A Theory of Condensation Model Reduction

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Abstract—A theory of condensation model reduction establishes conditions under which degrees-
of-freedom can be eliminated from semi-discrete models while retaining response-prediction fidelity
in those degrees-of-freedom that remain. In fact, the full-degree-of-freedom prediction for the original
model can be recovered from the corresponding prediction of the reduced-degree-of-freedom model.
Since continuum models can be made semi-discrete by common, well-understood techniques, the
method has broad applicability. By design, the method is directly implementable on a computer and
easily interfaces with current computational methods such as finite elements. A general condensation
scheme is given first and then specialized to the condensation of generic linear and quadratically
nonlinear dynamic models, the extension to higher order polynomial nonlinearities being straightfor-
ward. Exact results are obtained for the constant coefficient linear dynamic case. As an application,
degree-of-freedom reduction in a spatially discretized model of a deterministic, heterogeneous mater-
ial can be made to correspond to homogenization/smoothing of that material's behavior. In contrast
to the multiple scales and similar homogenization/smoothing methods, the condensation method
does not make use of a periodic media assumption and it fully and directly incorporates boundary
conditions. In continuous-frequency, spatially-discrete applications, such as the structural acoustics
of large, complex systems with realistic, finite-element-modeled geometries, the condensation method
can target specific regions of the spectrum, not necessarily near zero frequency, for which one would
like frequency-response fidelity. It can also function as an alternative to finite element modal decom-
position without the accompanying restrictions on damping. For eigenvalue problems it is shown that
all eigenvalues of the reduced model are also those of the original model. In addition, an eigenvalue
economizer condensation method in current use is shown to be an almost trivial special case of this
approach.

1. INTRODUCTION

Model reduction is the mathematical synthesis of a reduced, practicable mathematical model
from a known-but-intricate, complex mathematical model so that the "essential physics" of the
original model of the physical system is preserved. An operator-based framework was refined
by Flippen [1-3] to serve as a guide in the invention of new model reduction methods. Such
methods can be designed to fill those niches for which currently well-established methods are
either inappropriate or inefficient. All model reduction methods which are based on the framework
of [1-3] possess the important property that, with additional post processing of the reduced
model's response, a reconstruction of the corresponding response of the original, complex model
can be made. An outgrowth of this effort is the condensation model reduction method, which
reduces degrees-of-freedom while retaining response-prediction integrity for semi-discrete, linear
or nonlinear models. This method, which is the main topic of this paper, is an extension of a
method developed earlier [1] for linear, fully discrete models. (In a semi-discrete model, some of
the independent variables are discrete while some of the independent variables may be continuous.

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In the limiting case of a fully discretized model, all of the independent variables are discrete so that, in this paper, fully discrete is a special case of semi-discrete.) Additional methods [4,5] are available for degree-of-freedom reduction in fully discrete models, but there do not seem to be methods available for degree-of-freedom reduction on general semi-discrete models.

The model reduction application which motivated this work is the homogenization/smoothing of microscale constitutive models for deterministic, heterogeneous materials, such as composites. In relation to the model reduction process, if the complexity in the original model is due to intricate spatial heterogeneity and the desired response information is on a much larger scale, then the model reduction process corresponds to homogenization or smoothing [6,7]. Specifically, this includes the synthesis of effective macroscale constitutive relations from specified inclusion-matrix scale constitutive relations. The constitutive relations synthesized reflect an “equivalent,” spatially-smoothed or homogeneous (generally anisotropic) material which responds, on the macroscale, just as the original composite material does. Much of the literature devoted to this topic, however, seemed to be specialized to the point of being both esoteric and counter to the trend towards automated methods (for routine engineering use). A method of performing model reduction, the condensation method of this paper, was hence designed which takes, as input, models which are in the canonical form of semi-discreteness. This offers several advantages, one being that projecting the mathematical model of a physical system into a semi-discrete model is the first step in implementing most computational methods. Hence, the mechanism for translating any given mathematical model into canonical input (i.e., semi-discrete) form for the condensation model reduction method is well understood, commonly available, and easily coupled to existing numerical solution methods, such as finite elements. Another practical advantage is that, at the innermost, basic implementation level, such a model reduction process consists of manipulations on matrices. As matrices are a reasonably familiar topic to the rank and file engineer, this method allows for a more computer-code automated approach with minimal required expertise in homogenization/smoothing or, more generally, in model reduction. The intention was to develop, for working engineers, a method which is to reducing mathematical models what finite elements is to solving such models.

The next section defines more precisely what models are admissible to the general condensation model reduction method, including a qualitative discussion of homogenization/smoothing as it pertains to the method. Section 3 presents the general condensation model reduction method and its domain of applicability. This includes a discussion on smoothing-projector-matrix construction, which is useful for homogenization/smoothing applications. The derivation of the method, in terms of a proof to Theorem 1, and related preliminary background material, are found in the two appendices. In Sections 4 and 5, the development specializes to linear and quadratically nonlinear classes of dynamic models, respectively. These choices are practically important and yet sufficiently simple so as to avoid excessively obscuring details. In the linear dynamic case, the condensation of general first and second order (in time) transient systems is covered first with subsequent specialization to the constant coefficients case. The special but important case of constant coefficients is then more thoroughly covered by the transform condensation method and the transform-parameter neighborhood condensation method. The transform condensation method of Section 4.1.2 provides exact results for this case. In the (computationally less costly) transform-parameter neighborhood condensation method of Section 4.1.3, one can target specific regions of the spectrum, not necessarily near zero frequency, for which one would like frequency-response fidelity in the reduced model. In addition, the degree-of-freedom-reduction capability of transform-parameter neighborhood condensation allows it to be used to eliminate the degrees-of-freedom in substructure interiors and on global boundaries with homogeneous boundary conditions for finite-element-realizable geometries. This combination of frequency-response-fidelity and degree-of-freedom-elimination capabilities for realistic geometries makes transform-parameter neighborhood condensation unique when compared with other promising, currently-used methods, such as [8,9], for the structural acoustics of large, complex systems. A
Linear dynamic condensation is extended to quadratically-nonlinear dynamic condensation in Section 5. The extension of the quadratically nonlinear case to that of higher-order polynomial nonlinearity follows in a straightforward manner. The length of this paper precludes the inclusion of computed results for the application of the method to a specific model. However, demonstrations of the method can be found in [10–12] for the mechanical (transverse) response of a laminate.

2. CAVEATS AND ADMISSIBLE MODELS

The purpose of this section is to delineate precisely what constitutes valid input to the condensation model reduction process. The most important requirement is that the input model be semi-discrete. Some mathematical models are naturally discrete with respect to one (or more) of its independent variables and are hence already in semi-discrete form. Many mathematical models of interest, however, are continuous in all of their independent variables, and hence, each such model must be discretized with respect to one (or more) of its dependent variables before it can undergo condensation model reduction. The decision as to which independent variable (or variables) to discretize is tied to what motivated one to label the model as “complex” in the first place. Model reduction is the synthesis of a “simple” mathematical model from a known, “complex” mathematical model. The condensation model reduction setting provides a less subjective, more precise meaning to “simple” and “complex”; in this case “simple” corresponds to a small number of degrees-of-freedom and “complex” corresponds to a large number of degrees-of-freedom. (Of course “small” and “large” are still somewhat subjective.) Typically the complexity inherent in a physical system manifests itself as an intricate dependence of the corresponding mathematical model on a specific independent variable (or variables). This, in turn, means that discretizing with respect to such a variable (or variables) would typically require a large number of nodes, that is, a fine mesh or grid, in order to accurately capture the relevant physics and maintain adequate response-prediction fidelity with the original continuous model. There are times, however, when the complexity is “hidden” in the model, so that the current set of independent variables does not easily access it. In such a case, one may be able to transform the problem to a new set of independent variables so that the complexity is more directly manifested in terms of the model’s dependence on one (or more) of the new independent variables. As an example, a transform from spatial dependence to modal amplitude dependence may be appropriate in some given case.

2.1. Homogenization/Smoothing of a Dynamic Mechanical System

To illustrate the above reasoning, if one wanted to use the condensation model reduction method to globally homogenize the dynamic mechanical behavior of a composite material, one would discretize the composite spatially with sufficiently fine resolution so as to fully “capture” the heterogeneity of the composite, producing a finite, but large number of (spatial) degrees-of-freedom, with time left as continuous. The resulting coupled set of ordinary differential equations, exemplified by a second order matrix differential equation [13,14]

\[ M \frac{d^2 u}{dt^2} + D \frac{du}{dt} = Su + f, \]  

composed of a mass matrix \( M \), a stiffness matrix \( S \), possibly a damping matrix \( D \), a displacement response \( u \) (column matrix), a load \( f \) (column matrix), and a time derivative operator \( \tau \) (for time \( t \)) such that

\[ \tau = \frac{\partial}{\partial t} \]
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\[ T^2 = TT = \frac{\partial^2}{\partial t^2}, \]

is now in semi-discrete form suitable for input to the condensation model reduction method. (The possibly-time-dependent components of \( M, S, D, u, \) and \( f \) need not necessarily be scalars, the main requirement being that a "suitable" (algebraically at least a ring) addition and multiplication is defined over them.) Upon application of the condensation model reduction method, the resulting model will be another semi-discrete model, but with a reduction in the number of degrees-of-freedom; that is, for (1) it will have smaller mass, stiffness, and, if present, damping matrices. The resulting semi-discrete model in the composite example can be interpreted as a coarse-mesh, spatially-discretized formulation of an equivalent single-phase material whose predictions are smoothed versions of the corresponding original (fine-mesh) model's predictions.

The condensation model reduction method was designed to be more numerically oriented than many of those currently in the literature, but this is not the only niche for which the condensation method was designed. For the homogenization/smoothing of deterministic, heterogeneous materials models, an obvious gap in current capability is found in the periodic-media restriction [15–17]. As examples, there are significant problems, such as embedded actuator/sensor distributions in smart materials, individual specimen-histories of distributed damage evolution, flow of a known aggregate sample, or a given ensemble member in a stochastic material, which are deterministic but not necessarily periodic. In fact, without the periodic-media restriction one might approach stochastic materials via deterministic model reduction either on the "most probable" ensemble member or on each ensemble member (and then averaging over the ensemble). One model reduction method [18] was discovered for which periodicity is not essential and for which direct coupling with finite element methods seems feasible. The condensation model reduction method of this paper follows this same philosophy of excluding any periodic-media assumption from the method development. Hence, though the periodic-media assumption may still be used for computational expediency when it is applicable, it is not an essential, integral part of the condensation method itself, as is in multiple scales [15,16], in [17], or in [19, pp. 253-256] where it is implied by the use of Floquet theory there. Additionally, and in contrast to the global-boundary layers [20] present in the multiple scales method, the condensation model reduction method fully and directly incorporates boundary conditions. This is because boundary conditions are directly incorporated into semi-discrete models which result from the common methods of spatial discretization. The condensation process will hence directly operate on such boundary conditions simultaneously with the rest of the model.

### 2.2. Generic Semi-Discrete Form

Let an arbitrary \( n \times n \) semi-discrete system be generically represented by the equation

\[ Lu = f, \tag{2} \]

where \( L \) is an \( n \times n \) matrix, \( u \) is a \( n \times 1 \) column matrix (or vector), and \( f \) is also \( n \times 1 \). The \( f \) contains both the boundary and the internal (body forces, volumetric heat sources, etc.) stimuli which drive the system response \( u \). In the original discrete model reduction method of [1], equation (2) represents a linear, fully discrete system upon interpreting \( Lu \) as ordinary matrix multiplication of \( L \) times \( u \) and the components of \( L, u, \) and \( f \) as constants. In this paper, equation (2), \( L, u, \) and \( f \) are reinterpreted so that the components of \( u \) and \( f \) are functions of the remaining (continuous) independent variables, and so that each component of \( L \) is a mathematical operator which, in general, may itself depend upon the components of \( u \) (and perhaps their derivatives). This allows for overall nonlinearity in equation (2) if it is needed. The \( i^{th} \) row of (2) should hence be interpreted as

\[ (Lu)_i = L_{i1}(u_1) + L_{i2}(u_2) + \ldots + L_{in}(u_n), \tag{3} \]
where \( L_{i1}(u_1) \) denotes the component \( L_{i1} \) operating on the function \( u_1 \), for example, and where each \( L_{ij} \) could itself be dependent on one or more of the \( u_j \)'s. As an example, \( L_{ij}(g) \), for arbitrary function \( g \) (within the domain of component \( L_{ij} \)), could represent a specific linear combination of the \( u_j \)'s times \( g \). In order for (2) to qualify as a candidate for condensation model reduction, each component operator of \( L \) must have the property that it returns zero when operating on the constant function zero, that is

\[
L_{ij}(0) = 0
\]

(4)

for all \( i \) and \( j \). (This is automatically satisfied for linear operators.) The (original) special, limiting case of a linear, fully discrete system is obtained when each component operator of \( L \) reduces to multiplication by a constant and when there are no continuous independent variables, so that the components of \( u \) and \( f \) are constants.

The specific example of a semi-discrete system given by (1) can be put into the generic form of (2) upon taking \( L \) to be

\[
L = MT^2 + DT - S.
\]

(5)

The specific \( L \) given by (5) is clearly a matrix of operators upon interpreting the \( ij \)th component of (5) as

\[
M_{ij} \frac{\partial^2}{\partial t^2} + D_{ij} \frac{\partial}{\partial t} - S_{ij}.
\]

In this specific case, the components of \( u \) and \( f \) are then functions of time.

