tau_1\xi_1\,d\tau\,d\xi \\ &\leqslant M^2\|G\|^2_{L^2_{\tau,\xi}}\|H\|^2_{L^2_{\tau,\xi}} \leqslant M^2\|u\|^2_{X^{s,a,b}_{\beta,\gamma}}\|v\|^2_{X^{s,a,b}_{\beta,\gamma}}.\end{aligned}$$

(ii) We next consider the case that (2.9) holds. By the duality argument, (2.10) is equivalent to

$$\int_{\mathbf{R}^2} \widetilde{F_\Omega(u,v)}\,\overline{\widetilde{w}}\,d\tau\,d\xi \leqslant M\|u\|_{X^{s,a,b}_{\beta,\gamma}}\|v\|_{X^{s,a,b}_{\beta,\gamma}}\|w\|_{X^{-s,-a,1-b'}_{\beta,\gamma}}. \tag{2.11}$$

The left-hand side is bounded by

$$\begin{aligned}&\int_{\mathbf{R}^4} |\xi|\chi_\Omega\tilde{u}(\tau-\tau_1,\xi-\xi_1)\tilde{v}(\tau_1,\xi_1)\overline{\widetilde{w}}(\tau,\xi)\,d\tau\,d\xi\,d\tau_1\,d\xi_1 \\ &\quad\leqslant \left\|\int_{\mathbf{R}^2}|\xi|\chi_\Omega\tilde{u}(\tau-\tau_1,\xi-\xi_1)\overline{\widetilde{w}}(\tau,\xi)\,d\tau\,d\xi\right\|_{X^{-s,-a,-b}_{\beta,\gamma}}\|v\|_{X^{s,a,b}_{\beta,\gamma}}.\end{aligned}$$

By a similar argument as in (i), from (2.9), we have

$$\left\|\int_{\mathbf{R}^2}|\xi|\chi_\Omega\tilde{u}(\tau-\tau_1,\xi-\xi_1)\overline{\widetilde{w}}(\tau,\xi)\,d\tau\,d\xi\right\|_{X^{-s,-a,-b}_{\beta,\gamma}} \leqslant M\|u\|_{X^{s,a,b}_{\beta,\gamma}}\|w\|_{X^{-s,-a,1-b'}_{\beta,\gamma}}.$$

Thus, we obtain (2.11). □

3. Bilinear estimates and time local well-posedness

The following bilinear estimate was proved by Kenig, Ponce and Vega in [19].

Proposition 3.1. *Let $\beta = +1$ or -1 and $\min\{b+1/4+s/3, 1\} \geqslant b' \geqslant b > 1/2$. Then, for any $u, v \in Y^{s,b}_\beta$, we have*

$$\left\|\partial_x(uv)\right\|_{Y^{s,b'-1}_\beta} \lesssim \|u\|_{Y^{s,b}_\beta}\|v\|_{Y^{s,b}_\beta}. \tag{3.12}$$

The following bilinear estimate plays an important role in the present paper.

Proposition 3.2. *Let $\beta = +1$ or -1, $|\gamma| \leqslant \Gamma$, $0 \geqslant s \geqslant \max\{-a/2 + b'/2 - 1, b' - 5/4\}$, $0 \geqslant a \geqslant -1$ and $\min\{b + 1/4 + s/3, 2b - 1/2, 5/8\} \geqslant b' \geqslant b > 1/2$. Then, for any $u, v \in X^{s,a,b}_{\beta,\gamma}$, we have*

$$\left\| \partial_x(uv) \right\|_{X^{s,a,b'-1}_{\beta,\gamma}} \leqslant C_4 \|u\|_{X^{s,a,b}_{\beta,\gamma}} \|v\|_{X^{s,a,b}_{\beta,\gamma}}, \tag{3.13}$$

where $C_4 > 0$ depends only on s, a, b, b' and Γ when $-1/2 > a \geqslant -1$, and may depend not only on s, a, b and b', but also γ when $0 \geqslant a \geqslant -1/2$.

Proof. Without loss of generality, we can assume $\tilde{u} \geqslant 0$ and $\tilde{v} \geqslant 0$. By the Plancherel theorem, the left-hand side of (3.13) is equal to

$$\left\| F_{\mathbf{R}^4}(u, v) \right\|_{X^{s,a,b'-1}_{\beta,\gamma}}, \tag{3.14}$$

where $F_\Omega(u, v)$ is defined in Lemma 2.7. We divide the region $\mathbf{R}^4$ into six parts:

$$A_1 = \left\{(\tau, \xi, \tau_1, \xi_1) \in \mathbf{R}^4 \;\middle|\; \min\left\{|\xi_1|, |\xi|, |\xi - \xi_1|\right\} \geqslant |\gamma|/10\right\}, \tag{3.15}$$

