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Computing the extremal index of special Markov chains and queues

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Abstract

We consider extremal properties of Markov chains. Rootzén (1988) gives conditions for stationary, regenerative sequences so that the normalized process of level exceedances converges in distribution to a compound Poisson process. He also provides expressions for the extremal index and the compounding probabilities; in general it is not easy to evaluate these.

We show how in a number of instances Markov chains can be coupled with two random walks which, in terms of extremal behaviour, bound the chain from above and below. Using a limiting argument it is shown that the lower bound converges to the upper one, yielding the extremal index and the compounding probabilities of the Markov chain. An FFT algorithm by Grübel (1991) for the stationary distribution of a G/G/1 queue is adapted for the extremal index; it yields approximate, but very accurate results. Compounding probabilities are calculated explicitly in a similar fashion.

The technique is applied to the G/G/1 queue, G/M/c queues and ARCH processes, whose extremal behaviour de Haan et al. (1989) characterized using simulation.

Keywords: Extremal index; Clustering of extreme values; Harris chains

1. Introduction

Let $\{X_k\}$ be a stationary process with state space $\mathscr{S} \subseteq \mathbb{R}$ (the real numbers) and stationary (marginal) distribution function F. It is well-known (cf. Leadbetter, 1983; Leadbetter et al., 1983; O'Brien, 1987) that for a broad class of stationary sequences a number $\theta_1 \in [0, 1]$ exists such that the maximum $M_n = \max_{1 \leqslant k \leqslant n} X_k$ satisfies

$$[\boldsymbol{P}(M_n \leqslant u_n) - F^{n\theta_1}(u_n)] \to 0$$

for sequences $u_n = u_n(\tau)$ with $n(1 - F(u_n)) \to \tau > 0$, as $n \to \infty$. The parameter θ_1 is called the *extremal index* of $\{X_k\}$.

For stationary, regenerative sequences satisfying some additional requirements, Rootzén (1988) shows that the normalized process of level exceedances converges in

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distribution to a compound Poisson process on $\mathbb{R}^+$, and he provides expressions for the extremal index θ_1 and the compounding probabilities.

We will consider Harris recurrent Markov chains which are *asymptotically homogeneous*, that is, they satisfy the additional property that the conditional increment $X_{k+1} - X_k$, given $X_k = x$, converges in distribution to a random variable ξ with $E\xi < 0$, as $x \to \infty$. Perhaps the first one to note that the extremal properties for this class of Markov chains can be described in terms of level exceedances of the random walk $S_k = \xi_1 + \cdots + \xi_k$ was Aldous (1989) in his book on clumping heuristics. This feature was noticed by Rootzén for the G/G/1 queue (cf. Rootzén, 1988, Section 6), by de Haan et al. (1989) for ARCH processes, and in a more general context by Smith (1992) and Perfekt (1994).

In this article we present an algorithm, based on Grübel's (1991) algorithm for the stationary waiting time of stable G/G/1 queues. Apparently, this queueing result was already known to Ackroyd (1980), though Grübel, unaware of this, provided a firm theoretical basis. There is some connection with a paper by Woodroofe (1979), who derived an integral formula for the Laplace transform of the ladder height distribution of random walks; he used the transform to calculate probabilities and expectations by (sometimes double) numerical integration. Woodroofe and Grübel's paper have in common that they use the Spitzer–Baxter identities.

The algorithm allows one to compute the extremal index for a class of positive recurrent Harris chains where the marginal distribution F belongs to the domain of attraction of the Gumbel distribution, with some additional requirements on the tail of the distribution of the conditional increment $X_{k+1} - X_k$, given $X_k = x$, as $x \to \infty$. The compounding probabilities of the Poisson process of level exceedances can be computed as well. Numerical results are very accurate when the tails of the distribution of the limiting step size ξ are sufficiently flat. The algorithm is given in Section 3.

In order to determine whether for specific Harris recurrent chains this algorithm can be applied, we provide sufficient conditions that are easy to verify, in terms of the increment of the chain (Section 2, Theorem 1). These conditions imply the often used condition

$$\lim_{p \to \infty} \limsup_{n \to \infty} \boldsymbol{P}\left(\max_{p \leqslant k \leqslant p_n} X_k > u_n \,|\, X_0 > u_n \right) = 0$$

(for suitable sequences u_n and p_n, see Perfekt (1994) and Smith (1992)).

Finally, in Section 4, we apply the method to the following cases: (i) G/G/1 queues, (ii) G/M/c queues, and (iii) ARCH processes.

