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Hamiltonian double latin squares

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Abstract

A double latin square of order $2n$ on symbols $\sigma_1, \dots, \sigma_n$ is a $2n \times 2n$ matrix $A = (a_{ij})$ in which each a_{ij} is one of the symbols $\sigma_1, \dots, \sigma_n$ and each σ_k occurs twice in each row and twice in each column. For $k = 1, \dots, n$ let $B(A, \sigma_k)$ be the bipartite graph with vertices $\rho_1, \dots, \rho_{2n}, c_1, \dots, c_{2n}$ and $4n$ edges $[\rho_i, c_j]$ corresponding to ordered pairs (i,j) such that $a_{ij} = \sigma_k$. We say that A is Hamiltonian if $B(A, \sigma_k)$ is a cycle of length $4n$ for $k = 1, \dots, n$. Two double latin squares $(a_{ij}), (a'_{ij})$ of order $2n$ on symbols $\sigma_1, \dots, \sigma_n$ are said to be orthogonal if for each ordered pair (σ_h, σ_k) of symbols there are four ordered pairs (i,j) such that $a_{ij} = \sigma_h$, $a'_{ij} = \sigma_k$.

We explore ways of constructing Hamiltonian double latin squares (HLS), symmetric HLS, sets of mutually orthogonal HLS and pairs of orthogonal symmetric HLS. We identify those arrays which can be obtained from HLS by amalgamating rows and amalgamating columns in a certain sense, and we prove a similar result concerning symmetric arrays obtainable in this way from symmetric HLS. These results can be proved either by using matroids or by a more elementary method, and we illustrate both approaches. From these results we deduce a characterisation of those matrices which are submatrices of HLS on n symbols, a similar result concerning symmetric submatrices of symmetric HLS and some related results. Much of our discussion uses graph-theoretic language, since HLS on n symbols are equivalent to decompositions of $K_{2n,2n}$ into Hamiltonian cycles and symmetric HLS on n symbols are

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equivalent to decompositions of K_{2n} into Hamiltonian paths (and these are equivalent to decompositions of K_{2n+1} into Hamiltonian cycles).

1. Definition and elementary construction

A *double latin square* of *order* $2n$ is a $2n \times 2n$ matrix containing n symbols, such that each cell contains exactly one symbol and each symbol occurs exactly twice in each row and twice in each column. The occurrences of a symbol σ describe a set of disjoint cycles in a double latin square: if σ occurs in $2n$ distinct cells

$$(i_1,j_1),(i_1,j_2),(i_2,j_2),(i_2,j_3),(i_3,j_3),(i_3,j_4),\ldots,(i_\ell,j_\ell),(i_\ell,j_1)$$

then these cells are said to constitute a *cycle*, or more specifically a σ-*cycle*, of *length* 2ℓ. In a double latin square of order $2n$, the lengths of the cycles described by any one symbol have sum $4n$. A cycle of length $4n$, the maximum possible length, is called a *Hamiltonian* cycle of the double latin square.

In this paper we study double latin squares in which the occurrences of each symbol describe a Hamiltonian cycle. Such double latin squares are called *Hamiltonian double latin squares*. The expression "Hamiltonian double latin square(s) of order $2n$" will be abbreviated to HLS($2n$).

We let $A(i,j)$ denote the entry in the cell (i,j) of a matrix A. If A is an $n \times n$ matrix and γ is a permutation of the set $\{1,\ldots,n\}$ then $\pi_\gamma(A)$ will denote the matrix obtained from A by applying the permutation γ to its columns and $\pi^\gamma(A)$ will denote the matrix obtained from A by applying the permutation γ to its rows: thus $\pi_\gamma(A) = B$, $\pi^\gamma(A) = C$ where $B(i,\gamma(j)) = C(\gamma(i),j) = A(i,j)$ for $i,j = 1,\ldots,n$. The following theorem (which incorporates an improvement suggested by a referee) describes an easy way to construct several HLS($2n$) from two latin squares of order n.

Theorem 1.1. *If A, B are latin squares of order n on the same n symbols and γ is a permutation of $\{1,\ldots,n\}$ which has just one cycle* (*i.e.* $1,\gamma^1(1),\gamma^2(1),\gamma^3(1),\ldots,\gamma^{n-1}(1)$ *are distinct*) *then*

$$L(A,B;\gamma) = \begin{pmatrix} A & B \\ \pi_\gamma(A) & B \end{pmatrix}$$

is an HLS($2n$).

Proof. If a symbol σ occupies a cell (i,j) of A then it must also occupy the cell $(i,\gamma(j))$ of $\pi_\gamma(A)$ and some cell (i,k) of B and some cell $(h,\gamma(j))$ of A. Consequently, σ describes a cycle in $L = L(A,B;\gamma)$ in which five successive cells are $(i,j),(i,n+k)$, $(n+i,n+k)$, $(n+i,\gamma(j))$, $(h,\gamma(j))$. Hence, starting with the occurrence of σ in the first column of A, we find that in L there is a σ-cycle which visits in succession the columns

$$1, k_0, \gamma^1(1), k_1, \gamma^2(1), k_2, \gamma^3(1), k_3, \dots, \gamma^{n-1}(1), k_{n-1}, 1 \text{ of } L,$$

for some $k_0, k_1, \dots, k_{n-1} \in \{n+1, n+2, \dots, 2n\}$. Since $1, \gamma(1), \gamma^2(1), \dots, \gamma^{n-1}(1)$ are distinct, it follows that σ describes a Hamiltonian cycle in L. Since this argument applies to every symbol, L is Hamiltonian. $\square$

Example. If

$$A = \begin{array}{|cccc|}\hline 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \\ \hline\end{array}$$

and γ is the permutation $1 \mapsto 3 \mapsto 2 \mapsto 4 \mapsto 1$ then $L(A, A; \gamma)$ is

1	2	3	4	1	2	3	4
2	4	1	3	2	4	1	3
3	1	4	2	3	1	4	2
4	3	2	1	4	3	2	1
4	3	1	2	1	2	3	4
3	1	2	4	2	4	1	3
2	4	3	1	3	1	4	2
1	2	4	3	4	3	2	1

If we permute the rows and columns of an HLS($2n$) we obtain another HLS($2n$).

Proposition 1.2. *If A is an* HLS($2n$) *and γ, δ are permutations of* $\{1, 2, \dots, 2n\}$ *then* $\pi^\gamma(\pi_\delta(A))$ *is an* HLS($2n$).

Proof. If a symbol describes a Hamiltonian cycle

$$(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \dots, (i_{2n}, j_{2n}), (i_{2n}, j_1)$$

in A then it describes a Hamiltonian cycle

$$(\gamma(i_1), \delta(j_1)), (\gamma(i_1), \delta(j_2)), (\gamma(i_2), \delta(j_2)), (\gamma(i_2), \delta(j_3)),$$

$$\dots, (\gamma(i_{2n}), \delta(j_{2n})), (\gamma(i_{2n}), \delta(j_1))$$

in $\pi^\gamma(\pi_\delta(A))$. $\square$

2. Orthogonality

Let A, B be double latin squares of order $2n$ on the same symbols $\sigma_1, \ldots, \sigma_n$. We say that A, B are *orthogonal* if for each ordered pair (σ_i, σ_j) of symbols there are four ordered pairs (r, s) such that $A(r, s) = \sigma_i$ and $B(r, s) = \sigma_j$. We abbreviate "mutually orthogonal Hamiltonian double latin square(s) of order $2n$" to MOHLS($2n$).

For $n \geqslant 2$, let $H(2n)$ be the maximum number of MOHLS($2n$) and $N(n)$ be the maximum number of mutually orthogonal latin spares of order n, or MOLS(n).

Lemma 2.1. *$H(2n) \geqslant N(n)$ for all $n \geqslant 2$.*

Proof. Let γ be a permutation of $\{1, \ldots, n\}$ which has just one cycle. If $A_1, \ldots, A_{N(n)}$ are MOLS(n) then the $N(n)$ double latin squares $L(A_r, A_r; \gamma)$ $(r = 1, \ldots, N(n))$ are clearly mutually orthogonal, and are Hamiltonian by Theorem 1.1. $\square$

Problem 2.2. *It is well known that $N(n) \leqslant n - 1$, with equality occurring for some values of n. What is the comparable bound for $H(2n)$?*

A bound due to Hedayat et al. [13,14] for the maximum number of mutually orthogonal frequency squares implies that $H(2n) \leqslant (2n-1)^2/(n-1)$ (since double latin squares are special cases of frequency squares), but it seems unlikely that this bound is the right one.

In contrast to the fact that $N(n) = 1$ when n is 2 or 6, we have the following result:

Theorem 2.3. *$H(2n) \geqslant 2$ for all $n \geqslant 2$.*

Proof. When $n \notin \{2, 6\}$, Lemma 2.1 gives $H(2n) \geqslant N(n) \geqslant 2$, and so it remains to check that there exist two MOHLS(4) and two MOHLS(12). An example of the former is

$$\begin{array}{|cccc|}\hline 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ \hline\end{array}$$

To obtain a pair of MOHLS(12), start with the latin squares

$$A = \begin{array}{|cccccc|}\hline 1 & 2 & 5 & 6 & 3 & 4 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 5 & 6 & 3 & 4 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \\ \hline\end{array} \qquad B = \begin{array}{|cccccc|}\hline 1 & 2 & 5 & 6 & 3 & 4 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 4 & 1 & 2 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \\ 5 & 6 & 3 & 4 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ \hline\end{array} \qquad B' = \begin{array}{|cccccc|}\hline 2 & 1 & 6 & 5 & 4 & 3 \\ 1 & 2 & 5 & 6 & 3 & 4 \\ 4 & 3 & 2 & 1 & 6 & 5 \\ 3 & 4 & 1 & 2 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 6 & 3 & 4 & 1 & 2 \\ \hline\end{array}$$

Here A and B are obtained by using a direct product of a pair of MOLS(3) with an LS(2) and B' is obtained from B by interchanging three pairs of symbols. If γ is the permutation $1 \mapsto 6 \mapsto 5 \mapsto 4 \mapsto 3 \mapsto 2 \mapsto 1$ then $L(A, A; \gamma)$ and $L(B, B'; \gamma)$ are MOHLS(12). $\square$

Not surprisingly, orthogonality of HLS($2n$) is preserved by permutations of the rows or columns.

Theorem 2.4. *If* $\{A_1, \dots, A_r\}$ *is a set of* MOHLS($2n$) *and* γ, δ *are permutations of* $\{1, \dots, n\}$ *then* $\{\pi^\gamma(\pi_\delta(A_1)), \dots, \pi^\gamma(\pi_\delta(A_r))\}$ *is also a set of* MOHLS($2n$).

Proof. This is easy to see using Proposition 1.2. $\square$

3. Connections with graph theory

We use the following graph-theoretic language and conventions. As usual, $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively, of a graph G. A *spanning subgraph* of G is a subgraph S of G such that $V(S) = V(G)$. A $\{1, 2\}$*-factor* of G is a spanning subgraph S of G such that each vertex of G has degree 1 or 2 in S (i.e. such that each component of S is a path of non-zero length or a cycle). For the purposes of this paper, a *decomposition* of G is a set $\{S_1, \dots, S_r\}$ of spanning subgraphs of G such that each edge of G is in exactly one of them. This decomposition will be called (i) a *Hamiltonian decomposition* of G if $S_1, \dots, S_r$ are Hamiltonian cycles of G, (ii) a *Hamiltonian path decomposition* of G if $S_1, \dots, S_r$ are Hamiltonian paths of G, (iii) a $\{1, 2\}$*-factorisation* of G if $S_1, \dots, S_r$ are $\{1, 2\}$-factors of G. An edge of a graph joining vertices x, y will be denoted by $[x, y]$. A path or cycle in a graph will be denoted by $\langle x_1, x_2, \dots, x_m \rangle$ or $\langle x_1, x_2, \dots, x_m, x_1 \rangle$, respectively, if it has m distinct vertices $x_1, \dots, x_m$ and its edges are $[x_1, x_2], [x_2, x_3], \dots, [x_{m-1}, x_m]$ and, in the case of a cycle, $[x_m, x_1]$. We use the customary notations K_n and $K_{m,n}$ for complete graphs and complete bipartite graphs.

We describe here two connections between Hamiltonian double latin squares and graph theory.

For the first of these, consider any $2n \times 2n$ matrix A in which each cell contains exactly one of the symbols $\sigma_1, \dots, \sigma_n$. Let $K_{2n,2n}$ be a complete bipartite graph with vertices $\rho_1, \rho_2, \dots, \rho_{2n}, c_1, c_2, \dots, c_{2n}$ and edges $[\rho_i, c_j]$ $(i,j = 1, \dots, 2n)$: we think of the vertices ρ_i, c_j as representing the ith row and the jth column of A, respectively. Let S_k be the spanning subgraph of $K_{2n,2n}$ such that $[\rho_i, c_j] \in E(S_k)$ if and only if the cell (i,j) of A contains σ_k $(i,j = 1, \dots, 2n)$. Then $\{S_1, \dots, S_{2n}\}$ is a decomposition of $K_{2n,2n}$ which represents A in an obvious sense, and it is easily seen that $\{S_1, \dots, S_{2n}\}$ is a Hamiltonian decomposition of $K_{2n,2n}$ if and only if A is a Hamiltonian double latin square. Thus we have:

Lemma 3.1. *An* HLS($2n$) *is equivalent to a Hamiltonian decomposition of* $K_{2n,2n}$.

For a general reference about Hamiltonian decompositions, see [1], and for some conceptually similar current work, see [8,21].

The second connection with graph theory is less obvious. It concerns *symmetric Hamiltonian double latin squares*. We abbreviate "symmetric HLS($2n$)" to SHLS($2n$). We shall make use of $\{1,2\}$-factorisations of K_{2n} which comprise exactly n $\{1,2\}$-factors: these include Hamiltonian path decompositions of K_{2n} by the following (trivial) lemma:

Lemma 3.2. *If $\mathscr{D}$ is a Hamiltonian path decomposition of K_{2n} then $|\mathscr{D}| = n$ and each vertex of K_{2n} is an endvertex of exactly one member of $\mathscr{D}$.*

Proof. Since $|E(K_{2n})| = n(2n-1)$ and each member of $\mathscr{D}$ has $2n-1$ edges, it follows that $|\mathscr{D}| = n$. Since the degrees of a vertex in the n members of $\mathscr{D}$ add up to $2n-1$, it must be an endvertex of exactly one of them. $\square$

For any positive integer m, let $\mathbb{Z}_m$ denote the ring of residue classes modulo m. Expressions which denote integers will also be used as names for the corresponding residue classes modulo m, leaving the context to indicate the intended meaning. Throughout Sections 3–5, we shall for convenience take $V(K_{2n})$ to be $\mathbb{Z}_{2n}$. Consequently, expressions which denote integers can also serve as names for vertices of K_{2n}, and two such expressions serve as different names for the same vertex if they denote integers differing by a multiple of $2n$.

Given a symmetric double latin square A of order $2n$ on symbols $\sigma_1, \dots, \sigma_n$, let $H_r = H(A, \sigma_r)$ be the spanning subgraph of K_{2n} such that $E(H_r) = \{[i,j]\colon i \neq j \text{ and } A(i,j) = A(j,i) = \sigma_r\}$. Then $\{H_1, \dots, H_n\}$ is a decomposition of K_{2n}, and the presence of σ_r in the cells (i,j), (j,i) of A (where $i \neq j$) is witnessed by the edge $[i,j]$ of K_{2n} being in H_r. If $r \in \{1, \dots, n\}$ and $i \in \{1, \dots, 2n\}$, the symbol σ_r appears twice in the ith row of A, but only appearances of σ_r off the main diagonal of A give rise to edges of H_r. Therefore, the degree in H_r of the vertex i is 2 if $A(i,i) \neq \sigma_r$ and 1 if $A(i,i) = \sigma_r$. Consequently each H_r is a $\{1,2\}$-factor of K_{2n} whose vertices of degree 1 correspond to the occurrences of σ_r on the main diagonal of A. It follows that $\{H_1, \dots, H_n\}$ is a $\{1,2\}$-factorisation of K_{2n}, which we shall call the $\{1,2\}$-factorisation *corresponding* to A.

If a particular H_r is a Hamiltonian path $\langle i_1, i_2, \dots, i_{2n} \rangle$ of K_{2n} then σ_r describes a Hamiltonian cycle

$$(i_1, i_1), (i_1, i_2), (i_3, i_2), (i_3, i_4), (i_5, i_4), \dots, (i_{2n-1}, i_{2n-2}), (i_{2n-1}, i_{2n}), (i_{2n}, i_{2n}),$$
$$(i_{2n}, i_{2n-1}), (i_{2n-2}, i_{2n-1}), \dots, (i_4, i_5), (i_4, i_3), (i_2, i_3), (i_2, i_1)$$

in A. Conversely, if a particular symbol σ_r describes a Hamiltonian cycle in A then the corresponding $\{1,2\}$-factor H_r must clearly be connected, and so must be either a Hamiltonian path or a Hamiltonian cycle of K_{2n}; but H_r cannot be a Hamiltonian cycle $\langle i_1, i_2, \dots, i_{2n}, i_1 \rangle$ because then σ_r would describe two disjoint cycles

$$(i_1, i_2), (i_3, i_2), (i_3, i_4), (i_5, i_4), \dots, (i_{2n-1}, i_{2n-2}), (i_{2n-1}, i_{2n}), (i_1, i_{2n})$$

and

$$(i_2, i_1), (i_2, i_3), (i_4, i_3), (i_4, i_5), \ldots, (i_{2n-2}, i_{2n-1}), (i_{2n}, i_{2n-1}), (i_{2n}, i_1).$$

We conclude that σ_r describes a Hamiltonian cycle in A if and only if H_r is a Hamiltonian path of K_{2n}. Consequently, A is Hamiltonian if and only if $\{H_1, \ldots, H_r\}$ is a Hamiltonian path decomposition of K_{2n}; and we have established the following theorem:

Theorem 3.3. *A symmetric double latin square of order* $2n$ *is Hamiltonian if and only if the corresponding* $\{1, 2\}$*-factorisation of* K_{2n} *is a Hamiltonian path decomposition.*

Moreover, Lemma 3.2 implies that *any* Hamiltonian path decomposition $\{P_1, \ldots, P_n\}$ of K_{2n} is the $\{1, 2\}$-factorisation corresponding to a symmetric double latin square A on n symbols $\sigma_1, \ldots, \sigma_n$ such that

$$A(i,j) = \sigma_r \quad \text{when } i \neq j \text{ and } [i,j] \in E(P_r),$$

$$A(i,i) = \sigma_r \quad \text{when } i \text{ is an endvertex of } P_r.$$

From this observation and Theorem 3.3, we see that an SHLS$(2n)$ is equivalent to a Hamiltonian path decomposition of K_{2n}. This, in turn, implies the following further equivalence, which will be exploited in Section 8:

Corollary 3.4. *An* SHLS$(2n)$ *is equivalent to a Hamiltonian decomposition of* K_{2n+1}.

Proof. We may clearly regard K_{2n+1} as being obtained from K_{2n} by adding a new vertex v and edges joining v to the vertices of K_{2n}. By Lemma 3.2, any Hamiltonian path decomposition $\{H_1, \ldots, H_n\}$ of K_{2n} gives rise to a Hamiltonian decomposition $\{H'_1, \ldots, H'_n\}$ of K_{2n+1}, in which H'_r is obtained from H_r by adding v and the edges of K_{2n+1} joining v to the endvertices of H_r. Conversely, any Hamiltonian decomposition of K_{2n+1} becomes a Hamiltonian path decomposition of K_{2n} when we delete v and its incident edges from the Hamiltonian circuits concerned. Therefore, Hamiltonian decompositions of K_{2n+1} are equivalent to Hamiltonian path decompositions of K_{2n} and hence to symmetric Hamiltonian double latin squares of order $2n$. $\square$

4. Symmetry

We have just seen that an SHLS$(2n)$ is equivalent to a Hamiltonian path decomposition of K_{2n} and also to a Hamiltonian decomposition of K_{2n+1}. It is well known (see, for example, [7, Chapter1, Theorem 11]) that such decompositions exist for every positive integer n, and so we have:

Theorem 4.1. *An* SHLS$(2n)$ *exists for every positive integer* n.

In fact, each of Corollary 4.10, Theorems 8.2 and 9.1 below implies Theorem 4.1.

Lemma 4.2. *In a symmetric Hamiltonian double latin square, each symbol occurs exactly twice on the main diagonal.*

Proof. Let A be an SHLS($2n$) on symbols $\sigma_1, \dots, \sigma_n$. Then the corresponding $\{1,2\}$-factorisation $\{H_1, \dots, H_n\}$ of K_{2n} is obtained, as explained in Section 3, by taking H_r to be $H(A, \sigma_r)$ for $r = 1, \dots, n$. By Theorem 3.3, each H_r is a Hamiltonian path of K_{2n} and so has exactly two vertices of degree 1. It follows that each σ_r occurs exactly twice on the main diagonal of A because, as explained in Section 3, $A(i,i)$ is σ_r if and only if the vertex i has degree 1 in H_r. $\square$

If A is an SHLS($2n$) on symbols $\sigma_1, \dots, \sigma_n$ then there is by Lemma 4.2 a partition $\{\{i_1,j_1\}, \{i_2,j_2\}, \dots, \{i_n,j_n\}\}$ of $\{1, 2, \dots, 2n\}$ into n subsets of cardinality 2 such that $A(i_r, i_r) = A(j_r, j_r) = \sigma_r$ for $r = 1, \dots, n$. This partition will be called the *diagonal partition* induced by A.

For some purposes, it may be convenient to take the symbols in a double latin square of order $2n$ to be the numbers $1, \dots, n$ rather than arbitrary objects $\sigma_1, \dots, \sigma_n$. If A is an SHLS($2n$) on the symbols $1, \dots, n$ whose main diagonal is $(1, 2, \dots, n, 1, 2, \dots, n)$, we shall say that A is in *normal form*. Thus A is in normal form if $A(r,r) = A(n+r, n+r) = r$ for $r = 1, \dots, n$.

Example 4.3. The following SHLS(10) are both in normal form:

1	1	2	2	3	3	4	4	5	5
1	2	2	3	3	4	4	5	5	1
2	2	3	3	4	4	5	5	1	1
2	3	3	4	4	5	5	1	1	2
3	3	4	4	5	5	1	1	2	2
3	4	4	5	5	1	1	2	2	3
4	4	5	5	1	1	2	2	3	3
4	5	5	1	1	2	2	3	3	4
5	5	1	1	2	2	3	3	4	4
5	1	1	2	2	3	3	4	4	5

1	5	4	3	1	3	2	5	2	4
5	2	1	5	4	2	4	3	1	3
4	1	3	2	1	5	3	5	4	2
3	5	2	4	3	2	1	4	1	5
1	4	1	3	5	4	3	2	5	2
3	2	5	2	4	1	5	4	3	1
2	4	3	1	3	5	2	1	5	4
5	3	5	4	2	4	1	3	2	1
2	1	4	1	5	3	5	2	4	3
4	3	2	5	2	1	4	1	3	5

By Proposition 1.2 and Lemma 4.2, any SHLS($2n$) on the symbols $1, \dots, n$ can be transformed into an SHLS($2n$) in normal form by applying a suitable permutation to its rows and the same permutation to its columns.