$$A_2 = \left\{(\tau, \xi, \tau_1, \xi_1) \in \mathbf{R}^4 \;\middle|\; \max\left\{|\xi_1|, |\xi|, |\xi - \xi_1|\right\} \leqslant 11|\gamma|\right\}, \tag{3.16}$$

$$A_3 = \left\{(\tau, \xi, \tau_1, \xi_1) \in \mathbf{R}^4 \;\middle|\; |\xi| \geqslant 10|\gamma|,\ |\xi_1| \leqslant |\gamma|/10\right\}, \tag{3.17}$$

$$A_4 = \left\{(\tau, \xi, \tau_1, \xi_1) \in \mathbf{R}^4 \;\middle|\; |\xi| \geqslant 10|\gamma|,\ |\xi - \xi_1| \leqslant |\gamma|/10\right\}, \tag{3.18}$$

$$A_5 = \left\{(\tau, \xi, \tau_1, \xi_1) \in \mathbf{R}^4 \;\middle|\; |\xi| \leqslant |\gamma|/10,\ |\xi_1| \geqslant 10|\gamma|\right\}, \tag{3.19}$$

$$A_6 = \left\{(\tau, \xi, \tau_1, \xi_1) \in \mathbf{R}^4 \;\middle|\; |\xi| \leqslant |\gamma|/10,\ |\xi - \xi_1| \geqslant 10|\gamma|\right\}. \tag{3.20}$$

By symmetry, we have only to consider A_1, A_2, A_3 and A_5. We first consider the region A_1. From Lemma 2.1 and Proposition 3.1, we have

$$\begin{aligned}
\left\| F_{A_1}(u, v) \right\|_{X^{s,a,b'-1}_{\beta,\gamma}} &\lesssim \left\| \partial_x(u_{high} v_{high}) \right\|_{Y^{s,b'-1}_{\beta}} \\
&\lesssim \|u_{high}\|_{Y^{s,b}_{\beta}} \|v_{high}\|_{Y^{s,b}_{\beta}} \\
&\lesssim \|u_{high}\|_{X^{s,a,b}_{\beta,\gamma}} \|v_{high}\|_{X^{s,a,b}_{\beta,\gamma}} \\
&\leqslant \|u\|_{X^{s,a,b}_{\beta,\gamma}} \|v\|_{X^{s,a,b}_{\beta,\gamma}}.
\end{aligned}$$

We next consider the region A_2. From Hölder's inequality and Sobolev's inequality, we have

$$\left\| F_{A_2}(u, v) \right\|_{X^{s,a,b'-1}_{\beta,\gamma}} \lesssim \|u_l v_l\|_{L^2_{t,x}} \leqslant \|u_l\|_{L^4_{t,x}} \|v_l\|_{L^4_{t,x}} \lesssim \|u\|_{X^{s,a,b}_{\beta,\gamma}} \|v\|_{X^{s,a,b}_{\beta,\gamma}}.$$

We next consider the region A_3. In A_3, we have

$$\langle\xi\rangle^s \left\langle \gamma \xi^{-1} \right\rangle^{-a} \lesssim \langle\xi_1\rangle^s \left\langle \gamma \xi_1^{-1} \right\rangle^{-a} \langle \xi - \xi_1 \rangle^s \left\langle \gamma (\xi - \xi_1)^{-1} \right\rangle^{-a}.$$

Therefore, from Lemma 2.3, we have

$$\left\| F_{A_3}(u, v) \right\|_{X^{s,a,b'-1}_{\beta,\gamma}} \leqslant C_1 \|u\|_{X^{s,a,b}_{\beta,\gamma}} \|v\|_{X^{s,a,b}_{\beta,\gamma}},$$

where C_1 may depends on b, b' and γ. When $-a \geqslant b$, from Lemma 2.2, we have

$$\left\| F_{A_3}(u,v)\right\|_{X^{s,a,b'-1}_{\beta,\gamma}} \lesssim \left\|\partial_x(u_h v_{low})\right\|_{Y^{s,b'-1}_{\beta}} \lesssim \|u_h\|_{Y^{s,b}_{\beta}}\|v_{low}\|_{Y^{s,b}_{\beta}} \lesssim \|u\|_{X^{s,a,b}_{\beta,\gamma}}\|v\|_{X^{s,a,b}_{\beta,\gamma}},$$

where implicit constants do not depend on γ. Next, we consider A_5. In A_5, we have

$$\begin{gathered}
\left\langle \tau+\beta\xi^3+\gamma\xi^{-1}\right\rangle \sim \left\langle \tau+\gamma\xi^{-1}\right\rangle,\\
\left\langle \tau_1+\beta\xi_1^3+\gamma\xi_1^{-1}\right\rangle \sim \left\langle \tau_1+\beta\xi_1^3\right\rangle,\\
\left\langle (\tau-\tau_1)+\beta(\xi-\xi_1)^3+\gamma(\xi-\xi_1)^{-1}\right\rangle \sim \left\langle (\tau-\tau_1)+\beta(\xi-\xi_1)^3\right\rangle,\\
\langle\xi\rangle^s\left\langle\gamma\xi^{-1}\right\rangle^{-a}|\xi|\langle\xi-\xi_1\rangle^{-s}\left\langle\gamma(\xi-\xi_1)^{-1}\right\rangle^{a}\langle\xi_1\rangle^{-s}\left\langle\gamma\xi_1^{-1}\right\rangle^{a} \lesssim |\gamma|^{-a}\langle\xi_1\rangle^{-2s}|\xi|^{a+1}.
\end{gathered}$$