2. Theory

Let $\{X_k\}$ be a stationary Markov chain with state space $\mathscr{S} \subseteq \mathbb{R}$ and stationary distribution F. We consider the case where the right endpoint of F is $+\infty$. The chain can be represented recursively by

$$X_{k+1} = X_k + g(X_k, Y_{k+1}) \tag{1}$$

for some Borel measurable function g and an i.i.d. sequence $\{Y_k\}$, with Y_{k+1} independent of X_k. For $a \in \mathscr{S}$ define the random walk $\{S_k^a\}$ by $S_0^a = 0$, $S_k^a = \xi_1^a + \cdots + \xi_k^a$, for $k \geqslant 1$, where $\{\xi_i^a\}$ are independent and distributed as $g(a, Y_1)$.

Suppose there exists a sequence $\{a_m\}$ in $\mathscr{S}$ with $a_m \to \infty$, such that

$$\xi^{a_m} \xrightarrow{\mathrm{d}} \xi \quad \text{as } m \to \infty \text{ and } E\xi < 0 \tag{2}$$

and

$$\text{for all } m \text{ and } x \in \mathscr{S} \text{ with } x > a_m\text{: } \xi \leqslant_{\mathrm{st}} g(x, Y) \leqslant_{\mathrm{st}} \xi^{a_m}. \tag{3}$$

A simpler condition, sufficient for the sequel, is: ξ^u is stochastically non-increasing in u, i.e. $\boldsymbol{P}(\xi^u > x)$ is non-increasing in u for every x, and the distributional limit ξ of this sequence, which perforce exists – perhaps defective with mass at $-\infty$ –, has expectation $E\xi < 0$. The stated conditions (2) and (3) are slightly weaker: $g(x, Y) \leqslant_{\mathrm{st}} \xi^{a_m}$ for $x > a_m$ implies that ξ^{a_m} is stochastically non-increasing and has a (possibly defective) limit ξ; the left-hand inequality in (3) then implies $\xi^u \xrightarrow{\mathrm{d}} \xi$ as $u \to \infty$.

Let $\{S_k\}$ denote the random walk with step ξ and $S_0 = 0$.

Remark. A process structure as described here is often found in queueing theory. For instance, let X_k be the waiting time of the kth customer in a G/G/1 queue; then we have

$$g(x, Y) = \max(-x, Y), \quad x \geqslant 0,$$

where $Y = B - A$, the difference between a service time B and an interarrival time A. It is easily seen that conditions (2) and (3) are satisfied, and $\xi = Y$. Other queueing applications are given in Section 3.

An entirely different example is the autoregressive conditional heteroscedastic (ARCH) process (see Engle, 1982), used by economists to model financial data. It is defined by

$$Z_{k+1} = U_{k+1}(\beta + \lambda Z_k^2)^{1/2}. \tag{4}$$

where $U_1, U_2, \ldots,$ are i.i.d. standard normal random variables, $\beta > 0$ and $0 < \lambda < 1$. The transformation $X_k = \log(Z_k^2)$ simplifies the analysis (cf. de Haan et al., 1989) and yields the recursion

$$X_{k+1} = X_k + \log U_{k+1}^2(\lambda + \beta \mathrm{e}^{-X_k}),$$

which is of the type described, with $g(x, Y) = Y + \log(1 + (\beta/\lambda)\mathrm{e}^{-x})$, and $Y = \log \lambda U^2$.

Below we prove an easy variant of Corollary 4.2 of Rootzén (1988). In his corollary, Rootzén shows for regenerative processes satisfying some additional requirements on the mean cycle lengths, that the extremal index can be calculated as the limit of the conditional probability that the process does not exceed level u_n during the first delayed regeneration cycle, given that the process starts above level u_n. We show that in the present case it suffices to consider the process until it hits the regeneration set $(-\infty, a] \cap \mathscr{S}$ for the first time.

A set $R \subseteq \mathscr{S}$ is called a regeneration set if R is recurrent (for all $x \in \mathscr{S}$ we have that $\boldsymbol{P}_x(\tau_R < \infty) = 1$, where τ_R is the hitting time of R), and if there exists a number $p \in (0, 1]$, a positive integer r and a probability measure μ with the property that

$$\inf_{x \in R} \boldsymbol{P}(X_r \in A \mid X_0 = x) \geqslant p\mu(A),$$

for all measurable subsets A of $\mathscr{S}$.