As (3) might appear ambiguous with respect to the multiplication of two matrices, its definition is generalized here to

\[
(AB)_{ij} = \sum_k A_{ik} \circ B_{kj}
\]

(6)

for compatible matrices \( A \) and \( B \), where the symbol \( \circ \) denotes mapping composition, that is, \( A_{ik} \circ B_{kj} \) applied to some function \( g \) is interpreted as \( A_{ik}(B_{kj}(g)) \). In the special case where \( B \) is a matrix of functions (not operators), such as \( u \) in (2), then \( \circ \) in (6) is interpreted to mean \( A_{ik}(B_{kj}) = A_{ik}(B_{kj}) \) so that (6) is compatible with (3). Similarly, the requirement (4) is generalized here to

\[
A_{ij} \circ \bar{0} = \bar{0}
\]

(7)

for all \( i \) and \( j \) for any given matrix \( A \), where \( \bar{0} \) is the zero operator, for which everything in its domain is mapped into the zero function. The adoption of (6) and (7) precludes the use of ordinary matrix algebra as one is accustomed to. Appendix A gives the preliminary mathematical tools required in this setting to develop the general condensation method.

3. GENERAL CONDENSATION

The purpose of this section is to give the condensation model reduction method in theorem form so that the conditions under which the method is rigorous are clearly stated. The proof of the theorem, given in Appendix B, then constitutes a derivation of the method. The \( I \)'s and \( 0 \)'s used in this paper, which vary in size according to context, are analogous to the identity and zero matrices of ordinary matrix algebra, but are defined in Appendix A, Section A.2 so as to be compatible with (6) and (7). For those who are not familiar with permutation matrices, see Appendix A, Section A.2.1.

\[
\alpha
\]

Theorem 1. Assume that the square matrix \( L \) of (2) is given and that it obeys (6) and (7). Choose a matrix \( \alpha \) which also obeys (6) and (7) and from which the matrix

\[
\bar{P} = \begin{pmatrix} I & 0 \\ \alpha & 0 \end{pmatrix}
\]

(8)
can be formed such that \( \bar{P} \) itself and the \( I \) in (8) are both square, and such that \( \bar{P} \) is the same size as \( L \). Choose any permutation matrix \( \mathcal{P} \), along with its companion permutation matrix \( \mathcal{P}^{-1} \), such that \( \mathcal{P} \) is the same size as \( L \). Define the matrices

\[
P = \mathcal{P}^{-1} \bar{P} \mathcal{P} \\
\Pi = \begin{pmatrix} I & 0 \end{pmatrix} \mathcal{P} \\
\bar{L} = \mathcal{P} L \mathcal{P}^{-1}
\]

so that the submatrix \( I \) in (10) is the same \( I \) as in (8), that is, \( \Pi \) is the “top part” of \( \bar{P} \) times \( \mathcal{P} \).

Partition \( \bar{L} \)

\[
\bar{L} = \begin{pmatrix} \bar{L}_{11} & \bar{L}_{12} \\ \bar{L}_{21} & \bar{L}_{22} \end{pmatrix}
\]

the same way as in (8), that is, so that \( \bar{L}_{11} \) and the \( I \) in (8) are the same size, and similarly for the other submatrices. Solve the “condensation equations”

\[
\bar{L}_{21} \Omega + \bar{L}_{22} \beta = \alpha (\bar{L}_{11} + \bar{L}_{12} \beta)
\]

for the matrix (of operators) \( \beta \) and then define the matrices

\[
L_0 = \bar{L}_{11} + \bar{L}_{12} \beta \\
\Omega = \mathcal{P}^{-1} \begin{pmatrix} I \\ \beta \end{pmatrix}
\]

so that the submatrix \( I \) in (15) is the same \( I \) as in (8).

- The matrix \( \mathcal{P} \) is idempotent, that is, \( \mathcal{P} \mathcal{P} = \mathcal{P} \).
- For any \( f \) for which \( \mathcal{P} f = f \),

if \( v \) is a solution to

\[
L_0 v = \Pi f
\]

then

\[
u = \Omega v
\]

is a solution to (2).

The parentheses on the right side of (13) are there merely to signify that the \( \alpha \) cannot generally be distributed over the addition. There are no implications regarding multiplicative associativity intended. In fact, the rigorous interpretation of the left side of (13) is

\[
\bar{L}_{21} C + \bar{L}_{22} (\beta C)
\]

for a compatible (with respect to multiplication) but otherwise arbitrary matrix \( C \), and that of the right side of (13) is

\[
\alpha (\bar{L}_{11} C + \bar{L}_{12} (\beta C)).
\]

The equations (2) and (17) are labeled as the “original” and “reduced” models, respectively. Those \( u \) which are obtained from solving (17) and then using (18) are called “reconstructed” solutions, as opposed to “direct” solutions, which are obtained by directly solving (2). Additionally, it may be convenient elsewhere to refer to the “filtered” solution \( \tilde{u} \), defined as

\[
\tilde{u} = \mathcal{P}^{-1} \begin{pmatrix} v \\ \alpha v \end{pmatrix}
\]
for any given \( v \) which solves (17), so that \( \tilde{u} = Pu \) for solutions \( u \) to (2). (It could also be called the "smooth" solution when \( P \) represents a smoothing matrix.) It is named "filtered" solution because \( P\tilde{u} = \tilde{u} \), so that if \( P \) is considered as a filter then \( \tilde{u} \) is "passed" by the filter.

The size of \( \tilde{P} \) and \( P \) (both square) is fixed to be the same size as \( L \), and hence the user has no control over their size after discretization of the model has occurred. The size of \( L_0 \), and hence the size of the reduced model (17), is determined by the size of the (square) \( I \) submatrix in (8). This the user does have control over since the choice of \( \alpha \) by the user fully determines the \( I \) submatrix in (8). This is because the total number of rows in \( I \) and \( \alpha \) together, in (8), is constrained to be the same fixed number of rows as in \( L \), and hence there is a tradeoff between how big one chooses \( \alpha \) to be and how big or small the \( I \) submatrix in (8) becomes in response.

### 3.1. Tailoring General Condensation

The general condensation equations (13) are both fundamental and central to the method since their solution, in any given case, presents the main obstacle to the successful application of the method. The technique used to solve the condensation equations is mainly what differentiates how the method is applied from problem to problem, or from class of problems to class of problems. It must be emphasized that the unknown, to-be-determined entities in the general condensation equations, the components of \( \beta \), are operators. This is to be contrasted to the more familiar case where one solves a set of governing equations to obtain a function, or functions, as the solution. If (13) is not solved exactly, so that one only has an approximate \( \beta \), then the direct solution to (2) and the reconstructed solution will not exactly agree. The relationship between errors in (13) and the "distance" between the direct and reconstructed solutions (with respect to some metric) has not been investigated in this paper, although it certainly is an important topic. A similar statement holds regarding the relationship between the "distance" between \( Pf \) and \( f \), for some given \( f \), and that between the corresponding direct and reconstructed solutions.

One's choice of \( \alpha \) and \( P \), which in turn determino \( P \), is governed by what range of stimuli (set of \( f \)'s) one is going to allow to drive/excite the original model (2) and the reduced model (17). This is because of the requirement (16). The set of allowable stimuli (loads, sources, or whatever physics with which one labels \( f \)) corresponds mathematically to the range of \( P \). (The range of a mapping is the largest set onto which it maps. For an idempotent mapping, as \( P \) is, the set of all \( f \) such that \( Pf = f \) is identical with the range of \( P \) [1].) One's choice of \( P \) should hence be such that the range of \( P \) includes all, or as much as possible, of the desired stimuli. As an example, in homogenizing/smoothing a model of a heterogeneous material, it makes sense to consider only spatially smooth stimuli, and hence, a spatially smoothing \( P \) is appropriate. The components of \( \alpha \) are operators, and hence, one has a fair amount of flexibility in tailoring \( P \) to one's needs. There is a distinct advantage, however, in keeping a homomorphic (see Appendix A, Section A.1, Theorem 3 for a definition) since then (13) reduces to

\[
\left( \tilde{L}_{22} - \alpha \tilde{L}_{12} \right) \beta = \alpha \tilde{L}_{11} - \tilde{L}_{21}
\]

because of Theorem 3. There is one further point worth noting for the common case where the semi-discrete model results from a finite element or finite difference spatial discretization. If a node on a global boundary has a homogeneous boundary condition imposed there, then the corresponding component of \( f \) is, in such a case, always zero. The matrix \( \alpha \) can always be extended to encompass such a node by adding a row of zeros to \( \alpha \) at the row location corresponding to that component of \( f \). (Values in \( f \) and \( \alpha \) are related by (16), (8), and (9).) That means that such nodes are potential candidates for elimination in any substructuring scheme which is based on this condensation method.

#### 3.1.1. Smoothing-projector choices for \( P \)

An important class of choices for \( P \) consists of smoothing matrices. Such projector matrices are used in homogenization/smoothing applications, for example. For convenience and conciseness,
finite element terminology will be used in describing the construction of such matrices. Any given finite element discretization of the $k$ independent variables $r_j$, for $j = 1, 2, \ldots, k$, can be specified by $\hat{r}$, $N$, and $a$, where the components of $\hat{r}$ are defined by

$$
\hat{r}_i = (r_1 \times r_2 \times \cdots \times r_k)_{\text{value at node } i}
$$

for $i = 1, 2, \ldots, s$, $s$ being the total number of nodes. The $\times$ symbol represents a Cartesian product so that $\hat{r}_i$ is a $k$-tuple representing the “location” of the node labeled by $i$. The continuous dependent variables $u$ are approximated [14, p. 431] by the components of $N$ and $a$ as

$$
u(r_1, r_2, \ldots, r_k) = Na = \sum_{i=1}^{s} N_i(r_1, r_2, \ldots, r_k) a_i,
$$

where $N_i$ is a function of the $r_j$'s for each $i$ and where $a_i$ represents the value of $u$ at the $i^{th}$ node. The components of $N$ are called shape functions. The notation $N(\hat{r})$, defined by

$$
[N(\hat{r})]_{ij} = N_j(\hat{r}_i)
$$
on a component basis, is convenient. The fundamental property of shape functions can hence be compactly expressed as

$$
N(\hat{r}) = I
$$

when the set of nodes for $N$ and $\hat{r}$ are identical. Property (24) and equation (22) lead to

$$
u(\hat{r}) = N(\hat{r}) a = a
$$

when the set of nodes for $N$, $a$, and $\hat{r}$ are identical, which is consistent with the definition of the components of $a$.

With the above notation the construction of a smoothing $P$ can now be described. Let $V$ denote a fixed “region” of $r_1 \times r_2 \times \cdots \times r_k$. (A rigorous definition of “region” could be the union of a bounded, open subset of $r_1 \times r_2 \times \cdots \times r_k$ with its own boundary.) Let $N_f$, $a_f$, and $\hat{r}_f$ represent a “fine mesh” discretization over $V$. Similarly, let $N_c$, $a_c$, and $\hat{r}_c$ represent a “coarse mesh” discretisation over the same $V$ such that the nodes of $\hat{r}_c$ are a subset of the nodes of $\hat{r}_f$. Let $P$ be a permutation matrix such that

$$
P\hat{r}_f = \begin{pmatrix} \hat{r}_c \\ \hat{r}_b \end{pmatrix},
$$

$$
P a_f = \begin{pmatrix} a_c \\ a_b \end{pmatrix},
$$

and similarly for $N_f$. Define the matrix $\alpha$ by

$$
\alpha = N_c(\hat{r}_b)
$$

for $\hat{r}_b$ from (26), and note that one also has

$$
N_c(\hat{r}_c) = I
$$

by (24). With the above choices of $P$ and $\alpha$, the $P$ constructed by (8) and (9) is a matrix which smooths by interpolation on the “coarse mesh.” Defining $u_f$ by using $N_f$ and $a_f$ in (22), and similarly defining $u_c$, this can be proven by

$$
P u_f(\hat{r}_f) = P a_f
$$

$$
-P^{-1} \begin{pmatrix} I & 0 \\ \alpha & 0 \end{pmatrix} P a_f
$$
\[ \begin{align*}
&= p^{-1} \begin{pmatrix} N_c (\hat{\tau}_c) & 0 \\ N_c (\hat{\tau}_b) & 0 \end{pmatrix} \begin{pmatrix} a_c \\ a_b \end{pmatrix} \\
&= p^{-1} \begin{pmatrix} N_c (\hat{\tau}_c) a_c \\ N_c (\hat{\tau}_b) a_c \end{pmatrix} \\
&= p^{-1} N_c \begin{pmatrix} \hat{\tau}_c \\ \hat{\tau}_b \end{pmatrix} a_c \\
&= p^{-1} N_c (\hat{\tau}_f) a_c \\
&= N_c (\hat{\tau}_f) a_c \\
&= u_c (\hat{\tau}_f)
\end{align*} \]

using (25), (9), (8), (28), (29), (27), and then (26), in that order.