We divide A_5 into two parts as follows:

$$A_{51} = \left\{(\tau,\xi,\tau_1,\xi_1)\in A_5 \;\middle|\; 3|\xi|^2|\xi_1|^2 \leqslant 10|\gamma|\right\}, \tag{3.21}$$

$$A_{52} = \left\{(\tau,\xi,\tau_1,\xi_1)\in A_5 \;\middle|\; 3|\xi|^2|\xi_1|^2 \geqslant 10|\gamma|\right\}. \tag{3.22}$$

When $\Omega = A_{51}$, from Lemma 2.6, the left-hand side of (2.9) is bounded by

$$\begin{aligned}
&\frac{|\gamma|^{-2a}\langle\xi_1\rangle^{-4s}}{\langle\tau_1+\beta\xi_1^3\rangle^{2b}}\int_B \frac{|\xi|^{2a+2}}{\langle\tau_1+\gamma\xi^{-1}-\beta(\xi-\xi_1)^3\rangle^{2(1-b')}}\,d\xi\\
&\quad\lesssim |\gamma|^{-2a}\langle\xi_1\rangle^{-4s}\int_B \frac{|\xi|^{2a+2}}{\langle\gamma\xi^{-1}-3\beta\xi\xi_1^2\rangle^{2(1-b')}}\,d\xi,
\end{aligned} \tag{3.23}$$

where

$$B = \left\{\xi\in\mathbf{R} \;\middle|\; |\xi| \lesssim \min\left\{|\xi_1|^{-1}|\gamma|^{1/2}, |\gamma|\right\}\right\}.$$

Here, we consider the following two cases:

(i) $\beta\gamma<0$. In this case, we have $\langle\gamma\xi^{-1}-3\beta\xi\xi_1^2\rangle \geqslant \max\{|\gamma||\xi|^{-1}, |\xi||\xi_1|^2\}$. If $|\xi_1|\geqslant 1$, then (3.23) is bounded by

$$|\gamma|^{-2a}\langle\xi_1\rangle^{-4s}\int_{-C|\xi_1|^{-1}|\gamma|^{1/2}}^{C|\xi_1|^{-1}|\gamma|^{1/2}} \frac{|\xi|^{2a+2}}{(|\xi||\xi_1|^2)^{2(1-b')}}\,d\xi \lesssim |\gamma|^{-a+b'+1/2}|\xi_1|^{-4s-2a+2b'-5} \lesssim 1,$$

because $s \geqslant -a/2+b'/2-5/4$. If $|\xi_1|\leqslant 1$, then (3.23) is bounded by

$$|\gamma|^{-2a}\int_{-C|\gamma|}^{C|\gamma|} \frac{|\xi|^{2a+2}}{(|\gamma||\xi|^{-1})^{2(1-b')}}\,d\xi \lesssim |\gamma|^{-2a+2b'-2}\int_{-C|\gamma|}^{C|\gamma|} |\xi|^{2a-2b'+4}\,d\xi \lesssim 1.$$

(ii) $\beta\gamma > 0$. We put $x = \gamma\xi^{-1} - 3\beta\xi\xi_1^2$. Then, we have

$$\left|\frac{d\xi}{dx}\right| = \left|\frac{1}{-\gamma\xi^{-2} - 3\beta\xi_1^2}\right| \lesssim |\gamma|^{-1}|\xi|^2.$$

Therefore, (3.23) is bounded by

$$\int_B \frac{J(\gamma,\xi,\xi_1)|\gamma|}{\langle\gamma\xi^{-1} - 3\beta\xi\xi_1^2\rangle^{2(1-b')}|\gamma\xi^{-1} - 3\beta\xi\xi_1^2|^{\epsilon}|\xi|^2}\,d\xi$$
$$\lesssim \sup_{\xi\in B} J(\gamma,\xi,\xi_1)\int_{\mathbf{R}} \frac{1}{\langle x\rangle^{2(1-b')}|x|^{\epsilon}}\,dx \lesssim \sup_{\xi\in B} J(\gamma,\xi,\xi_1),$$

where $2b' - 1 < \epsilon \leqslant 4s + 2a + 4$ and

$$J(\gamma,\xi,\xi_1) = |\gamma|^{-2a-1}\langle\xi_1\rangle^{-4s}|\xi|^{2a+4}\left|\gamma\xi^{-1} - 3\beta\xi\xi_1^2\right|^{\epsilon}.$$

By a simple calculation, we have

$$\sup_{\xi\in B} J(\gamma,\xi,\xi_1) \lesssim |\gamma|^{-2a-1+\epsilon}\langle\xi_1\rangle^{-4s}|\xi|^{2a+4-\epsilon} \lesssim 1.$$

From (i) and (ii), we obtain (3.23) is bounded by C.

