Topology Vol. 13, pp. 219–224. Pergamon Press, 1974, Printed in Great Britain

THE STRUCTURE OF FOLIATIONS WITHOUT HOLONOMY ON NON-COMPACT MANIFOLDS WITH FUNDAMENTAL GROUP $\mathbb{Z}$

C. LAMOUREUX

(*Received* 4 *March* 1973; *revised* 12 *October* 1973)

INTRODUCTION

A CODIMENSION one transversally oriented transversally C^2 foliation $\mathscr{F}$ as in the title, has a structure classified by one of four topological types, according to the position and to the composition of the union U of the leaves of $\mathscr{F}$ met by a closed transversal. These four types can be shown for such foliations on $S^1 \times \mathbb{R}^2$.

The envelope $\bar{U} - U$ of U, if not empty, is a disconnecting closed subset made of closed leaves; the foliations $\mathscr{F}_U$ and $\mathscr{F}_S$ respectively induced by $\mathscr{F}$ on U and on its complementary S are glued along $\bar{U} - U$; the leaf space of $\mathscr{F}_U$ is S^1 and the leaf space of $\mathscr{F}_S$ is a (not Hausdorff) 1-manifold with boundary in the meaning of Haefliger–Reeb.

We study in this work the structure of the foliations $\mathscr{F}$ without holonomy on connected compact or non-compact manifolds X with fundamental group Z. The foliations considered are codimension one, and, say, transversally oriented and transversally C^2 foliations; if the boundary ∂X of X is not empty, the foliation $\mathscr{F}$ of X is supposed to be tangent to the boundary.

In the particular case of a compact manifold without boundary, Sacksteder proved that such a C^2-foliation $\mathscr{F}$ comes from a submersion onto S^1. We gave another proof of this property in [2], using a weaker hypothesis and different methods, see also the generalisations (i) and (ii) at the end of this introduction.

Even in the general case, such a foliation admits only proper leaves; furthermore, the envelope of every non-closed leaf is a non-empty union of closed leaves met by no transversal closed curve. In order to allow for example the generalisation (i) of the end of this introduction we shall rather consider these properties as a direct application of [2].

We first constructed and studied the following four essentially different foliations without holonomy on $S^1 \times \mathbb{R} \times \mathbb{R}$:

—$\mathscr{F}_I$ is just induced from projection to $\mathbb{R}$ and has the factors $S^1 \times \mathbb{R}$ as leaves; $\mathscr{F}_I$ has no closed transversal.

—$\mathscr{F}_{IIa}$ is just induced from projection to S^1 and has the factors $\mathbb{R}^2$ as leaves; $\mathscr{F}_{IIa}$ is defined by a submersion onto S^1 with connected level sets.

—$\mathscr{F}_{\mathrm{IIb}}$ is the foliation $\mathscr{F}(1/2)$ defined in (3) below: $\mathscr{F}_{\mathrm{IIb}}$ admits non-closed leaves all contained in the saturated set of a closed transversal: it can be defined neither by a submersion onto S^1 nor by a real-valued function.

—$\mathscr{F}_{\mathrm{IIc}}$ is the foliation $\mathscr{F}(1/4)$ constructed in (3) below; every leaf of $\mathscr{F}_{\mathrm{IIc}}$ is closed, but there exists a closed transversal meeting only some of its leaves; $\mathscr{F}_{\mathrm{IIc}}$ cannot be defined by a real-valued function nor by a submersion onto S^1 with connected level sets, only by a submersion to S^1 with disconnected level sets.

The main result is the following structure theorem:

THEOREM. *Let $\mathscr{F}$ be a foliation without holonomy of a connected manifold X with fundamental group Z. Let us write*

U *for the union of the leaves of $\mathscr{F}$ met by a closed transversal,*
T *for the union of the closed leaves of $\mathscr{F}$ met by a closed transversal,*
P *for the union of the (proper) non-closed leaves of $\mathscr{F}$,*
S *for the union of the (closed) leaves of $\mathscr{F}$ met by no closed transversal.*

Then:

(i) *There are only the four following types for the foliation $\mathscr{F}$:*

I. *The foliation $\mathscr{F}$ has no closed transversal; this implies $X = S$ and $P = T = U = \varnothing$.*

II. *The foliation $\mathscr{F}$ has a closed transversal meeting*

IIa. *Every leaf of $\mathscr{F}$; this implies $X = T = U$ and $P = S = \varnothing$.*

IIb. *A non-closed leaf of $\mathscr{F}$; this implies $U = P$, $T = \varnothing$ and $S = X - U \neq \varnothing$.*