We shall say that a double latin square A of order $2n$ on the symbols $1, \dots, n$ is *cyclic* if $A(i', j') \equiv A(i,j) + 1 \pmod{n}$ whenever $i' \equiv i + 1 \pmod{2n}$ and $j' \equiv j + 1 \pmod{2n}$. In other words, A is cyclic if $A(i+1, j+1) = A(i,j) + 1$ for $i,j = 1, \dots, 2n$, with $i+1, j+1$ interpreted modulo $2n$ and $A(i,j) + 1$ interpreted modulo

n. In a cyclic double latin square of order $2n$, if we start at a cell containing 1 and travel "South-East", we encounter the symbols $1, 2, \ldots, n, 1, 2, \ldots, n$ in that order, provided that on reaching a cell $(i, 2n)$ $(i<n)$, $(2n, j)$ $(j<n)$ or $(2n, 2n)$ we move next to the cell $(i+1, 1)$, $(1, j+1)$ or $(1, 1)$ respectively. An example of a cyclic HLS(10) which is not symmetric is

5	4	3	2	1	1	5	4	3	2
3	1	5	4	3	2	2	1	5	4
5	4	2	1	5	4	3	3	2	1
2	1	5	3	2	1	5	4	4	3
4	3	2	1	4	3	2	1	5	5
1	5	4	3	2	5	4	3	2	1
2	2	1	5	4	3	1	5	4	3
4	3	3	2	1	5	4	2	1	5
1	5	4	4	3	2	1	5	3	2
3	2	1	5	5	4	3	2	1	4

The SHLS(10) in Examples 4.3 are also both cyclic.

If a cyclic HLS($2n$) is also symmetric, it is by Theorem 3.3 associated with a Hamiltonian path decomposition of K_{2n}. We now examine those Hamiltonian path decompositions of K_{2n} which correspond to cyclic SHLS($2n$). We also seek ways of constructing such decompositions, which is tantamount to constructing examples of cyclic SHLS($2n$).

If e denotes an edge $[x, y]$ of K_{2n} and $r \in \mathbb{Z}_{2n}$ then $e + r$ will denote the edge $[x+r, y+r]$ of K_{2n}. If U is a subset of $V(K_{2n}) = \mathbb{Z}_{2n}$ or of $E(K_{2n})$ then $U + r$ will denote the set $\{u + r\colon u \in U\}$ and $U - r$ will denote $U + (-r)$. If S is a spanning subgraph of K_{2n} then $S + r$, $S - r$ will denote the spanning subgraphs of K_{2n} such that $E(S+r) = E(S) + r, E(S-r) = E(S) - r$, respectively. We shall call S a *cyclic* spanning subgraph and call $\{S+1, S+2, \ldots, S+n\}$ a *cyclic* decomposition of K_{2n} *generated* by S if each edge of K_{2n} belongs to exactly one of the spanning subgraphs $S+1, S+2, \ldots, S+n$.

Lemma 4.4. *If S is a cyclic spanning subgraph of K_{2n} then $S + n = S$.*

Proof. Since $\{E(S+1), E(S+2), \ldots, E(S+n)\}$ is a partition of $E(K_{2n})$, it follows that $\{E(S+1) - 1, E(S+2) - 1, \ldots, E(S+n) - 1\}$ is also a partition of $E(K_{2n})$, i.e. $\{E(S), E(S+1), \ldots, E(S+n-1)\}$ is a partition of $E(K_{2n})$. Since both $\{E(s+1), E(S+2), \ldots, E(S+n)\}$ and $\{E(S), E(S+1), \ldots, E(S+n-1)\}$ are partitions of $E(K_{2n})$, it follows that $E(S+n) = E(S)$, and so $S + n = S$. $\square$

Lemma 4.5. *If a symmetric double latin square of order $2n$ is cyclic then the corresponding $\{1, 2\}$-factorisation of K_{2n} is cyclic. Conversely, every cyclic $\{1, 2\}$-*

factorisation of K_{2n} is the $\{1,2\}$-factorisation corresponding to some cyclic symmetric double latin square of order $2n$.

Proof. Let

$$f(r) = r+1 \quad (r = 1, \dots, 2n-1), \; f(2n) = 1,$$

$$g(r) = r+1 \quad (r = 1, \dots, n-1), \; g(n) = 1.$$

Suppose that A is a cyclic symmetric double latin square of order $2n$. Then the corresponding $\{1,2\}$-factorisation of K_{2n} is $\{H_1, \dots, H_n\}$, where $H_r = H(A, r)$ for $r = 1, \dots, n$. Suppose that $i,j \in \{1, \dots, 2n\}$, $i \neq j$, $r \in \{1, \dots, n\}$ and e is the edge $[i,j]$ of K_{2n}. Then $e+1 = [i+1, j+1] = [f(i), f(j)]$ because $i+1 = f(i)$ and $j+1 = f(j)$ in $\mathbb{Z}_{2n} = V(K_{2n})$. Therefore $e+1 \in E(H_{g(r)})$ if and only if $A(f(i), f(j)) = g(r)$, which (since A is cyclic) is true if and only if $A(i,j) = r$, which is true if and only if $e = [i,j] \in E(H_r)$. Hence $E(H_{g(r)}) = E(H_r) + 1$. Since this is true for $r = 1, \dots, n$ it follows that $E(H_r) = E(H_n) + r$ for $r = 1, \dots, n$ and therefore $H_r = H_n + r$ for $r = 1, \dots, n$. Therefore the $\{1,2\}$-factorisation $\{H_1, \dots, H_n\}$ corresponding to A is cyclic.

Now suppose that $\mathscr{F}$ is a cyclic $\{1,2\}$-factorisation of K_{2n}. Then $\mathscr{F} = \{S+1, S+2, \dots, S+n\}$ for some cyclic spanning subgraph S of K_{2n}. Since $S+n+1 = S+1$ by Lemma 4.4, it follows that $S+r+1 = S+g(r)$ for $r = 1, \dots, n$. Each vertex of K_{2n} has degree $2n-1$, and so must have degree 1 in just one member of $\mathscr{F}$ and degree 2 in the others. Consequently, $\mathscr{F} = \{S+1, S+2, \dots, S+n\}$ is the $\{1,2\}$-factorisation of K_{2n} corresponding to the symmetric double latin square B defined by

$$B(i,j) = r \quad \text{when } i \neq j \text{ and } [i,j] \in E(S+r),$$

$$B(i,i) = r \quad \text{when the vertex } i \text{ has degree 1 in } S+r.$$

If $i,j \in \{1, \dots, n\}$ and $i \neq j$ and $B(i,j) = r$ then $[i,j] \in E(S+r)$ and so $[f(i), f(j)] = [i+1, j+1] \in E(S+r+1) = E(S+g(r))$ and therefore $B(f(i), f(j)) = g(r)$. If $i \in \{1, \dots, n\}$ and $B(i,i) = r$ then the vertex i has degree 1 in $S+r$ and so the vertex $i+1 = f(i)$ has degree 1 in $S+r+1 = S+g(r)$ and therefore $B(f(i), f(i)) = g(r)$. Hence $B(f(i), f(j)) = g(B(i,j))$ for $i,j = 1, \dots, 2n$. Therefore B is cyclic, and so $\mathscr{F}$ is the $\{1,2\}$-factorisation corresponding to a cyclic symmetric double latin square of order $2n$. □

Corollary 4.6. *If an* SHLS($2n$) *is cyclic then the corresponding $\{1,2\}$-factorisation of K_{2n} is a cyclic Hamiltonian path decomposition of K_{2n}. Conversely, every cyclic Hamiltonian path decomposition of K_{2n} is the $\{1,2\}$-factorisation corresponding to some cyclic* SHLS($2n$).

Proof. This follows from Theorem 3.3 and Lemma 4.5. □

Thus, searching for cyclic SHLS($2n$) is equivalent to searching for cyclic Hamiltonian path decompositions of K_{2n}, which is equivalent to searching for generators of such decompositions, i.e. cyclic Hamiltonian paths of K_{2n}.

We let ρ denotes the automorphism of K_{2n} such that $\rho(x) = x+1$ for each $x \in V(K_{2n}) = \mathbb{Z}_{2n}$. Clearly, ρ induces a permutation of $E(K_{2n})$ whose orbits are $E_1, \dots, E_n$, where

$$E_r = \{[1, 1+r], [2, 2+r], \dots, [2n, 2n+r]\}$$

for $r = 1, \dots, n-1$ and

$$E_n = \{[1, 1+n], [2, 2+n], \dots, [n, 2n]\}.$$

Lemma 4.7. *A spanning subgraph S of K_{2n} is cyclic if and only if $|E(S) \cap E_n| = 1$ and there are edges $e_1, \dots, e_{n-1}$ of K_{2n} such that $E(S) \cap E_r = \{e_r, e_r + n\}$ for $r = 1, \dots, n-1$.*

Proof. By definition, S is cyclic if and only if each edge of K_{2n} is in exactly one of $S+1, S+2, \dots, S+n$. This condition is satisfied by the edges in E_n if and only if $|E(S) \cap E_n| = 1$, and is satisfied by the edges in E_r, where $r \in \{1, \dots, n-1\}$, if and only if $E(S) \cap E_r = \{e_r, e_r + n\}$ for some edge e_r. $\square$

Definition. We shall say that a set A is a *transversal* of disjoint sets $B_1, \dots, B_m$ if $A \subseteq B_1 \cup \dots \cup B_m$ and $|A \cap B_r| = 1$ for $r = 1, \dots, m$. For $n \geqslant 2$, we define an *n-procession* to be a sequence $s_1, \dots, s_n$ of n elements of $\mathbb{Z}_{2n}$ which satisfies the conditions

(P1) $\{s_1, \dots, s_n\}$ is a transversal of the sets $\{0, n\}, \{1, n+1\}, \{2, n+2\}, \dots, \{n-1, 2n-1\}$;
(P2) $\{s_2 - s_1, s_3 - s_2, \dots, s_n - s_{n-1}\}$ is a transversal of the sets $\{1, -1\}, \{2, -2\}, \{3, -3\}, \dots, \{n-1, -(n-1)\}$.

We define an *n-gradation* $(n \geqslant 2)$ to be a sequence $a_1, \dots, a_{n-1}$ of $n-1$ elements of $\mathbb{Z}_{2n}$ which satisfies the conditions

(G1) $\{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_{n-1}\}$ is a transversal of the sets $\{1, n+1\}, \{2, n+2\}, \{3, n+3\}, \dots, \{n-1, 2n-1\}$;
(G2) $\{a_1, a_2, \dots, a_{n-1}\}$ is a transversal of the sets $\{1, -1\}, \{2, -2\}, \{3, -3\}, \dots, \{n-1, -(n-1)\}$.

These are, in a sense, equivalent concepts, since a sequence is an n-gradation if and only if it is $s_2 - s_1, s_3 - s_2, \dots, s_n - s_{n-1}$ for some n-procession $s_1, \dots, s_n$ and a sequence is an n-procession if and only if it is $x, x + a_1, x + a_1 + a_2, x + a_1 + a_2 + a_3, \dots, x + a_1 + \dots + a_{n-1}$ for some $x \in \mathbb{Z}_{2n}$ and some n-gradation $a_1, \dots, a_{n-1}$. Convenience will dictate whether we use n-processions or n-gradations in any particular part of our discussion.

Illustration. For any integer $n \geqslant 2$, an obvious example of an n-gradation is the sequence $1, -2, 3, -4, 5, -6, \dots, (-1)^n(n-1)$. An associated n-procession is the sequence $x, x+1, x-1, x+2, x-2, x+3, x-3, \dots$ ending with its nth term $x + \frac{n}{2}$ or $x - \frac{n-1}{2}$, where x is any element of $\mathbb{Z}_{2n}$.

It is easily checked that the only 4-gradations are the eight sequences

$$1,2,3; \quad 1,-2,3; \quad -1,2,-3; \quad -1,-2,-3;$$
$$3,2,1; \quad 3,-2,1; \quad -3,2,-1; \quad -3,-2,-1.$$

In the arithmetic of $\mathbb{Z}_8$, these are just the sequences $u, v, 3u$ where $u \in \{-3,-1,1,3\}$, $v \in \{-2,2\}$. It follows that there are just sixty-four 4-processions, namely the sequences x, $x+u$, $x+u+v, x+4u+v$ where $x \in \mathbb{Z}_8$, $u \in \{-3,-1,1,3\}, v \in \{-2,2\}$-or, more simply, the sequences $x, x+u, x+u+v, x-v$ with x, u, v as stated.

Definition. If a sequence $s_1, \ldots, s_n$ of elements of $\mathbb{Z}_{2n}$ satisfies (P1) then $H(s_1, \ldots, s_n)$ will denote the Hamiltonian path $\langle s_1, s_2, \ldots, s_n, s_n+n, s_{n-1}+n, s_{n-2}+n, \ldots, s_1+n \rangle$ of K_{2n}; if we let s_{i+n} denote $s_{n-i+1}+n$ for $i = 1, \ldots, n$, then $H(s_1, \ldots, s_n)$ is the Hamiltonian path $\langle s_1, s_2, \ldots, s_{2n} \rangle$. If a sequence $a_1, \ldots, a_{n-1}$ of elements of $\mathbb{Z}_{2n}$ satisfies (G1) then $H[a_1, \ldots, a_{n-1}]$ will denote the Hamiltonian path $H(0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_{n-1})$ of K_{2n}.

Lemma 4.8. *A Hamiltonian path Q of K_{2n} $(n \geqslant 2)$ is cyclic if and only if $Q = H(s_1, \ldots, s_n)$ for some n-procession $s_1, \ldots, s_n$.*

Proof. Assume first that $Q = H(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n$ is an n-procession. Then it follows from (P2) that $E(Q) \cap E_n = \{[s_n, s_n+n]\}$ and that $E(Q) \cap E_1$, $E(Q) \cap E_2, \ldots, E(Q) \cap E_{n-1}$ are the sets

$$\{[s_1, s_2], [s_1+n, s_2+n]\}, \{[s_2, s_3], [s_2+n, s_3+n]\}, \ldots, \{[s_{n-1}, s_n], [s_{n-1}+n, s_n+n]\}$$

in some order. Therefore Q is cyclic by Lemma 4.7.

Now assume that Q is cyclic. Then, by Lemma 4.4, the automorphism $x \mapsto x+n$ of K_{2n} induces an automorphism of Q. Since this automorphism of Q is not the identity automorphism, it must be the one which interchanges the endvertices of Q, and so Q must be $\langle s_1, s_2, \ldots, s_n, s_n+n, s_{n-1}+n, s_{n-2}+n, \ldots, s_1+n \rangle$ for some $s_1, \ldots, s_n \in \mathbb{Z}_{2n}$. Since the vertices $s_1, s_2, \ldots, s_n, s_n+n, s_{n-1}+n, \ldots, s_1+n$ of Q are distinct, the sequence $s_1, \ldots, s_n$ satisfies (P1). Since the edge $[s_n, s_n+n]$ of Q belongs to E_n, it follows from Lemma 4.7 that each of $E_1, \ldots, E_{n-1}$ includes two of the remaining $2n-2$ edges of Q, and so $s_1, \ldots, s_n$ must satisfy (P2). Hence $s_1, \ldots, s_n$ is an n-procession. Moreover $Q = \langle s_1, s_2, \ldots, s_n, s_n+n, s_{n-1}+n, \ldots, s_1+n \rangle = H(s_1, \ldots, s_n)$. $\square$

Corollary 4.9. *A Hamiltonian path Q of $K_{2n}(n \geqslant 2)$ is cyclic if and only if $Q = H[a_1, \ldots, a_{n-1}] + x$ for some n-gradation $a_1, \ldots, a_{n-1}$ and some $x \in \mathbb{Z}_{2n}$.*

Proof. If Q is cyclic then by Lemma 4.8 there is an n-procession $s_1, \ldots, s_n$ such that

$$Q = H(s_1, \ldots, s_n) = H[s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1}] + s_1,$$

which is of the required form since $s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1}$ is an n-gradation. Conversely, if $a_1, \ldots, a_{n-1}$ is an *n-gradation* and $x \in \mathbb{Z}_{2n}$ then

$$H[a_1, \ldots, a_{n-1}] + x = H(x, x + a_1, x + a_1 + a_2, \ldots, x + a_1 + \cdots + a_{n-1}),$$

which is a cyclic Hamiltonian path by Lemma 4.8 since $x, x + a_1, x + a_1 + a_2, \ldots, x + a_1 + \cdots + a_{n-1}$ is an n-procession. $\square$

Corollary 4.10. *There exists a cyclic* SHLS$(2n)$ *for every positive integer n.*

Proof. The double latin square $\boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}}$ is a cyclic SHLS(2). For $n \geqslant 2$, it follows from Corollary 4.9 that $H[1, -2, 3, -4, 5, -6, \ldots, (-1)^n(n-1)]$ is a cyclic Hamiltonian path of K_{2n} and so generates a cyclic Hamiltonian path decomposition of K_{2n}, which by Corollary 4.6 implies the existence of a cyclic SHLS$(2n)$. $\square$

(This proof of Corollary 4.10 is really just a disguised version of the usual way of proving that K_{2n} has a Hamiltonian path decomposition for every n.)

Searching for cyclic SHLS$(2n)$ is by Corollary 4.6 equivalent to searching for cyclic Hamiltonian paths of K_{2n}, which is by Corollary 4.9 equivalent (when $n \geqslant 2$) to searching for n-gradations. It is therefore worth noticing three simple transformations which generate new n-gradations from known ones.

Firstly, it is easily seen that if $a_1, \ldots, a_{n-1}$ is an n-gradation then so is $ka_1, \ldots, ka_{n-1}$ for any integer k coprime to $2n$. The multiplication by k is of course performed in $\mathbb{Z}_{2n}$. For example, the 10-gradation $1, -2, 3, -4, 5, -6, 7, -8, 9$ yields another 10-gradation $3, -6, 9, 8, -5, 2, 1, -4, 7$ when we multiply its terms by 3 in the arithmetic of $\mathbb{Z}_{20}$.

Secondly, if $a_1, a_2, \ldots, a_{n-1}$ is an n-gradation then so is $a_{n-1}, a_{n-2}, \ldots, a_1$. Reversing the order of the terms obviously preserves property (G2) of an n-gradation, and it also preserves (G1) because (G1) is equivalent to saying that none of the elements $a_i + a_{i+1} + a_{i+2} + \cdots + a_{j-1} + a_j$ $(1 \leqslant i \leqslant j \leqslant n-1)$ of $\mathbb{Z}_{2n}$ belongs to its subgroup $\{0, n\}$. Combining this observation with the preceding one, we see that if $a_1, \ldots, a_{n-1}$ is an n-gradation then so is $-a_{n-1}, -a_{n-2}, \ldots, -a_1$ and therefore $H[a_1, a_2, \ldots, a_{n-1}]$, $H[-a_{n-1}, -a_{n-2}, \ldots, -a_1]$ generate two cyclic Hamiltonian path decompositions $\mathscr{D}, \mathscr{D}^*$ of K_{2n}. For any Hamiltonian path $P = \langle x_1, x_2, \ldots, x_{2n} \rangle$ of K_{2n}, let $\mathscr{P}^*$ denote the Hamiltonian path $\langle x_n, x_{n-1}, \ldots, x_1, x_{2n}, x_{2n-1}, \ldots, x_{n+1} \rangle$ obtained from P by removing its middle edge $[x_n, x_{n+1}]$ and adding the edge $[x_1, x_{2n}]$ of K_{2n}. Then it is easily checked that

$$H[-a_{n-1}, -a_{n-2}, \ldots, -a_1] = H[a_1, a_2, \ldots, a_{n-1}]^* - (a_1 + \cdots + a_{n-1})$$

and consequently $\mathscr{D}^* = \{P^*\colon P \in \mathscr{D}\}$.

Thirdly, adding n (in the arithmetic of $\mathbb{Z}_{2n}$) to some of the terms of an n-gradation will preserve property (G1). It may not in general preserve (G2), but it will clearly do so if we add n to those terms which belong to $S \cup (-S)$ where S is a subset of $\{1, \ldots, n-1\}$ such that $S = n - S$. (As usual, $-S$ and $n - S$ mean $\{-r\colon r \in S\}$ and $\{n - r\colon r \in S\}$ respectively.) For example, taking $n = 12$ and $S = \{2, 3, 6, 9, 10\}$, the

12-gradation $5, -10, 3, -8, -11, 6, -1, -4, 9, -2, 7$ becomes a new 12-gradation $5, 2, -9, -8, -11, -6, -1, -4, -3, 10, 7$ when we add 12 (in $\mathbb{Z}_{24}$) to each of its terms $-10, 3, 6, 9, -2$.

Corollary 4.9 says that the sequence of vertices of a cyclic Hamiltonian path of K_{2n} can be derived from a shorter sequence, namely an n-gradation. In many cases, this in turn can be derived from an even shorter sequence, as indicated by Theorems 4.11 and 4.12 below. These theorems require a preliminary definition. If $m(>0)$ and x are integers, let $[x]_m$ denote the residue class of x modulo m (so that $[x]_m \in \mathbb{Z}_m$ and, in fact, $[x]_m$ is the element of $\mathbb{Z}_m$ which we commonly denote by just the symbol x). Then the *representatives* of $[x]_m$ in $\mathbb{Z}_{2m}$ are the elements $[x]_{2m}, [x+m]_{2m}$ of $\mathbb{Z}_{2m}$.

Theorem 4.11. *If $a_1, \dots, a_{n-1}$ is an n-gradation and $\bar{a}_i$ is a representative of a_i in $\mathbb{Z}_{4n}$ $(i = 1, \dots, n-1)$ and $\delta \in \{-1, 1\}$ then $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{n-1}, \delta n, 2n - \bar{a}_{n-1}, 2n - \bar{a}_{n-2}, \dots, 2n - \bar{a}_1$ is a $2n$-gradation.*

Proof. That $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{n-1}, \delta n, 2n - \bar{a}_{n-1}, 2n - \bar{a}_{n-2}, \dots, 2n - \bar{a}_1$ satisfies (G1) follows from the observation that in $\mathbb{Z}_{2n}$

$$\begin{aligned} \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_i + n = \bar{a}_1 + \cdots + \bar{a}_{n-1} + \delta n + (2n - a_{n-1}) \\ +(2n - a_{n-2}) + \cdots + (2n - a_{i+1}). \end{aligned}$$

That (G2) is satisfied follows by observing that in $\mathbb{Z}_{4n}$ one of $\bar{a}_i$ and $2n - \bar{a}_i$ is in the set $\{1, 2, \dots, n-1\} \cup \{-1, -2, \dots, -(n-1)\}$ and the other in the set $\{n+1, n+2, \dots, 2n-1\} \cup \{-(n+1), -(n+2), \dots, -(2n-1)\}$. $\square$

Theorem 4.12. *If $a_1, \dots, a_{n-1} \in \mathbb{Z}_{2n-1}$ and both $\{a_1, \dots, a_{n-1}\}$ and $\{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \cdots + a_{n-1}\}$ are transversals of the sets $\{1, -1\}, \{2, -2\}, \dots, \{n-1, -(n-1)\}$ and $\bar{a}_i$ is a representative of a_i in $\mathbb{Z}_{4n-2}$ $(i = 1, \dots, n-1)$ then $\bar{a}_{n-1}, \bar{a}_{n-2}, \dots, \bar{a}_2, \bar{a}_1, 2n-1+\bar{a}_1, 2n-1+\bar{a}_2, \dots, 2n-1+\bar{a}_{n-2}, 2n-1+\bar{a}_{n-1}$ is a $(2n-1)$-gradation.*

Proof. First, we prove that the sequence $\bar{a}_{n-1}, \bar{a}_{n-2}, \dots, \bar{a}_1, (2n-1) + \bar{a}_1, \dots, (2n-1) + \bar{a}_{n-1}$ satisfies (G2). We need to show that this sequence is a transversal of $\{1, -1\}, \{2, -2\}, \dots, \{2n-2, -(2n-2)\}$. Observe that the sequence has the correct number of elements. Therefore, to show that the sequence is a transversal, we only need to show that no two elements are in the same set. If one of $\bar{a}_i$ and $(2n-1) + \bar{a}_i$ equals one of $\bar{a}_j$ and $(2n-1) + \bar{a}_j$ for some $i \neq j$, $1 \leqslant i, j \leqslant n-1$ in $\mathbb{Z}_{4n-2}$, then $a_i = a_j$ in $\mathbb{Z}_{2n-1}$, a contradiction since $\{a_1, a_2, \dots, a_{n-1}\}$ is a transversal.

