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Exponents of 2-regular digraphs

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Abstract

A digraph G is called primitive if for some positive integer k, there is a walk of length exactly k from each vertex u to each vertex v (possibly u again). If G is primitive, the smallest such k is called the exponent of G, denoted by $\exp(G)$. A digraph G is said to be r-regular if each vertex in G has outdegree and indegree exactly r.

It is proved that if G is a primitive 2-regular digraph with n vertices, then $\exp(G) \leqslant (n-1)^2/4+1$. Also all 2-regular digraphs with exponents attaining the bound are characterized. This supports a conjecture made by Shen and Gregory. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction and notation

Let $G=(V,E)$ denote a digraph on n vertices. Loops are permitted but no multiple arcs. A $u \to v$ walk in G is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs $(u,u_1),(u_1,u_2),\ldots,(u_{p-1},v)$ where the vertices and the arcs are not necessarily distinct. A closed walk is a $u \to v$ walk where $u=v$. A path is a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u=v$. The length of a walk W is the number of arcs in W. The *girth* g of G is the length of a shortest cycle in G. An *r-cycle* is a cycle of length r. The digraph G is said to be *strongly connected* if there exists a path from u to v for all $u,v \in V$. The *distance* from u to v, denoted by $d_G(u,v)$ or $d(u,v)$ if G is specified, is the minimum k for which there is a $u \to v$ walk of length k. The *diameter* D of G is the maximum $d(u,v)$ among all ordered pairs $u,v \in V$.

The notation $u \stackrel{k}{\to} v$ (resp. $u \stackrel{k}{\nrightarrow} v$) is used to indicate that there is a $u \to v$ walk (resp. no $u \to v$ walk) of length k. A diagraph G is *primitive* if there exists some positive integer k such that $u \stackrel{k}{\to} v$ whenever $u,v \in V$. The minimum such k is called the *exponent* of G, denoted $\exp(G)$. The *local exponent* of G at a vertex $u \in V$, denoted $\exp(G{:}u)$, is

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the least integer k such that $u \xrightarrow{k} v$ for each $v \in V$. Much work has been done on finding upper bounds for $\exp(G)$ (see [14,3] for example). The diameter bound in Lemma 1 below was proved recently by Shen [9] and Neufeld [5] independently. Neufeld [5] characterized the case of equality in Lemma 1 with the following class of digraphs.

Let the family $\mathscr{F}_{\mathscr{D}}$ consist of the following digraphs $G=(V,E)$: The vertex set $V = \bigcup_{i=0}^{D} V_i$, where the V_i are pairwise disjoint and $|V_0|=1$. The arc set $E \supset \{(u,v)\colon u \in V_i, v \in V_{i+1}\}$, where addition is taken modulo $D+1$, and the remaining arcs in E may be any set of arcs from V_D to V_1 with the following properties: for each vertex $u \in V_D$, (u,w) is an arc for some $w \in V_1$, and for each vertex $v \in V_1$, (w,v) is an arc for some $w \in V_D$.

Lemma 1 (Neufeld [5], Shen [9]). *Suppose G is a primitive digraph with diameter D. Then*

$$\exp(G) \leqslant D^2 + 1$$

and equality holds if and only if $G \in \mathscr{F}_{\mathscr{D}}$.

We note that if G is a digraph with diameter D and girth g, then $g \leqslant D+1$. Consequently, when $g \leqslant D$, the diameter bound in Lemma 1 is implied by the following lemma.

Lemma 2 (Shen [8,10]). *Suppose G is a primitive digraph with diameter D and girth g. Then*

$$\exp(G) \leqslant D + 1 + g(D-1).$$

A digraph G is said to be *r-regular* if each vertex in G has outdegree and indegree exactly r. A digraph G is said to be *vertex-transitive* if, for each pair u, v of vertices, there is an automorphism of the digraph that takes u to v. Thus vertex-transitive digraphs are regular. In [12], Shen and Gregory studied the exponents of vertex-transitive digraphs and proved that

$$\exp(G) \leqslant \left\lceil \frac{n}{r} \right\rceil \left(\left\lceil \frac{n}{r} \right\rceil + 1 \right)$$

for all primitive vertex-transitive digraphs G. Based on this result, they raised the following conjecture in the same paper.