In the case of a single discrete independent variable, a reasonable choice of shape functions corresponds to a cubic spline interpolation. The shape functions for a cubic spline are the components of

\[ N_c = A + (C^{-1} E) \begin{pmatrix} A \end{pmatrix} B, \]  

where the components of \( A, B, C, \) and \( E \) are defined by

\[ A(r) \begin{cases} 
\frac{(\hat{\tau}_c)_{j+1} - r}{h_j} & \text{for } (\hat{\tau}_c)_{j+1} < r < (\hat{\tau}_c)_j \text{ and } 1 \leq j \leq s_c - 1 \\
\frac{r - (\hat{\tau}_c)_{j-1}}{h_{j-1}} & \text{for } (\hat{\tau}_c)_{j-1} < r < (\hat{\tau}_c)_j \text{ and } 2 \leq j \leq s_c \\
0 & \text{otherwise}
\end{cases} \]  

\[ B(r) \begin{cases} 
\frac{[(\hat{\tau}_c)_{j+1} - r]^3}{6h_j} - \frac{h_j}{6} (\hat{\tau}_c)_{j+1} & \text{for } (\hat{\tau}_c)_j < r < (\hat{\tau}_c)_{j+1} \text{ and } 1 \leq j \leq s_c - 1 \\
\frac{r - (\hat{\tau}_c)_{j-1}}{6h_{j-1}} - \frac{h_{j-1}}{6} (\hat{\tau}_c)_{j-1} & \text{for } (\hat{\tau}_c)_{j-1} < r < (\hat{\tau}_c)_j \text{ and } 2 \leq j \leq s_c \\
0 & \text{otherwise}
\end{cases} \]  

\[ C_{ij} = \begin{cases} 
1 & \text{for } j = i = 1 \text{ or } j = i = s_c \\
2(h_j + h_{j-1}) & \text{for } j = i \text{ and } 2 \leq i \leq s_c - 1 \\
h_j & \text{for } j = i - 1 \text{ and } 2 \leq i \leq s_c - 1 \\
h_{j-1} & \text{for } j = i + 1 \text{ and } 2 \leq i \leq s_c - 1 \\
0 & \text{otherwise}
\end{cases} \]  

\[ E_{ij} = \begin{cases} 
- \frac{6}{h_j} & \text{for } j = i \text{ and } 2 \leq i \leq s_c - 1 \\
\frac{6}{h_j} & \text{for } j = i - 1 \text{ and } 2 \leq i \leq s_c - 1 \\
0 & \text{otherwise}
\end{cases} \]  

\[ h_j \text{ is defined by } h_j = (\hat{\tau}_c)_{j+1} - (\hat{\tau}_c)_j, \]

and where \( s_c \) is the total number of nodes in \( \hat{\tau}_c \).

The boundary nodes corresponding to nonhomogeneous boundary conditions are usually included in \( \hat{\tau}_c \). The \( Pf = f \) requirement of (16) must be taken into account, however, in making the full choice of \( \hat{\tau}_c \). The components of \( f \) corresponding to the boundary and the
components of \( f \) corresponding to the interior are usually quite different in value and/or interpretation. Hence, interpolations over the interior for \( f \) will often need to exclude some or all of the boundary nodes since values of \( f \) there can interfere with the interpolation of interior values. From a practical point of view, this means that one sometimes has to carry along some adjacent-to-the-boundary interior nodes in \( \hat{r}_c \) as well, which will act as boundary nodes with respect to the interior interpolation.

### 3.2. Overview of Application Process

The equations of Theorem 1 can now be reorganized so as to present the condensation model reduction method as a procedure. Assuming that the model is semi-discrete, put the model to be processed into the form (2) so that \( L \) is known and satisfies (6) and (7). Choose the matrix \( \alpha \) and the permutation matrix \( P \), which in turn determine \( P \) via (8) and (9), such that the range of \( P \) includes all, or as much as possible, of the desired stimuli and such that \( \alpha \) satisfies (6) and (7). Next form \( \tilde{L} \) according to (11) and then partition it according to (12). Solve the condensation equations, which are given by (20) if \( \alpha \) is homomorphic or by (13) if it is not homomorphic, for \( \beta \) so that \( \beta \) is now known, at least approximately. Determine, using \( \beta \), the reduced model matrix \( L_0 \) and the reconstruction matrix \( \Omega \) by (14) and (15), respectively. The condensation process is now complete. For any given stimulus \( f \) from the range of \( P \) one obtains the reduced model stimulus \( \hat{f} \) using \( \hat{f} = \Pi u \). With \( u \) denoting the solution to (2) for the same \( f \), each component of \( v \) equals some component of \( u \) via \( v = \Pi u \). If this is all that one requires, one can stop with the \( v \) solution. If one desires a “filtered” solution one can use (19) on \( v \). If, instead, one desires the full solution \( u \) to the original problem, one can use (18) on \( v \) to “reconstruct” it.

### 3.3. Initial Conditions

The initial condition for \( v \), if required in solving (17), is

\[
v_0 = \Pi u_0,
\]

where the subscript 0 denotes “at time \( t = 0 \).” (Similar statements hold if initial values of \( T u, T^2 u, \) etc. are required, where \( T \) is the time derivative operator, so that \( (Tu)_0 = \Pi (Tu)_0 \) for example.)

If the example of the transform condensation method of Section 4.1.2 is representative, then including the initial conditions directly in the formulation of (2) apparently determines the initial value of \( \beta \) and, if required, that of one or more of its time derivatives. If the initial conditions are not explicitly included in (2) so as to be processed directly by the condensation procedure, then, assuming that the condensation equations corresponding to \( L \) have an evolutionary nature (as does \( L \)), one presumably has freedom in the initial value choice of \( \beta \) (and any required time derivatives). In such cases, one could utilize this resulting freedom to force the reconstructed and direct solutions to agree at time \( t = 0 \). In fact, the choice of initial \( \beta \) can be made to coincide with a whole set of desired initial-condition choices \( u_0 \). This is in the same spirit as \( P \) being chosen with respect to the desired set of possible stimuli. Similar statements could be made with respect to each initial value of any required time derivatives of \( \beta \).

Whenever \( v_0 \) is taken from (36), the set of possible initial conditions \( u_0 \) associated with any given \( \beta_0 \) corresponds to the null space of the square matrix

\[
\begin{pmatrix}
0 & 0 \\
\beta_0 & -I
\end{pmatrix} P,
\]

where \( \beta_0 \) is the initial value of \( \beta \). (The null space of a matrix \( A \) is the set of all column matrices \( x \) such that \( Ax = 0 \).) Alternatively, this null space corresponds precisely to the range of the right
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Side of (8) for which the value of $\alpha$ has been taken to be $\beta_0$. A typical element of this null space would have the form

$$\mathbf{p}^{-1}
\begin{pmatrix}
v_0 \\
\beta_0 v_0
\end{pmatrix}
$$

for some specific $v_0$. As an example, by taking $\beta_0$ to be the solution of the condensation equations (13) corresponding to the steady-state version of $L$, the $\beta$ one then finds for the transient version of $L$, with $\beta_0$ as its initial value, should be compatible with all $u$ satisfying the steady-state version of (2) for times $t \leq 0$. Another example consists of choosing $\beta_0$ to be $\alpha$ so that the set of eligible initial values $v_0$ corresponds to the set of eligible stimuli (the range of $P$). The initial condition issue probably deserves further investigation.

4. LINEAR DYNAMIC RESPONSE CONDENSATION

The condensation model reduction method will now be specialized to the case of a general second order, linear dynamic-response problem for which the spatial dependence has been discretized. The governing semi-discrete system has already been given as (1), so that the matrix $L$ is given by (5). All coupled sets of higher order ordinary differential equations can be put in the form of (1) with $M = 0$, that is, a coupled set of first order ordinary differential equations. In this sense all such (spatially discretized) linear dynamic-response problems are hence covered by considering (1) alone.

In this case, all of the component operators of $L$ are linear, so that $L$ is homomorphic (see Appendix A, Section A.1, Theorem 3 for a definition). If the $\alpha$ matrix in (8) is chosen to be homomorphic as well, then the condensation equations solution $\beta$ will, in such cases, also be homomorphic. One is then free to use the distributive and multiplicative associative properties of matrices in the familiar way in the development to follow. In order to keep this problem as simple as possible, the choice for the $\alpha$ matrix in (8) will, in this case, be that of an ordinary, constant matrix. Assuming that the $(\mu', \nu^{-1})$ pair is fixed (chosen), define the matrices

$$\tilde{M} = \mathcal{P}MP^{-1}$$

$$\tilde{D} = \mathcal{P}DP^{-1}$$

$$\tilde{S} = \mathcal{P}SP^{-1}$$

and insert (5) into (11) to get

$$L = \tilde{M}T^2 + \tilde{D}T - \tilde{S}$$

for the time derivative operator $T$ (for time $t$) such that

$$T = \frac{\partial}{\partial t}$$

$$T^2 = \frac{\partial^2}{\partial t^2}.$$

Similarly, the partitioning of $\tilde{L}$ in (12) translates into a corresponding partitioning of each of the matrices $\tilde{M}$, $\tilde{D}$, and $\tilde{S}$. The condensation equations (20) hence specialize, for this case, to

$$\begin{aligned}
\left(\tilde{M}_{22}T^2 + \tilde{D}_{22}T - \tilde{S}_{22}\right) - \alpha \left(\tilde{M}_{12}T^2 + \tilde{D}_{12}T - \tilde{S}_{12}\right) & = \beta \\
\end{aligned}$$

$$\begin{aligned}
\tilde{M}_{11}T^2 + \tilde{D}_{11}T - \tilde{S}_{11} & = \left(\tilde{M}_{21}T^2 + \tilde{D}_{21}T - \tilde{S}_{21}\right) \\
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\left(\tilde{M}_{22} - \alpha\tilde{M}_{12}\right)T^2\beta & + \left(\tilde{D}_{22} - \alpha\tilde{D}_{12}\right)T\beta + \left(\alpha\tilde{S}_{12} - \tilde{S}_{22}\right)\beta \\
& = \left(\alpha\tilde{M}_{11} - \tilde{M}_{21}\right)T^2 + \left(\alpha\tilde{D}_{11} - \tilde{D}_{21}\right)T + \left(\tilde{S}_{21} - \alpha\tilde{S}_{11}\right). \\
\end{aligned}$$
The components of $M$, $D$, and $S$, and hence of $\tilde{M}$, $\tilde{D}$, and $\tilde{S}$ and their submatrices, are generally functions of time. The corresponding system matrix $L_0$ for the reduced system (17) is

$$L_0 = \left( \tilde{M}_{11} T^2 | \tilde{D}_{11} T \cdot \tilde{S}_{11} \right) + \left( \tilde{M}_{12} T^2 + \tilde{D}_{12} T - \tilde{S}_{12} \right) \beta.$$  \hfill (43)

Further progress depends upon one being able to solve (42) for $\beta$.

One straightforward scheme for solving (42) is to assume that $\beta$ has the form of an infinite series

$$\beta = \sum_{k=0}^{\infty} B_k T^k,$$  \hfill (44)

where the components of each coefficient matrix $B_k$ are generally to-be-determined functions of time. Note that (44) and the linearity of $T$ guarantee that the requirement (7) is satisfied for $\beta$.

As the next step, insert the series (44) into (42), use

$$TB_i T^j = (TB_i) T^j + B_i T^{j+1}$$

in collecting terms, and equate coefficients of equal powers of $T$ (on each side of the equation) to each other. (The parentheses in $(TB_i)$ denote that the scope of the action of the $T$ is limited to the $B_i$ matrix components only.) The results of this process are

$$\left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) (T^2 B_0) + \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right) (TB_0) + \left( \alpha \tilde{S}_{12} - \tilde{S}_{22} \right) B_0 = \left( \tilde{S}_{21} - \alpha \tilde{S}_{11} \right)$$  \hfill (45)

as the governing equation for $B_0$,

$$\left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) (T^2 B_1) + \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right) (TB_1) + \left( \alpha \tilde{S}_{12} - \tilde{S}_{22} \right) B_1 = \left( \alpha \tilde{D}_{11} - \tilde{D}_{21} \right) - \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right) B_0 - 2 \left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) (TB_0)$$  \hfill (46)

as the governing equation for $B_1$, and

$$\left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) (T^2 B_2) + \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right) (TB_2) + \left( \alpha \tilde{S}_{12} - \tilde{S}_{22} \right) B_2 = \left( \alpha \tilde{M}_{11} - \tilde{M}_{21} \right) - \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right) B_1 - 2 \left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) (TB_1) - \left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) B_0$$

as the governing equation for $B_2$, and

$$\left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) (T^2 B_i) + \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right) (TB_i) + \left( \alpha \tilde{S}_{12} - \tilde{S}_{22} \right) B_i = - \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right) B_{i-1} - 2 \left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) (TB_{i-1}) - \left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) B_{i-2}$$  \hfill (48)

as the governing equation for $B_i$ for $i > 2$. This paper will not attempt to find the conditions under which the above solution scheme is guaranteed to converge, as useful as knowing such conditions may be, but will instead take the pragmatic viewpoint that extensive practical experience will ultimately determine their scope of utility.