We next consider A_{52}. We put

$$M = \max\left\{\left|\tau + \beta\xi^3 + \gamma\xi^{-1}\right|, \left|\tau_1 + \beta\xi_1^3 + \gamma\xi_1^{-1}\right|, \left|(\tau-\tau_1) + \beta(\xi-\xi_1)^3 + \gamma(\xi-\xi_1)^{-1}\right|\right\}.$$

From (3.19) and (3.22), we have

$$\left|\xi\xi_1^2\right| \geqslant \frac{10|\gamma|}{3|\xi|} \geqslant \frac{100}{3}, \qquad |\xi_1| \geqslant 10|\gamma| \geqslant 100|\xi|.$$

Therefore, by the triangle inequality, we obtain

$$\begin{aligned} M &\geqslant \left|-\beta\xi^3 - \gamma\xi^{-1} + \beta\xi_1^3 + \gamma\xi_1^{-1} + \beta(\xi-\xi_1)^3 + \gamma(\xi-\xi_1)^{-1}\right|/3 \\ &\geqslant \left|\xi\xi_1^2\right| - \left|\xi^2\xi_1\right| - \frac{|\gamma|}{3|\xi|} - \frac{|\gamma|}{3|\xi_1|} - \frac{|\gamma|}{3|\xi-\xi_1|} \\ &\geqslant \left(8\left|\xi\xi_1^2\right| - 1\right)/10. \end{aligned} \tag{3.24}$$

Thus, we have

$$M \geqslant \frac{7|\xi\xi_1^2|}{10} \geqslant \frac{7|\gamma|}{3|\xi|} \geqslant \frac{70}{3}, \qquad M \geqslant 7000\left|\xi^3\right|. \tag{3.25}$$

We divide A_{52} into three parts as follows:

$$A_{521} = \{(\tau, \xi, \tau_1, \xi_1) \in A_{52} \mid M = |\tau + \beta\xi^3 + \gamma\xi^{-1}|\},$$
$$A_{522} = \{(\tau, \xi, \tau_1, \xi_1) \in A_{52} \mid M = |\tau_1 + \beta\xi_1^3 + \gamma\xi_1^{-1}|\},$$
$$A_{523} = \{(\tau, \xi, \tau_1, \xi_1) \in A_{52} \mid M = |(\tau - \tau_1) + \beta(\xi - \xi_1)^3 + \gamma(\xi - \xi_1)^{-1}|\}.$$

By symmetry, we have only to consider A_{521} and A_{522}. We first consider A_{521}. In A_{521}, from (3.25), we have

$$|\tau + 3\beta\xi^3/4| \geqslant |\tau + \beta\xi^3 + \gamma\xi^{-1}| - |\xi^3|/4 - |\gamma||\xi|^{-1} \gtrsim M. \tag{3.26}$$

From Lemma 2.6, the left-hand side of (2.8) with $\Omega = A_{521}$ is bounded by

$$\begin{aligned} &\frac{|\gamma|^{-2a}|\xi|^{2a+2}}{\langle\tau + \beta\xi^3 + \gamma\xi^{-1}\rangle^{2(1-b')}} \int_{\mathbf{R}^2} \frac{\chi_{A_{521}}|\xi_1|^{-4s}}{\langle\tau_1 + \beta\xi_1^3\rangle^{2b}\langle\tau - \tau_1 + \beta(\xi - \xi_1)^3\rangle^{2b}}\, d\tau_1\, d\xi_1 \\ &\lesssim K \int \frac{|3\beta\xi^3 + 4\tau|^{1/2}|\xi|^{1/2}}{\langle\tau + 3\beta\xi\xi_1(\xi - \xi_1)\rangle^{2b}}\, d\xi_1, \end{aligned} \tag{3.27}$$

where

$$K = \sup_{(\tau,\xi,\tau_1,\xi_1)\in A_{521}} \frac{|\gamma|^{-2a}|\xi|^{2a+3/2}|\xi_1|^{-4s}}{\langle\tau + \beta\xi^3 + \gamma\xi^{-1}\rangle^{2(1-b')}|3\beta\xi^3 + 4\tau|^{1/2}}.$$