Conjecture 1. If G is a primitive r-regular digraph of order n, then

$$\exp(G) \leqslant \left\lfloor \frac{n}{r} \right\rfloor^2 + 1.$$

We remark that, by applying two results by Nishimura [6] and Soares [13], respectively, one can obtain a rough upper bound of $3n^2/r^2$ for the exponents of all r-regular digraphs of order n [12, Theorem 6].

In this paper, it is proved that if G is primitive and 2-regular, then $\exp(G) \leqslant (n-1)^2/4 + 1$. This confirms Conjecture 1 when $r = 2$. Also all 2-regular digraphs with exponents attaining the above bound are characterized.

2. Main results

Throughout the paper, we always assume that G is a digraph of order n, girth g and diameter D. Let $a_1 < a_2 < \cdots < a_k$ be positive integers such that $\gcd(a_1, a_2, \ldots, a_k) = 1$. By a result of Schur, it is known that if N is a sufficiently large integer, then for all $n \geqslant N$ the equation

$$b_1 a_1 + b_2 a_2 + \cdots + b_k a_k = n$$

has a solution in non-negative integers $b_1, b_2, \ldots, b_k$. The least such number N is called the *Frobenius–Schur index* and is denoted $\Phi(a_1, a_2, \ldots, a_k)$. For $k = 2$ it is well known that $\Phi(a_1, a_2) = (a_1 - 1)(a_2 - 1)$.

For a fixed positive integer c, let G^c be the digraph with the same vertex set as G and, for each pair w_1, w_2 of vertices, $w_1 \xrightarrow{1} w_2$ in G^c if and only if $w_1 \xrightarrow{c} w_2$ in G. It can be proved that, if G is primitive with diameter D, then G^c is also primitive with diameter at most D [7, Theorem 3.1]. We will repeatedly use this fact in the following lemma.

Lemma 3. *Suppose G is primitive and* $\exp(G{:}u) = D + 1 + g(D - 1)$. *Then*

1. *For any g-cycle $C_g = (w_1, w_2, \ldots, w_g)$, the distance from u to C_g is $D + 1 - g$.*
2. *For any g-cycle C_g, there is a vertex v on C_g such that the distance, $d(u, v)$, from u to v is D.*
3. *If $g + 1 \leqslant D \leqslant 2g - 1$, then the length of a shortest cycle containing u is $D + 1$.*

Proof. The first two statements follow easily from the proof of [8, Theorem 2]. For the third statement, it is a consequence of the diameter assumption that every vertex lies on a cycle of length at most $D + 1$. Suppose u lies on an l-cycle with $l \leqslant D$. Then $l \neq g$; otherwise, since G^l has diameter at most D [7, Theorem 3.1] and G^l has a loop at u, we have $u \xrightarrow{D} v$ in G^l for each v and so $\exp(G{:}u) \leqslant lD = gD < D + 1 + g(D - 1)$, a contradiction. Let $c = \gcd(g, l)$. Since $g + 1 \leqslant l \leqslant D \leqslant 2g - 1$, we have $1 \leqslant c \leqslant g/2$. By the first statement of the lemma, we may suppose that w is a vertex in C_g such that $u \xrightarrow{c\lceil (D+1-g)/c \rceil} w$ in G; i.e., $u \xrightarrow{\lceil (D+1-g)/c \rceil} w$ in G^c. Since u and w lie on two cycles of lengths l/c and g/c in G^c, respectively, $\exp(G^c{:}u) \leqslant d_{G^c}(u, w) + D + \Phi(g/c, l/c) \leqslant \lceil (D + 1 - g)/c \rceil + D + (g/c - 1)(l/c - 1)$ and $\exp(G{:}u) \leqslant c \cdot \exp(G^c{:}u) \leqslant c\lceil (D + 1 - g)/c \rceil + cD + (g - c)(l/c - 1) < D + 1 + g(D - 1)$, a contradiction. Therefore, the third statement of the lemma holds. □