Next we show that the given sequence satisfies (G1). We need to show that, writing $\bar{A} = \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_{n-1}$,

$$\begin{aligned} &\bar{a}_{n-1}, \bar{a}_{n-1} + \bar{a}_{n-2}, \dots, \bar{a}_{n-1} + \bar{a}_{n-2} + \cdots + \bar{a}_1, \bar{A} + (2n-1) + \bar{a}_1, \\ &\bar{A} + 2(2n-1) + \bar{a}_1 + \bar{a}_2, \dots, \bar{A} + (n-1)(2n-1) + \cdots + \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_{n-1} \end{aligned}$$

is a transversal in $\mathbb{Z}_{4n-2}$ of the sets

$$\{1,(2n-1)+1\},\{2,(2n-1)+2\},\dots,\{(2n-2),(2n-1)+(2n-2)\}.$$

The number of terms, $2n-2$, is the same as the number of sets. Therefore to show that the sequence is a transversal, we only need to show that no two elements are in the same set.

Suppose $\bar{a}_{n-1}+\cdots+\bar{a}_{n-i}\in\{\bar{a}_{n-1}+\cdots+\bar{a}_{n-j},\bar{a}_{n-1}+\cdots+\bar{a}_{n-j}+(2n-1)\}$ where $i\neq j$, $1\leqslant i,j\leqslant n-1$, in $\mathbb{Z}_{4n-2}$. Then in $\mathbb{Z}_{2n-1}$ we find that

$$a_{n-1}+\cdots+a_{n-i}=a_{n-1}+\cdots+a_{n-j}.$$

Subtracting these from $a_1+\cdots+a_{n-1}$ we obtain

$$a_1+\cdots+a_{n-i-1}=a_1+\cdots+a_{n-j-1},$$

contradicting the assumption that $\{a_1,a_1+a_2,\dots,a_1+a_2+\cdots+a_{n-1}\}$ is a transversal. A similar argument shows that $\bar{A}+i(2n-1)+\bar{a}_1+\cdots+\bar{a}_i\notin\{\bar{A}+j(2n-1)+\bar{a}_1+\cdots+\bar{a}_j,\bar{A}+j(2n-1)+\bar{a}_1+\cdots+\bar{a}_j+(2n-1)\}$ if $i\neq j$, $1\leqslant i,j\leqslant n-1$.

If $\bar{a}_{n-1}+\cdots+\bar{a}_{n-i}\in\{\bar{A}+j(2n-1)+\bar{a}_1+\cdots+\bar{a}_j,\bar{A}+j(2n-1)+\bar{a}_1+\cdots+\bar{a}_j+(2n-1)\}$, $1\leqslant i,j\leqslant n-1$, in $\mathbb{Z}_{4n-2}$, then, in $\mathbb{Z}_{2n-1}$,

$$a_{n-1}+\cdots+a_{n-i}=(a_1+a_2+\cdots+a_{n-1})+a_1+\cdots+a_j$$

so that

$$a_1+a_2+\cdots+a_j=-(a_1+\cdots+a_{n-i}).$$

But this contradicts the assumption that $a_1,a_1+a_2,\dots,a_1+a_2+\cdots+a_{n-1}$ is a transversal of the sets $\{1,-1\},\{2,-2\},\dots,\{n-1,-(n-1)\}$. $\square$

Recall that ρ denotes the automorphism $x\mapsto x+1$ of K_{2n}. Consequently, ρ^n is the automorphism of K_{2n} which interchanges each pair of vertices $x,x+n$. If, in K_{2n}, we identify pairs of vertices which are interchanged by ρ^n and identify pairs of edges which are interchanged by ρ^n, we obtain a multigraph K_n^* with n vertices, in which each vertex is incident with one loop and each pair of distinct vertices are joined by two edges. Under these identifications, the automorphism ρ of K_{2n} induces an automorphism ρ^* of K_n^* which permutes its vertices cyclically. Moreover, under the foregoing identifications, a cyclic Hamiltonian path decomposition of K_{2n} becomes a decomposition of K_n^* into spanning subgraphs each consisting of a Hamiltonian path with a loop attached to one of its endvertices, and ρ^* permutes the members of this decomposition cyclically. Some of our observations about n-gradations are interpretable in terms of such decompositions of K_n^*. Studying these decompositions might therefore yield further information and insight.

5. Orthogonality and symmetry

We shall abbreviate "mutually orthogonal symmetric HLS$(2n)$" to MOSHLS$(2n)$.

Example 5.1. For $n = 1, 2, 3, 4$ a pair A_n, B_n of MOSHLS$(2n)$ is given below:

$$A_1 = \begin{array}{|cc|}\hline 1 & 1 \\ 1 & 1 \\ \hline\end{array} \quad B_1 = \begin{array}{|cc|}\hline 1 & 1 \\ 1 & 1 \\ \hline\end{array} \quad A_2 = \begin{array}{|cccc|}\hline 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ \hline\end{array} \quad B_2 = \begin{array}{|cccc|}\hline 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ \hline\end{array}$$

$$A_3 = \begin{array}{|cccccc|}\hline 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 & 1 \\ 2 & 2 & 3 & 3 & 1 & 1 \\ 2 & 3 & 3 & 1 & 1 & 2 \\ 3 & 3 & 1 & 1 & 2 & 2 \\ 3 & 1 & 1 & 2 & 2 & 3 \\ \hline\end{array} \quad B_3 = \begin{array}{|cccccc|}\hline 1 & 3 & 3 & 2 & 1 & 2 \\ 3 & 2 & 1 & 1 & 3 & 2 \\ 3 & 1 & 3 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 3 & 3 \\ 1 & 3 & 2 & 3 & 2 & 1 \\ 2 & 2 & 1 & 3 & 1 & 3 \\ \hline\end{array}$$

$$A_4 = \begin{array}{|cccccccc|}\hline 1 & 1 & 3 & 2 & 2 & 3 & 4 & 4 \\ 1 & 2 & 4 & 2 & 3 & 3 & 4 & 1 \\ 3 & 4 & 3 & 1 & 4 & 1 & 2 & 2 \\ 2 & 2 & 1 & 4 & 1 & 4 & 3 & 3 \\ 2 & 3 & 4 & 1 & 1 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 4 & 2 & 1 & 2 \\ 4 & 4 & 2 & 3 & 2 & 1 & 3 & 1 \\ 4 & 1 & 2 & 3 & 3 & 2 & 1 & 4 \\ \hline\end{array} \quad B_4 = \begin{array}{|cccccccc|}\hline 1 & 1 & 4 & 2 & 3 & 2 & 3 & 4 \\ 1 & 2 & 1 & 4 & 4 & 3 & 2 & 3 \\ 4 & 1 & 3 & 2 & 2 & 4 & 1 & 3 \\ 2 & 4 & 2 & 4 & 3 & 3 & 1 & 1 \\ 3 & 4 & 2 & 3 & 1 & 1 & 4 & 2 \\ 2 & 3 & 4 & 3 & 1 & 2 & 4 & 1 \\ 3 & 2 & 1 & 1 & 4 & 4 & 3 & 2 \\ 4 & 3 & 3 & 1 & 2 & 1 & 2 & 4 \\ \hline\end{array}$$

We are indebted to a referee for the pair A_3, B_3 and to D.A. Pike who found the pair A_4, B_4 on a computer.

The two SHLS(10) in Example 4.3 are also mutually orthogonal.

Lemma 5.2. *If two* SHLS$(2n)$ *are orthogonal then they induce the same diagonal partition.*

Proof. Let A, B be MOSHLS$(2n)$ on symbols $\sigma_1, \ldots, \sigma_n$. If $p, q \in \{1, \ldots, 2n\}$, $A(p,p) = \sigma_r = A(q,q)$ and $B(p,p) = \sigma_s \neq B(q,q)$ then, by Lemma 4.2, p is the only value of i such that $A(i,i) = \sigma_r$ and $B(i,i) = \sigma_s$. Therefore the orthogonality of A, B requires that there be exactly three ordered pairs (i,j) such that $i \neq j$, $A(i,j) = \sigma_r$ and $B(i,j) = \sigma_s$, which contradicts the symmetry of A, B. This contradiction shows that if $p, q \in \{1, 2, \ldots, 2n\}$ and $A(p,p) = A(q,q)$ then $B(p,p) = B(q,q)$, thus proving Lemma 5.2. $\square$

Proposition 5.3. *If there exist p* MOSHLS$(2n)$ *then there exist p* MOSHLS$(2n)$ *in normal form.*

Proof. Let $A_1, \ldots, A_p$ be MOSHLS$(2n)$. Then $A_1, \ldots, A_p$ all induce the same diagonal partition by Lemma 5.2, and so there exists a permutation γ of $\{1, 2, \ldots, 2n\}$

such that $\pi^\gamma(\pi_\gamma(A_r)) = B_r$ (say) induces the diagonal partition $\{\{1, n+1\}, \{2, n+2\}, \ldots, \{n, 2n\}\}$ for $r = 1, \ldots, p$. Moreover $B_1, \ldots, B_p$ are Hamiltonian by Proposition 1.2, and are symmetric and mutually orthogonal since $A_1, \ldots, A_p$ have these properties. For $r = 1, \ldots, p$ let C_r denote the double latin square obtained from B_r when each symbol σ is replaced throughout B_r by the number $i \in \{1, \ldots, n\}$ such that $B_r(i,i) = B_r(n+i, n+i) = \sigma$. Then $C_1, \ldots, C_p$ are MOSHLS($2n$) since $B_1, \ldots, B_p$ are MOSHLS($2n$); and $C_1, \ldots, C_p$ are in normal form. □

Recalling that symmetric double latin squares of order $2n$ can be represented by $\{1,2\}$-factorisations of K_{2n}, we now consider how orthogonality of symmetric double latin squares translates into a property of the corresponding $\{1,2\}$-factorisations.

Definition. Let $V_1(G)$ denote the set of vertices which have degree 1 in a graph G. We shall say that two $\{1,2\}$-factors H, H' of K_{2n} are *orthogonal* if $2|E(H \cap H')| + |V_1(H) \cap V_1(H')| = 4$. We shall say that two $\{1,2\}$-factorisations $\mathscr{F}, \mathscr{F}'$ of K_{2n} are *orthogonal* if each member of $\mathscr{F}$ is orthogonal to each member of $\mathscr{F}'$.

For a general reference about orthogonality in graphs, see [2].

Lemma 5.4. *Two symmetric double latin squares of order* $2n$ *on the same symbols are orthogonal if and only if the corresponding* $\{1,2\}$*-factorisations of* K_{2n} *are orthogonal.*

Proof. Let A, B be symmetric double latin squares of order $2n$ on the same symbols $\sigma_1, \ldots, \sigma_n$. Let the corresponding $\{1,2\}$-factorisations of K_{2n} be $\{H_1, \ldots, H_n\}$ and $\{H'_1, \ldots, H'_n\}$ where $H_r = H(A, \sigma_r)$ and $H'_r = H(B, \sigma_r)$ for $r = 1, \ldots, n$. Suppose that $i, j \in \{1, 2, \ldots, 2n\}$ and $r, s \in \{1, \ldots, n\}$. If $i \neq j$ then

$$A(i,j) = \sigma_r \;\Leftrightarrow\; A(j,i) = \sigma_r \;\Leftrightarrow\; [i,j] \in E(H_r),$$

$$B(i,j) = \sigma_s \;\Leftrightarrow\; B(j,i) = \sigma_s \;\Leftrightarrow\; [i,j] \in E(H'_s).$$

Moreover, as explained in Section 3,

$$A(i,i) = \sigma_r \;\Leftrightarrow\; i \in V_1(H_r),$$

$$B(i,i) = \sigma_s \;\Leftrightarrow\; i \in V_1(H'_s).$$

Therefore, there are exactly $2|E(H_r \cap H'_s)| + |V_1(H_r) \cap V_1(H'_s)|$ ordered pairs (i,j) such that $A(i,j) = \sigma_r$ and $B(i,j) = \sigma_s$. It follows that A, B are orthogonal if and only if $2|E(H_r \cap H'_s)| + |V_1(H_r) \cap V_1(H'_s)| = 4$ for all $r, s \in \{1, \ldots, n\}$, i.e. if and only if the $\{1,2\}$-factorisations $\{H_1, \ldots, H_n\}$ and $\{H'_1, \ldots, H'_n\}$ are orthogonal. □

In particular, Lemma 5.4 implies that two SHLS($2n$) are orthogonal if and only if the corresponding Hamiltonian path decompositions of K_{2n} are orthogonal. In this connection, the following further lemma is helpful.

Lemma 5.5. *Two Hamiltonian path decompositions of K_{2n} are orthogonal if and only if they can be expressed in the forms $\{H_1, \dots, H_n\}$ and $\{H'_1, \dots, H'_n\}$ where*

(i) *for each $r \in \{1, \dots, n\}, H_r$ has the same endvertices as H'_r;*
(ii) $|E(H_r \cap H'_r)| = 1$ *for* $r = 1, \dots, n$;
(iii) $|E(H_r \cap H'_s)| = 2$ *when* $r, s \in \{1, \dots, n\}$ *and* $r \neq s$.

Proof. Assume first that $\{H_1, \dots, H_n\}$ and $\{H'_1, \dots, H'_n\}$ are Hamiltonian path decompositions of K_{2n} satisfying (i)–(iii). Then $|V_1(H_r) \cap V_1(H'_r| = 2$ for $r = 1, \dots, n$ by (i); and $V_1(H_r) \cap V_1(H'_s) = \emptyset$ when $r \neq s$ by (i) and Lemma 3.2. These observations and (ii) and (iii) imply that $2|E(H_r \cap H'_s)| + |V_1(H_r) \cap V_1(H'_s)| = 4$ for all $r, s \in \{1, \dots, n\}$ and so $\{H_1, \dots, H_n\}$ and $\{H'_1, \dots, H'_n\}$ are orthogonal.

Now assume that $\mathscr{D}, \mathscr{D}'$ are orthogonal Hamiltonian path decompositions of K_{2n}. Then $|\mathscr{D}| = |\mathscr{D}'| = n$ by Lemma 3.2, and $|E(H \cap H'| \leqslant 2$ for all $H \in \mathscr{D}$, $H' \in \mathscr{D}'$ since $\mathscr{D}, \mathscr{D}'$ are orthogonal. Since each Hamiltonian path of K_{2n} has $2n - 1$ edges, it follows that each member of $\mathscr{D}$ shares one edge with one member of $\mathscr{D}'$ and two edges with each of the other $n - 1$ members of $\mathscr{D}'$ and that the same is true with $\mathscr{D}, \mathscr{D}'$ interchanged. Therefore, we can choose an ordering $H_1, \dots, H_n$ of the members of $\mathscr{D}$ and an ordering $H_1, \dots, H'_n$ of the members of $\mathscr{D}'$ such that (ii) and (iii) are true. Then (i) follows from (ii) and the fact that $2|E(H_r \cap H'_r)| + |V_1(H_r) \cap V_1(H'_r)| = 4$ for $r = 1, \dots, n$. $\square$

By Corollary 4.6 and Lemma 5.4, two cyclic SHLS($2n$) are orthogonal if and only if the corresponding cyclic Hamiltonian path decompositions of K_{2n} are orthogonal. Moreover, a cyclic Hamiltonian path decomposition of K_{2n} is generated by a cyclic Hamiltonian path which, by Lemma 4.8, is $H(s_1, \dots, s_n)$ for some n-procession $s_1, \dots, s_n$. So we might ask what conditions on two n-processions $s_1, \dots, s_n$ and $t_1, \dots, t_n$ ensure that $H(s_1, \dots, s_n)$ and $H(t_1, \dots, t_n)$ generate orthogonal cyclic Hamiltonian path decompositions of K_{2n}. This will in effect provide a test for orthogonality of two cyclic SHLS($2n$).

This question is answered by Theorem 5.7, whose statement is slightly simplified by assuming that $s_1 = t_1$. This assumption involves no real loss of generality, in view of the following simple observation:

Lemma 5.6. *If $\mathscr{D}$ is a cyclic Hamiltonian path decomposition of K_{2n} and $x \in \mathbb{Z}_{2n}$ then there exists an n-procession $s'_1, \dots, s'_n$ such that $s'_1 = x$ and $H(s'_1, \dots, s'_n)$ generates $\mathscr{D}$.*

Proof. Since $\mathscr{D}$ is a cyclic Hamiltonian path decomposition, it is generated by some cyclic Hamiltonian path H, and $H = H(s_1, \dots, s_n)$ for some n-procession $s_1, \dots, s_n$ by Lemma 4.8. Let $y = x - s_1$. By Lemma 4.4 (or by an easy inference from the definition of $H(s_1, \dots, s_n)$), $H + n = H$ and so $\{H + y + 1, H + y + 2, \dots, H + y + n\} = \{H + 1, H + 2, \dots, H + n\} = \mathscr{D}$. Therefore $\mathscr{D}$ is generated by $H + y = H(s_1 +$

$y, s_2 + y, \dots, s_n + y)$, where $s_1 + y, s_2 + y, \dots, s_n + y$ is an n-procession with first term x. $\square$

Definition. If $s, t \in \mathbb{Z}_{2n}$, the statement $s \equiv t \pmod{n}$ will mean that $s = t$ or $s = t + n$. We recall that $U + r$ means $\{u + r\colon u \in U\}$ when $U \subseteq \mathbb{Z}_{2n}$ and $r \in \mathbb{Z}_{2n}$.

Recall that, in $\mathbb{Z}_{2n}$, $(s_1, s_2, \dots, s_{2n}) = (s_1, s_2, \dots, s_n, s_n + n, s_{n-1} + n, \dots, s_1 + n)$.

Theorem 5.7. *Let $s_1, \dots, s_n$ and $t_1, \dots, t_n$ be n-processions such that $s_1 = t_1$. Then $H(s_1, \dots, s_n)$ and $H(t_1, \dots, t_n)$ generate orthogonal cyclic Hamiltonian path decompositions of K_{2n} if and only if*

(i) $s_n \equiv t_n \pmod{n}$,

and

(ii) *for each $k \in \{1, \dots, n-1\}$ there exists exactly one pair (x, y) with $x \in \{1, \dots, n-1\} \cup \{n+1, \dots, 2n-1\}$ and $y \in \{1, \dots, n-1\}$ such that $\{s_x, s_{x+1}\} \equiv \{t_y, t_{y+1}\} + k \pmod{2n}$.*

Proof. Let $H = H(s_1, \dots, s_n), H' = H(t_1, \dots, t_n)$ and let $H_i = H + i, H'_i = H' + i$ for each $i \in \mathbb{Z}_{2n}$. The cyclic Hamiltonian path decompositions generated by $H(s_1, \dots, s_n)$ and $H(t_1, \dots, t_n)$ are $\{H_1, \dots, H_n\}$ and $\{H'_1, \dots, H'_n\}$. Since $s_1 = t_1$ and H_i has endvertices $s_1 + i, s_1 + n + i$ and H'_j has endvertices $t_1 + j,\ t_1 + n + j$, it follows that H_i, H'_j have the same endvertices when $i = j$ and have no common endvertex when $i \not\equiv j \pmod{n}$. Therefore $\{H_1, \dots, H_n\}$ and $\{H'_1, \dots, H'_n\}$ are orthogonal if and only if

(a) $|E(H_i \cap H'_i)| = 1$ for $i = 1, \dots, n$

and

(b) $|E(H_i \cap H'_j)| = 2$ when $i, j \in \{1, \dots, n\}$ and $i \neq j$.

It therefore remains to be proved that (a) and (b) are both true if and only if (i) and (ii) are both true.

Assume first that (i) and (ii) are true. By (i), the edge $[s_n + i, s_n + n + i]$ of H_i coincides with the edge $[t_n + i, t_n + n + i]$ of H'_i for $i = 1, \dots, n$. If $i, j \in \{1, \dots, n\}$ and $i \neq j$ then, by (ii), there exist $x \in \{1, \dots, n-1\} \cup \{n+1, \dots, 2n-1\}$ and $y \in \{1, \dots, n-1\}$ such that $\{s_x, s_{x+1}\} = \{t_y, t_{y+1}\} + j - i \pmod{2n}$. Therefore, the edges $[s_x + i, s_{x+1} + i]$ of H_i and $[t_y + j, t_{y+1} + j]$ of H'_j coincide, and the edges $[s_x + n + i, s_{x+1} + n + i]$ of H_i and $[t_y + n + j, t_{y+1} + n + j]$ of H'_j coincide. Hence $|E(H_i \cap H'_j)| \geqslant 1$ for $i = 1, \dots, n$ and $|E(H_i \cap H'_j)| \geqslant 2$ when $i, j \in \{1, \dots, n\}$ and $i \neq j$. This implies (a) and (b) because $\sum_{j=1}^{n} |E(H_j \cap H'_j)| = |E(H_i)| = 2n - 1$ for each i.