From (3.25) and (3.26), we have

$$K \lesssim |\gamma|^{-2a}\left(\frac{|\xi||\xi_1|^2}{M}\right)^{-2s}|\xi|^{2s+2a+3/2}M^{-2s+2b'-5/2} \lesssim 1$$

when $s \geqslant -a - 3/4$ and

$$K \lesssim |\gamma|^{2s+3/2}\left(\frac{|\xi||\xi_1|^2}{M}\right)^{-2s}\left(\frac{|\gamma|}{|\xi|M}\right)^{-2s-2a-3/2}M^{-4s-2a+2b'-4} \lesssim 1$$

when $s < -a - 3/4$ because $0 \geqslant s \geqslant \max\{-a/2 + b'/2 - 1, b' - 5/4\}$.

We put $x = \tau + 3\beta\xi\xi_1(\xi - \xi_1)$. Then, we have

$$\xi_1 = \frac{\xi}{2} \pm \frac{1}{2\beta}\sqrt{\frac{3\xi^3 + 4\beta(\tau - x)}{3\xi}},$$
$$\left|\frac{d\xi_1}{dx}\right| = \frac{1}{|3\xi(\xi - 2\xi_1)|} = \frac{1}{|3\xi|^{1/2}|3\beta\xi^2 + 4\tau - 4x|^{1/2}}.$$

Therefore, from Lemma 2.6, (3.27) is bounded by

$$\int \frac{|3\beta\xi^3 + 4\tau|^{1/2}}{\langle x\rangle^{2b}|3\beta\xi^3 + 4\tau - 4x|^{1/2}}\, dx \lesssim 1.$$

We next consider A_{522}. From Lemma 2.6, the left-hand side of (2.9) with $\Omega = A_{522}$ is bounded by

$$\frac{|\gamma|^{-2a}|\xi_1|^{-4s}}{\langle \tau_1 + \beta\xi_1^3 + \gamma\xi_1^{-1}\rangle^{2(1-b')}} \int \chi_{A_{522}} \frac{|\xi|^{2a+2}}{\langle \tau - \tau_1 + \beta(\xi - \xi_1)^3\rangle^{2b} \langle \tau + \gamma\xi^{-1}\rangle^{2b}}\, d\tau\, d\xi$$
$$\lesssim L \int \frac{|3\beta(\xi - \xi_1)^2 + \gamma\xi^{-2}|}{\langle -\tau_1 + \beta(\xi - \xi_1)^3 - \gamma\xi^{-1}\rangle^{2b}}\, d\xi, \tag{3.28}$$

where

$$L = \sup_{(\tau,\xi,\tau_1,\xi_1)\in A_{522}} \frac{|\gamma|^{-2a}|\xi_1|^{-4s}|\xi|^{2a+2}}{\langle \tau_1 + \beta\xi_1^3 + \gamma\xi_1^{-1}\rangle^{2(1-b')}|3\beta(\xi - \xi_1)^2 + \gamma\xi^{-2}|}.$$

Since $|\tau_1 + \beta\xi_1^3 + \gamma\xi_1^{-1}| \gtrsim |\xi\xi_1^2|$ and $|3\beta(\xi - \xi_1)^2 + \gamma\xi^{-2}| \gtrsim |\xi_1|^2$ in A_{522}, we have

$$L \lesssim \big(|\gamma|/|\xi|\big)^{-2a}\big(|\xi||\xi_1|^2\big)^{-2s+2b'-3}|\xi|^{2s+3}.$$

Therefore, from (3.25), we have $L \lesssim (|\gamma|/|\xi|)^{-2a-2s-3}(|\xi||\xi_1|^2)^{-2s+2b'-3}|\gamma|^{2s+3} \lesssim 1$ because $-2s + 2b' - 3 \leqslant 0$ and $-2a - 2s - 3 \leqslant 0$. Here, we put $x = -\tau_1 + \beta(\xi - \xi_1)^3 - \gamma\xi^{-1}$. Then, we have

$$\left|\frac{d\xi}{dx}\right| = \frac{1}{|3\beta(\xi - \xi_1)^2 + \gamma\xi^{-2}|}.$$

Therefore, (3.28) is bounded by

$$\int \frac{1}{\langle x\rangle^{2b}}\, dx \lesssim 1. \qquad \Box$$

Now, we prove Theorem 1.1. Precisely speaking, we have the following proposition.