Before introducing some results from [11], we need some notation. Let $\delta^+(G)$ (resp. $\delta^-(G)$) denote the minimum outdegree (resp. indegree) of G. Let $\deg^+(u)$ (resp. $\deg^-(u)$) denote the outdegree (resp. indegree) of u. For all $i \geqslant 0$, let $R_i(u) = \{v \in G{:}\ u \xrightarrow{i} v\}$, $R'_i(u) = \{v \in G{:}\ v \xrightarrow{i} u\}$, $N_i(u) = \bigcup_{j=0}^{i} R_j(u)$ and $N'_i(u) = \bigcup_{j=0}^{i} R'_j(u)$. Then $N_i(u)$ is the set of vertices that can be reached from u by a path of length at most i. Recall that $R_0(u) = R'_0(u) = \{u\}$ since a vertex is at distance 0 from itself. Let

$D_0(u)=\{u\}$ and $D_i(u)=N_i(u)\setminus N_{i-1}(u)$ for all $i\geqslant 1$. In other words, $D_i(u)$ is the set of vertices v such that $d(u,v)=i$. Thus $N_i(u)=N_{i-1}(u)\cup D_i(u)$ and $N_{i-1}(u)\cap D_i(u)=\emptyset$.

Lemma 4 (Shen [11, Theorems 1,2]). *Suppose G is a digraph with $\delta^+(G)\geqslant 1$. Let $t=|\{u\in G\colon \deg^+(u)=1\}|$. Then*

$$g\leqslant\begin{cases}\lceil n/2\rceil & \text{if } t=0,\\ \lceil (n+t-1)/2\rceil & \text{if } t\geqslant 1.\end{cases}$$

Moreover, if G is strongly connected, then $D\leqslant n-g+t$

Lemma 5. *Suppose G is a digraph with $\delta^+(G)\geqslant 2$. Then for all u and all i, $0\leqslant i\leqslant g-1$,*

$$|N_i(u)|\geqslant 2i+1.$$

Furthermore, if $D\geqslant 2g-1$ and $\exp(G{:}u)=D+1+g(D-1)$ *for some u, then $|N_g(u)|\geqslant 2g+1$.*

Proof. Suppose, contrary to the theorem, that there exists a smallest i, $0\leqslant i\leqslant g-2$, such that $|N_i(u)|\geqslant 2i+1$ while $|N_{i+1}(u)|\leqslant 2i+2$. Then $|N_i(u)|=2i+1$ and $|D_{i+1}(u)|=1$ since $D_{i+1}(u)\neq\emptyset$. Also we have $|D_i(u)|\leqslant 2$; otherwise $|N_{i-1}(u)|=|N_i(u)|-|D_i(u)|<2i-1$, contradicting the choice of i. Let G_1 be the subdigraph of G induced by $N_i(u)\setminus\{u\}$. Since there is no arc from $N_i(u)\setminus\{u\}$ to u, we have $\delta^+(G_1)\geqslant\delta^+(G)-|D_{i+1}(u)|\geqslant 1$. Also the number of vertices with outdegree 1 in G_1 is at most $|D_i(u)|\leqslant 2$. By Lemma 4, $g\leqslant g(G_1)\leqslant\lceil(|N_i(u)|+1)/2\rceil=i+1\leqslant g-1$, a contradiction. Thus the first part of Lemma 5 follows.

Now suppose $D\geqslant 2g-1$ and $\exp(G{:}u)=D+1+g(D-1)$. If $|N_g(u)|\leqslant 2g$, then $|N_{g-1}(u)|=2g-1$, $|D_{g-1}(u)|=2$ and $|N_g(u)|=1$ by the above argument. There is no arc from $N_{g-1}(u)\setminus\{u\}$ to u; otherwise u is contained in a g-cycle and thus it is easy to prove that $\exp(G{:}u)\leqslant gD<D+1+g(D-1)$, a contradiction. By applying Lemma 4 to the subdigraph G_2 of G induced by $N_{g-1}(u)\setminus\{u\}$, we have $g(G_2)\leqslant\lceil(|N_{g-1}(u)\setminus\{u\}|+|D_{g-1}(u)|)/2\rceil=g$. Thus there exists a g-cycle C_g in G_2. It can be verified that the distance from u to C_g is at most $\lfloor g/2\rfloor<D+1-g$, which contradicts Lemma 3(1). Therefore Lemma 5 follows. □