Now assume (a) and (b). From the definition of $H(s_1, \dots, s_n)$ it follows that $H = H + n = H_n$ and $H' = H' + n = H'_n$. Therefore $|E(H \cap H')| = |E(H_n \cap H'_n)| = 1$ by (a). Since $H = H + n$ and $H' = H' + n$ it follows that $E(H \cap H') = E(H \cap H') + n$ and so the unique edge of $H \cap H'$ must be $[p, p + n]$ for some p. By (P2), the only edge of this form in H is $[s_n, s_n + n]$ and the only such edge in H' is $[t_n, t_n + n]$. Therefore $[s_n, s_n + n]$ and $[t_n, t_n + n]$ must be the same edge and so (i) is true. Now let $k \in \{1, \dots, n-1\}$. Since $H = H_n$, it follows from (b) that at least one edge of H other than $[s_n, s_n + n]$ must be in H'_k. Therefore there exists $x \in \{1, \dots, n-1\} \cup \{n+1, \dots, 2n-1\}$ such that $[s_x, s_{x+1}]$ is in H'_k, and consequently both of the edges $[s_x, s_{x+1}]$ and $[s_x + n, s_{x+1} + n]$ must be in H'_k because $H'_k = H' + k = (H' + n) + k = H'_k + n$. By (b) there can be only one such x. By (P2), $[s_x, s_{x+1}]$ and $[s_x + n, s_{x+1} + n]$ must be edges of H'_k other than $[t_n + k, t_n + n + k]$, and so the pair $\{[s_x, s_{x+1}], [s_x + n, s_{x+1} + n]\}$ of edges of H must coincide with a pair $\{[t_y + k, t_{y+1} + k], [t_y + n + k, t_{y+1} + n + k]$ of edges of H'_k for some $y \in \{1, \dots, n-1\}$. Clearly there is precisely one such y. If $[s_x, s_{x+1}] = [t_y + k, t_{y+1} + k]$ then $\{s_x, s_{x+1}\} \equiv \{t_y, t_{y+1}\} + k \pmod{2n}$, as asserted. If $[s_x + n, s_{x+1} + n] = [t_y + k, t_{y+1} + k]$ and $x < n$ then $s_x + n = s_{2n-x+1}$ and $s_{x+1} + n = s_{2n-x}$, so $[s_x + n, s_{x+1} + n] = [s_{2n-x}, s_{2n-x+1}]$. Therefore $\{s_{x'}, s_{x'+1}\} \equiv \{t_y, t_{y+1}\} + k \pmod{2n}$, with $x' = 2n - x$, where $x' \in \{1, \dots, n-1\} \cup \{n+1, \dots, 2n-1\}$. If $[s_x + n, s_{x+1} + n] = [t_y + k, t_{y+1} + k]$ with $x > n$ then the same holds since $s_{2n-x+1} + n = s_{2n-(2n-x+1)+1} = s_x$, so $s_x + n = s_{2n-x+1}$ in $\mathbb{Z}_{2n}$. This proves (ii). $\square$

Call two n-gradations $(a_1, \dots, a_{n-1})$ and $(b_1, \dots, b_{n-1})$ *orthogonal* if $H[a_1, \dots, a_{n-1}]$ and $H[b_1, \dots, b_{n-1}]$ generate orthogonal cyclic Hamiltonian path decompositions of K_{2n} (or, equivalently, generate a pair of MOSHLS$(2n)$). Similarly, if $(s_1, \dots, s_n)$ and $(t_1, \dots, t_n)$ are the n-processions corresponding to $(a_1, \dots, a_{n-1})$ and $(b_1, \dots, b_{n-1})$, also call $(s_1, \dots, s_n)$ and $(t_1, \dots, t_n)$ orthogonal.

Lemma 5.8. *The following pairs of n-gradations are either all orthogonal, or all not orthogonal*:

(i) $(a_1, \dots, a_{n-1})$ *and* $(b_1, \dots, b_{n-1})$,
(ii) $(ka_1, \dots, ka_{n-1})$ *and* $(kb_1, \dots, kb_{n-1})$, *where k is any integer coprime to* $2n$, *the multiplication being performed in* $\mathbb{Z}_{2n}$,
(iii) $(a_{n-1}, \dots, a_1)$ *and* $(b_{n-1}, \dots, b_1)$.
(iv) $(a'_1, \dots, a'_{n-1})$ *and* $(b'_1, \dots, b'_{n-1})$, *where these are obtained from* $(a_1, \dots, a_{n-1})$ *and* $(b_1, \dots, b_{n-1})$ *by adding n in* $\mathbb{Z}_{2n}$ *to those terms which belong to* $S \cup (-S)$, *where S is a subset of* $\{1, \dots, n-1\}$ *such that* $S = n - S$.

Proof. (i) $\Leftrightarrow$ (ii): Clearly (i) is just a special case of (ii), so (ii) $\Rightarrow$ (i). To show the converse, suppose that the n-gradations $(a_1, \dots, a_{n-1})$ and $(b_1, \dots, b_{n-1})$ are orthogonal. Let $(s_1, \dots, s_n)$ and $(t_1, \dots, t_n)$ be corresponding n-processions. Then $H(s_1, \dots, s_n)$ and $H(t_1, \dots, t_n)$ generate orthogonal cyclic Hamiltonian path decompositions $\{H^a_1, \dots, H^a_n\}$ and $\{H^b_1, \dots, H^b_n\}$ of K_n. As observed in Section 4, since k is coprime to $2n$, $(ka_1, \dots, ka_{n-1})$ and $(kb_1, \dots, kb_{n-1})$ are n-gradations, and it follows that $H(ks_1, \dots, ks_n)$ and $H(kt_1, \dots, kt_n)$ are cyclic Hamiltonian paths generating Hamiltonian path decompositions $\{H^{ka}_1, \dots, H^{ka}_n\}$ and $\{H^{kb}_1, \dots, H^{kb}_n\}$, respectively.

Suppose that $|E(H^{ka}_r \cap H^{kb}_s)| \geqslant 3$ for some $r \neq s, r, s \in \{1, \dots, n\}$. Then there are at least three pairs (i,j) $(i \neq j, i,j \in \{1, \dots, n\})$ such that $\{ks_{i-1} + r, ks_i + r\} = \{kt_{j-1} + s, kt_j + s\}$. But then $\{s_{i-1} + k^{-1}r, s_i + k^{-1}r\} = \{t_{j-1} + k^{-1}s, t_j + k^{-1}s\}$, so that $|E(H^a_{k^{-1}r} \cap H^b_{k^{-1}s}| \geqslant 3$, contradicting Lemma 5.5. Therefore $|E(H^{ka}_r \cap H^{kb}_r)| \leqslant 2$.

Similarly $|\mathrm{E}(H_r^{ka} \cap H_s^{kb})| \leqslant 1$. By counting edges, it follows that $|E(H_r^{ka} \cap H_s^{kb})| = 2$ and $|E(H_r^{ka} \cap H_s^{kb})| = 1$, so that by Lemma 5.5 $\{H_1^{ka}, \dots, H_n^{ka}\}$ and $\{H_1^{kb}, \dots, H_n^{kb}\}$ are orthogonal.

(i)$\Leftrightarrow$(iii): This is similar to the proof that (i)$\Leftrightarrow$(ii), and may be left to the reader.

(i)$\Leftrightarrow$(iv): Suppose first that $(a_1, \dots, a_{n-1})$ and $(b_1, \dots, b_{n-1})$ are orthogonal. We wish to show that $(a'_1, \dots, a'_{n-1})$ and $(b'_1, \dots, b'_{n-1})$ are orthogonal.

We may set $s_1 = t_1 = 0$. Since $(a_1, \dots, a_{n-1})$ and $(b_1, \dots, b_{n-1})$ are orthogonal, by Theorem 5.7, $s_n \equiv t_n \pmod n$ and, for each $k \in \{1, \dots, n-1\}$, there exists exactly one pair (x, y) with $x \in \{1, \dots, n-1\} \cup \{n+1, \dots, 2n-1\}$ and $y \in \{1, \dots, n-1\}$ such that $\{s_x, s_{x+1}\} = \{t_y, t_{y+1}\} + k$ (in $\mathbb{Z}_2$). Then

$$\{\{s_x, s_{x+1}\}, \{s_{2n-x}, s_{2n-x+1}\}\} = \{\{t_y, t_{y+1}\} + k, \{t_{2n-y}, t_{2n-y+1}\} + k\}.$$

For such a pair (x, y), set $z = t_{y+1} - t_y$. Then if $z \notin S \cup (-S)$, it follows that

$$t'_{y+1} - t'_y = t_{y+1} - t_y = z$$

and

$$s'_{x+1} - s'_x = s_{x+1} - s_x = z,$$

and if $z \in S \cup (-S)$, it follows that

$$t'_{y+1} - t'_y = t_{y+1} - t_y + n = z + n$$

and

$$s'_{x+1} - s'_x = s_{x+1} - s_x + n = z + n.$$

Since this holds for any value of x and y, it follows by summation that

$$s'_x = s_x \quad \text{or} \quad s'_x = s_x + n = s_{2n-x+1}$$

and that

$$t'_y = t_y \quad \text{or} \quad t'_y = t_y + n = t_{2n-y+1}.$$

Thus, if $z \notin S \cup (-S)$

$$\{s'_x, s'_{x+1}\} = \{s_x, s_{x+1}\} \quad \text{or} \quad \{s_{2n-x}, s_{2n-x+1}\}$$

and

$$\{t'_y, t'_{y+1}\} = \{t_y, t_{y+1}\} \quad \text{or} \quad \{t_{2n-y}, t_{2n-y+1}\}$$

so

$$\{\{s'_x, s'_{x+1}\}, \{s'_{2n-x}, s'_{2n-x+1}\}\} = \{\{t'_y, t'_{y+1}\} + k, \{t'_{2n-y}, t'_{2n-y+1}\} + k\}. \qquad (*)$$

Similarly, if $z \in S \cup (-S)$,

$$\{s'_x, s'_{x+1}\} = \{s_x, s_{2n-x}\} \quad \text{or} \quad \{s_{x+1}, s_{2n-x+1}\}$$

and

$$\{t_y, t'_{y+1}\} = \{t_y, t_{y+1}\} \quad \text{or} \quad \{t_{y+1}, t_{2n-y+1}\},$$

so again $(*)$ holds.

It follows that there is a pair (x', y') with $x' \in \{1, \dots, n-1\} \cup \{n+1, \dots, 2n-1\}$ and $y' \in \{1, \dots, n-1\}$ such that

$$\{s'_{x'}, s'_{x'+1}\} = \{t'_y, t'_{y+1}\} + k,$$

and, by reversing the argument, it follows that there is exactly one such pair.

Clearly $s'_1 = t'_1 = 0$ and $s'_n \equiv t'_n \equiv s_n \equiv t_n \pmod n$, so it follows from Theorem 5.7 that $(a'_1, \dots, a'_{n-1})$ and $(b'_1, \dots, b'_{n-1})$ are orthogonal.

The n-gradations $(a_1, \dots, a_{n-1})$ and $(b_1, \dots, b_{n-1})$ may be obtained from $(a'_1, \dots, a'_{n-1})$ and $(b'_1, \dots, b'_{n-1})$ by adding n in $\mathbb{Z}_{2n}$ to those terms which belong to $S \cup (-S)$, so the argument above shows that if $(a'_1, \dots, a'_{n-1})$ and $(b'_1, \dots, b'_{n-1})$ are orthogonal, then so are $(a_1, \dots, a_{n-1})$ and $(b_1, \dots, b_{n-1})$. $\square$

Illustration. Examples of pairs of orthogonal n-processions are $(0, 1)$ and $(0, -1)$ when $n = 2$, $(0, 1, -1)$ and $(0, -2, -1)$ when $n = 3$, and $(0, 1, -1, 2, -2)$ and $(0, 4, 2, 1, -2)$ when $n = 5$. In fact these could be used to generate Example 5.1 (A_2 and B_2) when $n = 2$, Example 5.1 (A_3 and B_3) when $n = 3$, and Example 4.3 when $n = 5$. An example of such a pair when $n = 7$ is $(0, 1, -1, 2, -2, 3, -3)$ and $(0, -2, -6, 2, 3, 6, -3)$, an example when $n = 9$ is $(0, 1, -1, 2, -2, 3, -3, 4, -4)$ and $(0, -1, 3, -8, 4, 6, -2, -7, -4)$, an example when $n = 11$ is $(0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5)$ and $(0, -1, -4, -10, 2, 4, 8, 3, -6, 9, -5)$, and an example when $n = 13$ is $(0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6)$ and $(0, 3, 4, 12, -7, 2, 8, 10, -2, -12, 9, 5, -6)$. Here, of course, in each example, whichever of a pair of sequences is chosen to be $(s_1, \dots, s_n)$, then $(s_{n+1}, \dots, s_{2n})$ is found using the equation $s_i = s_{2n-i+1}$ $(1 \leqslant i \leqslant n)$. For further information about such pairs of sequences, see [18] or: http://www.math.wvu.edu/2mays/moshls.htm.

Theorem 5.7 can be used to show, for example, that there do not exist two orthogonal cyclic Hamiltonian path decompositions of K_8. To see this, suppose that two such decompositions exist. Then by Lemma 5.6 there must exist 4-processions s_1, s_2, s_3, s_4 and t_1, t_2, t_3, t_4 such that $s_1 = t_1 = 0$ and $H(s_1, s_2, s_3, s_4)$, $H(t_1, t_2, t_3, t_4)$ generate orthogonal cyclic Hamiltonian path decompositions of K_8. This requires s_1, s_2, s_3, s_4 and t_1, t_2, t_3, t_4 to satisfy the conditions of Theorem 5.7. However, there are only 64 4-processions, which were specified in Section 4, and the only ones with first term 0 are the eight sequences $0, u, u+v, -v$ where $u \in \{-3, -1, 1, 3\}$, $v \in \{-2, 2\}$. It is easily checked that no two of these eight sequences satisfy the conditions of Theorem 5.7. Therefore K_8 does not have two orthogonal cyclic Hamiltonian path decompositions and so, by Corollary 4.6 and Lemma 5.4, there do not exist two cyclic MOSHLS(8).

Theorem 5.9. *If there exist two orthogonal Hamiltonian path decompositions of K_{2n} then there exist two orthogonal Hamiltonian path decompositions of K_{4n}.*

Proof. By associating two vertices v^*, v^{**} of K_{4n} with each $v \in V(K_{2n})$, we can take $V(K_{4n})$ to be $\{v^*\colon v \in V(K_{2n})\} \cup \{v^{**}\colon v \in V(K_{2n})\}$. For any edge $e = [u, v]$ of K_{2n} we define subsets X_e, Y_e, Z_e of $E(K_{4n})$ by

$$X_e = \{[u^*, v^*], [u^{**}, v^{**}]\}, \quad Y_e = \{[u^*, v^{**}], [u^{**}, v^*]\}, \; Z_e = X_e \cup Y_e.$$

For each $v \in V(K_{2n})$ let $b(v)$ be the edge $[v^*, v^{**}]$ of K_{4n}. If H is a Hamiltonian path of K_{2n} with endvertices u, v, let $\tilde{H}$ denote the spanning subgraph of K_{4n} such that $E(\tilde{H}) = \{b(u), b(v)\} \cup \bigcup_{e \in E(H)} Z_e$. If, for each $e \in E(H)$, we take S_e to be one of X_e, Y_e and T_e to be the other, then clearly $\tilde{H}$ has a Hamiltonian path decomposition $\{P, Q\}$ such that

$$E(P) = \{b(u)\} \cup \bigcup_{e \in E(H)} S_e, \quad E(Q) = \{b(v)\} \cup \bigcup_{e \in E(H)} T_e.$$

There are altogether 2^{2n-1} such Hamiltonian path decompositions of $\tilde{H}$ because for each $e \in E(H)$ we can choose whether S_e is X_e or Y_e. Moreover, if $\{H_1, \dots, H_n\}$ is a Hamiltonian path decomposition of K_{2n}, then $\{\tilde{H}_1, \dots, \tilde{H}_n\}$ is a decomposition of K_{4n} in view of the second assertion of Lemma 3.2. Consequently, taking $\{P_r, Q_r\}$ to be one of the 2^{2n-1} Hamiltonian path decompositions of $\tilde{H}_r$ arising from the above construction for $r = 1, \dots, n$ yields $2^{n(2n-1)}$ different Hamiltonian path decompositions $\{P_1, Q_1, P_2, Q_2, \dots, P_n, Q_n\}$ of K_{4n}. This provides the key to our proof.

Now assume that there exist two orthogonal Hamiltonian path decompositions of K_{2n}. Then, by Lemma 5.5, we can take these to be $\{H_1, \dots, H_n\}$ and $\{H'_1, \dots, H'_n\}$, where H_r and H'_r have two common endvertices u_r, v_r and just one common edge $c(r)$ for $r = 1, \dots, n$ and H_r, H'_s have just two common edges $f(r,s)$, $g(r,s)$ when $r \neq s$. For $r = 1, \dots, n$ let P_r, Q_r be the Hamiltonian paths of K_{4n} such that

$$E(P_r) = \{b(u_r)\} \cup \bigcup_{e \in E(H_r)} X_e, \quad E(Q_r) = \{b(v_r)\} \cup \bigcup_{e \in E(H_r)} Y_e.$$

Let $\mathscr{D}$ denote the Hamiltonian path decomposition $\{P_1, Q_1, P_2, Q_2, \dots, P_n, Q_n\}$ of K_{4n}. For each $e \in E(K_{2n})$, define S_e, T_e as follows:

$$S_e = Y_e, \quad T_e = X_e \quad \text{if } e = c(r) \text{ for some } r \text{ or}$$
$$e = f(r,s) \text{ for some } r, s (r \neq s);$$
$$S_e = X_e, \quad T_e = Y_e \quad \text{if } e = g(r,s) \text{ for some } r, s (r \neq s).$$

For $r = 1, \dots, n$ let P'_r, Q'_r be the Hamiltonian paths of K_{4n} such that

$$E(P'_r) = \{b(u_r)\} \cup \bigcup_{e \in E(H'_r)} S_e, \quad E(Q'_r) = \{b(v_r)\} \cup \bigcup_{e \in E(H'_r)} T_e.$$

Let $\mathscr{D}'$ denote the Hamiltonian path decomposition $\{P'_1, Q'_1, P'_2, Q'_2, \dots, P'_n, Q'_n\}$ of K_{4n}. Since $u_1, v_1, u_2, v_2, \dots, u_n, v_n$ are by Lemma 3.2 distinct, it is clear that

(I) for $r = 1, \dots, n$ we have

$$V_1(P_r) \cap V_1(P'_r) = \{v_r^*, v_r^{**}\}, \quad V_1(Q_r) \cap V_1(Q'_r) = \{u_r^*, u_r^{**}\},$$

$$V_1(P_r) \cap V_1(Q'_r) = V_1(Q_r) \cap V_1(P'_r) = \emptyset,$$

$$E(P_r \cap P'_r) = \{b(u_r)\}, \quad E(Q_r \cap Q'_r) = \{b(v_r)\},$$

$$E(P_r \cap Q'_r) = X_{c(r)}, \quad E(Q_r \cap P'_r) = Y_{c(r)};$$

(II) when $r, s \in \{1, \dots, n\}$ and $r \neq s$ we have

$$V_1(P_r) \cap V_1(P'_s) = V_1(Q_r) \cap V_1(Q'_s) = V_1(P_r) \cap V_1(Q'_s) \\ = V_1(Q_r) \cap V_1(P'_s) = \emptyset,$$

$$E(P_r \cap P'_s) = X_{g(r,s)}, \quad E(Q_r \cap Q'_s) = Y_{g(r,s)},$$

$$E(P_r \cap Q'_s) = X_{f(r,s)}, \quad E(Q_r \cap P'_s) = Y_{f(r,s)}.$$

Therefore $|V_1(J) \cap V_1(J')| + 2|E(J \cap J')| = 4$ for every pair $J \in \mathscr{D}$, $J' \in \mathscr{D}'$ and so $\mathscr{D}$, $\mathscr{D}'$ are orthogonal Hamiltonian path decompositions of K_{4n}. □

Corollary 5.10. *If there exist two* MOSHLS($2n$) *then there exist two* MOSHLS($4n$).

Proof. This follows from Theorem 3.3, Lemma 5.4 and Theorem 5.9. □

From Examples 4.3 and 5.1 and the examples after Lemma 5.8, we know that two MOSHLS($2n$) exist when $n \in \{1, 3, 5, 7, 9, 11\}$. Consequently, by repeated application of Corollary 5.10, two MOSHLS($2n$) exist whenever n is 2^m or 3.2^m or 5.2^m or 7.2^m or 9.2^m or 11.2^m or 13.2^m for some non-negative integer m. The first value of $2n$ for which the existence of two MOSHLS($2n$) has not been demonstrated is 30.

6. Amalgamation and embedding: introductory remarks

We define an *unfilled matrix* (*on symbols* $\sigma_1, \dots, \sigma_n$) to be a matrix in which certain cells are left unoccupied and each remaining cell contains one symbol (belonging to the set $\{\sigma_1, \dots, \sigma_n\}$). (For example, the cells which contain symbols might be those of a specified submatrix.) We shall sometimes, for clarity, use the term "*ordinary* matrix" for a matrix in which each cell contains exactly one symbol. We shall regard an ordinary matrix as a special kind of unfilled matrix, i.e. the set of unoccupied cells in an "unfilled" matrix may be empty. If we insert a symbol into each unoccupied cell of an unfilled matrix M, we obtain an ordinary matrix M' and we shall say that M has been *embedded* in M'. This leads to questions about which unfilled square matrices can be embedded in a latin square, or a symmetric latin square, or some other desired type of array: see, for example, [3–5,10,12,16,20], etc. In Section 9, we shall prove some results about embeddability in (i) Hamiltonian double latin squares, and (ii) more specifically, symmetric Hamiltonian double latin squares.

These results will be deduced, somewhat in the spirit of [16], from two Theorems 7.2 and 8.2 concerning "amalgamation" of Hamiltonian double latin squares, i.e. a process of "amalgamating" certain rows and "amalgamating" certain columns in a way explained below. Very roughly, Theorem 7.2 says that any array which looks as though it might have been obtained from a Hamiltonian double latin square by such amalgamation can in fact be obtained in this way. Theorem 8.2 is a similar result concerning *symmetric* Hamiltonian double latin squares.

Theorems 7.2 and 8.2 are fairly easily deducible from two graph-theoretic propositions, Propositions 7.5 and 8.6, respectively, whose proofs will therefore be our main task. Proposition 7.5 is in fact a special case of [17, Theorem 1] but, to make the required ideas more accessible, we shall here prove Proposition 7.5 in a slightly different way and without complications arising from the greater generality of the treatment in [17]. (Actually, it has recently come to light that the proof of Theorem 1 in [17] is flawed.) Proposition 8.6 is [15, Theorem 1] but again it may be helpful to present a different proof here. Propositions 7.5 and 8.6 and some similar statements can be proved either by using matroids or by a somewhat more elementary method. We have somewhat arbitrarily chosen to present an elementary proof of Proposition 7.5 and a matroid proof of Proposition 8.6, thus enabling the reader to see these two different methods of proof side by side. Although we have for some while been aware of the possible use of matroids in such proofs, we believe that it has not hitherto been mentioned in print. Illustrating it here may help to make known a possible tool for tackling future amalgamation problems. We may also note that our elementary proof makes use of laminar sets, and so is different from the elementary proofs in [15,17] and elsewhere which use de Werra's theorem [23]; another elementary proof could also be found by using some results of Buchanan [9].

Definition. A *composition* of a positive integer n is a sequence of positive integers whose sum is n. A *multiple-entry matrix on symbols* $\sigma_1, \ldots, \sigma_u$ is a matrix in which each cell contains finitely many symbols drawn from the set $\{\sigma_1, \ldots, \sigma_u\}$, the same symbol being allowed to occur more than once in a cell. For example,

$$M = \begin{array}{|l|l|l|} \hline \sigma_1\,\sigma_1\,\sigma_3\,\sigma_3 & \sigma_1\,\sigma_4\,\sigma_4\,\sigma_4\,\sigma_4\,\sigma_4 & \sigma_2\,\sigma_3\,\sigma_3\,\sigma_4 \\ \hline \sigma_2\,\sigma_2 & \sigma_1\,\sigma_5\,\sigma_5\,\sigma_5\,\sigma_5 & \sigma_2\,\sigma_3\,\sigma_3\,\sigma_5 \\ \hline \sigma_1\,\sigma_1\,\sigma_1\,\sigma_1 & \sigma_4\,\sigma_4\,\sigma_5\,\sigma_5 & \sigma_1\,\sigma_3\,\sigma_3\,\sigma_3\,\sigma_4 \\ \hline \sigma_2 & \sigma_1\,\sigma_4\,\sigma_4\,\sigma_5 & \sigma_2\,\sigma_2\,\sigma_2\,\sigma_3\,\sigma_3 \\ \hline \end{array}$$

is a 4×3 multiple-entry matrix on symbols $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$. We regard the symbols in any one cell as being unordered: for example, changing the top left-hand entry in the above matrix M to $\sigma_3, \sigma_1, \sigma_3, \sigma_1$ would merely give a different notation for the same multiple-entry matrix.