IIc. *A closed leaf of $\mathscr{F}$, without meeting every leaf of $\mathscr{F}$; this implies $U = T$, $P = \varnothing$ and $S = X - U \neq \varnothing$.*

(ii) *The leaf space X_U of the foliation $\mathscr{F}_U$ induced by $\mathscr{F}$ on U is empty or S^1, and the canonical projection $U \to X_U$ is a submersion with connected level sets.*

(iii) *The leaf space of the foliation $\mathscr{F}_S$ induced by $\mathscr{F}$ on the submanifold with boundary $S = X - U$ is a simply-connected, generally not Hausdorff, 1-manifold with boundary X_S, in the sense of Haefliger–Reeb*[1].

In reading the theorem, one must pay attention to the fact that S (and so X_S) may be empty or reduced to its boundary. In some explicit examples, S may have only one leaf, which has to be considered as the boundary of S. When ∂X is not empty, S and X_S have a non-empty boundary; the same property holds in cases IIb and IIc.

With the notations of the theorem, the foliations $\mathscr{F}_{\mathrm{I}}$, $\mathscr{F}_{\mathrm{IIa}}$, $\mathscr{F}_{\mathrm{IIb}}$ and $\mathscr{F}_{\mathrm{IIc}}$ already defined on $S^1 \times \mathbb{R}^2$ are in fact indexed by their type.

Some preliminary lemmas are given in (1); the structure theorem is proved in (2); in (3) we construct the foliations $\mathscr{F}_{\mathrm{IIb}}$ and $\mathscr{F}_{\mathrm{IIc}}$ on $S^1 \times \mathbb{R}^2$, we add some remarks for the case when X is compact (stability and fibration theorem) and for the existence of seemingly messy examples when X is not compact.

Let us remark that each non-compact manifold X without boundary does admit a foliation with closed leaves, which has then no holonomy. It would be interesting to solve the problem of the existence on a given manifold X, of course non-compact and with fundamental group for example Z, of foliations of types IIa, IIb and IIc: a foliation of type IIc always exists, as kindly communicated by W. P. Thurston.

These results extend without great modifications:

(i) To foliations without holonomy of manifolds with fundamental group for example $Z \oplus G$, when G is finite or without elements of infinite order, and to foliations without holonomy satisfying for example the following condition: the subgroup of $\pi_1(X)$ generated by the free homotopy classes of all the closed transversals is finite or is Z;

(ii) To C^1-foliations (and to many C^0-foliations; of course to any C^0-foliation on $S^1 \times \mathbb{R}$), because the transversality and Poincaré–Bendixson properties, which come in only via [4] and [5], are still valid for C^1-foliations.

§1. PRELIMINARY LEMMAS

In that paragraph, $\mathscr{F}$ is a foliation without holonomy on a connected manifold X with fundamental group Z.

LEMMA 1. *For every closed transversal t and for every leaf F, the set $t \cap F$ is finite.*

Proof. In order to allow generalisation (i) quoted at the end of the introduction, we shall rather consider the number $d(F, \mathscr{F})$ defined in [2]. We have F non-trapped and $d(F, \mathscr{F}) \leq 1$ for any leaf F of $\mathscr{F}$, where we translate "captée" by "trapped"; Lemma 1 is then an assertion already given during the proof of Theorem 1 of [2].

Let us remark that Lemma 1 was a crucial step in the proof of the Theorem 1 of [2]; in fact a (transversally oriented) foliation verifies Lemma 1 *if and only if* each leaf of $\mathscr{F}$ is proper and has an envelope made of closed leaves met by no closed transversal.

LEMMA 2. *Every closed transversal t meets every leaf of U.*

Proof. Because we defined U as the union of the leaves of $\mathscr{F}$ met by a closed transversal, we just have to prove that two closed transversals t and t' meet the same set of leaves. As the fundamental group of X if Z, there exist integers n and n', not both zero, such that the closed curve t^n is freely homotopic to $t'^{n'}$. Because $\mathscr{F}$ has no holonomy, n and n' are both non-zero [5]; then t^n and $t'^{n'}$ build a simple bounding family [5] in X and meet the same set of leaves, after Theorem 1 of [5] or Corollary 3, Chapter V of [3]. In fact we just need here our Proposition 2 in the *Comptes Rendus*, **270**, A1970, p. 1719. Then t and t' meet the same set of leaves because n and n' are both non-zero.