Proposition 3.3. *Assume that $\gamma \neq 0$, $\beta = +1$ or -1 and $\varphi \in H_\gamma^{s,a}$. Let $s > -a/2 - 3/4$, $0 \geqslant a \geqslant -1$, $b > 1/2$ and b be sufficiently close to $1/2$. Then, (1.1) is time locally well-posed and the lifetime of the solution t_0 satisfies*

$$t_0 \gtrsim \big\langle C_4\|\varphi\|_{H_\gamma^{s,a}}\big\rangle^{-1/\epsilon}$$

for some $\epsilon \gtrsim 1$, where C_4 is defined in Proposition 3.2. Moreover, the solution is in $C([0,t_0] : H_\gamma^{s,a}) \cap X_{\beta,\gamma}^{s,a,b}$ and satisfies the following estimates:

$$\sup_{0\leqslant t\leqslant t_0} \|u\|_{H_\gamma^{s,a}} \lesssim \|\varphi\|_{H_\gamma^{s,a}},$$
$$\|u\|_{X_{\beta,\gamma}^{s,a,b}} \lesssim \|\varphi\|_{H_\gamma^{s,a}}.$$

The proof of this proposition follows from the standard argument of the Fourier restriction norm method. Therefore, we mention only the outline of the proof. For more detail, see e.g. [3,11].

Proof. We consider the following map:

$$N(u) = \rho(t)V_{\beta,\gamma}(t)\varphi + \frac{\rho_{t_0}(t)}{2}\int_0^t V_{\beta,\gamma}(t-t')\partial_x\bigl(u^2\bigr)\,dt'.$$

Put $X_r = \{u \in X^{s,a,b}_{\beta,\gamma} \mid \|u\|_{X^{s,a,b}_{\beta,\gamma}} \leqslant r\|\varphi\|_{H^{s,a}_\gamma}\}$. We shall prove that N is a map from X_r into X_r for sufficiently large $r > 0$. From Lemma 2.4, we have

$$\bigl\|\rho(t)V_{\beta,\gamma}(t)\varphi\bigr\|_{X^{s,a,b}_{\beta,\gamma}} \lesssim \|\varphi\|_{H^{s,a}_\gamma}.$$

From Lemma 2.5, we have

$$\left\|\frac{\rho_{t_0}(t)}{2}\int_0^t V_{\beta,\gamma}(t-t')\partial_x\bigl(u^2\bigr)\,dt'\right\|_{X^{s,a,b}_{\beta,\gamma}} \lesssim t_0^{b'-b}\bigl\|\partial_x\bigl(u^2\bigr)\bigr\|_{X^{s,a,b'-1}_{\beta,\gamma}}.$$

Therefore, from Proposition 3.2, we have

$$\bigl\|N(u)\bigr\|_{X^{s,a,b}_{\beta,\gamma}} \leqslant C\bigl(\|\varphi\|_{H^{s,a}_\gamma} + C_4 t_0^{b'-b}\|u\|^2_{X^{s,a,b}_{\beta,\gamma}}\bigr).$$

We take $r > 2C$ and $0 < t_0^\epsilon < \min\{(2CC_4 r\|\varphi\|_{H^{s,a}_\gamma})^{-1}, 1\}$ where $\epsilon = b' - b$. Then, for $u \in X_r$, we have

$$\bigl\|N(u)\bigr\|_{X^{s,a,b}_{\beta,\gamma}} < r\|\varphi\|_{H^{s,a}_\beta}.$$

We can easily check that N is a contraction map, too. Thus, we obtain the existence of the solution by the fix point argument. The remaining estimate follows from the well-known embedding inequality:

$$\sup_t \|u\|_{H^{s,a}_\gamma} \lesssim \|u\|_{X^{s,a,b}_{\beta,\gamma}}. \qquad \square$$

In the same manner, we can prove the following proposition from Proposition 3.1. This result was originally proved by Kenig, Ponce and Vega in [19].

Proposition 3.4. *Assume that $\gamma = 0$, $\beta = +1$ or -1 and $\varphi \in H^s$. Let $s > -3/4$, $b > 1/2$ and b be sufficiently close to $1/2$. Then, (1.1) is time locally well-posed and the lifetime of the solution t_1 satisfies*

$$t_1 \gtrsim \bigl\langle\|\varphi\|_{H^s}\bigr\rangle^{-1/\epsilon}$$

for some $\epsilon \gtrsim 1$. Moreover, the solution is in $C([0,t_1] : H^s) \cap Y^{s,b}_\beta$ and satisfies the following estimates

$$\sup_{0\leqslant t\leqslant t_1} \|u\|_{H^s} \lesssim \|\varphi\|_{H^s},$$

$$\|u\|_{Y^{s,b}_\beta} \lesssim \|\varphi\|_{H^s}.$$

4. A priori estimates and weak rotation limit

We first prove Proposition 1.2.