We can obtain a multiple-entry matrix from an ordinary matrix by "amalgamating" rows and "amalgamating" columns in a certain sense. Before defining this process formally, we illustrate it by an example. Let A be the 9×8

matrix

1	2	3	4	1	2	3	4
2	4	1	3	2	4	1	2
3	1	4	2	3	1	4	2
4	3	2	1	4	3	2	1
4	1	2	3	1	2	3	4
3	3	4	1	2	4	1	3
2	3	1	4	3	1	4	2
1	4	3	2	4	3	2	1
3	4	3	1	2	2	2	5

and consider the composition $S = (3, 4, 2)$ of 9 and the composition $T = (2, 3, 1, 2)$ of 8. We shall use S to decide which rows of A to amalgamate and use T to decide which columns to amalgamate. First, since $S = (3, 4, 2)$, we amalgamate the first three rows and then amalgamate the next four rows and finally amalgamate the last two rows, to produce the 3×8 multiple-entry matrix

123	124	134	234	123	124	134	224
2344	1333	1224	1134	1234	1234	1234	1234
13	44	33	12	24	23	22	15

.

Then, since $T = (2, 3, 1, 2)$, we amalgamate the first two columns, then amalgamate the next three columns, leave the next column alone, and finally amalgamate the last two columns. This produces the 3×4 multiple-entry matrix

112234	112233344	124	122344
12333344	111122233444	1234	11223344
1344	122334	23	1225

,

which we call the (S, T)-*amalgamation* of A.

The general definition is as follows. Let m, n be positive integers and let $S = (p_1, \ldots, p_s)$ be a composition of m and $T = (q_1, \ldots, q_t)$ be a composition of n. Let A be an $m \times n$ matrix with one of the symbols $\sigma_1, \ldots, \sigma_u$ in each of its cells. Then by partitioning A into submatrices, we can write

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1t} \\ A_{21} & A_{22} & \ldots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \ldots & A_{st} \end{pmatrix},$$

where $A_{\alpha\beta}$ is a $p_\alpha \times q_\beta$ submatrix of A with the cell $(p_1 + \cdots + p_\alpha, q_1 + \cdots + q_\beta)$ of A in its bottom right-hand corner. We define the (S, T)-*amalgamation* of A to be the

$s \times t$ multiple-entry matrix on symbols $\sigma_1, \ldots, \sigma_u$ such that the number of occurrences of σ_k in the cell (α, β) of A^* is equal to the number of occurrences of σ_k in $A_{\alpha\beta}$ for $\alpha = 1, \ldots, s$ and $\beta = 1, \ldots, t$ and $k = 1, \ldots, u$.

In discussing this concept, we shall continue to use graph theory, but we shall need multigraphs, i.e. graphs which may have loops and/or multiple edges. Where necessary, we extend our graph-theoretic language and notation to multigraphs in obvious ways. We shall refer to "subgraphs" (rather than "submultigraphs") of multigraphs, but it will be understood that a "subgraph" of a multigraph may have loops and/or multiple edges. If u, v are vertices of a multigraph G then $d_G(u, v)$ will denote the number of edges joining them and $d_G(v)$ will denote the degree of v in G: thus $d_G(v) = 2p + q$ if v is incident with p loops and q other edges. We let $G - v$ denote the subgraph obtained from G by removing v and the edges incident with it. The *order* of G is $|V(G)|$. The set of components of G will be denoted by (G). A *bridge* of a multigraph is an edge which is not in any cycle.

If D, G are multigraphs and $E(D) = E(G)$, we define a *DG- amalgamator* to be a surjection $\Omega : V(D) \to V(G)$ which, for each $e \in E(D) = E(G)$, maps the vertices joined by e in D to those joined by e in G (so that, in particular, e must be a loop of G if in D it joins vertices x, y such that $\Omega(x) = \Omega(y)$). Thus a DG-amalgamator is, more informally, an operation which transforms D into G by identifying (or "amalgamating") vertices. Amalgamating rows and columns of Hamiltonian double latin squares will give rise to operations of this kind on certain graphs associated with these double latin squares.

For the purposes of this paper, we define an *n-edge-coloured multigraph* to be an ordered pair (G, ϕ) such that G is a finite multigraph and ϕ is a function from $E(G)$ into the set $\{1, \ldots, n\}$. We shall say that an edge e has *colour* $\phi(e)$ in (G, ϕ). We let $G\langle i \rangle$ denote the spanning subgraph of G such that $E(G\langle i \rangle)$ is the set of edges of G which have colour i. (Of course, this notation only makes sense in contexts where the use of some particular 'colouring function' ϕ is understood.)

7. Amalgamating Hamiltonian double latin squares: an elementary proof

If A is an $s \times t$ multiple-entry matrix on symbols $\sigma_1, \ldots, \sigma_u$ and $k \in \{1, \ldots, u\}$, we define $B(A, \sigma_k)$ to be a bipartite multigraph on two sets of vertices $\{\rho_1, \ldots, \rho_s\}$ and $\{c_1, \ldots, c_t\}$ such that the number of edges joining ρ_i to c_j is equal to the number of occurrences of σ_k in the cell (i,j) of A for $i = 1, \ldots, s$ and $j = 1, \ldots, t$. For example, the multiple-entry matrix M in Section 6 gives rise to bipartite multigraphs $B(M, \sigma_k)$ $(k = 1, 2, 3, 4, 5)$, of which $B(M, \sigma_1)$ is shown in Fig. 1.

It is easy to verify the following proposition.

Proposition 7.1. *If* $S = (p_1, \ldots, p_s)$, $t = (q_1, \ldots, q_t)$ *are compositions of* $2n$ *and* A^* *is the* (S, T)*-amalgamation of an* HLS$(2n)$ *on symbols* $\sigma_1, \ldots, \sigma_n$ *then*

(OH1) *row* α *of* A^* *contains each symbol* $2p_\alpha$ *times, for* $\alpha = 1, \ldots, s$;

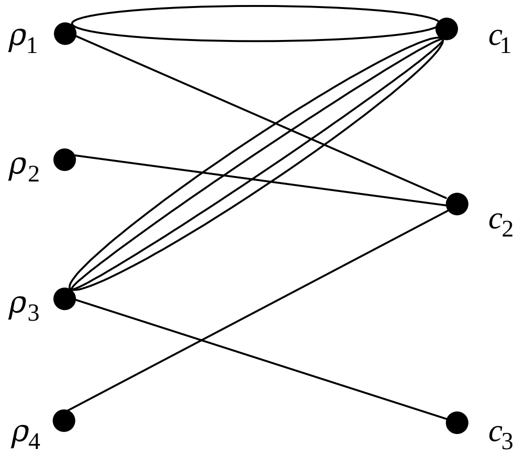

Fig. 1.

(OH2) *column β of A^* contains each symbol $2q_\beta$ times, for $\beta = 1, \ldots, t$;*
(OH3) *cell (α, β) of A^* contains $p_\alpha q_\beta$ symbols, counting repetitions, for $\alpha = 1, \ldots, s$ and $\beta = 1, \ldots, t$;*
(OH4) *$B(A^*, \sigma_k)$ is connected for $k = 1, \ldots, n$.*

The truth of (OH4) follows from the fact that, if A is the relevant HLS($2n$), then $B(A^*, \sigma_k)$ is obtained from $B(A, \sigma_k)$ by identifying vertices and $B(A, \sigma_k)$ is a cycle (the cycle S_k in the notation of the paragraph preceding Lemma 3.1).

If $S = (p_1, \ldots, p_s), T = (q_1, \ldots, q_t)$ are compositions of $2n$ and if A^* is an $s \times t$ multiple-entry matrix on symbols $\sigma_1, \ldots, \sigma_n$ satisfying the above conditions (OH1)–(OH4), then we shall call A^* an *(S, T)-outline Hamiltonian double latin square*. By Proposition 7.1, an (S, T)-amalgamation of an HLS($2n$) is an (S, T)-outline Hamiltonian double latin square. The main result of this section if the following:

Theorem 7.2. *If S, T are compositions of $2n$ then each (S, T)-outline Hamiltonian double latin square is the (S, T)-amalgamation of an* HLS($2n$).

In order to give our elementary proof of this theorem, we need a lemma from [15]. A set $\mathscr{F}$ of sets will be said to be *laminar* if, for every pair X, Y of sets belonging to $\mathscr{F}$, one of the statements $X \subseteq Y$, $Y \subseteq X$, $X \cap Y = \emptyset$ is true. If x, y are real numbers then $\lfloor y \rfloor$, $\lceil y \rceil$ denote (as usual) the integers such that $y - 1 < \lfloor y \rfloor \leqslant y \leqslant \lceil y \rceil < y + 1$ and the statement $x \approx y$ will mean that $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$. We observe that the relation $\approx$ is reflexive and transitive but not symmetric.

Lemma 7.3. *If $\mathscr{F}$, $\mathscr{G}$ are two laminar sets of subsets of a finite set M and h is a positive integer then M has a subset L such that*

$$|L \cap X| \approx |X|/h \quad \textit{for every } X \in \mathscr{F} \cup \mathscr{G}. \tag{1}$$

A proof of Lemma 7.3 may be found in [19]. It uses a fairly simple network flow argument essentially contained in the paper [6] of Baranyai, who in turn attributes the underlying idea to Lovász.

We define an *n-bimultigraph* to be an ordered quadruple $(G, \phi; P, Q)$ such that (G, ϕ) is an n-edge-coloured multigraph and P, Q are disjoint non-empty sets with union $V(G)$ and each edge of G joins an element of P to an element of Q.

We shall say that an n-bimultigraph $(D, \phi; P', Q')$ is a *detachment* of an n-bimultigraph $(G, \phi; P, Q)$ if $E(D) = E(G)$ and there exists a DG-amalgamator Ω such that $\Omega(P') = P$, $\Omega(Q') = Q$. An important special case will be that in which $V(D) = V(G) \cup \{v^*\}$ for some element $v^* \notin V(G)$ and there is a DG-amalgamator Ω such that $\Omega(P') = P$, $\Omega(Q') = Q$ and $\Omega(x) = x$ for every $x \in V(G)$. Then $\Omega(v^*)$ must be some vertex v of G, and we shall say that the detachment $(D, \phi; P', Q')$ of $(G, \phi; P, Q)$ is *obtained by splitting off a new vertex v^* from v*. Clearly, in this case either $v \in P$, $P' = P \cup \{v^*\}$ and $Q' = Q$ or $v \in Q$, $P' = P$ and $Q' = Q \cup \{v^*\}$. In more informal language, a detachment of $(G, \phi; P, Q)$ is obtained by splitting each $x \in V(G)$ into one or more vertices (the elements of $\Omega^{-1}(\{x\})$). In this process, an edge joining vertices x, y in G becomes an edge joining one of the vertices into which x splits to one of the vertices into which y splits. The process does not change colours of edges, since $(G, \phi; P, Q)$ and $(D, \phi; P', Q')$ involve the same 'colouring function' ϕ. If we merely split one vertex v of G into two vertices v, v^*, leaving all other vertices intact, then the resulting detachment is "obtained by splitting off v^* from v".

Let $(G, \phi; P, Q)$ be an n-bimultigraph. We shall say that $(G, \phi; P, Q)$ is (i) *$2n$-bicomplete* if $|P| = |Q| = 2n$ and $d_G(x, y) = 1$ for all $x \in P, y \in Q$ (ii) *Hamiltonian* if $G\langle 1 \rangle, G\langle 2 \rangle, \ldots, G\langle n \rangle$ are all Hamiltonian cycles of G. We shall say that $(G, \phi; P, Q)$ is *n-admissible* if it satisfies the following conditions:

(A1) $d_G(x)/2n$ is a positive integer for every $x \in V(G)$;
(A2) each vertex x of G is incident with $d_G(x)/n$ edges of each colour;
(A3) $d_G(x, y) = d_G(x)d_G(y)/4n^2$ for all $x \in P$, $y \in Q$;
(A4) $G\langle 1 \rangle, G\langle 2 \rangle, \ldots, G\langle n \rangle$ are connected.

It is easily seen (although we shall not need this fact in any of our proofs) that (A1)–(A4) are necessary conditions for $(G, \phi; P, Q)$ to have a $2n$-bicomplete Hamiltonian detachment, the integer $d_G(x)/2n$ in (A1) being the number of vertices into which x must be split in forming such a detachment. In Proposition 7.5, we shall see that these necessary conditions are also sufficient.

Lemma 7.4. *If $(G, \phi; P, Q)$ is an n-admissible n-bimultigraph, $v \in V(G)$ and $d_G(v) > 2n$ then an n-admissible detachment of $(G, \phi; P, Q)$ is obtainable by splitting off a new vertex from v.*

Proof. Assume without loss of generality that $v \in P$. (Clearly, a similar argument can be given when $v \in Q$.) Let M be the set of edges incident with v in G. For each $y \in Q$ let M^y be the set of edges joining v to y in G, and let $\mathscr{F} = \{M^y; y \in Q\}$. For $k = 1, \ldots, n$ let M_k be the set of edges of colour k in M and for each component C of $G\langle k \rangle - v$

let M_k^C be the set of those edges in M_k which join v to elements of $V(C)$ in G. Let $\mathcal{M}_k$ denote the set $\{M_k^C : C \in (G\langle k\rangle - v)\}$ of subsets of M_k and let $\mathcal{G}$ be the set $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_n \cup \{M_1, M_2, \ldots, M_n, M\}$ of subsets of M.

By (A1), $d_G(v) = 2hn$ for some integer h; and $h \geqslant 2$ by our hypothesis that $d_G(v) > 2n$. Since $\mathcal{F}$, $\mathcal{G}$ are laminar sets of subsets of M, there exists by Lemma 7.3 a subset L of M such that (1) is true. Let $(D, \phi; P \cup \{v^*\}, Q)$ be the detachment of $(G, \phi; P, Q)$ obtained by splitting off a new vertex v^* from v and taking the set of edges incident with v^* in D to be the set of edges of L. We will prove that $(D, \phi; P \cup \{v^*\}, Q)$ is n-admissible.

Since $M \in \mathcal{G}$ and $|M| = d_G(v) = 2hn$, taking $X = M$ in (1) gives $d_D(v*) = |L| = 2n$ and so $d_D(v) = |M| - |L| = 2(h-1)n$. Moreover G satisfies (A1) and all vertices in $V(D)\backslash\{v, v^*\}$ have the same degrees in D as in G. Therefore D satisfies (A1). For $k = 1, \ldots, n$ we have $|M_k| = d_G(v)/n = 2h$ by (A2) and so taking $X = M_k \in \mathcal{G}$ in (1) gives $|L \cap M_k| = 2$ and consequently $|M_k \backslash L| = 2h - 2$. Therefore v^* is incident in D with $2 = d_D(v^*)/n$ edges of each colour and v is incident in D with $2h - 2 = d_D(v)/n$ edges of each colour. Moreover (G, ϕ) satisfies (A2) and all vertices in $V(D)\backslash\{v, v^*\}$ are incident with the same edges in D as in G. Therefore (D, ϕ) satisfies (A2). If $y \in Q$ then (A3) gives $|M^y| = d_G(v, y) = d_G(v)d_G(y)/4n^2 = hd_G(y)/2n$ and so (since $d_G(y)/2n$ is an integer by (A1)) taking $X = M^y \in \mathcal{F}$ in (1) gives $|L \cap M^y| = d_G(y)/2n$ and consequently $|M^y \backslash L| = (h-1)d_G(y)/2n$. Therefore $d_D(v^*, y) = d_G(y)/2n = d_D(y)/2n = d_D(v^*)d_D(y)/4n^2$ and $d_D(v, y) = (h-1)d_G(y)/2n = (h-1)d_D(y)/2n = d_D(v)d_D(y)/4n^2$ for every $y \in Q$. Moreover $(G, \phi; P, Q)$ satisfies (A3) and $d_D(x, y) = d_G(x, y)$, $d_D(x) = d_G(x)$, $d_D(y) = d_G(y)$ whenever $x \in P\backslash\{v\}, y \in Q$. Therefore $(D, \phi; P \cup \{v^*\}, Q)$ satisfies (A3).

Let $k \in \{1, \ldots, n\}$. By (A1) and (A2), each vertex of G has even degree in $G\langle k\rangle$ and so $G\langle k\rangle$ has no bridges. Since $G\langle k\rangle$ is connected by (A4) and has no bridges, $|M_k^C| \geqslant 2$ for each component C of $G\langle k\rangle - v$. Therefore, for each such C, taking $x = M_k^C \in \mathcal{G}$ in (1) gives $|L \cap M_C^k| < |M_k^C|$, and so not all edges joining v to vertices of C in $G\langle k\rangle$ become incident with v^* in D. Therefore v is adjacent in $D\langle k\rangle$ to at least one vertex of each component of $G\langle k\rangle - v = (D\langle k\rangle - v^*) - v$ and so $D\langle k\rangle - v^*$ is connected. Moreover, v^* is adjacent in $D\langle k\rangle$ to at least one vertex of $D\langle k\rangle - v^*$ since we have seen that v^* is incident in D with two edges of each colour. Therefore $D\langle k\rangle$ is connected. We have thus proved that (D, ϕ) satisfies (A4).

We conclude that $(D, \phi; P \cup \{v^*\}, Q)$ is n-admissible, as required. □

Proposition 7.5. *Every n-admissible n-bimultigraph has a* 2*n-bicomplete Hamiltonian detachment.*

Proof. Let $(G, \phi; P, Q)$ be an n-admissible n-bimultigraph. Let $(D, \phi; P', Q')$ be an n-admissible detachment of $(G, \phi; P, Q)$ such that $|V(D)|$ is as large as possible. If any vertex had degree exceeding $2n$ in D then $(D, \phi; P', Q')$ would by Lemma 7.4 have an n-admissible detachment involving a graph with more vertices than D and, since this detachment would also be a detachment of $(G, \phi; P, Q)$, it would contradict the maximality of $|V(D)|$. Therefore no vertex has degree exceeding $2n$ in D and so, by

(A1), $d_D(x) = 2n$ for every $x \in V(D)$. Therefore, by (A3), $d_D(x, y) = 1$ for all $x \in P'$, $y \in Q'$. Considering any fixed $x \in P'$ now gives

$$2n = d_D(x) = \sum_{y \in Q'} d_D(x, y) = \sum_{y \in Q'} 1 = |Q'|$$

and a similar argument gives $|P'| = 2n$. Therefore $(D, \phi; P', Q')$ is $2n$-bicomplete. Since all vertices have degree $2n$ in D, it follows from (A2) and (A4) that each $D\langle k \rangle$ is a connected graph in which every vertex of D has degree 2, i.e. a Hamiltonian cycle of D. Therefore $(D, \phi; P', Q')$ is Hamiltonian. $\square$

Proof of Theorem 7.2. Let $S = (p_1, \dots, p_s)$, $T = (q_1, \dots, q_t)$ be compositions of $2n$ and let A^* be an (S, T)-outline Hamiltonian double latin square on symbols $\sigma_1, \dots, \sigma_n$. Then A^* satisfies (OH1)–(OH4). Let Γ denote an n-bimultigraph $(G, \phi; \{\rho_1, \dots, \rho_s\}, \{c_1, \dots, c_t\})$ such that $d_{G\langle k \rangle}(\rho_\alpha, c_\beta)$ is the number of occurrences of σ_k in the cell (α, β) of A^* for $\alpha = 1, \dots, s$ and $\beta = 1, \dots, t$ and $k = 1, \dots, n$. Thus $G\langle k \rangle$ is precisely the multigraph $B(A^*, \sigma_k)$, and we can think of Γ as being obtained by superposing $B(A^*, \sigma_1), \dots, B(A^*, \sigma_n)$ with their edges coloured $1, \dots, n$, respectively, to distinguish between them.

For $\alpha = 1, \dots, s$ it follows from (OH1) that ρ_α is incident in G with $2p_\alpha$ edges of each colour, which implies that $d_G(\rho_\alpha) = 2p_\alpha n$ and that ρ_α is incident in G with $d_G(\rho_\alpha)/n$ edges of each colour. Similarly (OH2) implies that $d_G(c_\beta) = 2q_\beta n$ and c_β is incident in G with $d_G(c_\beta)/n$ edges of each colour for $\beta = 1, \dots, t$. Therefore Γ satisfies (A1) and (A2). By (OH3), $d_G(\rho_\alpha, c_\beta) = p_\alpha q_\beta = d_G(\rho_\alpha) d_G(c_\beta)/4n^2$ for $\alpha = 1, \dots, s$ and $\beta = 1, \dots, t$ and so Γ satisfies (A3). Since $G\langle k \rangle = B(A^*, \sigma_k)$ for $k = 1, \dots, n$, it follows from (OH4) that Γ satisfies (A4). Therefore Γ is n-admissible and so has, by Proposition 7.5, a $2n$-bicomplete Hamiltonian detachment $(D, \phi; P, Q)$.

By the definition of detachment, there exists a DG-amalgamator Ω such that $\Omega(P) = \{\rho_1, \dots, \rho_s\}$, $\Omega(Q) = \{c_1, \dots, c_t\}$. The definition of a DG-amalgamator implies that $d_G(\rho_\alpha) = \sum (d_D(x)\colon x \in \Omega^{-1}(\{\rho_\alpha\}))$, which is $2n|\Omega^{-1}(\{\rho_\alpha\})|$ since $(D, \phi; P, Q)$ is $2n$-bicomplete. Therefore $|\Omega^{-1}(\{\rho_\alpha\})| = d_G(\rho_\alpha)/2n = p_\alpha$ ($\alpha = 1, \dots, s$). Consequently, the elements of P can be arranged in an order $\rho'_1, \dots, \rho'_{2n}$ such that

$$\Omega(\rho'_i) = \rho_\alpha \quad \text{when} \quad \bar{p}_{\alpha-1} < i \leqslant \bar{p}_\alpha \quad (\alpha = 1, \dots, s),$$

where $\bar{p}_0 = 0, \bar{p}_\alpha = p_1 + \cdots + p_\alpha$ ($\alpha = 1, \dots, s$). For similar reasons, the elements of Q can be arranged in an order $c'_1, \dots, c'_{2n}$ such that

$$\Omega(c'_j) = c_\beta \quad \text{when} \quad \bar{q}_{\beta-1} < j \leqslant \bar{q}_\beta \quad (\beta = 1, \dots, t),$$

where $\bar{q}_0 = 0, \bar{q}_\beta = q_1 + \cdots + q_\beta$ ($\beta = 1, \dots, t$). Let A be the $2n \times 2n$ matrix such that $A(i,j) = \sigma_k$ whenever the edge joining ρ'_i to c'_j has colour k in (D, ϕ). Then $B(A, \sigma_k)$ is the graph obtained from $D\langle k \rangle$ on replacing ρ'_i by ρ_i and c'_j by c_j for $i, j = 1, \dots, 2n$. Since $(D, \phi; P, Q)$ is Hamiltonian, each $D\langle k \rangle$ is a Hamiltonian cycle of D. Therefore each $B(A, \sigma_k)$ is a cycle with vertices $\rho_1, \rho_2, \dots, \rho_{2n}, c_1, c_2, \dots, c_{2n}$ and so A is a HLS$(2n)$.