In order to get reduced equations of the same form as the original governing equations (1), equations (44) and (43) will be truncated at the order of the original governing equations. The truncated (44) becomes

$$\beta \approx B_0 + B_1 T + B_2 T^2,$$  \hfill (49)

which leads to

$$T \beta \approx (TB_0) + [B_0 + (TB_1)] T + [B_1 + (TB_0)] T^2$$

$$T^2 \beta \approx (T^2 B_0) + [(T^2 B_1) + 2 (TB_0)] T + [B_0 + 2 (TB_1) + (T^2 B_2)] T^2.$$
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Inserting these into (43) and truncating to second order leads to

\[ L_0 \approx \tilde{M}_{\text{red}} T^2 + \tilde{D}_{\text{red}} T - \tilde{S}_{\text{red}} \]  

as the reduced version of (5), where

\[ \tilde{M}_{\text{red}} = \left( \tilde{M}_{11} - \tilde{S}_{12} B_2 \right) + \tilde{M}_{12} \left[ D_0 + 2 (T B_1) + (T^2 B_2) \right] + \tilde{D}_{12} \left[ B_1 + (T B_2) \right] \]  

\[ \tilde{D}_{\text{red}} = \left( \tilde{D}_{11} - \tilde{S}_{12} B_1 \right) + \tilde{D}_{12} \left[ (T^2 B_1) + 2 (T B_0) \right] + \tilde{D}_{12} \left[ B_0 + (T B_1) \right] \]  

\[ \tilde{S}_{\text{red}} = \left( \tilde{S}_{11} + \tilde{S}_{12} B_0 \right) - \tilde{M}_{12} \left( T^2 B_0 \right) - \tilde{D}_{12} \left( T B_0 \right) \]

(51)  

(52)  

(53)

gives the reduced counterpart \( \tilde{M}_{\text{red}} \) to \( \tilde{M} \), and similarly for \( \tilde{D}_{\text{red}} \) and \( \tilde{S}_{\text{red}} \). One would determine \( B_0, B_1, \) and \( B_2 \) from (45), (46), and (47), respectively, and then insert the results into (51), (52), and (53) to obtain the reduced versions of the (permuted) mass, damping, and stiffness matrices, respectively.

For the sake of more efficient and convenient solution reconstruction, the solution reconstruction process for times \( t > 0 \) is explicitly developed. If \( u \) is a solution to (17) for which a reconstructed solution \( \tilde{u} \) is desired, then \( u \) satisfies

\[ T^2 u \approx \tilde{M}_{\text{red}}^{-1} \tilde{S}_{\text{red}} v - \tilde{M}_{\text{red}}^{-1} \tilde{D}_{\text{red}} T v + \tilde{M}_{\text{red}}^{-1} \Pi f \]

to second order, for times \( t > 0 \), because of (50). This can be substituted into (49) operating on \( v \) to get

\[ \beta v \approx C_0 \Pi f + C_1 T v + C_2 v \]

upon defining the matrices

\[ C_0 = B_2 \tilde{M}_{\text{red}}^{-1} \]  

\[ C_1 = B_1 - C_0 \tilde{D}_{\text{red}} \]  

\[ C_2 = B_0 + C_0 \tilde{S}_{\text{red}}. \]

(54)  

(55)  

(56)

The reconstruction process given by

\[ u \approx P^{-1} \begin{pmatrix} v \\ C_0 \Pi f + C_1 T v + C_2 v \end{pmatrix} \]

(57)

follows from inserting (15) into (18) and then using the above result for \( \beta v \). Equation (57), valid for \( t > 0 \), is convenient for \( L_0 \) given by (50) because one then usually has direct access to \( T v \) as well as \( v \) in the process of obtaining \( v \) from (17). It is especially convenient when the matrices involved are constant, since \( C_0, C_1, \) and \( C_2 \) need be computed only once prior to solving (17). The analogous expression for the reconstruction of the velocity \( T u \) is

\[ T u \approx P^{-1} \begin{pmatrix} T v \\ W_0 \Pi f + W_1 T v + W_2 v \end{pmatrix} \]

(58)

for which the coefficient matrices

\[ W_0 = \left[ B_1 + (T B_2) \right] \tilde{M}_{\text{red}}^{-1} \]  

\[ W_1 = B_0 + (T B_1) - \left( W_0 \tilde{D}_{\text{red}} \right) \]  

\[ W_2 = (T B_0) + \left( W_0 \tilde{S}_{\text{red}} \right) \]

(59)  

(60)  

(61)

were obtained from applying the expression for \( T \beta \) to \( v \) and then substituting the above expression for \( T^2 v \).
4.1. Constant Matrices Special Case

The common and important special case of constant $M$, $D$, and $S$, and hence constant $\tilde{M}$, $\tilde{D}$, and $\tilde{S}$, will now be investigated. There are three approaches which will be developed, the choice between them in a given case depending upon what one is after in that particular application. If one wants to follow a transient in the time domain, with the emphasis on the lower part of the frequency spectrum, one might use the time-derivative series method. It is basically a specialization of the time-derivative series method already given by (44). Exact, full-spectrum results for this (constant matrices) case are provided by the transform condensation method, but the cost is probably high for all but modest sized problems. If, instead, one wants to target specific regions in the frequency spectrum, not necessarily near zero frequency, for which one would like frequency-response fidelity in the reduced model at more modest cost, then one might prefer the transform-parameter neighborhood condensation method. For much of the spectrum the latter method avoids complex computation, which is also one of the main advantages in the use of modal decomposition [14, pp. 561–564]. Unlike modal decomposition [14, pp. 532, 563–564], however, transform-parameter neighborhood condensation does not place any restrictions on the damping matrix. By targeting the lower end of the spectrum, the transform-parameter neighborhood condensation method can hence function as an alternative to modal decomposition. In addition, as an alternative to globally reducing the entire model, each of the three methods could be used locally as a substructuring technique.

4.1.1. Time-derivative series approach

It makes sense to look for constant $B_i$ solutions to (44), for $i = 0, 1, 2, \ldots$, when the special case of constant $M$, $D$, and $S$, and hence constant $\tilde{M}$, $\tilde{D}$, and $\tilde{S}$, arises. The results for (45) through (48) are

$$B_0 = \left(\alpha \tilde{S}_{12} - \tilde{S}_{22}\right)^{-1} \left(\tilde{S}_{21} - \alpha \tilde{S}_{11}\right)$$  \hspace{1cm} (62)

as the governing equation for $B_0$,

$$B_1 = \left(\alpha \tilde{S}_{12} - \tilde{S}_{22}\right)^{-1} \left[(\alpha \tilde{D}_{11} - \tilde{D}_{21}) - (\tilde{D}_{22} - \alpha \tilde{D}_{12}) B_0\right]$$  \hspace{1cm} (63)

as the governing equation for $B_1$,

$$B_2 = \left(\alpha \tilde{S}_{12} - \tilde{S}_{22}\right)^{-1} \left[(\alpha \tilde{M}_{11} - \tilde{M}_{21}) - (\tilde{D}_{22} - \alpha \tilde{D}_{12}) B_1 - (\tilde{M}_{22} - \alpha \tilde{M}_{12}) B_0\right]$$  \hspace{1cm} (64)

as the governing equation for $B_2$, and

$$B_i = -\left(\alpha \tilde{S}_{12} - \tilde{S}_{22}\right)^{-1} \left[(\tilde{D}_{22} - \alpha \tilde{D}_{12}) B_{i-1} + (\tilde{M}_{22} - \alpha \tilde{M}_{12}) B_{i-2}\right]$$  \hspace{1cm} (65)

as the governing equation for $B_i$ for $i > 2$. The results for (51) through (53) are

$$\tilde{M}_{\text{red}} = \left(\tilde{M}_{11} \tilde{S}_{12} B_2\right) \parallel \tilde{M}_{12} B_0 \parallel \tilde{D}_{12} B_1,$$  \hspace{1cm} (66)

$$\tilde{D}_{\text{red}} = \left(\tilde{D}_{11} \tilde{S}_{12} B_1\right) + \tilde{D}_{12} B_0,$$  \hspace{1cm} (67)

$$\tilde{S}_{\text{red}} = \left(\tilde{S}_{11} + \tilde{S}_{12} B_0\right).$$  \hspace{1cm} (68)

Note that the results for $B_0$ and $\tilde{S}_{\text{red}}$, given by (62) and (68), respectively, are the static, discrete model reduction results of [1].

It should be noted that the constant $B_i$ solutions inherent in (62) through (65) have broken the link between an initial value at $t = 0$ to be prescribed for $\beta$ (and its first derivative) and the solution for $\beta$ for $t > 0$. This is a possible indication that a sort of time-domain “boundary layer”
exists near \( t = 0 \), with the conjecture that the actual history of \( \beta \) undergoes an initial transient part, which is better described by the transform condensation method (to follow) and over which the initial value of \( \beta \) (and its first derivative) has influence, and then it settles into the steady solutions given by (62) through (65). Future investigations and an accrual of experience should eventually settle this issue.

**ZERO DAMPING CASE.** If in addition to the above assumption of constant \( M, D, \) and \( S \), one also assumes that the damping matrix \( D \), and hence \( \tilde{D} \), is zero, one gets

\[
B_{2i+1} = 0
\]  

for \( i = 0, 1, 2, \ldots \), so that one has

\[
B_2 = \left( \alpha \tilde{S}_{12} - \tilde{S}_{22} \right)^{-1} \left[ (\alpha \tilde{M}_{11} - \tilde{M}_{21}) - (\tilde{M}_{22} - \alpha \tilde{M}_{12}) B_0 \right]
\]  

as the governing equation for \( B_2 \) and

\[
B_i = -\left( \alpha \tilde{S}_{12} - \tilde{S}_{22} \right)^{-1} \left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right) B_{i-2}
\]  

as the governing equation for \( B_i \) for \( i > 2 \), where \( B_0 \) is given by (62), as before. The reduced (permuted) mass matrix becomes

\[
\tilde{M}_{\text{red}} = \tilde{M}_{11} - \tilde{S}_{12} B_2 + \tilde{M}_{12} B_0,
\]

and the reduced (permuted) stiffness matrix is given by (68), as before. The matrices \( C_0 \) and \( C_2 \) are still found from (54) and (56), respectively, but \( C_1 \) reduces to zero, so that

\[
u \approx \mathcal{P}^{-1} \left( C_0 \mathcal{F} + C_2 \mathcal{V} \right)
\]

follows from (57) as the appropriate equation for solution reconstruction.

**4.1.2. Exact results: Transform condensation**

The case of constant \( M, D, \) and \( S \) is suggestive of transform techniques. Let \( \mathcal{F} \) denote any linear transform with respect to time such that

\[
\mathcal{T} = \nu \mathcal{F} + \eta \mathcal{T},
\]

where \( \nu \) and \( \eta \) are functions of the transform parameter only, and where \( \mathcal{T} \) is an operator such that \( \mathcal{T} g(t) = g(0) \) for any "reasonable" function of time \( g \). (It is assumed that the inverse transform \( \mathcal{F}^{-1} \) exists for \( \mathcal{F} \) to have any utility.) For example, if \( \mathcal{F} \) is the (exponential form of the) Fourier transform, as will often be the case, then \( \nu = i \omega \) and \( \eta = 0 \) if the transform parameter is denoted by \( \omega \). Another example is the Laplace transform, for which \( \nu = s \) and \( \eta = -1 \), where the transform parameter is denoted by \( s \). Equation (74) leads to

\[
\mathcal{T} \mathcal{T}^2 \mathcal{T} = \nu^2 \mathcal{T} + \eta \mathcal{T} + \nu \eta \mathcal{T},
\]

which, along with the linearity of \( \mathcal{F} \), allows one to apply \( \mathcal{F} \) to (2) and (5) so as to get

\[
(\mathcal{F}L) u = (\mathcal{F}f)
\]

and

\[
\mathcal{F}L = (\nu^2 M + \nu D - S) \mathcal{F} + \eta [M \mathcal{T} + (\nu M + D) \mathcal{T}],
\]

respectively.
The condensation model reduction process will be applied to (75) as opposed to (2). Permute (76) according to (11), partition the results according to (12), and define the matrices

\begin{align*}
G_2 &= \left( \tilde{M}_{22} - \alpha \tilde{M}_{12} \right), \\
G_1 &= \left( \tilde{D}_{22} - \alpha \tilde{D}_{12} \right), \\
G_0 &= \left( \alpha \tilde{S}_{12} - \tilde{S}_{22} \right), \\
H_2 &= \left( \alpha \tilde{M}_{11} - \tilde{M}_{21} \right), \\
H_1 &= \left( \alpha \tilde{D}_{11} - \tilde{D}_{21} \right), \\
H_0 &= \left( \tilde{S}_{21} - \alpha \tilde{S}_{11} \right),
\end{align*}

so that the condensation equations (20) take the form

\begin{equation}
\{(\nu^2 G_2 + \nu G_1 + G_0) F + \eta \left[ G_2 \gamma T T + (\nu G_2 + G_1) \gamma \right] \}\beta \\
= (\nu^2 H_2 + \nu H_1 + H_0) F + \eta \left[ H_2 \gamma T T + (\nu H_2 + H_1) \gamma \right].
\end{equation}

One can uncouple the requirements on \( \beta \) in (83) using

\begin{align*}
\Upsilon \beta &= (\Upsilon T \beta) T \\
\Upsilon T \beta &= \Upsilon \left[ (\Upsilon T \beta) + \beta T \right] \\
&= (\Upsilon T \beta) T \gamma (\Upsilon T \beta) TT,
\end{align*}

so that if

\begin{align*}
(\nu^2 G_2 + \nu G_1 + G_0) F \beta &= (\nu^2 H_2 + \nu H_1 + H_0) F \\
G_2 (\Upsilon T \beta) &= H_2, \\
G_2 (\Upsilon T \beta) &= (\nu H_2 + H_1) - (\nu G_2 + G_1) (\Upsilon T \beta)
\end{align*}

are satisfied, then (83) is satisfied. Upon noting that the second equation leads to \( \nu H_2 - \nu G_2 (\Upsilon T \beta) = 0 \) in the third equation, these three equations reduce to

\begin{align*}
\beta &= \mathcal{F}^{-1} \left( \nu^2 G_2 + \nu G_1 + G_0 \right)^{-1} \left( \nu^2 H_2 + \nu H_1 + H_0 \right) F, \\
(\Upsilon T \beta) &= G_2^{-1} H_2, \\
(\Upsilon T \beta) &= G_2^{-1} \left( H_1 - G_1 G_2^{-1} H_2 \right),
\end{align*}

for \( M \neq 0 \). For the special case of \( M = 0 \), the value of \( \Upsilon T \beta \) is no longer required, and the value of \( \Upsilon T \beta \) is then given by

\begin{equation}
(\Upsilon T \beta) = G_1^{-1} H_1.
\end{equation}

The main remaining difficulty centers on making progress with the \( (\nu^2 G_2 + \nu G_1 + G_0)^{-1} \) term of (86).