Lemma 6. *Suppose G is strongly connected with* $\min\{\delta^+(G),\delta^-(G)\}\geqslant 2$. *Then*

$$D\leqslant\begin{cases}n-g & \text{if } D\leqslant g,\\ \lfloor n/2\rfloor & \text{if } g+1\leqslant D\leqslant 2g-1,\\ n-2g+1 & \text{if } D\geqslant 2g.\end{cases}$$

Furthermore, if G is primitive and $\exp(G)=D+1+g(D-1)$, *then*

$$D\leqslant\begin{cases}\lfloor (n-1)/2\rfloor & \text{if } g+1\leqslant D\leqslant 2g-1,\\ n-2g & \text{if } D\geqslant 2g.\end{cases}$$

Proof. By Lemma 4, it may be supposed that $D \geqslant g+1$. Let u, v be two vertices with distance $d(u,v)=D$. In case G is primitive with exponent $D+1+g(D-1)$, by Lemma 3(2), we may properly choose u such that $\exp(G{:}u)=D+1+g(D-1)$. Since $\delta^-(G)\geqslant 2$, applying Lemma 5 to the digraph obtained from G by reversing all its arcs, we have $|N_i'(v)|\geqslant 2i+1$ for all $i, 1\leqslant i\leqslant g-1$.

Case 1: $g+1\leqslant D\leqslant 2g-1$. Then $N_{g-1}(u)\cap N'_{D-g}(v)=\emptyset$; otherwise $d(u,v)\leqslant D-1$. Thus by Lemma 5, $n\geqslant |N_{g-1}(u)|+|N'_{D-g}(v)|\geqslant 2(g-1)+1+2(D-g)+1=2D$; i.e., $D\leqslant \lfloor n/2\rfloor$. Now suppose G is primitive and $\exp(G{:}u)=D+1+g(D-1)$. Then $N_{D-g+1}(u)\cap N'_{g-1}(u)=\{u\}$; otherwise u is contained in a cycle of length at most D, contradicting Lemma 3(3). Thus $n\geqslant |N_{D-g+1}(u)|+|N'_{g-1}(u)|-1\geqslant 2(D-g+1)+1+2(g-1)=2D+1$; i.e., $D\leqslant \lfloor (n-1)/2\rfloor$.

Case 2: $D\geqslant 2g$. Then similarly to Case 1, we have $N_{D-g}(u)\cap N'_{g-1}(v)=\emptyset$ and $n\geqslant |N_{D-g}(u)|+|N'_{g-1}(v)|=|N_{g-1}(u)|+\sum_{i=g}^{D-g}|D_i(u)|+|N'_{g-1}(v)|\geqslant 2(g-1)+1+(D-2g+1)+2(g-1)+1=D+2g-1$; i.e., $D\leqslant n-2g+1$. Now suppose G is primitive and $\exp(G{:}u)=D+1+g(D-1)$. Then, by Lemma 5, we can similarly obtain $n\geqslant |N_g(u)|+\sum_{i=g+1}^{D-g}|D_i(u)|+|N'_{g-1}(v)|\geqslant 2g+1+(D-2g)+2g-1=D+2g$; i.e., $D\leqslant n-2g$.