Let A be partitioned into submatrices as

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{st} \end{pmatrix},$$

where $A_{\alpha\beta}$ is the $p_\alpha \times q_\beta$ submatrix of A formed by the entries $A(i,j)$ with $\bar{p}_{\alpha-1} < i \leqslant \bar{p}_\alpha, \bar{q}_{\beta-1} < j \leqslant \bar{q}_\beta$. Then the number of occurrences of σ_k in $A_{\alpha\beta}$ is the number of pairs (i,j) such that $\rho'_i \in \Omega^{-1}(\{\rho_\alpha\})$, $c'_j \in \Omega^{-1}(\{c_\beta\})$ and the edge joining ρ'_i to c'_j has colour k in (D, ϕ). This in turn is equal to the number of edges of colour k joining ρ_α to c_β in (G, ϕ), which is the number of occurrences of σ_k in the cell (α, β) of A^*. Therefore A^* is the (S, T)-amalgamation of A. $\square$

8. Amalgamating symmetric Hamiltonian double latin squares: a matroid proof

Definition. A cell (i,j) of a square matrix will be called a *diagonal* cell if $i = j$ and an *off-diagonal* cell if $i \neq j$. Let A be an $s \times s$ multiple-entry matrix on symbols $\sigma_1, \ldots, \sigma_n$ and let $N(k; \alpha, \beta)$ denote the number of occurrences of σ_k in the cell (α, β) of A. We shall say that σ_k *appears* in the cell (α, β) of A if $N(k; \alpha, \beta) > 0$ and that σ_k *appears oddly* in this cell if $N(k; \alpha, \beta)$ is odd. If $\alpha \in \{1, \ldots, s\}$ then $q_\alpha(A)$ will denote the number of symbols which appear oddly in the cell (α, α) of A. A symbol σ_k is *diagonally even* in A if $N(k; \alpha, \alpha)$ is even for $\alpha = 1, \ldots, s$ and is *diagonally odd* in A if $N(k; \alpha, \alpha)$ is odd for at least one $\alpha \in \{1, \ldots, s\}$. A symbol σ_k is *diagonally confined* (in A) to a subset X of $\{1, \ldots, s\}$ if $N(k; \alpha, \alpha) = 0$ for every $\alpha \in \{1, \ldots, s\} \backslash X$. For $k = 1, \ldots, n$ we define $F(A, \sigma_k)$ to be a multigraph with s vertices $\rho_1, \ldots, \rho_s$ in which ρ_α and ρ_β are joined by $N(k; \alpha, \beta)$ edges for $\alpha, \beta = 1, \ldots, s$. (In particular, ρ_α is incident with $N(k; \alpha, \alpha)$ loops.)

Proposition 8.1. *If $S = (p_1, \ldots, p_s)$ is a composition of $2n$ and A^* is the (S, S)-amalgamation of an* SHLS($2n$) *on symbols $\sigma_1, \ldots, \sigma_n$ then*

(OS1) *row α of A^* contains each symbol $2p_\alpha$ times, for $\alpha = 1, \ldots, s$;*
(OS2) *cell (α, β) of A^** contains *$p_\alpha p_\beta$ symbols, counting repetitions, for $\alpha, \beta = 1, \ldots, s$;*
(OS3) $\mathrm{dec}(X) \leqslant \frac{1}{2}\sum_{\alpha \in X} (p_\alpha - q_\alpha(A^*))$ *for every subset X of* $\{1, \ldots, s\}$, *where* $\mathrm{dec}(X)$ *is the number of symbols which are diagonally even and diagonally confined to X in A^*;*
(OS4) *$F(A^*, \sigma_k)$ is connected for $k = 1, \ldots, n$.*

We remark that in Proposition 8.7 and Theorem 8.8 we show that (OS3) can be replaced by an alternative condition (OS3*) which does not involve a set of inequalities.

Proof. It is easy to see that A^* satisfies (OS1) and (OS2).

Let A be the SHLS($2n$) of which A^* is the (S, S)-amalgamation. Since A is Hamiltonian, $F(A, \sigma_k)$ is clearly connected for each symbol σ_k. Since $F(A^*, \sigma_k)$ is obtained from $F(A, \sigma_k)$ by identifying vertices, (OS4) follows.

To prove (OS3), recall the definition of (S, T)-amalgamation in Section 6. When $T = S = (p_1, \ldots, p_s)$, this definition involves partitioning A into submatrices $A_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, s)$. Let X be a subset of $\{1, \ldots, s\}$ and let D_X be the set of all cells on the main diagonal of A which are in the submatrices $A_{\alpha\alpha}$ $(\alpha \in X)$. Each occurrence of a symbol in $A_{\alpha\alpha}$ gives rise to an occurrence of that symbol in the cell (α, α) of A^*. Therefore, if a symbol σ_k is diagonally confined to X in A^*, then all occurrences of σ_k on the main diagonal of A must be in cells belonging to D_X and so, by Lemma 4.2, two of these cells must contain σ_k. Consequently, at least $2\mathrm{dec}(X)$ cells in D_X contain symbols which are diagonally even in A^*. (We say 'at least' here since a symbol σ_k that is diagonally even and occurs twice in D_X may not be diagonally confined, as it could occur an even number of times in $A_{\beta\beta}$ but not on the main diagonal of $A_{\beta\beta}$, for some $\beta \notin X$). If a symbol σ_k appears oddly in a cell (α, α) of A^* then the number of occurrences of σ_k in $A_{\alpha\alpha}$ is odd and so, since A is symmetric, at least one cell on the main diagonal of $A_{\alpha\alpha}$ must contain σ_k; in view of Lemma 4.2, exactly one cell on the main diagonal of $A_{\alpha\alpha}$ contains σ_k. Consequently, $\sum_{\alpha \in X} q_\alpha(A^*)$ cells in D_X contain symbols which are diagonally odd in A^*. Hence

$$2\mathrm{dec}(X) + \sum_{\alpha \in X} q_\alpha(A^*) \leqslant |D_X| = \sum_{\alpha \in X} p_\alpha,$$

and (OS3) is proved. $\square$

If $S = (p_1, \ldots, p_s)$ is a composition of $2n$ and if A^* is a symmetric $s \times s$ multiple-entry matrix on symbols $\sigma_1, \ldots, \sigma_n$ satisfying conditions (OS1)–(OS4), then we shall call A^* a *symmetric S-outline Hamiltonian double latin square*. By Proposition 8.1, an (S, S)-amalgamation of an SHLS($2n$) is a symmetric S-outline Hamiltonian double latin square. The main result of this section is the following:

Theorem 8.2. *If S is a composition of* $2n$ *then each symmetric S-outline Hamiltonian double latin square is the* (S, S)*-amalgamation of an* SHLS($2n$).

Theorem 7.2 could be used to show that any symmetric S-outline Hamiltonian double latin square is the (S, S)-amalgamation of an HLS($2n$). (Hint: deduce (OH4) from (OS3) and (OS4); taking $X = \emptyset$ in (OS3) shows that every symbol occurs on the diagonal of A^*.) However, this approach would not guarantee that the HLS($2n$) concerned was symmetric. So we need a different argument, although it will bear some resemblances to the *proof* of Theorem 7.2.

As already stated, our proof of Theorem 8.2 will use matroids. We recall that a *matroid* is an ordered pair $(M, \mathfrak{I})$ such that M is a finite set, $\mathfrak{I}$ is a set of subsets of M (which are called *independent* sets) and the following axioms are satisfied:

(i) $\emptyset \in \mathfrak{I}$;

(ii) if $I \in \mathfrak{I}$ and $J \subseteq I$ then $J \in \mathfrak{I}$;
(iii) for each subset A of M, all maximal independent subsets of A have the same cardinality (which is called the *rank* of A and denoted by $r(A)$).

We shall need the following Matroid Intersection Theorem of Edmonds.

Theorem 8.3 (Edmonds [11,22, Section 69; Section 8.5]). *Let $(M, \mathfrak{I})$, $(M, \mathfrak{I}')$ be matroids with the same underlying set M and with rank functions r, r', respectively. Then these matroids have a common independent set of cardinality c if and only if $r(A) + r'(M \backslash A) \geqslant c$ for every subset A of M.*

We shall say that an n-edge-coloured multigraph (D, ϕ) is a *detachment* of an n-edge-coloured multigraph (G, ϕ) if $E(D) = E(G)$ and there exists a DG-amalgamator. An important special case will be that in which $V(D) = V(G) \cup \{v^*\}$ for some element $v^* \notin V(G)$ and there is a DG-amalgamator Ω such that $\Omega(x) = x$ for every $x \in V(G)$. Then $\Omega(v^*)$ must be some vertex v of G, and we shall say that the detachment (D, ϕ) of (G, ϕ) is *obtained by splitting off a new vertex v^* from v*. In more informal language, a detachment (D, ϕ) of (G, ϕ) is obtained by splitting each $x \in V(G)$ into one or more vertices (the elements of $\Omega^{-1}(\{x\})$ for some DG-amalgamator Ω). In this process, an edge joining vertices x, y in G becomes an edge joining one of the vertices into which x splits to one of the vertices into which y splits. The process does not change colours of edges, since (G, ϕ) and (D, ϕ) involve the same 'colouring function' ϕ. A loop ℓ incident with a vertex x in G becomes an edge of D joining two of the vertices into which x splits. These two vertices need not be distinct, and so ℓ may become a loop of D incident with one of the vertices into which x splits. If we merely split one vertex v of G into two vertices v, v^*, leaving all other vertices intact, then the resulting detachment is "obtained by splitting off v^* from v".

Let (G, ϕ) be an n-edge-coloured multigraph. We shall say that (G, ϕ) is (i) $(2n+1)$-*complete* if G is a complete graph of order $2n+1$, (ii) *Hamiltonian* if $G\langle 1 \rangle$, $G\langle 2 \rangle, \dots, G\langle n \rangle$ are all Hamiltonian cycles of G. We shall say that (G, ϕ) is n-*helpful* if it satisfies the following conditions:

(H0) $|E(G)| = 2n^2 + n$;
(H1) $d_G(x)/2n$ is a positive integer for every $x \in V(G)$;
(H2) $d_{G\langle k \rangle}(x) = d_G(x)/n$ for each $x \in V(G)$ and for $k = 1, \dots, n$;
(H3) $d_G(x, y) = d_G(x)d_G(y)/4n^2$ for every pair x, y of distinct vertices of G;
(H4) $G\langle 1 \rangle, G\langle 2 \rangle, \dots, G\langle n \rangle$ are connected.

It is easily seen (although we shall not need this fact) that (H0)–(H4) are necessary conditions for (G, ϕ) to have a $(2n+1)$-complete Hamiltonian detachment, the integer $d_G(x)/2n$ in (H1) being the number of vertices into which x must be split in forming such a detachment. In Proposition 8.6, we shall see that these necessary conditions are in fact sufficient.

Lemma 8.4. *If* (G, ϕ) *is an n-helpful n-edge-coloured multigraph and* $x \in V(G)$ *and* $d_G(x) = 2hn$ *then x is incident in G with exactly* $\binom{h}{2}$ *loops.*

Proof. By (H0), $\sum_{y \in V(G)} d_G(y) = 4n^2 + 2n$ and so $\sum_{y \in V(G) \setminus \{x\}} d_G(y) = 4n^2 + 2n - 2hn$. By (H3) it follows that $d_G(x, y) = \frac{h}{2n} d_G(y)$, so $\sum_{y \in V(G) \setminus \{x\}} d_G(x, y) = h(2n + 1 - h)$. Since $d_G(x) = 2hn$, it follows that x must be incident with exactly $h(h-1)/2$ loops. $\square$

Lemma 8.5. *If* (G, ϕ) *is an n-helpful n-edge-coloured multigraph,* $v \in V(G)$ *and* $d_G(v) > 2n$ *then an n-helpful detachment of* (G, ϕ) *is obtainable by splitting off a new vertex from v.*

Proof. By (H1), $d_G(v) = 2hn$ for some integer h; and $h \geqslant 2$ by our hypothesis that $d_G(v) > 2n$. Let M be the set of edges incident with v in G. For each $y \in V(G)$ let M^y be the set of edges joining v to y in G. (In particular, M^v is the set of loops incident with v in G.) Let $\mathfrak{I}$ be the set of all subsets X of M such that $|X \cap M^v| \leqslant h - 1$ and $|X \cap M^y| \leqslant d_G(y)/2n$ for each $y \in V(G) \setminus \{v\}$. It is easy to see that $(M, \mathfrak{I})$ is a matroid.

For $k = 1, \dots, n$ let M_k be the set of edges of colour k in M and for each component C of $G\langle k \rangle - v$ let M_k^C be the set of those edges in M_k which join v to elements of $V(C)$ in G. Let $\mathscr{C}_k$ be the set of all components C of $G\langle k \rangle - v$ such that $|M_k^C| = 2$. Let $\mathfrak{I}'_k$ be the set of all subsets I of M_k such that $|I \cap M_k^C| \leqslant 1$ for each $C \in \mathscr{C}_k$. Let $\mathfrak{I}'$ be the set of all sets of the form $I_1 \cup I_2 \cup \cdots \cup I_n$ where $I_k \in \mathfrak{I}'_k$ for $k = 1, \dots, n$. Since for each k the sets M_k^C $(C \in \mathscr{C}_k)$ are disjoint, it is easily seen that $(M'_k, \mathfrak{I}'_k)$ is a matroid for $k = 1, \dots, n$. Therefore $(M, \mathfrak{I}')$ is a matroid.

Let r, r' be the rank functions of the matroids $(M, \mathfrak{I})$, $(M, \mathfrak{I}')$, respectively. Let A be a subset of M. Since $|A \cap M^v| \leqslant |M^v| = h(h-1)/2$ by Lemma 8.4 and $h \geqslant 2$ it follows that $\min(|A \cap M^v|, h-1) \geqslant 2|A \cap M^v|/h$. If $y \in V(G) \setminus \{v\}$ then $|A \cap M^y| \leqslant |M^y| = hd_G(y)\ /2n$ by (H3) and so $\min(|A \cap M^y|, d_G(y)/2n) \geqslant |A \cap M^y|/h$. Therefore

$$\begin{aligned} r(A) &= \min(|A \cap M^v|, h-1) + \sum_{y \in V(G) \setminus \{v\}} \min(|A \cap M^y|, d_G(y)/2n) \\ &\geqslant \left(2|A \cap M^v| + \sum_{y \in V(G) \setminus \{v\}} |A \cap M^y| \right) /h = (2|A \cap M^v| + |A \setminus M^v|)/h. \end{aligned}$$

For each $k \in \{1, \dots, n\}$ the sets M_k^C $(C \in \mathscr{C}_k)$ are disjoint subsets of $M_k \setminus M^v$ each of which has cardinality 2, and so $A \cap M_k$ has a subset S_k such that $|S_k| \geqslant |(A \cap M_k) \cap M^v| + \frac{1}{2}|(A \cap M_k) \setminus M^v|$ and $|S_k \cap M_k^C| \leqslant 1$ for each $C \in \mathscr{C}_k$. Therefore any subset of S_k of cardinality $\min(|S_k|, 2)$ is a set $I_k \subseteq A \cap M_k$ such that $I_k \in \mathfrak{I}'_k$ and

$$|I_k| \geqslant \min(|A \cap M_k \cap M^v| + \tfrac{1}{2}|(A \cap M_k) \setminus M^v|, 2).$$

Moreover (H2) gives

$$\begin{aligned}2|A\cap M_k\cap M^v|+|(A\cap M_k)\backslash M^v| &\leqslant 2|M_k\cap M^v|+|M_k\backslash M^v|\\ &= d_{G\langle k\rangle}(v)\\ &= d_G(v)/n=2h.\end{aligned}$$

Therefore $|I_k|\geqslant(2|A\cap M_k\cap M^v|+|(A\cap M_k)\backslash M^v|)/h$. Since $I_1\cup\cdots\cup I_n\in\mathfrak{I}'$ and $I_1\cup\cdots\cup I_n\subseteq A$ it follows that

$$r'(A)\geqslant|I_1\cup\cdots\cup I_n|\geqslant(2|A\cap M^v|+|A\backslash M^v|)/h.$$

Since $r(A), r'(A)$ are both at least $(2|A\cap M^v|+|A\backslash M^v|)/h$ for every $A\subseteq M$, it follows that

$$r(A)+r'(M\backslash A)\geqslant(2|M^v|+|M\backslash M^v|)/h=d_G(v)/h=2n$$

for every $A\subseteq M$, and so there is by Theorem 8.3 a set $L\in\mathfrak{I}\cap\mathfrak{I}'$ such that $|L|=2n$. Let (D,ϕ) be the detachment of (G,ϕ) obtained by splitting off a new vertex v^* from v, taking L to be the set of edges incident with v^* in D and requiring edges in $L\cap M^v$ to join v to v^* in D (so that v^* is not incident with any loops in D). We will prove that (D,ϕ) is n-helpful.

Since G satisfies (H0) and $E(D)=E(G)$ it follows that D satisfies (H0). Since $d_G(v)=2hn$ and $d_D(v^*)=|L|=2n$ it follows that $d_D(v)=d_G(v)-d_D(v^*)=2(h-1)n$. Moreover G satisfies (H1) and all vertices in $V(D)\backslash\{v,v^*\}$ have the same degrees in D as in G. Therefore D satisfies (H1). Since $L\in\mathfrak{I}'$ it follows that $L=L_1\cup\cdots\cup L_n$ for some sets $L_1\in\mathfrak{I}'_1,\ldots,L_n\in\mathfrak{I}'_n$. Since $|L|=2n$ and no set in any $\mathfrak{I}'_k$ has cardinality exceeding 2, it follows that $|L_1|=\cdots=|L_n|=2$ and so $d_{D\langle k\rangle}(v^*)=|L_k|=2=|L|/n=d_D(v^*)/n$ for $k=1,\ldots,n$. Moreover, for $k=1,\ldots,n$ we have $d_{G\langle k\rangle}(v)=d_G(v)/n=2h$ since (G,ϕ) satisfies (H2) and consequently $d_{D\langle k\rangle}(v)=2h-d_{D\langle k\rangle}(v^*)=2h-2=d_D(v)/n$. Furthermore, since (G,ϕ) satisfies (H2) it follows that $d_{D\langle k\rangle}(x)=d_{G\langle k\rangle}(x)=d_G(x)/n=d_D(x)/n$ for all $x\in V(D)\backslash\{v,v^*\}$. Hence (D,ϕ) satisfies (H2).

Since $L\in\mathfrak{I}$ it follows that

$$|L\cap M^v|\leqslant h-1=(d_G(v)/2n)-1 \quad\text{and}$$

$$|L\cap M^y|\leqslant d_G(y)/2n \quad \text{for every } y\in V(G)\backslash\{v\}.$$

Since

$$(d_G(v)/2n)-1+\sum_{y\in V(G)\backslash\{v\}} d_G(y)/2n=(|E(G)|/n)-1=2n=|L|$$

by (H0), it follows that $|L\cap M^v|=(d_G(v)/2n)-1=h-1$ and $|L\cap M^y|=d_G(y)/2n$ for every $y\in V(G)\backslash\{v\}$. Therefore $d_D(v,v^*)=|L\cap M^v|=h-1=d_D(v)d_D(v^*)/4n^2$ and

$$d_D(v^*,y)=|L\cap M^y|=d_G(y)/2n=d_D(y)/2n=d_D(v^*)d_D(y)/4n^2$$

for every $y \in V(G)\backslash\{v\} = V(D)\backslash\{v, v^*\}$. Moreover, since G satisfies (H3),

$$d_D(v,y) = d_G(v,y) - d_D(v^*,y) = (d_G(v)d_G(y) - d_D(v^*)d_D(y))/4n^2$$
$$= (d_G(v) - d_D(v^*))d_D(y)/4n^2 = d_D(v)d_D(y)/4n^2$$

for every $y \in V(G)\backslash\{v\} = V(D)\backslash\{v, v^*\}$ and

$$d_D(x,y) = d_G(x,y) = d_G(x)d_G(y)/4n^2 = d_D(x)d_D(y)/4n^2$$

for every two distinct elements x, y of $V(G)\backslash\{v\} = V(D)\backslash\{v, v^*\}$. Hence D satisfies (H3).

Let $k \in \{1, \dots, n\}$. By (H1) and (H2), each vertex of G has even degree in $G\langle k \rangle$ and so $G\langle k \rangle$ has no bridges. Since $G\langle k \rangle$ is connected by (H4) and has no bridges, $|M_k^C| \geqslant 2$ for each component C of $G\langle k \rangle - v$ and so $|M_k^C| > 2 = |L_k| \geqslant |L_k \cap M_k^C|$ for each component C of $G\langle k \rangle - v$ such that $C \notin \mathscr{C}_k$. Since $L_k \in \mathfrak{I}'_k$ it follows that $|L_k \cap M_k^C| \leqslant 1 < 2 = |M_k^C|$ for every $C \in \mathscr{C}_k$. Hence, for each component C of $G\langle k \rangle - v$, we have $|M_k^C| > |L_k \cap M_k^C| = |L \cap M_k^C|$, and so not all edges joining v to vertices of C in $G\langle k \rangle$ become incident with v^* in D. Therefore v is adjacent in $D\langle k \rangle$ to at least one vertex of each component of $G\langle k \rangle - v = (D\langle k \rangle - v^*) - v$ and so $D\langle k \rangle - v^*$ is connected. Moreover, v^* is adjacent in $D\langle k \rangle$ to at least one vertex of $D\langle k \rangle - v^*$ since we have seen that $d_{D\langle k \rangle}(v^*) = 2$ and no loops are incident with v^* in D. Therefore $D\langle k \rangle$ is connected. We have thus proved that (D, ϕ) satisfies (H4).