Using Lemmas 1 and 2 we can now prove the following technical lemma:

LEMMA 3. *Any open distinguished neighborhood ϑ of the foliation $\mathscr{F}$ has the following properties:*

(α) *Two different plaques of $F \cap \vartheta$ are separated by a plaque of $G \cap \vartheta$ for any leaf G in U.*

(β) *If the set of the plaques of $F \cap \vartheta$ is not finite, it is countable and has at most two accumulation-plaques in ϑ, which are then respectively the upper and lower bounds of $F \cap \vartheta$ in ϑ.*

Proof. If $F \cap \vartheta$ contains two different plaques P and P', we can "smooth" a transversal arc A joining P to P' in ϑ, into a closed transversal τ_A meeting F: then any leaf meeting A is in U. Lemma 2 implies that τ_A and A meet every leaf G of U: this proves (α). Lemma 1 implies that $A \cap F$ is finite, this proves that $F \cap \vartheta$ is discrete and at most countable. We already saw that any plaque in ϑ lying between two plaques of $F \cap \vartheta$ belongs to a leaf in U: then an accumulation-plaque of $F \cap \vartheta$ in ϑ does not separate $F \cap \vartheta$ because it belongs to a leaf of the envelope of F, which is met by no closed transversal after the property recalled in the introduction. For the same reason an accumulation-plaque of $F \cap \vartheta$ may not lie between two other accumulation-plaques of $F \cap \vartheta$ in ϑ; this proves (β). Lemma 3 implies the following three geometrical lemmas:

LEMMA 4. *If U is not empty, there exists in U a closed transversal meeting every leaf of $\mathscr{F}$ contained in U at exactly one point.*

LEMMA 5. *The face of a leaf G which is in the envelope $\bar{F} - F$ of a leaf F is in the envelope $\bar{H} - H$ of every leaf H contained in U.*

LEMMA 6. *A leaf in $\bar{U} - U$ has only one face in $\bar{U} - U$.*

Proof of Lemmas 4–6. There is in U a closed transversal t meeting a given leaf F in U at exactly one point, after Lemma 1. This closed transversal t meets every leaf G of $\mathscr{F}$ contained in U after Lemma 2, at exactly one point after Lemma 3 and the definition of t. This proves Lemma 4.

For a suitable distinguished neighbourhood ϑ meeting the G of Lemma 5, $G \cap \vartheta$ is the unique accumulation-plaque of $F \cap \vartheta$ in ϑ, after Lemma 3. Furthermore F lies in U. From Property (β) of Lemma 3, $G \cap \vartheta$ is also the unique accumulation-plaque of $H \cap \vartheta$ in ϑ for any leaf H in U. This proves Lemma 5.

Lemma 6 follows directly from the definition of U and from Property (α) in Lemma 3.

§2. PROOF OF THE STRUCTURE THEOREM

Using the notations of the theorem, we may have the four following cases:

I. $U = \varnothing$; this gives $X = S$, $P = T = U = \varnothing$. This happens (only) in case I of the theorem.

IIa. $U = X$; from Lemma 4, we get $T = U$, $P = \varnothing$. Because $S = \varnothing$, this happens (only) in case IIa of the theorem.

IIb. $X \neq U \neq \varnothing$. If U does contain a non-closed leaf of the foliation, Lemma 5 implies that every leaf in U is non-closed. Then $T = \varnothing$. But $X - U = S \neq \varnothing$ because there is a leaf having a non-empty envelope. This happens (only) in case IIb of the theorem.

IIc. $X \neq U \neq \varnothing$ and U contains only closed leaves. After case IIb, $P = \varnothing$. Lemma 4 implies $U = T$. This happens (only) in case IIc of the theorem.

These are the only possible cases, and this proves Part (i) of the theorem. Lemma 4 gives Part (ii) directly. Because of Property (β) in Lemma 3, each distinguished open set ϑ in

X contains at most two plaques which are not in the interior of S. Then the closed set S is a submanifold of X, with boundary if U is not empty and not equal to X. Such a ϑ meets every leaf of $\mathscr{F}_S$ in at most one plaque because S is $X - U$. Then X_S is a 1-manifold with boundary in the sense of Haefliger–Reeb [1]. Furthermore X_S is simply-connected because it is 1-dimensional and because the transversally oriented $\mathscr{F}_S$ has no closed transversal. This proves Part (iii).

§3. REMARKS AND EXAMPLES

Remark 3. When X is *compact*, the closed set S must be also an open set in X, after the local stability theorem of Reeb. Because X is connected, we may have either $X = U$ or $X = S$, i.e. only cases I and IIa may occur. The theorem implies then that P is empty, so we get a *global* stability theorem.