By combining the above two cases, Lemma 6 follows. □

The *Cayley digraph* $\mathrm{Cay}(\boldsymbol{Z}_n,\{1,2\})$ is the digraph with vertex set $\boldsymbol{Z}_n$, the cyclic group of order n, and arc set $E=\{(i,j){:}\ j-i=1 \text{ or } 2\}$, where subtraction is taken modulo n. The *lexicographic product* $G\otimes G'$ of a digraph $G=(V,E)$ with a digraph $G'=(V',E')$ is the digraph with vertex set $V\times V'$ and arc set

$$\{((x_1,y_1),(x_2,y_2)){:}\ (x_1,x_2)\in E\}\cup\{((x,y_1),(x,y_2)){:}\ x\in V,\ (y_1,y_2)\in E'\}.$$

It is well known (see [2] for example) that if G is primitive, then the greatest common divisor of the lengths of the cycles in G is one. Let C_g denote a g-cycle and $\overline{K}_2$ denote two isolated vertices. Then $C_g\otimes\overline{K}_2$ is not primitive when $g\geqslant 2$, since it only contains cycles of lengths g and $2g$.

Lemma 7. *Suppose G is 2-regular. If $D=g=n/2$, then G is isomorphic to either* $\mathrm{Cay}(\boldsymbol{Z}_n,\{1,2\})$ *or* $C_g\otimes\overline{K}_2$.

Proof. Let u, v be two vertices such that $d(u,v)=D$. Then $N_{i-1}(u)$, $D_i(u)$ and $N'_{D-i-1}(v)$ are pairwise disjoint for all i, $1\leqslant i\leqslant D-1$. by Lemma 5, $|D_i(u)|\leqslant n-|N_{i-1}(u)|-|N'_{D-i-1}(v)|\leqslant n-(2i-1)-(2D-2i-1)=2$. On the other hand, since $N_{i-1}(u)\cap N'_{D-i}(v)=\emptyset$, we have $|D_i(u)|=|N_i(u)|-|N_{i-1}(u)|\geqslant |N_i(u)|+|N'_{D-i}(v)|-n\geqslant 2i+1+2(D-i)+1-n=2$. Thus $|D_i(u)|=2$ for all i, $1\leqslant i\leqslant D-1=g-1$. Since $1\leqslant |D_g(u)|\leqslant n-\sum_{i=0}^{g-1}|D_i(u)|=n-2(g-1)-1=1$, we have $|D_g(u)|=1$ and $V=\dot{\bigcup}_{i=0}^{g} D_i(u)$. Let $D_i(u)=\{w_i,w_i'\}$, $1\leqslant i\leqslant g-1$.

Claim 1. *For any i,j such that $i\leqslant g-1$ and $1\leqslant i-j\leqslant g-3$, there are no arcs from $D_i(u)$ to $D_j(u)$.*

Proof. Otherwise suppose i is the smallest number for which Claim 1 fails. Then $g \geqslant 4$ and $i \geqslant 2$. For all $1 \leqslant j \leqslant i-1$, let G_j be the subdigraph of G induced by $D_j(u) \cup D_{j+1}(u)$. Then $\deg^+_{G_j}(w_j) = \deg^+_{G_j}(w'_j) = 2$ by the choice of i. We define the following two types of arc sets:

Type I: $\{(w_j, w_{j+1}), (w_j, w'_{j+1}), (w'_j, w_{j+1}), (w'_j, w'_{j+1})\}$.
Type II: $\{(w_j, w_{j+1}), (w_j, w'_{j+1}), (w'_j, w_j), (w'_j, w'_{j+1})\}$.

Let Out($D_j(u)$) denote the set of arcs coming from the vertices in $D_j(u)$. By re-ordering w_1, w'_1 and re-ordering w_2, w'_2 if necessary, Out($D_1(u)$) is of type I or II. Now suppose $j \leqslant i-2$. If Out($D_j(u)$) is of type I, then Out($D_{j+1}(u)$) is of type I as well since G is 2-regular. Similarly if Out($D_j(u)$) is of type II, then by re-ordering w_{j+2}, w'_{j+2} if necessary, it can be verified that Out($D_{j+1}(u)$) is of type I or II. Thus by induction each Out($D_j(u)$), $1 \leqslant j \leqslant i-1$, is of type I or II. It can be seen that the distance from each vertex in $D_j(u)$ to each vertex in $D_i(u)$ is at most $i-j+1$. Since there is an arc from $D_i(u)$ to some $D_j(u)$ by the choice of i, G contains a cycle of length at most $i-j+2 \leqslant g-1$, a contradiction. Therefore Claim 1 follows.