We conclude that (D, ϕ) is n-helpful, as required. □

Proposition 8.6. *Every n-helpful n-edge-coloured multigraph has a $(2n+1)$-complete Hamiltonian detachment.*

Proof. Let (G, ϕ) be an n-helpful n-edge-coloured multigraph. Let (D, ϕ) be an n-helpful detachment of (G, ϕ) such that $|V(D)|$ is as large as possible. If any vertex had degree exceeding $2n$ in D then (D, ϕ) would by Lemma 8.5 have an n-helpful detachment involving a graph with more vertices than D and, since this detachment would also be a detachment of (G, ϕ), it would contradict the maximality of $|V(D)|$. Therefore no vertex has degree exceeding $2n$ in D and so, by (H1), $d_D(x) = 2n$ for every $x \in V(D)$. From this and (H3) and Lemma 8.4, it follows that D is a complete graph, which must have order $2n+1$ since each of its vertices has degree $2n$. Therefore (D, ϕ) is $(2n+1)$-complete. By (H2) and (H4), each $D\langle k \rangle$ is a connected graph in which every vertex of D has degree 2, i.e. a Hamiltonian cycle of D. Therefore (D, ϕ) is Hamiltonian. □

Proof of Theorem 8.2. Let $S = (p_1, \dots, p_s)$ be a composition of $2n$ and let A^* be a symmetric S-outline Hamiltonian double latin square on symbols $\sigma_1, \dots, \sigma_n$. Then A^* satisfies (OS1)–(OS4), and $\{\sigma_1, \dots, \sigma_n\} = \Phi \cup \Psi$, $\Psi = \Psi_1 \cup \cdots \cup \Psi_s$ where Φ, Ψ are, respectively, the sets of diagonally even and diagonally odd symbols in A^* and Ψ_α is the set of those symbols which appear oddly in the cell (α, α) of A^* for $\alpha = 1, \dots, s$. For simplicity write $q_\alpha = q_\alpha(A^*)$ $(\alpha = 1, \dots, s)$. By (OS2), any diagonal cell (α, α) of A^* contains p_α^2 symbols (counting repetitions) and so q_α must have the same parity as

p_α^2. Therefore $\frac{1}{2}(p_\alpha - q_\alpha)$ is an integer, which is non-negative since $\text{dec}(\{\alpha\}) \leqslant \frac{1}{2}(p_\alpha - q_\alpha)$ by (OS3). Let r_α denote the non-negative integer $\frac{1}{2}(p_\alpha - q_\alpha)$ for $\alpha = 1, \dots, s$.

Since A^* is symmetric, each symbol occurs an even number of times in the union of its non-diagonal cells, and by (OS1) each symbol occurs an even number of times in the whole of A^*. Therefore each symbol occurs an even number of times in the union of the diagonal cells of A^*, and so each member of Ψ must appear oddly in at least two diagonal cells of A^*. Therefore $|\Psi| \leqslant \frac{1}{2}(q_1 + \cdots + q_s)$ and, if this inequality is an equality, each member of Ψ must appear oddly in exactly two diagonal cells of A^*. However, by (OS3),

$$|\Phi| = \text{dec}(\{1, \dots, s\}) \leqslant \tfrac{1}{2}(p_1 + \cdots + p_s) - \tfrac{1}{2}(q_1 + \cdots + q_s) = n - \tfrac{1}{2}(q_1 + \cdots + q_s)$$

and so $\frac{1}{2}(q_1 + \cdots + q_s) \leqslant n - |\Phi| = |\Psi|$. Therefore $|\Psi| = \frac{1}{2}(q_1 + \cdots + q_s)$ and each member of Ψ appears oddly in exactly two diagonal cells of A^*. In other words each member of Ψ belongs to exactly two of the sets $\Psi_1, \dots, \Psi_s$.

We observe that

$$r_1 + \cdots + r_s = \tfrac{1}{2}(p_1 + \cdots + p_s) - \tfrac{1}{2}(q_1 + \cdots + q_s) = n - |\Psi| = |\Phi|. \tag{2}$$

For $\alpha = 1, \dots, s$ let Π_α be the set of those symbols in Φ which appear in the cell (α, α) of A^*. Let Z be a subset of $\{1, \dots, s\}$. A symbol in Φ is diagonally confined to $\{1, \dots, s\} \backslash Z$ in A^* if and only if it does not appear in the cell (α, α) of A^* for any $\alpha \in Z$, i.e. if and only if it does not belong to $\bigcup_{\alpha \in Z} \Pi_\alpha$. Therefore $\text{dec}(\{1, \dots, s\} \backslash Z) = |\Phi| - |\bigcup_{\alpha \in Z} \Pi_\alpha|$, and so (OS3) and (2) give

$$|\Phi| - \left| \bigcup_{\alpha \in Z} \Pi_\alpha \right| \leqslant \sum_{\alpha \in \{1, \dots, s\} \backslash Z} r_\alpha = |\Phi| - \sum_{\alpha \in Z} r_\alpha.$$

Hence $|\bigcup_{\alpha \in Z} \Pi_\alpha| \geqslant \sum_{\alpha \in Z} r_\alpha$ for every subset Z of $\{1, \dots, s\}$ and so, by Hall's Theorem, there exist distinct representatives of the sets $\Pi_1, \Pi_1, \dots, \Pi_1, \Pi_2, \Pi_2, \dots, \Pi_2, \dots, \Pi_s, \Pi_s, \dots, \Pi_s$, where Π_1 is listed r_1 times and Π_2 is listed r_2 times and … and Π_s is listed r_s times. From this and (2), it follows that Φ is the union of disjoint sets $\Phi_1, \dots, \Phi_s$ such that $|\Phi_\alpha| = r_\alpha$ and $\Phi_\alpha \subseteq \Pi_\alpha$ for $\alpha = 1, \dots, s$.

Since the sets $\Phi = \Phi_1 \cup \cdots \cup \Phi_s$, $\Psi = \Psi_1 \cup \cdots \cup \Psi_s$ are disjoint, we can define $t_{\alpha k}$ to be 2 if $\sigma_k \in \Phi_\alpha$ and 1 if $\sigma_k \in \Psi_\alpha$ and 0 if $\sigma_k \notin \Phi_\alpha \cup \Psi_\alpha$ for $k = 1, \dots, n$ and $\alpha = 1, \dots, s$. Let $N(k; \alpha, \beta)$ denote the number of occurrences of a symbol σ_k in a cell (α, β) of A^*. We observe that $N(k; \alpha, \alpha)$ and $t_{\alpha k}$ are, by the definitions of Ψ_α and $t_{\alpha k}$, both odd if $\sigma_k \in \Psi_\alpha$ and both even if $\sigma_k \notin \Psi_\alpha$. Moreover, $N(k; \alpha, \alpha) \geqslant t_{\alpha k}$ because $N(k; \alpha, \alpha)$ is odd when $\sigma_k \in \Psi_\alpha$ and (since $\Psi_\alpha \subseteq \Pi_\alpha \subseteq \Phi$) is even and non-zero when $\sigma_k \in \Phi_\alpha$. We may therefore define (G, ϕ) to be an n-edge-coloured multigraph with $s + 1$ vertices $\rho_0, \rho_1, \dots, \rho_s$ such that

(i) $d_{G\langle k \rangle}(\rho_\alpha, \rho_\beta) = N(k; \alpha, \beta)$ when $k \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, s\}$ and $\alpha \neq \beta$;
(ii) $d_{G\langle k \rangle}(\rho_0, \rho_\alpha) = t_{\alpha k}$ for $k = 1, \dots, n$ and $\alpha = 1, \dots, s$;
(iii) $d_{G\langle k \rangle}(\rho_\alpha, \rho_\alpha) = \frac{1}{2}(N(k; \alpha, \alpha) - t_{\alpha k})$ for $k = 1, \dots, n$ and $\alpha = 1, \dots, s$;
(iv) no loops of G are incident with ρ_0.

We will now prove that (G, ϕ) is n-helpful.

If $k \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, s\}$ then

$$d_{G\langle k\rangle}(\rho_\alpha) = 2d_{G\langle k\rangle}(\rho_\alpha, \rho_\alpha) + d_{G\langle k\rangle}(\rho_0, \rho_\alpha) + \sum_{\beta \in \{1,\dots,s\}\setminus\{\alpha\}} d_{G\langle k\rangle}(\rho_\alpha, \rho_\beta)$$

$$= \sum_{\beta=1}^{s} N(k; \alpha, \beta) = 2p_\alpha$$

by (i)–(iii) and (OS1). Since $\Phi = \Phi_1 \cup \dots \cup \Phi_s$, $\Psi = \Psi_1 \cup \dots \cup \Psi_s$ and $\Phi_1, \dots, \Phi_s$ are disjoint and each member of Ψ belongs to exactly two of $\Psi_1, \dots, \Psi_s$ it follows that $\sum_{\alpha=1}^{s} t_{\alpha k} = 2$ and consequently, by (ii) and (iv), $d_{G\langle k\rangle}(\rho_0) = 2$ for $k = 1, \dots, n$. Since the degree of any vertex in G is the sum of its degrees in $G\langle 1\rangle, \dots, G\langle n\rangle$, it follows that

$$d_G(\rho_\alpha) = 2p_\alpha n \ (\alpha = 1, \dots, s), \quad d_G(\rho_0) = 2n \tag{3}$$

and consequently $|E(G)| = (p_1 + \dots + p_s + 1)n = (2n+1)n$. These calculations show that (G, ϕ) satisfies (H0)–(H2). For $\alpha, \beta = 1, \dots, s$ it follows from (ii) that

$$d_G(\rho_0, \rho_\alpha) = \sum_{k=1}^{n} t_{\alpha k} = 2|\Phi_\alpha| + |\psi_\alpha| = 2r_\alpha + q_\alpha = p_\alpha$$

and from (i) and (OS2) that

$$d_G(\rho_\alpha, \rho_\beta) = \sum_{k=1}^{n} N(k; \alpha, \beta) = p_\alpha p_\beta \quad \text{if } \alpha \neq \beta.$$

From this and (3), it follows that (G, ϕ) satisfies (H3). If $k \in \{1, \dots, n\}$ then, by (i), every two *distinct* vertices in the set $\{\rho_1, \dots, \rho_s\}$ are joined by the same number of edges in $G\langle k\rangle$ as in $F(A^*, \sigma_k)$, which is connected by (OS4), and so $G\langle k\rangle - \rho_0$ is connected. From this and (iv) and the fact that $d_{G\langle k\rangle}(\rho_0) = 2$, it follows that $G\langle k\rangle$ is connected. Hence (G, ϕ) satisfies (H4). This completes the proof that (G, ϕ) is n-helpful and so has by Proposition 8.6 a $(2n+1)$-complete Hamiltonian detachment (D, ϕ).

By the definition of detachment, there exists a DG-amalgamator Ω. The definition of a DG-amalgamator implies that $d_G(\rho_\alpha) = \sum d_D(x)$: $x \in \Omega^{-1}(\{\rho_\alpha\})$, which is $2n|\Omega^{-1}(\{\rho_\alpha\})|$ since (D, ϕ) is $(2n+1)$-complete, and so (3) implies that $|\Omega^{-1}(\{\rho_0\})| = 1$ and $|\Omega^{-1}(\{\rho_\alpha\})| = p_\alpha$ $(\alpha = 1, \dots, s)$. We can clearly take any $2n+1$ objects to be the vertices of D: so we may suppose that $\Omega^{-1}(\{\rho_\alpha\}) = \bar{p}_{\alpha-1} \nearrow \bar{p}_\alpha$ for $\alpha = 1, \dots, s$, where $\bar{p}_0 = 0, \bar{p}_\alpha = p_1 + \dots + p_\alpha$ $(\alpha = 1, \dots, s)$ and $x \nearrow y$ denotes the subset $\{x+1, x+2, \dots, y\}$ of $\mathbb{Z}_{2n}$. Let v denote the unique element of $\Omega^{-1}(\{\rho_0\})$. Then $V(D - v) = \mathbb{Z}_{2n}$ and so $D - v$ can be identified with the graph K_{2n} considered in Section 3. Let $[x, y]$ denote the edge joining any two distinct vertices x, y in D, and let A be the symmetric $2n \times 2n$ matrix such that for $i, j = 1, \dots, 2n$ and $k = 1, \dots, n$,

$$A(i,j) = \sigma_k \quad \text{when } i \neq j \text{ and } [i,j] \in E(D\langle k\rangle),$$

$$A(i,i) = \sigma_k \quad \text{when } [v, i] \in E(D\langle k\rangle).$$

Since (D, ϕ) is Hamiltonian, each $D\langle k\rangle$ is a Hamiltonian cycle of D: therefore each symbol σ_k occurs exactly twice in each row of A and twice in each column of A, and so A is a double latin square. For any two distinct elements i,j of $\mathbb{Z}_{2n}$, the definition of A implies that $[i,j]$ is in $D\langle k\rangle$ and consequently in $D\langle k\rangle - v$ if and only if $A(i,j) = A(j,i) = \sigma_k$: therefore, in the notation of Section 3, $D\langle k\rangle - v = H(A, \sigma_k)$. Since each $D\langle k\rangle$ is a Hamiltonian cycle of D, it follows that $D\langle k\rangle - v = H(A, \sigma_k)$ is a Hamiltonian path of K_{2n} for $k = 1, \dots, n$ and so, by Theorem 3.3, A is an SHLS$(2n)$.

Consider any $\alpha, \beta \in \{1, \dots, s\}$ and any $k \in \{1, \dots, n\}$. Let $e_{\alpha\beta}\langle k\rangle$ denote the number of edges of $D\langle k\rangle$ which join elements of $\Omega^{-1}(\{\rho_\alpha\})$ to elements of $\Omega^{-1}(\{\rho_\beta\})$ and $e_{v\alpha}\langle k\rangle$ denote the number of edges of $D\langle k\rangle$ which join v to elements of $\Omega^{-1}(\{\rho_\alpha\})$. Let $A_{\alpha\beta}$ be the $p_\alpha \times p_\beta$ submatrix of A formed by the entries $A(i,j)$ with $\bar{p}_{\alpha-1} < i \leqslant \bar{p}_\alpha$, $\bar{p}_{\beta-1} < j \leqslant \bar{p}_\beta$. Since $\Omega^{-1}(\{\rho_\alpha\}) = \bar{p}_{\alpha-1} \nearrow \bar{p}_\alpha$ and $\Omega^{-1}(\{\rho_\beta\}) = \bar{p}_{\beta-1} \nearrow \bar{p}_\beta$, the definition of A implies that σ_k occurs exactly $e_{\alpha\beta}\langle k\rangle$ times in $A_{\alpha\beta}$ when $\alpha \neq \beta$ and exactly $2e_{\alpha\alpha}\langle k\rangle + e_{v\alpha}\langle k\rangle$ times in $A_{\alpha\alpha}$. Moreover $e_{\alpha\alpha}\langle k\rangle = d_{G\langle k\rangle}(\rho_\alpha, \rho_\alpha)$, $e_{\alpha\beta}\langle k\rangle = d_{G\langle k\rangle}(\rho_\alpha, \rho_\beta)$ and $e_{v\alpha}\langle k\rangle = d_{G\langle k\rangle}(\rho_0, \rho_\alpha)$ since Ω is a DG-amalgamator, and so (i)–(iii) give $e_{\alpha\beta}\langle k\rangle = N(k; \alpha, \beta)$ when $\alpha \neq \beta$ and $2e_{\alpha\alpha}\langle k\rangle + e_{v\alpha}\langle k\rangle = N(k; \alpha, \alpha)$. Hence σ_k occurs exactly $N(k; \alpha, \beta)$ times in $A_{\alpha\beta}$ for all $\alpha, \beta \in \{1, \dots, s\}$ and all $k \in \{1, \dots, n\}$, and so A^* is the (S, S)-amalgamation of A. $\square$

Condition (OS3) in the definition of a symmetric S-outline Hamiltonian double latin square can be replaced by the following condition (OS3*):

(OS3*) There is a multiset Σ of $2n$ ordered pairs $(\sigma_k, (\alpha, \alpha))$ such that if Σ contains $(\sigma_k, (\alpha, \alpha))$ x times then symbol σ_k occurs at least x times in cell (α, α) of A^*, and:

(A) each symbol σ_k occurs twice in ordered pairs of Σ,
(B) each diagonal cell (α, α) occurs p_α times in ordered pairs of Σ, and
(C) for $1 \leqslant \alpha \leq s$, if a symbol σ_k occurs an odd number of times in cell (α, α) of A^*, then $(\sigma_k, (\alpha, \alpha))$ occurs exactly once in Σ.

Proposition 8.7. *If $S = (p_1, p_2, \dots, p_n)$ is a composition of $2n$ and A^* is the (S, S)-amalgamation of an* SHLS$(2n)$ *on symbols $\sigma_1, \sigma_2, \dots, \sigma_n$, then A^* satisfies condition* (OS3*).

Proof. Let A be the SHLS$(2n)$ of which A^* is the (S, S)-amalgamation. Recall that in the definition of an (S, T)-amalgamation in Section 6, when $S = T = (p_1, \dots, p_s)$, the matrix A is partitioned into submatrices $A_{\alpha\beta}$ $(\alpha, \beta = 1, \dots, s)$. The multiset Σ corresponds to the set of $2n$ ordered pairs $(\sigma_k, (\alpha, \alpha))$ where σ_k occurs in a diagonal cell d of A and d occurs in the submatrix $A_{\alpha\alpha}$. (A) follows from Lemma 4.2, (B) is true since $A_{\alpha\alpha}$ is a $p_\alpha \times p_\alpha$ submatrix of A, and (C) follows from (A) and the symmetry of the submatrix A_{ii}. $\square$

Theorem 8.8. *Let $S = (p_1, p_2, \dots, p_n)$ be a composition of* $2n$ *and* A^* *be a symmetric* $s \times s$ *multiple entry matrix on symbols* $\sigma_1, \dots, \sigma_n$ *satisfying conditions* (OS1), (OS2) *and* (OS4). *Then* A^* *is a symmetric S-outline Hamiltonian double latin square if and only if* A^* *satisfies* (OS3*). [*Thus* A^* *satisfies* (OS3) *if and only if* A^* *satisfies* (OS3*).]

Proof. *Necessity*: If A^* satisfies (OS3) then, by Theorem 8.2, A^* is the (S, S)-amalgamation of an SHLS($2n$), so by Proposition 8.7, A^* satisfies (OS3*).

Sufficiency: Let X be a subset of $\{1, \dots, s\}$ and let Σ_X be the submultiset of Σ consisting of all ordered pairs of Σ, occurring with the same multiplicity as in Σ, of the form $(\sigma_k, (\alpha, \alpha))$ with $\alpha \in X$. If a symbol α_k is diagonally confined to X in A^*, then all occurrences of σ_k in Σ must actually occur in Σ_X and so, by (A), occur exactly twice in Σ_X. Therefore at least $2dec(X)$ elements in Σ_X (counting repetitions) contain symbols which are diagonally even in A^*. If a symbol σ_k appears oddly in a cell (α, α) of A^*, then by (C) the symbol $(\sigma_k, (\alpha, \alpha))$ occurs exactly once in $\sum$. Therefore at least $\sum_{\alpha \in X} q_\alpha(A^*)$ entries in Σ_X contain symbols which are diagonally odd in A^*. By (B), $\Sigma_X = \sum_{\alpha \in X} p_\alpha$. Therefore

$$2\text{dec}(X) + \sum_{\alpha \in X} q_\alpha(A^*) \leqslant \Sigma_X = \sum_{\alpha \in X} p_\alpha,$$

proving (OS3). □

9. Embedding

If A' is an $s' \times t'$ matrix and $s \in \{1, \dots, s'\}$, $t \in \{1, \dots, t'\}$ then $A'[s, t]$ will denote the $s \times t$ submatrix of A' in its top left-hand corner, i.e. obtained from A' by deleting its last $s' - s$ rows and its last $t' - t$ columns. We shall say that an $s \times t$ ordinary matrix A can be *extended* to A' if $A = A'[s, t]$. More generally, we shall say that an $s \times t$ unfilled matrix M can be *extended* to A' if M can be converted into $A'[s, t]$ by inserting a symbol into each unoccupied cell of M. We shall allow the possibility of extending an $s \times t$ ordinary matrix A to an $s' \times t'$ *multiple-entry* matrix A': this will be taken to mean that $A'[s, t] = A$ but any cell (i, j) of A' for which $i > s$ or $j > t$ may contain more than one symbol. Extending an $s \times t$ ordinary matrix (or more generally an $s \times t$ unfilled matrix) A to an $s' \times t'$ ordinary matrix A' can be viewed as an instance of the notion of "embedding" mentioned in Section 6: it amounts to embedding in A' an $s' \times t'$ matrix $\hat{A}$ such that $\hat{A}[s, t] = A$ and all cells of $\hat{A}$ outside $\hat{A}[s, t]$ are unoccupied.

Let R be an $s \times t$ unfilled matrix on the symbols $\sigma_1, \dots, \sigma_n$ and let $\mathscr{S}_m$ be the statement that each symbol occurs at most m times in each row of R and at most m times in each column of R. We shall call R (i) an *unfilled sublatin rectangle* (*on* $\sigma_1, \dots, \sigma_n$) if $\mathscr{S}_1$ is true, (ii) an *unfilled latin square* (*on* $\sigma_1, \dots, \sigma_n$) if $\mathscr{S}_1$ is true and $s = t = n$, (iii) an *unfilled subdouble latin rectangle* (*on* $\sigma_1, \dots, \sigma_n$) if $\mathscr{S}_2$ is true. In each case, the word "unfilled" may be omitted if R has no unoccupied cells, i.e. is an ordinary matrix. An unfilled subdouble latin rectangle R on $\sigma_1, \dots, \sigma_n$ is *acyclic* if for

$k = 1, \ldots, n$ there is no σ_k-cycle in R, the term "σ_k-cycle" being defined as in Section 1. The number of occurrences of a symbol σ in a matrix R will be denoted by $N_R(\sigma)$ and, if R is a square matrix, $D_R(\sigma)$ will denote the number of occurrences of σ on its main diagonal. If R is an unfilled matrix then $N_R(\rho_i)$, $N_R(c_j)$ will denote the number of occupied cells in its ith row and jth column, respectively, and U_R will denote the number of unoccupied cells in R. We shall say that two cells (i,j), (i',j') of a matrix are *contiguous* if either $(i = i'$ and $j \neq j')$ or $(i \neq i'$ and $j = j')$.

Ryser [20] has proved that an $s \times t$ sublatin rectangle on symbols $\sigma_1, \ldots, \sigma_n$ can be extended to a latin square of order n if and only if $N_R(\sigma_k) \geqslant s + t - n$ for $k = 1, \ldots, n$. Our next result is a similar theorem concerning Hamiltonian double latin squares. It is essentially equivalent to [17, Theorem 7], but we now present its statement and proof in the language of the present paper.

Theorem 9.1. *Suppose that* $s, t \in \{1, \ldots, 2n-1\}$ *and* R *is an* $s \times t$ *subdouble latin rectangle on symbols* $\sigma_1, \ldots, \sigma_n$. *Then* R *can be extended to an* HLS$(2n)$ *if and only if* R *is acyclic and for each* $k \in \{1, \ldots, n\}$ *either*

(a) $N_R(\sigma_k) > 2(s + t - 2n)$ *or*
(b) $N_R(\sigma_k) = 2(s + t - 2n)$ *and* $B(R, \sigma_k)$ *has at least one component of even order.*

Proof. Assume first that R can be extended to an HLS$(2n)$ L on the symbols $\sigma_1, \ldots, \sigma_n$. For $k \in \{1, \ldots, n\}$ there can be no σ_k-cycle in R since there is no σ_k-cycle of length less than $4n$ in L. Therefore R must be acyclic. Now let $k \in \{1, \ldots, n\}$ and let $N(i > s, j > t) = x$, where $N(\wp, \mathcal{Q})$ denotes the number of cells (i,j) such that $L(i,j) = \sigma_k$ and i,j satisfy conditions $\wp, \mathcal{Q}$. Then, since σ_k occurs twice in each row and twice in each column of L, we have $N(i \leqslant s, j > t) = 2(2n - t) - x$ and hence

$$N_R(\sigma_k) = N(i \leqslant s, j \leqslant t) = 2s - N(i \leqslant s, j > t) = 2(s + t - 2n) + x. \tag{4}$$

Let P be a shortest path in the cycle $B(L, \sigma_k)$ such that P connects some $\rho_u (u > s)$ to some $c_v (v > t)$. Then $|V(P)|$ is even since each edge of $B(L, \sigma_k)$ joins some ρ_i to some c_j. If $x = 0$ then ρ_u, c_v cannot be joined by an edge of $B(L, \sigma_k)$ and so $(P - \rho_u) - c_v$ is a component of $B(R, \sigma_k)$ of even order. From this and (4), it follows that (a) or (b) is true.