Proof. By the density argument, without loss of generality, we can assume u is sufficiently smooth, $u, \partial_x u, \partial_x^2 u \to 0$ as $|x| \to \infty$ and $\hat{u}$ vanishes at the origin $\xi = 0$. Calculating

$$\int_0^T \int \left(\partial_t u - \beta \partial_x^3 u + u \partial_x u - \gamma \partial_x^{-1} u\right) \cdot u \, dx \, dt = 0,$$

we obtain (1.2). Calculating

$$\int \partial_x^{-1} \left(\partial_t u - \beta \partial_x^3 u + u \partial_x u - \gamma \partial_x^{-1} u\right) \cdot \partial_x^{-1} u \, dx = 0,$$

we have

$$\frac{1}{2} \partial_t \left\| u(\cdot, t) \right\|_{\dot{H}^{-1}}^2 \leqslant \gamma \int u^2 \partial_x^{-1} u \, dx.$$

By the Schwarz and the Sobolev inequalities, the right-hand side is bounded by

$$\gamma \|u\|_{L^2}^2 \left\| \partial_x^{-1} u \right\|_{L^\infty} \leqslant \gamma \|\varphi\|_{L^2}^{5/2} \|u\|_{\dot{H}^{-1}}^{1/2}.$$

Thus, we obtain (1.3). We put

$$\widehat{Pu} = |\gamma|^{-a} |\xi|^a \hat{u}(\xi) \big|_{|\xi| \leqslant |\gamma|}.$$

Calculating

$$\int P\left(\partial_t u - \beta \partial_x^3 u + u \partial_x u - \gamma \partial_x^{-1} u\right) \cdot Pu \, dx = 0,$$

we obtain

$$\int \partial_t Pu \cdot Pu \, dx - \beta \int P \partial_x^3 u \cdot Pu \, dx + \frac{1}{2} \int P \partial_x u^2 \cdot Pu \, dx = \gamma \int P \partial_x^{-1} u \cdot Pu \, dx.$$

The second term of the left-hand side and the right-hand side vanish. The third term is bounded by

$$\left\| u^2 \right\|_{L^1} \left\| P^2 \partial_x u \right\|_{L^\infty} \lesssim \|u\|_{L^2}^2 \left\| P^2 \partial_x u \right\|_{L^2}^{1/2} \left\| P^2 \partial_x u \right\|_{\dot{H}^1}^{1/2} \leqslant \|u\|_{L^2}^2 |\gamma|^{3/2} \|Pu\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2}.$$

Here, we used

$$|\widehat{P\partial_x}| \leqslant |\gamma|^{-a} |\xi|^{1+a} \big|_{|\xi| \leqslant |\gamma|} \leqslant |\gamma|.$$

Therefore, we have

$$\partial_t \|Pu\|_{L^2}^2 \lesssim |\gamma|^{3/2} \|\varphi\|_{L^2}^{5/2} \|Pu\|_{L^2}^{1/2}.$$

Therefore, we obtain

$$\sup_{0\leqslant t\leqslant T} \left\| Pu(\cdot,t)\right\|_{L^2}^{3/2} \leqslant \|P\varphi\|_{L^2}^{3/2} + CT|\gamma|^{3/2}\|\varphi\|_{L^2}^{5/2}. \tag{4.29}$$

Since

$$\left\|u(\cdot,t)\right\|_{H_\gamma^{0,a}} \lesssim \left\|Pu(\cdot,t)\right\|_{L^2} + \left\|u(\cdot,t)\right\|_{L^2},$$

we have (1.4) from (1.2) and (4.29). $\square$

We next prove Theorem 1.5.

Proof. Let u_n be the solution of (1.5) obtained in Proposition 3.3 and v be the solution of (1.6) obtained in Proposition 3.4, namely, u_n and v satisfy the following integral equations:

$$u_n = \rho(t)V_{\beta,\gamma_n}(t)\varphi_n + \frac{\rho_{t_0}(t)}{2}\int_0^t V_{\beta,\gamma_n}(t-t')\partial_x\left(u_n^2\right)dt',$$

$$v = \rho(t)U_\beta(t)\psi + \frac{\rho_{t_1}(t)}{2}\int_0^t U_\beta(t-t')\partial_x\left(v^2\right)dt',$$

where t_0, t_1 is the lifetimes obtained in Propositions 3.3, 3.4. By density argument, we have only to prove estimate (1.7) for sufficiently smooth solutions. Since u_n satisfies (1.5) on $t\in[0,t_0]$, the following equation holds on $t\in[0,t_0]$:

$$u_n(t) = U_\beta(t)\varphi_n + \frac{1}{2}\int_0^t U_\beta(t-t')\left\{\partial_x\left(u_n^2\right) + \gamma_n\partial_x^{-1}u_n\right\}dt'.$$

Put $w_n = u_n - v$. Then, for $0\leqslant t<\min\{t_0,t_1\}$, w_n satisfies the following integral equation:

$$w_n(t) = U_\beta(t)w_n(0) + \frac{1}{2}\int_0^t U_\beta(t-t')\left\{\partial_x\left(w_n(u_n+v)\right) + \gamma_n\partial_x^{-1}u_n\right\}dt'.$$