By Claim 1 and its proof, it is easy to obtain the following two further claims: After re-ordering the vertices within each $D_i(u)$, $1 \leqslant i \leqslant g-2$, if necessary,

Claim 2. *All Out($D_i(u)$), $1 \leqslant i \leqslant g-2$, are of the same type.*

Claim 3. $d(w_1, w_{g-1}) = d(w_1, w'_{g-1}) = d(w'_1, w'_{g-1}) = g-2$ and $g-2 \leqslant d(w'_1, w_{g-1}) \leqslant g-1$.

Case 1: All Out($D_i(u)$), $1 \leqslant i \leqslant g-2$, are of type I. Then the distance from any vertex in $D_j(u)$ to any vertex in $D_i(u)$ is $i-j$ whenever $0 \leqslant j < i \leqslant g$. Since G is 2-regular and has girth $g = n/2$, the remaining arcs are uniquely determined, which are $\{(w_{g-1}, v), (w'_{g-1}, v), (w_{g-1}, u), (w'_{g-1}, u), (v, w_1), (v, w'_1)\}$. Thus G is isomorphic to $C_g \otimes \overline{K}_2$.

Case 2: All Out($D_i(u)$), $1 \leqslant i \leqslant g-2$, are of type II. Then $d(w_j, w_i) = d(w_j, w'_i) = d(w'_j, w'_i) = i-j$ and $d(w'_j, w_i) = i-j+1$ whenever $1 \leqslant j < i \leqslant g-1$. Similarly, the remaining arcs are also uniquely determined, which are $\{(w_{g-1}, v), (w'_{g-1}, v), (v, u), (v, w'_1), (w_{g-1}, u), (w'_{g-1}, w_{g-1})\}$. Thus G is isomorphic to Cay($\boldsymbol{Z}_n, \{1, 2\}$). This completes the proof of Lemma 7. □

Before stating our main theorem, we recall from the definition of the digraph set $\mathscr{F}_{\mathscr{D}}$ that, if $G \in \mathscr{F}_{\mathscr{D}}$ and G is 2-regular, then G is unique (up to isomorphism).

Theorem 1. *If G is 2-regular and primitive with order $n \geqslant 8$, then*

$$\exp(G) \leqslant \frac{(n-1)^2}{4} + 1$$

and equality holds if and only if G is the unique 2-regular digraph (up to isomorphism) in $\mathscr{F}_{\mathscr{D}}$.

Proof. *Case* 1: Either $D=g-1$ or $D=g \leqslant (n-1)/2$. Then by Lemma 6, $D \leqslant (n-1)/2$ is always true. By Lemma 1, $\exp(G) \leqslant D^2+1 \leqslant (n-1)^2/4+1$ with equality if and only if G is the unique 2-regular digraph (up to isomorphism) in $\mathscr{F}_{\mathscr{D}}$.

Case 2: $D=g \geqslant n/2$. Then by Lemma 6, $D=g=n/2$. Since $C_1 \otimes \overline{K_2}$ is isomorphic to $\mathrm{Cay}(\boldsymbol{Z}_2, \{1,2\})$ and $C_g \otimes \overline{K_2}$ is not primitive when $g \geqslant 2$, G is isomorphic to $\mathrm{Cay}(\boldsymbol{Z}_n\{1,2\})$ by Lemma 7. Then $\exp(G) \leqslant n-1 < (n-1)^2/4+1$ by a theorem of Kim and Krabill [4].

Case 3: $g+1 \leqslant D \leqslant 2g-1$. It may be supposed that $\exp(G) \leqslant D+g(D-1)$. Otherwise $\exp(G)=D+1+g(D-1)$ by Lemma 2. Then by Lemma 6, $D \leqslant \lfloor (n-1)/2 \rfloor$ and this case follows from Case 1. Thus $\exp(G) \leqslant D+g(D-1) \leqslant D(D-1)+1 \leqslant \lfloor n/2 \rfloor(\lfloor n/2 \rfloor - 1)+1 < (n-1)^2/4+1$ by Lemma 6 again.