Let \( \phi \) be the matrix function of \( \nu \) defined by

\begin{equation}
\phi(\nu) = \nu^2 G_2 + \nu G_1 + G_0
\end{equation}

so that

\begin{equation}
\beta = \mathcal{F}^{-1} \phi^{-1} \left( \nu^2 H_2 + \nu H_1 + H_0 \right) F
\end{equation}

from (86) and (90). For \( M \neq 0 \), the standard triple [21, pp. 493-495] result for inverting \( \phi \) is

\begin{equation}
\phi^{-1} = U \left( \nu I - F \right)^{-1} V,
\end{equation}
upon defining

\begin{align*}
U &= \begin{pmatrix} I & 0 \end{pmatrix}, \\
V &= \begin{pmatrix} 0 \\ G_2^{-1} \end{pmatrix}, \\
F &= \begin{pmatrix} 0 & I \\ -G_2^{-1}G_0 & -G_2^{-1}G_1 \end{pmatrix},
\end{align*}

(93) (94) (95)

where the $F$ of (95) is square and where all of the submatrices 0 and $I$ in (93) through (95) are the same size. For the special case of $M = 0$, for which $G_2 = 0$, equation (92) still holds, but with

\begin{align*}
U - I, \\
V &= G_1^{-1}, \\
F &= -G_1^{-1}G_0,
\end{align*}

(96) (97) (98)

instead. (One could more efficiently handle the special case of $M \neq 0$ and $D = 0$, so that $G_1 = 0$ and $H_1 = 0$, in a similar manner, but with $\nu^2$ replacing $\nu$ on the right side of (92) and with $G_2^{-1}$ replacing $G_1^{-1}$ in (97) and (98). So as to avoid carrying along another special case, however, only (92) through (98) will be used here.) The $M = 0$ case above is easily verified. The $M \neq 0$ case can be verified as follows, starting with the verification of

\begin{equation}
(\nu I - F)^{-1} = \begin{pmatrix} \frac{I - \phi^{-1}G_0}{\nu} & \phi^{-1}G_2 \\ \frac{-\phi^{-1}G_0}{\nu} & \nu \phi^{-1}G_2 \end{pmatrix},
\end{equation}

from

\begin{equation}
\nu I - F = \begin{pmatrix} \nu I - G_1^{-1}G_0 & -I \\ G_2^{-1}G_0 & \nu I + G_2^{-1}G_1 \end{pmatrix},
\end{equation}

which came from (95) and (90). Substitution of the above expression for $(\nu I - F)^{-1}$, along with (93) and (94), into the right side of (92) completes the verification.

Equation (92) shows that, for this case, the problem of solving the condensation equations has finally been reduced to that of finding $(\nu I - F)^{-1}$ as a function of $\nu$ for a given constant, square matrix $F$. The expression $(\nu I - F)^{-1}$ for a given square matrix $F$ is so important and common in matrix (particularly spectral) theory that it has been given a name; it is called the \textit{resolvent} of $F$. The resolvent-of-$F$ problem can be solved exactly using the theory of spectral resolution of a function of a matrix [21, pp. 314–320]. Defining the function $f$ by $f(\lambda, \nu) = (\nu - \lambda)^{-1}$, one sees that $(\nu I - F)^{-1} = f(F, \nu)$, that is, that the resolvent of $F$ is simply the function $f$ of $F$, where $f$ also depends upon the parameter $\nu$. The spectral resolution of this particular function $f$ is given [21, p. 315] by

\begin{equation}
(\nu I - F)^{-1} = \sum_{k=1}^{s} \sum_{j=0}^{m_k-1} \frac{j!}{(\nu - \lambda_k)^{j+1}} Z_{kj},
\end{equation}

(99)

where the $\lambda_k$’s, for $k = 1, 2, \ldots, s$, are the \textit{distinct} eigenvalues of $F$, and $m_k$ is the index [21, p. 226] of the corresponding $\lambda_k$. (The index of an eigenvalue is its multiplicity as a root of the \textit{minimal} polynomial of the matrix from which the eigenvalue originated.) The constant matrices $Z_{kj}$ are the components of $F$ and are independent of the function of $F$ being generated. The problem has hence been transformed into one of finding the distinct eigenvalues $\lambda_k$ and the corresponding $m_k$ and $Z_{kj}$’s (for $j = 0, 1, \ldots, m_k - 1$) for each $k$, of the matrix $F$. If this information can be obtained for $F$, then the condensation equations are solved by (77) through (82), (91), (92), (99), and either (93) through (95), (87), and (88) or (96) through (98) and (89).
The $L_0$ which determines the reduced model (17) is now found to be

$$
\mathcal{F} L_0 = \left( \nu^2 \tilde{M}_{11} + \nu \tilde{D}_{11} - \tilde{S}_{11} \right) \mathcal{F} + \eta \left[ \tilde{M}_{11} \Upsilon T + \left( \nu \tilde{M}_{11} + \tilde{D}_{11} \right) \Upsilon \right]
$$

$$
+ \left( \nu^2 \tilde{M}_{12} + \nu \tilde{D}_{12} - \tilde{S}_{12} \right) \mathcal{F} \beta + \eta \left[ \tilde{M}_{12} \Upsilon T + \left( \nu \tilde{M}_{12} + D_{12} \right) \Upsilon \right] \beta
$$

by (14) and (76). Using (84), (85), and (91), this becomes

$$
\mathcal{F} L_0 = \left( \nu^4 \theta_4 + \nu^3 \theta_3 + \nu^2 \theta_2 + \nu \theta_1 + \theta_0 \right) \mathcal{F}
$$

$$
+ \eta \left( D_0 + \nu M_0 \right) \Upsilon + \eta M_0 \Upsilon T,
$$

(100)

upon defining

$$
\theta_4 = \tilde{M}_{12} \phi^{-1} H_2
$$

(101)

$$
\theta_3 = \tilde{M}_{12} \phi^{-1} H_1 + \tilde{D}_{12} \phi^{-1} H_2
$$

(102)

$$
\theta_2 = \tilde{D}_{11} + \tilde{D}_{12} \phi^{-1} H_0 + \tilde{D}_{12} \phi^{-1} H_1 - \tilde{S}_{12} \phi^{-1} H_2
$$

(103)

$$
\theta_1 = \tilde{D}_{11} + \tilde{D}_{12} \phi^{-1} H_0 - \tilde{S}_{12} \phi^{-1} H_1
$$

(104)

$$
\theta_0 = -\tilde{S}_{11} - \tilde{S}_{12} \phi^{-1} H_0
$$

(105)

$$
D_0 = \tilde{D}_{11} + \tilde{D}_{12} \left( \Upsilon \beta \right) + \tilde{M}_{12} \left( \Upsilon \Upsilon \beta \right)
$$

(106)

$$
M_0 = \tilde{M}_{11} + \tilde{M}_{12} \left( \Upsilon \beta \right).
$$

(107)

The value of $(\Upsilon \beta)$ is substituted from (87) or (89), and that of $(\Upsilon \Upsilon \beta)$, when required, is substituted from (88). Further simplification of (100) into a sum of ($(\text{constant-matrix-coefficient}) \times (\text{scalar-function-of-}\nu)$) terms requires explicit knowledge of $\phi^{-1}$ in the form of (92) and (99), the details of which vary from problem to problem. The resemblance between equation (76) and equation (100) suggests that (100) can be inverse transformed into

$$
L_0 = M_0 \Upsilon^2 + D_0 \Upsilon + \mathcal{F}^{-1} \kappa \mathcal{F}
$$

(108)

where

$$
\kappa = \nu^4 \theta_4 + \nu^3 \theta_3 + \nu^2 \left( \theta_2 - M_0 \right) + \nu \left( \theta_1 - D_0 \right) + \theta_0.
$$

(109)

In general, $\mathcal{F}^{-1} \kappa \mathcal{F}$ is a nonlocal operator in time.

The results of the transform method of this section are exact, so that if they could be performed without error then the reconstruction processing of the reduced model solutions should exactly produce the frequency responses for the original problem (75). Accurately finding all of the distinct eigenvalues, their indices, and the component matrices of $F$ may be feasible for modest sized $M$, $D$, and $S$ if one is reducing the entire model at once (global reduction). A computationally less demanding “spin-off” of this section is the transform-parameter neighborhood condensation method of the next section. For much larger problems, however, either transform method (of this section or the next) may be more useful if they are applied locally, say to a contiguous subcollection of elements in a finite element model or to each substructure, in turn, in a large complex system of coupled substructures. This is similar in spirit to ordinary substructuring [14, pp. 162–164] in finite elements, a useful and common procedure.

4.1.3. Transform-parameter neighborhood condensation

Transform-parameter neighborhood condensation parallels that of the previous section up to equation (99), at which point one approximates $(\nu I - F)^{-1}$ by

$$
(\nu I - F)^{-1} \approx \sum_{j=0}^{\mu-1} (\nu - \nu_0)^{-j+1} \zeta_j + \Gamma_0 + \nu \Gamma_1
$$

(110)
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in a neighborhood of $\nu$ about the \textit{(a priori)} user-chosen value $\nu_0$. The terms in (110) are eventually determined by first solving

$$
(\nu_0 I - F) E = I - \zeta_0,
$$

(111)

for the square matrix $E$. The attempt to solve (111) should establish whether the determinant of $(\nu_0 I - F)$, denoted by $\det(\nu_0 I - F)$, is essentially zero or not. A singular value decomposition on $(\nu_0 I - F)$, for example, would simultaneously determine this and solve (111) via a (Moore-Penrose) pseudo-inverse \cite[pp. 242-243]{22}. If $\det(\nu_0 I - F)$ is \textit{not} zero, then take

$$
\zeta_j = 0,
$$

(112)

for all $j$, the value of $\mu$ in (110) then being irrelevant. If $\det(\nu_0 I - F)$ is zero, so that $\nu_0$ corresponds to an eigenvalue of $F$, say $\lambda_k$ for some specific $k$, then take

$$
\zeta_j - (j!) Z_{kj},
$$

(113)

$$
\mu = m_k,
$$

(114)

for the eigenvalue $\nu_0 = \lambda_k$, for $j = 0, 1, \ldots, (\mu - 1)$, where the notation of (99) is being used. (The $Z_{kj}$ and $m_k$ are only needed for this \textit{one} value of $k$, not all $k$.) Using $E$ from (111), one solves for the square matrix $\Gamma_1$ in

$$
(\nu_0 I - F) \Gamma_1 = -E.
$$

(115)

With $E$ and $\Gamma_1$ known, $\Gamma_0$ can finally be found from

$$
\Gamma_0 = E - \nu_0 \Gamma_1.
$$

(116)

Finally, to complete the model reduction process, equation (110) is used in (92), which, in turn, is used in (91) and (100) through (105), as before.

The above process, though not trivial, is much less work than trying to find $\lambda_k$, $m_k$, and $Z_{kj}$ for all $k$ and $j$ for $F$, as was required in the previous transform approach. In fact, for many choices of $\nu_0$ one would not need to find \textit{any} of the $Z_{kj}$ or $m_k$. In such cases, one can easily avoid the use of complex numbers and complex computation (if the original system is real). Even for those choices of $\nu_0$ which are near an eigenvalue, for which one might as well take $\nu_0$ to be the eigenvalue, many times the eigenvalue is such that $m_k = 1$, so that only $Z_{k0}$ is needed. In such a case, $Z_{k0}$ can be found simply and directly from the associated left eigenvector \cite[p. 154]{21} $y_k$ and right eigenvector $x_k$ as

$$
Z_{k0} = x_k y_k^\top,
$$

where $\top$ denotes transpose. The trade-off in this method is that the above results do not hold for all values of $\nu$, but only for some neighborhood of $\nu_0$, and that one must always solve (111) and (115). It now remains to justify (110).

If $\det(\nu_0 I - F)$ is not zero, then (111) and (115) are solved by

$$
E = (\nu_0 I - F)^{-1},
$$

$$
\Gamma_1 = - (\nu_0 I - F)^{-1} E
$$

$$
= - (\nu_0 I - F)^{-2}
$$

upon using $\zeta_0 = 0$ from (112). When these values of $E$ and $\Gamma_1$ are substituted into (110), along with (112) and (116), it yields

$$
(\nu I - F)^{-1} \approx (\nu_0 I - F)^{-1} - (\nu - \nu_0) (\nu_0 I - F)^{-2}.
$$
The known [21, p. 331] relation

$$\frac{\partial}{\partial \nu} \left[ (\nu I - F)^{-1} \right] = - (\nu I - F)^{-2}$$

(117)

allows one to recognize the above approximation to \((\nu I - F)^{-1}\) as being nothing more than the Taylor series expansion of \((\nu I - F)^{-1}\) about the value \(\nu_0\), truncated after the first two terms.