We put

$$\mu_n = \rho(t)U_\beta(t)w_n(0) + \frac{\rho_{t_2}(t)}{2}\int_0^t U_\beta(t-t')\left\{\partial_x\left(w_n(u_n+v)\right) + \gamma_n\partial_x^{-1}u_n\right\}dt'$$

for $0<t_2<\min\{t_0,t_1\}/2$. Then, it follows that $w_n=\mu_n$ on $t\in[0,t_2]$. From Lemma 2.5, for $b>1/2$, we have

$$\begin{aligned}\left\|\rho_{t_2}(t)\int_0^t U_\beta(t-t')\gamma_n\partial_x^{-1}u_n\,dt'\right\|_{Y_\beta^{0,b}} &= \left\|\rho_{t_2}(t)\int_0^t U_\beta(t-t')\rho_{2t_2}(t')\gamma_n\partial_x^{-1}u_n\,dt'\right\|_{Y_\beta^{0,b}}\\ &\lesssim t_2^{1-b}|\gamma_n|\left\|\rho_{2t_2}\partial_x^{-1}u_n\right\|_{Y_\beta^{0,0}}\\ &\lesssim t_2^{1-b}\|\rho_{2t_2}\|_{L_t^2}\sup_{0\leqslant t\leqslant 2t_2}|\gamma_n|\left\|u_n(t)\right\|_{\dot H_x^{-1}}\\ &\lesssim t_2^{3/2-b}\sup_{0\leqslant t\leqslant 2t_2}|\gamma_n|\left\|u_n(t)\right\|_{\dot H_x^{-1}}.\end{aligned}$$

Therefore, in the same manner as the proof of Proposition 3.3, we obtain

$$\begin{aligned}\|\mu_n\|_{Y_\beta^{0,b}} \lesssim \left\|w_n(0)\right\|_{L^2} + t_2^{\epsilon}\|w_n\|_{Y_\beta^{0,b}}\left(\|u_n\|_{Y_\beta^{0,b}}+\|v\|_{Y_\beta^{0,b}}\right)\\ + t_2^{3/2-b}\sup_{0\leqslant t\leqslant 2t_2}|\gamma_n|\left\|u_n(t)\right\|_{\dot H_x^{-1}}\end{aligned}$$

from Proposition 3.1 and Lemmas 2.4, 2.5. We have $\|u_n\|_{Y_\beta^{0,b}} \leqslant \|u_n\|_{X_{\beta,\gamma_n}^{0,-1,b}} \lesssim \|\varphi_n\|_{H_{\gamma_n}^{0,-1}}$ from Proposition 3.3 and Lemma 2.2 and we have $\|v\|_{Y_\beta^{0,b}} \lesssim \|\psi\|_{L^2}$ from Proposition 3.4. We take $t_2 > 0$ such that $t_2^{\epsilon}M$ is sufficiently small. Then, we obtain

$$\sup_{0\leqslant t\leqslant t_2}\left\|w_n(t)\right\|_{L^2} \lesssim \|\mu_n\|_{Y_\beta^{0,b}} \lesssim \left\|w_n(0)\right\|_{L^2} + t_2^{3/2-b}\sup_{0\leqslant t\leqslant 2t_2}|\gamma_n|\left\|u_n(t)\right\|_{\dot H_x^{-1}}.$$

Fix $T > 1$. Then, from a priori estimate (1.3), we obtain

$$\sup_{0\leqslant t\leqslant t_2}\left\|w_n(t)\right\|_{L^2} \lesssim \left(\left\|w_n(0)\right\|_{L^2} + |\gamma_n|\|\varphi_n\|_{\dot H^{-1}} + T^{2/3}|\gamma_n|^{5/3}\|\varphi_n\|_{L^2}^{5/3}\right).$$

In the same manner, we obtain

$$\sup_{jt_2\leqslant t\leqslant (j+1)t_2}\left\|w_n(t)\right\|_{L^2} \lesssim \left(\left\|w_n(jt_2)\right\|_{L^2} + |\gamma_n|\|\varphi_n\|_{\dot H^{-1}} + T^{2/3}|\gamma_n|^{5/3}\|\varphi_n\|_{L^2}^{5/3}\right)$$

for $0 < (j+1)t_2 \leqslant T$ and $j \in \mathbf{Z}$. Note that we can take the same size of $t_2 > 0$ in this process because we have a priori estimates (1.2) and (1.4). Finally, we obtain

$$\sup_{0\leqslant t\leqslant T}\left\|w_n(t)\right\|_{L^2} \leqslant C_0\left(\left\|w_n(0)\right\|_{L^2} + |\gamma_n|\|\varphi_n\|_{\dot H^{-1}} + T^{2/3}|\gamma_n|^{5/3}\|\varphi_n\|_{L^2}^{5/3}\right),$$

where C_0 depends only on Γ, T and M. □

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