Now assume that R is acyclic and that for each $k \in \{1, \ldots, n\}$ either (a) or (b) is true. Let u_{ki} $(\leqslant 2)$, $v_{kj} (\leqslant 2)$ be the number of occurrences of σ_k in the ith row and jth column, respectively, of R. Extend R to an $(s+1) \times (t+1)$ multiple-entry matrix A^* on $\sigma_1, \ldots, \sigma_n$ by making σ_k occur $2 - u_{ki}$ times in the cell $(i, t+1)$ of A^* and $2 - v_{kj}$ times in the cell $(s+1, j)$ of A^* and $N_R(\sigma_k) - 2(s + t - 2n)$ times in the cell $(s+1, t+1)$ of A^* for $k = 1, \ldots, n$ and $i = 1, \ldots, s$ and $j = 1, \ldots, t$. Let S, T be the compositions $(1, 1, \ldots, 1, 2n - s)$ and $(1, 1, \ldots, 1, 2n - t)$ of $2n$, respectively. We will show that A^*, S, T satisfy (OH1)–(OH4).

It is clear that any symbol σ_k occurs exactly twice in each of the first s rows of A^* and, since $v_{k1} + v_{k2} + \cdots + v_{kt} = N_R(\sigma_k)$, the number of occurrences of σ_k in the

$(s+1)$th row of A^* is $2(2n-s)$. Therefore A^*, S satisfy (OH1), and for similar reasons A^*, T satisfy (OH2). Clearly the cell (i,j) of A^* contains exactly one symbol if $i \leqslant s$ and $j \leqslant t$. Since $u_{1i} + u_{2i} + \cdots + u_{ni}$ is the number t of symbols in the ith row of R, the cell $(i, t+1)$ of A^* contains exactly $2n-t$ symbols for $i = 1, \dots, s$. Similarly, the cell $(s+1, j)$ of A^* contains exactly $2n-s$ symbols for $j = 1, \dots, t$. Since $\sum_{k=1}^{n} N_R(\sigma_k)$ is the total number st of symbols in R, the cell $(s+1, t+1)$ of A^* contains exactly $st - n.2(s+t-2n) = (2n-s)(2n-t)$ symbols. Hence A^*, S, T satisfy (OH3). To prove (OH4), let $k \in \{1, \dots, n\}$ and let $B_R = B(R, \sigma_k)$, $B^* = B(A^*, \sigma_k)$, so that $B_R = (B^* - \rho_{s+1}) - c_{t+1}$. Since σ_k occurs exactly twice in each of the rows $1, \dots, s$ and columns $1, \dots, t$ of A^*, each vertex of B_R has degree 2 in B^*. Moreover B_R contains no cycle since R is acyclic. Therefore each component of B_R is a path, each of whose endvertices is adjacent in B^* to ρ_{s+1} or c_{t+1}. In case (b), one of these components is a path P of even order, which must be contained in a path from ρ_{s+1} to c_{t+1} in B^* because each endvertex of P is adjacent to ρ_{s+1} or c_{t+1} and every edge of B^* joins some ρ_i to some c_j. In case (a), our definition of A^* implies that σ_k occurs at least once in its cell $(s+1, t+1)$ and so ρ_{s+1}, c_{t+1} are adjacent in B^*. In both cases, we infer that ρ_{s+1}, c_{t+1} are in the same component of B^*. Since we have seen that each component of $B_R = B^* - \rho_{s+1} - c_{t+1}$ has a vertex adjacent in B^* to ρ_{s+1} or c_{t+1}, it follows that B^* is connected. This proves (OH4).

We have thus shown that A^* is an (S, T)-outline Hamiltonian double latin square. Therefore A^* is by Theorem 7.2 the (S, T)-amalgamation of an HLS$(2n)$ and so R can be extended to an HLS$(2n)$. $\square$

Corollary 9.2. *For $s \in \{1, \dots, 2n-1\}$, an $s \times 2n$ subdouble latin rectangle on n symbols can be extended to an* HLS$(2n)$ *if and only if it is acyclic.*

Proof. Let $s \in \{1, \dots, 2n-1\}$ and R be an $s \times 2n$ subdouble latin rectangle on symbols $\sigma_1, \dots, \sigma_n$. Then each σ_k occurs exactly twice in each row of R. If R can be extended to an HLS$(2n)$ L then it is acyclic since for each $k \in \{1, \dots, n\}$ there is no σ_k-cycle of length less than $4n$ in L and consequently no σ_k-cycle in R.

Now assume that R is acyclic. Let $Q = R[s, 2n-1]$ and let $k \in \{1, \dots, n\}$. Since R is acyclic and σ_k occurs exactly twice in each of its rows and at most twice in each of its columns, $B(R, \sigma_k)$ contains no cycle and has no vertex of degree greater than 2 and $\rho_1, \dots, \rho_s$ all have degree 2 in $B(R, \sigma_k)$. Therefore the component of $B(R, \sigma_k)$ containing c_{2n} is a path P from some c_u to some c_v. Each component of $P - c_{2n}$ has even order since each edge of $B(R, \sigma_k)$ joins some ρ_i to some c_j; and any component of $P - c_{2n}$ is a component of $B(R, \sigma_k) - c_{2n} = B(Q, \sigma_k)$. Therefore $B(Q, \sigma_k)$ has a component of even order provided that $V(P) \neq \{c_{2n}\}$, i.e. provided that c_{2n} has non-zero degree in $B(R, \sigma_k)$, i.e. provided that σ_k occurs at least once in the $2n$th column of R. Moreover since σ_k occurs twice in each row of R and at most twice in its $2n$th column, $N_Q(\sigma_k) \geqslant 2s - 2 = 2(s + (2n-1) - 2n)$ and this inequality is strict unless σ_k occurs twice in the last column of R, in which case we have seen that $B(Q, \sigma_k)$ has a component of even order. Therefore $B(Q, \sigma_k)$ can by Theorem 9.1 be extended to an HLS$(2n)$ L. Since each σ_k occurs exactly twice in each row of R and in each row of L and since $L[s, 2n-1] = Q = R[s, 2n-1]$, it follows that $L[s, 2n] = R$ and so R can be extended to an HLS$(2n)$. $\square$

We can alternatively prove Corollary 9.2 by a simplified version of the proof of Theorem 9.1 in which R is extended to an $(s+1) \times 2n$ multiple-entry matrix, which is proved using Theorem 7.2 to be the (S, T)-amalgamation of an HLS$(2n)$ where $S = (1, 1, \dots, 1, 2n-s)$, $T = (1, 1, \dots, 1)$.

Corollary 9.3. *Suppose that* $1 \leqslant s \leqslant 2n-1$ *and* R *is an* $s \times (2n-s)$ *unfilled acyclic subdouble latin rectangle on symbols* $\sigma_1, \dots, \sigma_n$. *Then* R *can be extended to an* HLS$(2n)$ *if and only if* $N_R(\sigma_k) = 0$ *for at most* U_R *symbols* σ_k.

Proof. Assume first that R can be extended to an HLS$(2n)$ L on $\sigma_1, \dots, \sigma_n$. Since the matrix $M = L[s, 2n-s]$ can be extended to L, each σ_k must satisfy condition (a) or (b) of Theorem 9.1 with R, t replaced by $M, 2n-s$. This implies that $N_M(\sigma_k) > 0$ for $k = 1, \dots, n$, since $B(M, \sigma_k)$ would have no edges if $N_M(\sigma_k)$ were 0 and thus all components of $B(M, \sigma_r)$ would be single vertices, and thus have odd order. Therefore $N_R(\sigma_k)$ can only be 0 for symbols σ_k in the U_R cells of M which are unoccupied in R.

Now assume that $|\Omega| \leqslant U_R$, where Ω is the set of symbols σ_k with $N_R(\sigma_k) = 0$. Inserting each element of Ω into a different one of the U_R unoccupied cells of R will convert R into an $s \times (2n-s)$ unfilled acyclic subdouble latin rectangle S with $U_R - |\Omega|$ unoccupied cells and $N_S(\sigma_k) > 0$ for $k = 1, \dots, n$. Then transform S into an $s \times (2n-s)$ subdouble latin rectangle T on $\sigma_1, \dots, \sigma_n$ by filling its $U_R - |\Omega|$ unoccupied cells one by one and, when filling any cell (i, j), using a symbol which is already present in at most one of the $2n-2$ cells contiguous to (i, j). This rule ensures that inserting a symbol σ_k into a cell never completes a σ_k-cycle, and so T is acyclic. Since $N_T(\sigma_k) \geqslant N_S(\sigma_k) > 0$ for $k = 1, \dots, n$, we can by Theorem 9.1 extend T, and hence also R, to an HLS$(2n)$. $\square$

Corollary 9.4. *If* s, t *are positive integers and* $s + t < 2n$ *then every* $s \times t$ *unfilled acyclic subdouble latin rectangle on* n *symbols can be extended to an* HLS$(2n)$.

Proof. Let R be an $s \times t$ unfilled acyclic subdouble latin rectangle on symbols $\sigma_1, \dots, \sigma_n$. Let S be the $s \times (2n-s)$ unfilled acyclic subdouble latin rectangle on $\sigma_1, \dots, \sigma_n$ such that $S[s, t] = R$ and all cells in the last $2n-s-t$ columns of S are unoccupied. If the first row of R contains m *distinct* symbols and u unoccupied cells then $2m + u \geqslant t$ since R is subdouble. Since $N_S(\sigma_k) = 0$ for at most $n-m$ symbols σ_k and $U_S \geqslant u + s(2n-s-t) \geqslant \frac{1}{2}u + \frac{1}{2}s + \frac{1}{2}(2n-s-t) = n + \frac{1}{2}(u-t) \geqslant n-m$, it follows from Corollary 9.3 that S can be extended to an HLS$(2n)$ and therefore so can R. $\square$

We note also the following consequence of Corollary 9.3:

Corollary 9.5. *Any $n \times n$ unfilled latin square on n symbols can be extended to an* HLS($2n$).

An $n \times n$ unfilled latin square on symbols $\sigma_1, \ldots, \sigma_n$ can be extended to a $2n \times 2n$ latin square on symbols $\sigma_1, \ldots, \sigma_{2n}$. This was observed by Evans [12], and can be proved by an argument somewhat like the latter part of the proof of Corollary 9.3, using Ryser's theorem on extending sublatin rectangles in place of Theorem 9.1. One wonders whether this observation and Corollary 9.5 can be subsumed in a single statement in the following way.

Problem 9.6. Can every $n \times n$ unfilled latin square on symbols $\sigma_1, \ldots, \sigma_n$ be extended to a $2n \times 2n$ latin square on symbols $\sigma_1, \ldots, \sigma_{2n}$ which becomes an HLS($2n$) when σ_{n+k} is replaced by σ_k for each $k \in \{1, \ldots, n\}$?

The following statement is contained in [16, Theorem 11]:

Proposition 9.7. *Suppose that $s, t \in \{1, \ldots, 2n\}$ and $u \in \{1, \ldots, n\}$ and R is an $s \times t$ unfilled subdouble latin rectangle on symbols $\sigma_1, \ldots, \sigma_u$. Then R can be extended to a $2n \times 2n$ double latin square on symbols $\sigma_1, \ldots, \sigma_n$ without inserting any of $\sigma_1, \ldots, \sigma_u$ into unoccupied cells of R if and only if*

(i) $U_R \leqslant st + 2su + 2tu - 2n(s + t + 2u - 2n)$;
(ii) $N_R(\sigma_k) \geqslant 2(s + t - 2n)$ *for* $k = 1, \ldots, u$;
(iii) $N_R(\rho_i) \geqslant 2u + t - 2n$ *for* $i = 1, \ldots, s$;
(iv) $N_R(c_j) \geqslant 2u + s - 2n$ *for* $j = 1, \ldots, t$.

If $s, t \in \{1, \ldots, 2n\}$ and $s + t < 4n$ and $u \in \{1, \ldots, n\}$ and R is an $s \times t$ unfilled subdouble latin rectangle on $\sigma_1, \ldots, \sigma_u$ then (i)–(iv) are by Proposition 9.7 *necessary* conditions for R to be extendible to an HLS($2n$) without inserting any of $\sigma_1, \ldots, \sigma_u$ into unoccupied cells of R. Since we are now assuming that $s + t < 4n$, a further necessary condition is that R be acyclic. In view of Theorem 9.1, these necessary conditions seem unlikely to be sufficient when some of the inequalities in (i)–(iv) are actually equalities, but we propose the following conjecture.

Conjecture 9.8. *Suppose that $s, t \in \{1, \ldots, 2n\}$ and $s + t < 4n$ and $u \in \{1, \ldots, n\}$ and R is an $s \times t$ unfilled acyclic subdouble latin rectangle on symbols $\sigma_1, \ldots, \sigma_u$. If*

$$U_R < st + 2su + 2tu - 2n(s + t + 2u - 2n),$$

$$N_R(\sigma_k) > 2(s + t - 2n) \quad \textit{for } k = 1, \ldots, u,$$

$$N_R(\rho_i) > 2u + t - 2n \quad \textit{for } i = 1, \ldots, s$$

and

$$N_R(c_j) > 2u + s - 2n \quad \textit{for } j = 1, \ldots, t$$

then R can be extended to an HLS($2n$) *without inserting any of $\sigma_1, \dots, \sigma_u$ into unoccupied cells of R.*

We now consider embedding problems like the foregoing, with the additional condition of symmetry imposed. We begin with the following counterpart of Theorem 9.1:

Theorem 9.9. *Suppose that $s < 2n$ and R is an $s \times s$ symmetric subdouble latin rectangle on symbols $\sigma_1, \dots, \sigma_n$. Then R can be extended to an* SHLS($2n$) *if and only if R is acyclic and for $k = 1, \dots, n$ we have*

(a) $D_R(\sigma_k) \leqslant 2$,
(b) $N_R(\sigma_k) + D_R(\sigma_k) > 4(s-n)$.

Proof. If R can be extended to an SHLS($2n$) L on $\sigma_1, \dots, \sigma_n$ then R is acyclic by Theorem 9.1 and $D_R(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$ by Lemma 4.2. Moreover, if σ_k occurs in exactly x cells (i,j) of L with $i > s, j > s$ then the argument leading to (4) (with $s = t$) gives $N_R(\sigma_k) = 4(s-n) + x$, which implies (b) since $x \geqslant 2 - D_R(\sigma_k)$ by Lemma 4.2.

Now assume that R is acyclic and (a) and (b) hold for $k = 1, \dots, n$. Since R is symmetric, $N_R(\sigma_k) - D_R(\sigma_k)$ is even and so (a) and (b) imply that $N_R(\sigma_k) \geqslant 4(s-n)$ for $k = 1, \dots, n$. Let $u_{ki} (\leqslant 2)$ be the number of occurrences of σ_k in the ith row of R. Extend R to a symmetric $(s+1) \times (s+1)$ multiple-entry matrix A^* on $\sigma_1, \dots, \sigma_n$ by making σ_k occur $2 - u_{ki}$ times in each of the cells $(i, s+1), (s+1, i)$ of A^* and $N_R(\sigma_k) - 4(s-n)$ times in the cell $(s+1, s+1)$ of A^* for $k = 1, \dots, n$ and $i = 1, \dots, s$. Let $p_1 = \cdots = p_s = 1$, $p_{s+1} = 2n - s$ and S be the composition $(p_1, \dots, p_{s+1}) = (1, 1, \dots, 1, 2n-s)$ of $2n$. We must verify that A^* and S satisfy (OS1)–(OS4).

The verification of (OS1) and (OS2) resembles the verification of (OH1)–(OH3) in the proof of Theorem 9.1, and may be left to the reader. To verify (OS3), define $\mathrm{dec}(X)$ as in (OS3) for every set $X \subseteq \{1, \dots, s+1\}$. Let Φ be the set of symbols σ_k which diagonally even in A^* and Ω_m be the set of symbols σ_k for which $D_R(\sigma_k) = m$ and let $w_m = |\Omega_m|$. Then $\{\sigma_1, \dots, \sigma_n\} = \Omega_0 \cup \Omega_1 \cup \Omega_2$ by (a) and so $w_0 + w_1 + w_2 = n$, $w_1 + 2w_2 = s$ and consequently $w_0 = n - \frac{1}{2}s - \frac{1}{2}w_1 = \frac{1}{2}(p_{s+1} - w_1)$. If σ_k appears oddly in the cell $(s+1, s+1)$ of A^* then $N_R(\sigma_k) - 4(s-n)$ is odd by the definition of A^* and so $N_R(\sigma_k)$ is odd and therefore, since $N_R(\sigma_k) - D_R(\sigma_k)$ is even, $D_R(\sigma_k)$ is odd and so $\sigma_k \in \Omega_1$ by (a). Therefore $q_{s+1}(A^*) \leqslant w_1$. If $\sigma_k \in \Phi$ then σ_k cannot occur exactly once in any diagonal cell of A^* and so cannot occur in any diagonal cell of $R = A^*[s,s]$ and consequently $D_R(\sigma_k) = 0$. Therefore $\Phi \subseteq \Omega_0$ and for each $\sigma_k \in \Phi$ we have $N_R(\sigma_k) > 4(s-n)$ by (b) and so σ_k appears in the cell $(s+1, s+1)$ of A^*. Therefore $\mathrm{dec}(X) = 0$ for every set $X \subseteq \{1, \dots, s\}$ and $\mathrm{dec}(X) \leqslant |\Phi| \leqslant w_0 = \frac{1}{2}(p_{s+1} - w_1) \leqslant \frac{1}{2}(p_{s+1} - q_{s+1}(A^*))$ for every set $X \subseteq \{1, \dots, s+1\}$. Moreover, since each diagonal cell of $A^*[s,s]$ contains exactly one symbol, $q_\alpha(A^*) = 1 = p_\alpha$ for $\alpha = 1, \dots, s$. Hence $\mathrm{dec}(X) \leqslant \frac{1}{2}\sum_{\alpha \in X} (p_\alpha - q_\alpha(A^*))$ for every set $X \subseteq \{1, \dots, s+1\}$ and (OS3) is verified. To verify (OS4), suppose that $F(A^*, \sigma_k)$ is disconnected. Then it has a component C which does not include ρ_{s+1}. Since σ_k occurs exactly twice in each of the first s rows of A^*, each of $\rho_1, \dots, \rho_s$ is incident with exactly two edges (one of which may be a loop) in $F(A^*, \sigma_k)$. Therefore C is either a cycle or a path augmented by adding two loops, one incident with each of its endvertices. In each of these cases, it is easily seen that $A^*[s, s] = R$ contains a σ_k-cycle, contradicting the hypothesis that it is acyclic.

We conclude that A^* is a symmetric S-outline Hamiltonian double latin square. Therefore A^* is by Theorem 8.2 the (S, S)-amalgamation of an SHLS$(2n)$ and so R can be extended to an SHLS$(2n)$. $\square$

Corollary 9.10. *Suppose that R is an $n \times n$ symmetric unfilled acyclic subdouble latin rectangle on symbols $\sigma_1, \dots, \sigma_n$ with d unoccupied diagonal cells and $2e$ unoccupied off-diagonal cells. Then R can be extended to an* SHLS$(2n)$ *if and only if $D_R(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$ and $N_R(\sigma_k) = 0$ for at most d+e symbols σ_k.*

Proof. Assume first that R can be extended to an SHLS$(2n)$ L on $\sigma_1, \dots, \sigma_n$. Moreover $D_R(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$ by Lemma 4.2. Since the matrix $M = L[n, n\}$ can be extended to L, each σ_k must satisfy condition (b) of Theorem 9.9 with R, s replaced by M, n. Since $D_M(\sigma_k) \leqslant N_M(\sigma_k)$, this implies that $N_M(\sigma_k) > 0$ for $k = 1, \dots, n$. Therefore $N_R(\sigma_k)$ can only be 0 for symbols σ_k in the $d + 2e$ cells of $M = L[n, n]$ which are unoccupied in R, and there are at most $d + e$ such symbols since L is symmetric.

Now assume that $D_R(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$ and $|\Omega| \leqslant d + e$, where Ω is the set of symbols σ_k with $N_R(\sigma_k) = 0$. Convert R into an $n \times n$ symmetric unfilled acyclic subdouble latin rectangle S with $N_S(\sigma_k) > 0$ for $k = 1, \dots, n$ by inserting each $\sigma_k \in \Omega$ into either one unoccupied diagonal cell of R or two unoccupied off-diagonal cells (i_k, j_k), (j_k, i_k) of R. Then transform S into an $n \times n$ symmetric subdouble latin rectangle T on $\sigma_1, \dots, \sigma_n$ by a succession of operations each of which either (i) inserts into an unoccupied diagonal cell (i, i) a symbol which is already present in at most one of the $2n - 2$ cells (i, j) $(j \neq i)$, (j, i) $(j \neq i)$ or (ii) inserts into each of two unoccupied off-diagonal cells $(i, j), (j, i)$ a symbol which is already present in at most one of the $2n - 2$ cells contiguous to (i, j). If $k \in \{1, \dots, n\}$ then, since T is a symmetric subdouble latin rectangle, it is easily seen that any σ_k-cycle in T which included a diagonal cell (i, i) would have to include both another diagonal cell and a cell (i, j) $(j \neq i)$. Moreover any σ_k-cycle in T which included a cell (i, j) would have to include two cells contiguous to (i, j). Consequently, neither of procedures (i), (ii) can complete a σ_k-cycle and so T is acyclic. Since $D_R(\sigma_k) \leqslant 2$ and consequently $D_S(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$, procedure (i) ensures that $D_T(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$. Moreover $N_T(\sigma_k) \geqslant N_S(\sigma_k) > 0$ for $k = 1, \dots, n$. Therefore, by Theorem 9.9, we can extend T, and hence also R, to an SHLS$(2n)$. $\square$

Corollary 9.11. *For $s<n$, an $s \times s$ symmetric unfilled acyclic subdouble latin rectangle R on symbols $\sigma_1, \dots, \sigma_n$ can be extended to an* SHLS($2n$) *if and only if $D_R(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$.*

Proof. If R can be extended to an SHLS($2n$) then $D_R(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$ by Lemma 4.2. Now assume that $D_R(\sigma_k) \leqslant 2$ for $k = 1, \dots, n$. Let S be the $n \times n$ symmetric unfilled acyclic subdouble latin rectangle on $\sigma_1, \dots, \sigma_n$ such that $S[s, s] = R$ and all cells of S outside $S[s, s]$ are unoccupied. This S has at least one unoccupied diagonal cell and at least $2n - 2$ unoccupied off-diagonal cells. Consequently, by Corollary 9.10, S can be extended to an SHLS($2n$) and therefore so can R. $\square$

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