The situation is more complicated when \(\det(\nu_0 I - F)\) is zero. As stated before, \(\nu_0\) now corresponds to an eigenvalue of \(F\), say \(\lambda_k\) for some specific \(k\). It can be shown [21, p. 321], using the orthogonality properties of the component matrices \(Z_{ij}\), that

$$\frac{1}{(\nu I - F)^{-1}} (I - Z_{k0}) = \sum_{i=1, \neq k}^s \sum_{j=0}^{m_i - 1} \frac{j!}{(\nu - \lambda_i)^{j+1}} Z_{ij}.$$  

(118)

Substituting (118) into (99) yields

$$(\nu I - F)^{-1} - \sum_{j=0}^{\mu-1} (\nu - \nu_0)^{-j+1} \zeta_j \approx (\nu I - F)^{-1} (I - Z_{k0}),$$

which becomes

$$(\nu I - F)^{-1} (I - Z_{k0}) = \sum_{j=0}^{\mu-1} (\nu - \nu_0)^{-j+1} \zeta_j \approx (\nu I - F)^{-1} (I - Z_{k0}),$$

(119)

upon substituting \(\nu_0 = \lambda_k\), (113), and (114). Equation (118) shows that \((\nu I - F)^{-1} (I - Z_{k0})\) is analytic at \(\nu_0 = \lambda_k\), and hence a Taylor expansion of this term about the value \(\nu_0\) is justified. Carrying this out to two terms gives

$$(\nu I - F)^{-1} (I - Z_{k0}) \approx E + (\nu - \nu_0) \Gamma_1,$$

upon defining

$$E = \lim_{\nu \to \nu_0} \left[ (\nu I - F)^{-1} (I - Z_{k0}) \right],$$

(120)

$$\Gamma_1 = \lim_{\nu \to \nu_0} \frac{\partial}{\partial \nu} \left[ (\nu I - F)^{-1} (I - Z_{k0}) \right],$$

(121)

the last result following from (117). Substituting a two term approximation for \((\nu I - F)^{-1} (I - Z_{k0})\) into (119) gives (110) upon using the definition for \(\Gamma_0\) given by (116). It now remains to show that the \(E\) and \(\Gamma_1\) of (120) and (121), respectively, are also given by (111) and (115).

The argument for \(E\) and \(\Gamma_1\) used here is a modified extension of the one given by [21, p. 322] for \(E\) only. Multiply the identities

$$(\nu_0 I - F) = (\nu_0 - \nu) I + (\nu I - F),$$

$$\frac{1}{(\nu_0 I - F)} = (\nu_0 - \nu)^{-1} I + 2 (\nu_0 - \nu) (\nu I - F) + (\nu I - F)^2,$$

by \((\nu I - F)^{-1} (I - Z_{k0})\) and \(- (\nu I - F)^{-2} (I - Z_{k0})\) on the right, respectively, to get

$$(\nu_0 I - F) (\nu I - F)^{-1} (I - Z_{k0}) = (\nu_0 - \nu) (\nu I - F)^{-1} (I - Z_{k0}) + (I - Z_{k0}),$$

$$- (\nu_0 I - F)^2 (\nu I - F)^{-2} (I - Z_{k0}) = - (\nu_0 - \nu)^2 (\nu I - F)^{-2} (I - Z_{k0})$$

$$- 2 (\nu_0 - \nu) (\nu I - F)^{-1} (I - Z_{k0}) - (I - Z_{k0}).$$

(120)  (121)
Taking the limit of each as \( \nu \) goes to \( \nu_0 \), one gets

\[
\begin{align*}
(n_0 I - F) E &= 0E + (I - Z_{ko}) \\
&= (I - Z_{ko}) ,
\end{align*}
\]

\[
(n_0 I - F)^2 \Gamma_1 = 0^2 \Gamma_1 - (2)(0) E - (I - Z_{ko})
\]

\[
- (I - Z_{ko}) ,
\]

upon using the definitions (120) and (121). The first of the above equations is just (111) upon noting that \( \zeta_0 = Z_{ko} \). Multiplying (115) by \((n_0 I - F)\) and substituting the first of the above equations reproduces the second of the above equations.

### 4.1.4. Eigenvalue preservation in condensation process

If one considers linear transforms with respect to time such that (74) holds and \( \eta = 0 \), then, whenever \( FLF^{-1} \) is a polynomial in \( \nu \) with constant matrix coefficients, one has the associated general eigenvalue problem

\[
\text{det } [FLF^{-1}(\nu)] = 0.
\]

The dependence of \( FLF^{-1} \) on \( \nu \) is explicitly shown as \( FLF^{-1}(\nu) \). Taking \( \eta = 0 \) in (76) gives

\[
FLF^{-1}(\nu) = \nu^2 M + \nu D - S
\]

as an example. The eigenproblem is directly associated with the free response of the system, such as the free vibrations of a solid structure, but it can also be used, via superposition, to build solutions to nonfree (driven) problems. The following theorem is directly relevant to such eigenproblems.

**Theorem 2.** Assume that \( L \) and \( L_0 \) satisfy the hypotheses of Theorem 1 and that each are functions of a parameter \( \nu \) only. For every value \( \nu_0 \) such that

\[
\text{det } [L_0(\nu_0)] = 0
\]

is true,

\[
\text{det } [L(\nu_0)] = 0
\]

is also true.

Theorem 2 essentially says that the eigenvalues of the reduced system are also eigenvalues of the original system.

The proof of Theorem 2 starts with the hypothesis that

\[
\text{det } [L_0(\nu_0)] = 0.
\]

This means that there is a nonzero \( \nu \) such that \([L_0(\nu_0)] \nu = 0\), so that \( \Pi f = 0 \) has the solution \( \nu \) to (17). The last part of the proof of Theorem 1 shows that

\[
f = P^{-1} \begin{pmatrix} (\Pi f) \\ \alpha (\Pi f) \end{pmatrix}
\]

because of (16). By Theorem 1, however, (18) and (2) must combine to

\[
[L(\nu_0)] \Omega \nu = f = P^{-1} \begin{pmatrix} 0 \\ \alpha \nu \end{pmatrix} = 0,
\]

where \( \Omega \nu \) cannot be zero because \( \Pi \Omega \nu = \nu \) and \( \nu \) is nonzero. This implies that \( \text{det } [L(\nu_0)] = 0 \) and the theorem is proved.
4.1.5. Eigenvalue economizer as special case

One currently used method to condense eigenvalue problems [14, pp. 547-549] to computationally economical sizes can be shown to be a special case of the above methods. Taking $\nu$ as zero in the transform condensation method, so as to recover only the static parts, one gets

$$\phi^{-1} = G_0^{-1}$$

from (90), and hence

$$\beta = G_0^{-1} H_0$$

from (91). This, in turn, gives

$$\beta = \tilde{S}_{22}^{-1} \tilde{S}_{21}$$

upon taking $\alpha = 0$. This also corresponds to the lowest order term for $\beta$ in the time-derivative series approach with $\alpha = 0$, as seen by (62). The “master/slave” notational scheme in [14] is inverted with respect to this paper in the sense that, in this paper, the “master” variables are at the top of the total displacement vector, whereas in [14, Equation (20.59)] they are at the bottom. This means that, in translating from this paper to [14], 1 corresponds to $m$ for “master” and 2 corresponds to $s$ for “slave.” Substituting this into the above result for $\beta$ gives

$$\beta = -\tilde{S}_{22}^{-1} \tilde{S}_{sm},$$

which is exactly the result [14, Equation (20.65)] for the eigenvalue economizer method. (The stiffness matrix $K$ of [14, Equation (20.62)] corresponds to $-S$ here and the $T$ of [14, Equation (20.61)] corresponds to $\beta$ here. It should be noted that (20.59) and (20.61) are inconsistent in [14]; the $I$ and $T$ in (20.61) should be switched.)

5. QUADRATICALLY NONLINEAR CONDENSATION

This section was included as a demonstration that one can use the general condensation approach of this paper to develop specific condensation methods for nonlinear semi-discrete problems (or classes of such problems). The problem to be attacked here is that of a generic second order dynamic-response problem for which the stiffness term is quadratically nonlinear and the spatial dependence has been discretized. The approach is easily extended to higher order polynomial nonlinearities. (The use of symbolic processing software, such as Mathematica, may be indespensible in such cases due to the rapid escalation in the tedious details as one moves to higher order.) This quadratically nonlinear semi-discrete problem is probably the simplest nonlinear extension of (1) of practical significance, hence excessive details which would otherwise obscure the main ideas of the approach are minimized. Cartesian tensor notation with the Einstein summation convention will be used for convenience.

Consider the second order dynamic system with quadratically nonlinear stiffness which is generically represented by the equation

$$M_{ij} T^2 u_{ij} + D_{ij} u_{ij} = (S_0)_{ij} u_j + (S_1)_{ijk} u_j u_k + f_i, \quad (122)$$

where each integer index ranges from 1 to $n$ and where the components of the matrices $M$, $D$, $S_0$, and $S_1$ are at most functions of time. Equation (122) is in the component form of (2) if one takes $L$ as

$$L_{ij} = M_{ij} T^2 + D_{ij} T - (S_0)_{ij} - (S_1)_{ijk} u_k \quad (123)$$

so that (122) can be written as

$$L_{ij} u_j = f_i. \quad (124)$$

This is a case where the components of $L$ are functions of the components of $u$, a possibility described in Section 2.2. As in the development of the linear dynamic case, in order to keep this
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problem as simple as possible, the choice for the $\alpha$ matrix in (8) will be that of an ordinary, constant matrix. Assuming that the $(\mathcal{P}, \mathcal{P}^{-1})$ pair is also fixed (chosen), define the matrices

$$
\begin{align*}
\left(\tilde{S}_0\right)_{ij} &= \mathcal{P}_{i\ell} (S_0)_{\ell m} (\mathcal{P}^{-1})_{mj}, \\
\left(\tilde{S}_1\right)_{ijk} &= \mathcal{P}_{i\ell} (S_1)_{\ell mr} (\mathcal{P}^{-1})_{mi} (\mathcal{P}^{-1})_{rk},
\end{align*}
$$

so that (38), (39), (123), (125), and (126) determine $\tilde{L}$ as

$$
\tilde{L}_{ij} = \tilde{M}_{ij} T^2 + \tilde{D}_{ij} T - \left(\tilde{S}_0\right)_{ij} - \left(\tilde{S}_1\right)_{ijk} (\mathcal{P} u)_k
$$

(127)
on a component basis. For any fixed $k$, $\left(\tilde{S}_1\right)_{ijk}$ is a matrix of the same size as $\tilde{S}_0$. Partitioning $\tilde{S}_1$ as in (12) for each value $k$ of its third index, and partitioning $M$, $D$, and $\tilde{S}_0$ as in (12), one gets

$$
\begin{bmatrix}
(G_{20})_{i\ell} T^2 + (G_{10})_{i\ell} T + (G_{00})_{i\ell} + (G_{01})_{i\ell k} (\mathcal{P} u)_k
\end{bmatrix} \beta_{\ell j}
$$

(128)
for the condensation equations (20) upon defining

$$
\begin{align*}
G_{20} &= (M_{22} - \alpha \tilde{M}_{12}), \\
G_{10} &= (D_{22} - \alpha \tilde{D}_{12}), \\
G_{00} &= \alpha \left(\tilde{S}_0\right)_{12} - \left(\tilde{S}_0\right)_{22}, \\
G_{01} &= \alpha \left(\tilde{S}_1\right)_{12} - \left(\tilde{S}_1\right)_{22}, \\
H_{20} &= (\alpha \tilde{M}_{11} - \tilde{M}_{21}), \\
H_{10} &= (\alpha \tilde{D}_{11} - \tilde{D}_{21}), \\
H_{00} &= \left(\tilde{S}_0\right)_{21} - \alpha \left(\tilde{S}_0\right)_{11}, \\
H_{01} &= \left(\tilde{S}_1\right)_{21} - \alpha \left(\tilde{S}_1\right)_{11}.
\end{align*}
$$

The next immediate goal is to put (128) into the form of a power series in $T$ and $v$. Substituting (15) into (18) and matrix multiplying the results by $\mathcal{P}$ gives

$$
\mathcal{P} u = \begin{pmatrix} v \\ \beta v \end{pmatrix}
$$

(137)
If it is assumed that $\beta$ has the form

$$
\beta_{ij} = (B_{00})_{ij} v_k + (B_{01})_{ijk} v_k + (B_{10})_{ij} T + (B_{02})_{ijk} T v_k + (B_{20})_{ij} T^2 + (B_{11})_{ijk} (T v_k) + \cdots
$$

(138)
on a component basis, then (137) can be put into the form of a power series in $T$ and $v$ as

$$
(\mathcal{P} u)_k = \begin{pmatrix} I \\ B_{00} \end{pmatrix} v_m + \begin{pmatrix} 0 \\ B_{01} \end{pmatrix} v_m v_r + \begin{pmatrix} 0 \\ B_{10} \end{pmatrix} (T v_m) + \cdots
$$