If U is X, $\mathscr{F} = \mathscr{F}_U$ is given by a submersion onto S^1 with connected level sets, after Part (ii) of the theorem, so we get a *fibration* theorem. This fibration theorem is due to Tischler–Sacksteder–Reeb for the C^2-foliations, but is new for C^1-foliations, as well as the method of proof.

If S is X, then X has a non-empty boundary; Part (iii) of the theorem says that the connected (Hausdorff) X_S is a non-empty simply-connected 1-dimensional manifold with boundary which is orientable. So X_S is I, X is a product $X' \times I$ and $\mathscr{F}$ is the product foliation up to a topological conjugacy. We have again stability and a fibration theorem.

Remarks 4.1. In cases IIb and IIc, the leaf space of $\mathscr{F}_S$ is generally not connected; it is always with boundary; it may be very complicated and have for example ends in infinite number.

Remarks 4.2. The foliation $\mathscr{F}$ appears in cases IIb and IIc as glued along the envelope of U, which is a closed set filled in by closed leaves, from the nice foliations $\mathscr{F}_U$ and $\mathscr{F}_S$ in a nice manner due to Lemma 6.

Remarks 4.3. The number of the closed leaves contained in the envelope of U may be any integer equal or superior to 1, it may be infinite. Let us also remark that the Theorem 1 of Chapter III of [3] implies that no leaf of the envelope $\overline{U} - U$ of U is compact. The foliation $\mathscr{F}_U$ behaves like a foliation of type IIa, the foliation $\mathscr{F}_S$ like a foliation of type I.

Remarks 4.4. The leaves of the envelope $\overline{U} - U$ of U may be, all or only some of them, with ends in not finite number and with fundamental group without finite presentation.

From the following foliations $\mathscr{F}(r)$ of the examples below, one can produce new examples illustrating all the affirmations of Remarks 4.1, 4.3 and 4.4:

Examples. Let $\mathscr{R}$ be a C^∞ Reeb foliation on $S^1 \times D^2$, parametrized by $(\theta; \rho, \Psi)$, where $0 \leq \rho \leq 1/2$ and where θ and Ψ are defined modulo 2π. The foliation $\mathscr{R}$ is C^∞-tangent to the boundary. Let $\mathscr{R}'$ be the C^∞-foliation of $S^1 \times \mathbb{R}^2 = S^1 \times D^2 - S^1 \times \partial D^2$, parametrized by (θ, ρ, Ψ), where $0 \leq \rho < 1$ and where θ and Ψ are defined modulo 2π, such that: the restriction of $\mathscr{R}'$ on $\{(\theta, \rho, \Psi); 1/2 \leq \rho < 1\}$ has the compact leaves with equation $\rho = C$; the restriction of $\mathscr{R}'$ on $\{(\theta, \rho, \Psi); \rho \leq 1/2\}$ is the foliation $\mathscr{R}$. Let $G(r)$ be the subset of $S^1 \times \mathbb{R}^2$ consisting of the points (θ, ρ, Ψ) such that $\theta \equiv 0 \bmod 2\pi$ and $0 < r \leq \rho < 1$. Let $\mathscr{F}(r)$ be the foliation induced by $\mathscr{R}'$ on $S^1 \times (D^2 - \partial D^2) - G(r)$, which is homeomorphic to $S^1 \times \mathbb{R}^2$.

The foliation $\mathscr{F}(r)$ is then a foliation on $S^1 \times \mathbb{R}^2$, which is without holonomy when $0 < r \leq 1/2$. The foliation $\mathscr{F}(1/2)$ clearly has type IIb, and the foliations $\mathscr{F}(r)$ have type IIc when $0 < r < 1/2$.

REFERENCES

1. A. Haefliger and G. Reeb: Variétés (non séparées) à une dimension et structures feuilletées du plan, *Enseignement Mathématique* **3** (1957), 107–125.
2. C. Lamoureux: Sur quelques phénoménes de captage, *Annls Inst. Fourier* 23 (1973) 229. Cf. [4].
3. C. Lamoureux: Feuilletages de codimension un des variétés compactes et non compactes. Thesis (1974).
4. C. Lamoureux: *C.r. Acad. Sci. Paris* **274** (1972), 31.
5. C. Lamoureux: Foliations of codimension one of not necessarily compact spaces, *Differentialtopologie*: speziell Blätterungen, Oberwolfach, May 1971.

5 rue Broussais, Paris