Case 4: $D \geqslant 2g \geqslant 2$. Similarly to Case 3, it may be supposed that $\exp(G) \leqslant D+g(D-1)$. Otherwise, if $\exp(G)=D+1+g(D-1)$, then by Lemma 6, $\exp(G)=(g+1)(D-1)+2 \leqslant (g+1)(n-2g-1)+2 < (n-1)^2/4+1$. Thus $\exp(G) \leqslant D+g(D-1) \leqslant (g+1)(n-2g)+1 < (n-1)^2/4+1$ by Lemma 6 again.

By combining Cases 1–4 above, Theorem 1 follows. □

If $n \leqslant 7$, a routine computer check shows that $\exp(G) \leqslant (n-1)^2/4+1$ is still true. However, for the second part of Theorem 1, besides the unique 2-regular digraph (up to isomorphism) in $\mathscr{F}_{\mathscr{D}}$, there is one more digraph with exponent $(n-1)^2/4+1$. The adjacency matrix of the digraph is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

By using the techniques presented in this paper, one can prove the following theorem. Here we only outline its proof.

Theorem 2. *Suppose G is primitive.*

1. *If* $\min\{\delta^+(G), \delta^-(G)\} \geqslant 2$ *and* $n \geqslant 6$, *then*

$$\exp(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor^2 + 1.$$

2. *If* $\delta^+(G) \geqslant 2$, *then*

$$\exp(G) \leqslant \frac{n^2}{4} + 1.$$

Proof (*outline*). The proof of Theorem 1 works for proving the first statement except for Case 2, where we have $D=g=n/2$ and so $2/n$ and $\exp(G) \leqslant D^2+1=\lfloor n/2 \rfloor^2+1$.

For the second statement, by Lemmas 2 and 4, we have $\exp(G) \leqslant D+1+g(D-1)=(g+1)(D-1)+2 \leqslant (g+1)(n-g-1)+2 \leqslant n^2/4+2$. Thus, in order to prove (2), we may suppose, on the contrary, that $\exp(G) = n^2/4 + 2$. This implies $D = n - g$, $D = g + 1, g + 2, g + 3$ and $\exp(G{:}u) = D + 1 + g(D - 1)$ for some u. By Lemma 3(2), there exists a vertex v in a g-cycle such that $d(u,v) = D$. By Lemma 5, $n = |N_{g-1}(u)| + \sum_{i=g}^{D} |D_i(u)| \geqslant 2g - 1 + D - g + 1 = D + g$. Since $D = n - g$, again by Lemma 5, we have $|D_i(u)| = 2$ for all $1 \leqslant i \leqslant g - 1$ and $|D_i(u)| = 1$ for all $g \leqslant i \leqslant D$. By using similar proof techniques employed in Claim 1 of Lemma 7, we can show that for any i, j such that $i \leqslant g - 1$ and $1 \leqslant i - j \leqslant g - 3$, there are no arcs from $D_i(u)$ to $D_j(u)$. Thus there is at least an arc from $D_{g-1}(u)$ to $N_1(u)$, and this arc lies on a g-cycle. Therefore, the distance from u to this g-cycle is at most $1 < D + 1 - g$ (recall that $D = g + 1, g + 2$ or $g + 3$), contradicting Lemma 3(1). □

3. Closing remarks

Conjecture 1 is still open when $r \geqslant 3$. As it has been mentioned in [12], the conjectured bound cannot be decreased since $\exp(G) = \lfloor n/r \rfloor^2 + 1$ for all r-regular digraphs in $\mathscr{F}_{\mathscr{D}}$.

In order to settle Conjecture 1 completely, we believe that first of all a good upper bound on the girth g of all r-regular digraphs should be found so that one can use Lemma 2. In 1970, Behzad et al. [1] made the following conjecture.

Conjecture 2 (*Behzad et al.* [1]). *Let G be an r-regular digraph of order n. Then* $g \leqslant \lceil n/r \rceil$.

This conjecture has been proved for $n \leqslant 5$. For more details on the conjecture and two more related conjectures, we refer the reader to [11] and references therein.

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