(139)
where the context determines the number of indices of each 0 within each parentheses. Substituting (138) and (139) into (128), collecting coefficients of powers of \( v \), powers of \( T \), and their products, and equating corresponding coefficients on each side to each other gives

\[
\mathcal{R}_{i\ell} (B_{00})_{\ell j} = (H_{00})_{ij}
\]

(140)

as the governing equation for \( B_{00} \) upon defining the matrix of operators \( \mathcal{R} \) as

\[
\mathcal{R}_{i\ell} (B_{00})_{\ell j} = (G_{20})_{i\ell \ell} \left[ T^2 (B_{00})_{\ell j} \right] + (G_{10})_{i\ell \ell} \left[ T (B_{00})_{\ell j} \right] + (G_{00})_{i\ell \ell} (B_{00})_{\ell j}.
\]

(141)

At the same time, one obtains

\[
\mathcal{R}_{i\ell} (B_{10})_{\ell j} = (H_{10})_{ij} - 2 (G_{20})_{i\ell \ell} \left[ T (B_{00})_{\ell j} \right] - (G_{10})_{i\ell \ell} (B_{00})_{\ell j}
\]

(142)

as the governing equation for \( B_{10} \),

\[
\mathcal{R}_{i\ell} (B_{01})_{\ell j} = (H_{01})_{ij} \left( \frac{I}{B_{01}} \right)_{km} - (G_{01})_{i\ell \ell \ell} (B_{00})_{\ell j} \left( \frac{I}{B_{01}} \right)_{km}
\]

(143)

as the governing equation for \( B_{01} \),

\[
\mathcal{R}_{i\ell} (B_{20})_{\ell j} = (H_{20})_{ij} - (G_{20})_{i\ell \ell} (B_{00})_{\ell j} - 2 (G_{20})_{i\ell \ell} \left[ T (B_{10})_{\ell j} \right] - (G_{10})_{i\ell \ell} (B_{10})_{\ell j}
\]

(144)

as the governing equation for \( B_{20} \),

\[
\mathcal{R}_{i\ell} (B_{11})_{\ell j} = (H_{11})_{ij} \left( \frac{I}{B_{10}} \right)_{km} - (G_{01})_{i\ell \ell \ell} (B_{00})_{\ell j} \left( \frac{I}{B_{10}} \right)_{km} - 2 (G_{20})_{i\ell \ell} \left[ T (B_{01})_{\ell j} \right] - (G_{10})_{i\ell \ell} (B_{10})_{\ell j}
\]

(145)

as the governing equation for \( B_{11} \), and

\[
\mathcal{R}_{i\ell} (B_{11})_{\ell j} = (H_{11})_{ij} \left( \frac{I}{B_{10}} \right)_{km} - (G_{01})_{i\ell \ell \ell} (B_{00})_{\ell j} \left( \frac{I}{B_{10}} \right)_{km}
\]

(146)

as the governing equation for \( B_{11} \), which is as far as the series will be carried.

Inserting (127), (139), and (138) into (14), the reduced model matrix \( L_0 \) is found to be

\[
(L_0)_{ij} = \left( \tilde{M}_{\text{red}} \right)_{ij} \cdot T^2 + \left\{ \left( \tilde{D}_0 \right)_{\text{red}} \right\}_{ij} \cdot v_k + \left\{ \left( \tilde{D}_1 \right)_{\text{red}} \right\}_{ijk} \cdot v_k \cdot v_r + \cdots
\]

(148)

On a component basis, where

\[
\left( \tilde{M}_{\text{red}} \right)_{ij} = \left( \tilde{M}_{11} \right)_{ij} - \left\{ \left( \tilde{S}_0 \right)_{12} \right\}_{ij} \cdot (B_{20})_{kj} + \left( \tilde{D}_{12} \right)_{ik} \cdot (B_{10})_{kj} + \left[ T (B_{00})_{kj} \right] \]

(149)
\[
\left\{ (\widetilde{D}_0)_{\text{red}} \right\}_{ij} = (\widetilde{D}_{11})_{ij} - \left\{ (\widetilde{S}_0)_{12} \right\}_{ik} (B_{10})_{kj} + (\widetilde{D}_{12})_{ik} \left\{ (B_{00})_{kj} + \left[ T (B_{10})_{kj} \right] \right\} \\
\quad + (\overline{M}_{12})_{ik} \left\{ 2 \left[ T (B_{00})_{kj} \right] + \left[ T^2 (B_{10})_{kj} \right] \right\} \\
\left\{ (\widetilde{D}_1)_{\text{red}} \right\}_{ijm} = - \left\{ \left\{ (\widetilde{S}_1)_{11} \right\}_{imk} + \left\{ (\widetilde{S}_1)_{12} \right\}_{i \ell k} (B_{00})_{\ell m} \right\} (0 \quad B_{10})_{kj} + (\overline{M}_{12})_{ik} \\
\quad \times \left[ \left\{ T^2 (B_{11})_{kjm} \right\} + \left\{ T^2 (B_{11})_{kmj} \right\} + 2 \left\{ T (B_{01})_{kjm} \right\} + 2 \left\{ T (B_{01})_{kmj} \right\} \right] \\
\quad + (\widetilde{D}_{12})_{ik} \left\{ (B_{01})_{kjm} + (B_{01})_{kmj} + [T (B_{11})_{kjm}] + [T (B_{11})_{kmj}] \right\} \\
\quad - \left\{ \left\{ (\widetilde{S}_1)_{12} \right\}_{i \ell k} (B_{10})_{kj} \right\} \left\{ (B_{11})_{kjm} + (B_{11})_{kmj} \right\} (151)
\]

\[
\left\{ (\widetilde{S}_0)_{\text{red}} \right\}_{ij} = \left\{ (\widetilde{S}_0)_{11} \right\}_{ij} + \left\{ (\widetilde{S}_0)_{12} \right\}_{ik} (B_{00})_{kj} \\
\quad - (\overline{M}_{12})_{ik} \left[ T^2 (B_{00})_{kj} \right] - (\widetilde{D}_{12})_{ik} \left[ T (B_{00})_{kj} \right] \right\} \\
\left\{ (\widetilde{S}_1)_{\text{red}} \right\}_{ijm} = \left\{ \left\{ (\widetilde{S}_1)_{11} \right\}_{ijk} + \left\{ (\widetilde{S}_1)_{12} \right\}_{i \ell k} (B_{00})_{\ell m} \right\} \left( I \quad B_{00} \right)_{km} \\
\quad - (\overline{M}_{12})_{ik} \left[ T^2 (B_{01})_{kjm} \right] - (\widetilde{D}_{12})_{ik} \left[ T (B_{01})_{kjm} \right] \\
\quad + \left\{ (\widetilde{S}_0)_{12} \right\}_{ik} (B_{01})_{kjm} \right\} \\
\left\{ (\widetilde{S}_2)_{\text{red}} \right\}_{ijkr} = \left\{ \left\{ (\widetilde{S}_1)_{11} \right\}_{ijk} + \left\{ (\widetilde{S}_1)_{12} \right\}_{i \ell k} (B_{00})_{\ell m} \right\} \left( 0 \quad B_{01} \right)_{km} \\
\quad - (\overline{M}_{12})_{ik} \left[ T^2 (B_{02})_{kjm} \right] - (\widetilde{D}_{12})_{ik} \left[ T (B_{02})_{kjm} \right] \\
\quad + \left\{ \left\{ (\widetilde{S}_0)_{12} \right\}_{ik} (B_{02})_{kjm} + \left\{ (\widetilde{S}_0)_{12} \right\}_{i \ell k} (B_{01})_{kjr} \right\} \left( I \quad B_{00} \right)_{\ell m} \right\} (154)
\]

As a check, the results for the limiting case of \( S_1 = 0 \) correspond to those of Section 4. Note that (148) and (17) together represent a degree-of-freedom-reduced nonlinear model. In fact, if the \( (\widetilde{D}_1)_{\text{red}}, (\widetilde{S}_2)_{\text{red}} \), and higher order terms are discarded, it will be directly analogous to the original model (122).

### 6. CONCLUDING REMARKS

The above application to a quadratic nonlinearity, and its implied extension to higher order polynomial nonlinearities, should not be construed so as to imply that these are the only nonlinearities that can be handled by the general condensation method of Section 3. They are the only forms of nonlinearities for which solutions to the condensation equations have been found so far. This does not preclude, however, the possibility that other undiscovered solutions exist for other forms of nonlinearity. In a similar vein, the use of constant-component \( \alpha \) matrices in the linear dynamic case of Section 4, and the quadratically nonlinear case of Section 5, does not preclude the development of these same (or other) cases with more general versions of \( \alpha \). For example, one is free to propose a frequency-dependent \( \alpha \) for Fourier-transformed, frequency-dependent models if that is appropriate for the types of sources (loads) \( f \) to be imposed on the system. (Recall that \( \alpha \) and \( f \) are linked by (8), (9), and (16).) The diversity and potential of the condensation model reduction method has only been partly explored in this paper. The eventual determination of its boundary of utility, however, depends upon gaining much more extensive...
computational experience in its use against realistic problems. This would ultimately require the interfacing of the method with a commercially significant finite element code.

The computational costs of carrying out the condensation procedure of this paper have not been discussed up to this point. Detailed studies of computational costs in terms such as operation counts and run-time comparisons with alternative methods ("brute force attack" being one) have not been performed here. Such studies should eventually be performed, but as the methodology is so new, the initial emphasis should be on the exploration of the more basic question as to whether it works, irrespective of the costs. Practical implementation considerations can sometimes overcome methods that theoretically appeared worthy. Besides, there are scenarios for which even computationally expensive methods are useful if they work well. For example, the condensation model reduction method is most effective when the reduced model is to be used a large number of times, making the initial cost of performing the reduction worthwhile. Another scenario for which the method is more likely to find immediate success are those niches in which current condensation methods are already successfully used. In such cases, one could apply the method

- once globally (entire model) with a large degree-of-freedom reduction,
- recursively and globally with a small degree-of-freedom reduction per recursion, or
- locally (as in substructuring) wherever it is needed most in the model.

The middle procedure is somewhat reminiscent of multilevel/multigrid methods [23] used in the numerical solution of partial differential equations.

APPENDIX A

TOOLS FOR OPERATOR-COMPONENT MATRICES

The purpose of this appendix is to provide the preliminary mathematical background required for a rigorous derivation of the condensation model reduction method in Appendix B to follow. (There are no claims as to originality for the material here, but there did not seem to be a reasonable reference for which the bulk of the results given here would be self-contained.) As ordinary matrix multiplication has been abandoned in favor of (6), some of the consequences of this need to be investigated. In particular, some additional mathematical tools will be needed to "make up" for the deficiencies that result.

To illustrate that partitioning of matrices is still possible under (6), a fact that will have future utility, let $A$ and $B$ be $n \times n$ matrices partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $A_{11}$ and $B_{11}$ are $m \times m$ with $m < n$. As a multiplication example,

$$(AB)_{11} = (AB)_{ij} = \sum_{k=1}^{n} A_{ik} \circ B_{kj} = \left( \sum_{k=1}^{m} A_{ik} \circ B_{kj} + \sum_{k=m+1}^{n} A_{ik} \circ B_{kj} \right) = A_{11} B_{11} + A_{12} B_{21},$$

where $i < m$ and $j < m$ on the right-hand sides, and similarly for $(AB)_{12}$, $(AB)_{21}$, and $(AB)_{22}$. 
A.1. Associative and Distributive Conditions

Ordinary matrix multiplication and ordinary matrix addition together form an algebraic structure called a ring. (In a ring, addition is commutative and associative, every element has an inverse (negative), and an identity exists (zero), making it an abelian group with respect to addition. In a ring, multiplication is associative and a multiplicative identity exists, making it a monoid with respect to multiplication. Additionally, multiplication is distributive (on both sides) over addition in a ring.) Definition (6) does not satisfy the associative law with respect to multiplication and it does not satisfy the $A(B + C) = AB + AC$ distributive law. The following theorem provides the conditions, in this new setting, which allow one to still make selective use of the associative and distributive laws which are ordinarily taken for granted.

**Theorem 3.** Matrices whose operator components consist entirely of homomorphisms with respect to the addition operation are defined to be “homomorphic.” The associative rule $A(BC) = (AB)C$ and the distributive rule $A(B + C) = AB + AC$ hold for multiplication defined by (6), and for any compatible matrices $A$, $B$, and $C$, if $A$ is “homomorphic.”

An operator is a homomorphism with respect to the addition operation if it preserves the addition operation in its mapping action, that is, if $A_{ik} \circ (B_{mj} + C_{mj}) = A_{ik} \circ B_{mj} + A_{ik} \circ C_{mj}$ for any components of $A$, $B$, and $C$, respectively. A linear operator is a homomorphism with respect to the addition operation, for example (as well as with respect to multiplication by a scalar). (The other relation, $(A_{ik} + B_{ik}) \circ C_{mj} = A_{ik} \circ C_{mj} + B_{ik} \circ C_{mj}$ for any components of $A$, $B$, and $C$, respectively, holds generally since it is basically nothing more than the natural definition of how to composite the sum of two operators.) That Theorem 3 is true for the distributive rule should be obvious. The associative rule of Theorem 3 follows from

$$ (A(BC))_{ij} = \sum_k A_{ik} \circ \left( \sum_m B_{km} \circ C_{mj} \right) $$

$$ = \sum_k \sum_m A_{ik} \circ B_{km} \circ C_{mj} $$

$$ ((AB)C)_{ij} = \sum_m \left( \sum_k A_{ik} \circ B_{km} \right) \circ C_{mj} $$

$$ = \sum_m \sum_k A_{ik} \circ B_{km} \circ C_{mj}, $$

the hypothesis concerning the components of $A$, and the finiteness of the above summations (so that the summations signs can be interchanged).

Theorem 3 allows one to group, as one matrix, any multiplicative sequence of homomorphic matrices. This follows inductively from $A_{j+1}((A_j \cdots A_1)B) = (A_{j+1}(A_j \cdots A_1))B = (A_{j+1} \cdots A_1)B$, which is clearly true for $j = 1$ and each subsequent value of $j$, for any compatible matrix $B$ and for any multiplicative sequence of homomorphic matrices $A_N \cdots A_1$. In addition, one can associate the matrices, within the multiplicative sequence of homomorphic matrices, as one pleases by a suitable sequence of applications of Theorem 3. Finally, as compositions of homomorphisms are themselves homomorphisms and sums of homomorphisms (with respect to addition) are themselves homomorphisms, any given multiplicative sequence of homomorphic matrices, taken as one matrix, is itself a homomorphic matrix.

A.2. Matrices of Zero and Identity Components

Special matrices whose components consist entirely of $0$’s and $1$’s, such as the matrices $0$ and $I$, where

- $\tilde{0}$ is the zero operator, for which everything in its domain is mapped into the zero function,
· $I$ is the identity operator, for which each function is mapped into itself,

· $0$ denotes a matrix whose components consist entirely of $0$'s, and

· $I$ denotes a matrix whose components consist of $I$'s on the diagonal and of $0$'s elsewhere, deserve special attention. The matrices $0$ and $I$, for which multiplication is defined by (6), are analogous to the ordinary zero and identity matrices of ordinary matrix multiplication. Since $0$ and $I$ are trivially seen to be homomorphisms with respect to addition, all matrices whose components consist entirely of $0$'s and $I$'s are homomorphic.

The requirement (7) ultimately allows one to utilize familiar properties of the zero, identity, and permutation matrices ordinarily taken for granted. The additional relations

$$\tilde{0} \circ A_{ij} = \tilde{0}$$  \hspace{1cm} (155)

$$A_{ij} \circ I = A_{ij}$$  \hspace{1cm} (156)

$$I \circ A_{ij} = A_{ij},$$  \hspace{1cm} (157)

are trivially true for all $i$ and $j$ for any given matrix $A$, and together with (7) and (6) they prove the following theorem:

**Theorem 4.** For all matrices $A$ whose components obey (6) and (7), the following are true

$$A0 = 0$$

$$0A = 0$$

$$AI = A$$

$$IA = A$$

for all compatible $0$'s and $I$'s.

### A.2.1. Permutation matrices

Permutation matrices form an important class of matrices which, in this setting, have components consisting entirely of $0$'s and $I$'s. An arbitrary elementary permutation matrix $\Pi_{ij}$, designed to permute only rows or columns $i$ and $j$ of the matrix it multiplies, can be given as

$$(\Pi_{ij})_{\ell m} = \begin{cases} 
I & \text{for } \ell = m, \ell \neq i, \ell \neq j \\
I & \text{for } \ell = i \text{ and } m = j \\
I & \text{for } \ell = j \text{ and } m = i \\
0 & \text{otherwise.}
\end{cases}$$

The result of $\Pi_{ij} A$, using (6), (155), and (157), is $A$ with rows $i$ and $j$ interchanged. The result of $A \Pi_{ij}$, using (6), (7), and (156), is $A$ with columns $i$ and $j$ interchanged. Note that $\Pi_{ij}$ has the properties:

$$\Pi_{ij}^2 = \Pi_{ij} \Pi_{ij} = I$$  \hspace{1cm} (158)

$$\Pi_{ij}^\top = \Pi_{ij}$$  \hspace{1cm} (159)

$$A(\Pi_{ij} R) = (A \Pi_{ij}) R,$$  \hspace{1cm} (160)

where the superscript $T$ denotes the transpose of the matrix, and $A$ and $B$ are arbitrary compatible matrices. Properties (158) and (159) are readily verified. Property (160) is also readily verified once one realizes that the $\ell m$th element of $AB$ with the rows $i$ and $j$ of $B$ interchanged before multiplication, and the $\ell m$th element of $A B$ with the columns $i$ and $j$ of $A$ interchanged before multiplication, are both equal to

$$\sum_{k, k \neq i, k \neq j} A_{\ell k} \circ B_{km} + A_{\ell i} \circ B_{jm} + A_{\ell j} \circ B_{im}.$$
One can build up more complex permutation matrices as a multiplicative sequence of elementary permutation matrices. Since each elementary permutation matrix is a homomorphic matrix, this would also be a multiplicative sequence of homomorphic matrices. As noted at the end of Section A.1, such a multiplicative sequence can be taken as one matrix which is itself a homomorphic matrix. Let \( \mathcal{P} \) be a permutation matrix formed from the multiplicative sequence

\[
\mathcal{P} = \mathcal{P}_1 \cdots \mathcal{P}_N,
\]

where each \( \mathcal{P}_j \) is an elementary permutation matrix. (The subscript \( j \) on \( \mathcal{P}_j \) is merely a sequence labeling; it does not represent the row or column numbers being permuted as was previously the case in \( \Pi_{ij} \).) Reversing the order of the multiplicative sequence produces the inverse of \( \mathcal{P} \), that is,

\[
\mathcal{P}^{-1} = \mathcal{P}_N \cdots \mathcal{P}_1.
\]

This can be seen inductively by

\[
(\mathcal{P}_N \cdots \mathcal{P}_j)(\mathcal{P}_j \cdots \mathcal{P}_N) = (\mathcal{P}_N \cdots \mathcal{P}_{j-1}) (\mathcal{P}_N^{2} \cdots \mathcal{P}_N) = (\mathcal{P}_N \cdots \mathcal{P}_{j-1})(\mathcal{P}_N \cdots \mathcal{P}_N)
\]

upon using \( \mathcal{P}_j^2 = I \), which follows from (158), starting from \( j = 1 \). The proof also works with the above sequences reversed as \( (\mathcal{P}_1 \cdots \mathcal{P}_j)(\mathcal{P}_j \cdots \mathcal{P}_1) \) and starting from \( j = N \). (One is free to make associations within these multiplicative sequences as needed because the matrices are all homomorphic.) That \( \mathcal{P}^{-1} \) is a permutation matrix itself also follows since it is a multiplicative sequence of elementary permutation matrices. In summary, permutation matrices always come in pairs as \( \mathcal{P} \) and \( \mathcal{P}^{-1} \) (algebraically they form a group).

The last result of this section simply extends (160) to include all permutation matrices:

**Theorem 5.** For arbitrary matrices \( A \) and \( B \) satisfying (6) and (7), and for arbitrary permutation matrix \( \mathcal{P} \), one has

\[
A(\mathcal{P}B) = (AP) B
\]

whenever the matrix sizes are compatible with the indicated multiplications.

The relation

\[
(...(A)\mathcal{P}_1)\cdots)\mathcal{P}_{j-1})(\mathcal{P}_j(\mathcal{P}_{j+1}(\cdots(\mathcal{P}_N(C)\cdots)
\]

\[
= (...(A)\mathcal{P}_1)\cdots)\mathcal{P}_j)(\mathcal{P}_{j+1}(\cdots(\mathcal{P}_N(C)\cdots)
\]

(161)

follows from (160), where \( \mathcal{P}_j \), for each value of \( j \), is an elementary permutation matrix. For \( j = 1 \) and \( C \) taken as \( B \), the left side of (161) is \( A(\mathcal{P}B) \) for \( \mathcal{P} = \mathcal{P}_1 \cdots \mathcal{P}_N \). Using (161) inductively, with \( C \) taken as \( B \), one eventually gets

\[
A(\mathcal{P}D) = (...(A)\mathcal{P}_1)\cdots)\mathcal{P}_N) B.
\]

For \( j = N \) and \( C \) taken as \( I \), the right side of (161) is \( (...(A)\mathcal{P}_1)\cdots)\mathcal{P}_N) \). Using (161) inductively in the other direction, with \( C \) taken as \( I \), one eventually gets

\[
(...(A)\mathcal{P}_1)\cdots)\mathcal{P}_N) = A(\mathcal{P}_1(\mathcal{P}_2(\cdots(\mathcal{P}_N)\cdots)) = AP,
\]

which completes the proof.
This preliminary part of the proof involves mainly $P$ and $\tilde{L}$. With $P^{-1}$ being homomorphic, the definition for $P$ in (9) does not require explicitly giving an association since

$$P^{-1} \left( \tilde{P}P \right) = \left( P^{-1} \tilde{P} \right) P$$

follows from Theorem 3. An analogous statement

$$P \left( L P^{-1} \right) = (PL) P^{-1}$$

holds for the definition of $\tilde{L}$ given by (11), with $L$ playing the role of $\tilde{P}$, since $P$ is homomorphic. That the matrices defining $P$ in (9) can always be grouped together as one matrix is shown by

$$P^{-1} \left( \tilde{P}(P B) \right) = P^{-1} \left( (\tilde{P} P) B \right)$$

$$= \left( P^{-1} (\tilde{P} P) \right) B$$

$$= \left( P^{-1} \tilde{P} P \right) B$$

$$= P B \quad (162)$$

for a compatible (with respect to multiplication) but otherwise arbitrary matrix $B$, where Theorem 5, with $P$ being a permutation matrix, and then Theorem 3, with $P^{-1}$ being homomorphic, were used. Again, an analogous statement

$$P \left( L (P^{-1} B) \right) = \tilde{L} B \quad (163)$$

holds for the definition of $\tilde{L}$ given by (11), with $L$ playing the role of $\tilde{P}$, but with $P$ and $P^{-1}$ swapping roles.

To prove that $P$ is idempotent, one first shows that $\tilde{P}$ is idempotent, so that

$$\tilde{P} (P B) = \tilde{P} B \quad (164)$$

for a compatible but otherwise arbitrary matrix $B$. Upon row partitioning $B$

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

so that $B_1$ has the same number of rows as the number of columns of $\alpha$ in (8), one gets

$$\tilde{P} B = \begin{pmatrix} B_1 \\ \alpha B_1 \end{pmatrix}.$$ 

This can be used recursively, since $B$ is arbitrary, to get

$$\tilde{P} (\tilde{P} B) = \begin{pmatrix} D_1 \\ \alpha B_1 \end{pmatrix},$$

which then leads to (164). One gets the interim result

$$\tilde{P} (P (P B)) = \tilde{P} \left( P \left( P^{-1} \left( \tilde{P} (P B) \right) \right) \right)$$

$$= \tilde{P} \left( P \left( P^{-1} \tilde{P} (P B) \right) \right)$$

$$= \tilde{P} \left( P (P B) \right)$$

$$= \tilde{P} (P B),$$
upon using (162), Theorem 3 (since \( P \) is homomorphic), and then using (164) with \( B \) replaced by \( PB \). This, in turn, can be used in

\[
P(PB) = P^{-1} \left( \tilde{P} \left( P(PB) \right) \right)
= P^{-1} \left( \tilde{P} \left( PB \right) \right)
= PB,
\]

upon using (162) again, and hence \( P \) is idempotent.

The second part of the proof begins with

\[
Lu = P^{-1} \left( \begin{pmatrix} v \\ \beta v \end{pmatrix} \right),
\]

which follows from

\[
Lu = (P^{-1}P) (L(\Omega v))
= P^{-1} \left( P \left( L \left( P^{-1} \left( \begin{pmatrix} v \\ \beta v \end{pmatrix} \right) \right) \right) \right)
= P^{-1} \left( \begin{pmatrix} v \\ \beta v \end{pmatrix} \right)
\]

upon using (18) for \( u \), Theorem 3 (since \( P^{-1} \) is homomorphic), (15), and then (163) with \( B \) replaced by

\[
\left( \begin{array}{c}
v \\
\beta v \end{array} \right).
\]

Using (12) in (165) gives

\[
Lu = P^{-1} \left( \begin{pmatrix} \tilde{L}_{11} v + \tilde{L}_{12} \beta v \\ \tilde{L}_{21} v + \tilde{L}_{22} \beta v \end{pmatrix} \right)
= P^{-1} \left( \begin{pmatrix} L_0 v \\ \alpha (L_0 v) \end{pmatrix} \right)
= P^{-1} \left( \begin{pmatrix} \Pi f \\ \alpha (\Pi f) \end{pmatrix} \right)
\]

upon using (13), (14), and then (17). Partitioning

\[
P f = \begin{pmatrix} (P f)_1 \\ (P f)_2 \end{pmatrix}
\]

so that \((P f)_1\) has the same number of rows as the number of columns of \( \alpha \) in (8), it is straightforward to see that

\[
\Pi f = (P f)_1
\]

\[
\tilde{F} (P f) = \begin{pmatrix} (P f)_1 \\ \alpha (P f)_1 \end{pmatrix}
\]

using (10) for the first equation and using (8) for the second equation. Combining the two equations gives

\[
\tilde{F} (P f) = \begin{pmatrix} \Pi f \\ \alpha (\Pi f) \end{pmatrix}.
\]
so that

\[ Lu = \mathcal{P}^{-1} \left( \alpha (\Pi f) \right) \]

\[ = \mathcal{P}^{-1} \left( \tilde{P} (\mathcal{P} f) \right) \]

\[ = f \]

upon using (162) with \( B \) replaced by \( f \) and then (16).

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