It is well-known that if $\{X_k\}$ has a regeneration set then one can construct a renewal process $T_1, T_2, \ldots$ which makes $\{X_k\}$ regenerative if $r = 1$ and 1-dependent regenerative if $r > 1$. We will denote the cycle lengths of this renewal process by $C_0, C_1, \ldots$, where C_0 denotes the length of the first regeneration cycle and $C_k = T_{k+1} - T_k$, $k \geqslant 1$, are the i.i.d. times between subsequent regenerations (see Asmussen, 1987, Section VI.3 for background on regenerative processes).

Let

$$N_a = \inf\{k \geqslant 0: X_k \leqslant a\}. \tag{5}$$

Note that for Lemma 1 not all of assumption (3) is needed; it is sufficient that above a certain level the Markov chain is dominated by a random walk with negative drift.

Lemma 1. *Suppose $(-\infty, a] \cap \mathscr{S}$ is regenerative for $\{X_k\}$, with aperiodic cycle length, and suppose that for all $x \in \mathscr{S}$ with $x > a$,*

$$g(x, Y) \leqslant_{\mathrm{st}} g(a, Y) \stackrel{\mathrm{d}}{=} \xi^a,$$

where $E\xi^a < 0$. If the stationary distribution F satisfies $EX_0^+ = \int_{(0,\infty)} (1 - F(x))\,\mathrm{d}x < \infty$, and the chain satisfies

$$\sup_{x \in (-\infty, a] \cap \mathscr{S}} E[x + g(x, Y)]^+ < \infty, \tag{6}$$

then $\{X_k\}$ has extremal index $\theta_1 \in [0, 1]$ if and only if

$$\theta_1 = \lim_{n \to \infty} \boldsymbol{P}\left(\sup_{1 \leqslant k \leqslant N_a} X_k \leqslant u_n \mid X_0 > u_n \right), \tag{7}$$

for some sequence $u_n = u_n(\tau)$ with $n(1 - F(u_n)) \to \tau$, for some $\tau > 0$.

Proof. We first show that $EN_a < \infty$ and $EC_0 < \infty$. This follows from Gut (1988, Theorem 8.1), since for $k < N_a$ the Markov chain is dominated by the random walk $X_0 + S_k^a$, which has negative drift and so

$$EN_a \leqslant cE(X_0 - a)^+ < \infty,$$

where the constant $c > 0$ depends only on the value of a and the distribution of ξ^a. Let $\tau_1 = N_a$, and define

$$\tau_{l+1} = \inf\{k > \tau_l + r: X_k \leqslant a\}, \quad l = 1, 2, \ldots$$

Since the Markov chain $\{X_k\}$ is dominated by a random walk with negative drift, it follows from (6) that $\tau_2, \tau_3, \ldots$ are well defined and have finite expectation. In fact, $E\tau_l \leqslant lc'$, for some constant c'. Because $(-\infty, a] \cap \mathscr{S}$ is regenerative, each visit to this set leads to a regeneration r time units later with probability p. Now let N be the number of the first visit that leads to regeneration. Then $C_0 \leqslant \tau_{N+1}$, $P(N = i) = p(1-p)^{i-1}$, $i = 1, 2, \ldots$ and $EC_0 \leqslant c'E(N+1) < \infty$.

By Corollary 4.2 of Rootzén (1988) it is therefore sufficient to prove that

$$\lim_{u\to\infty} \boldsymbol{P}\left(\sup_{N_a \leqslant k \leqslant C_0} X_k > u \mid X_0 > u\right) = 0. \tag{8}$$

For $i = 1, 2, \ldots$ define the blocks $B_i = \{k: \ \tau_i < k \leqslant \tau_{i+1}\}$, and let $F_i = \{\sup_{k\in B_i} X_k > u\}$. Then

$$\begin{aligned}\boldsymbol{P}\left(\sup_{N_a \leqslant k \leqslant C_0} X_k > u \mid X_0 > u\right) &\leqslant \boldsymbol{P}\left(\bigcup_{i=1}^{N} F_i \mid X_0 > u\right)\\ &= \boldsymbol{P}\left(N > n_0, \bigcup_{i=1}^{N} F_i \mid X_0 > u\right) + \boldsymbol{P}\left(N \leqslant n_0, \bigcup_{i=1}^{N} F_i \mid X_0 > u\right)\\ &\leqslant \boldsymbol{P}(N > n_0) + \sum_{i=1}^{n_0} \boldsymbol{P}(F_i \mid X_0 > u).\end{aligned} \tag{9}$$

Now observe that by the strong Markov property for each $i \geqslant 1$,

$$\boldsymbol{P}(F_i \mid X_0 > u) \leqslant \sup_{x\in(-\infty,a]\cap\mathscr{S}} \boldsymbol{P}\left(\sup_{k\leqslant\tau} X_k > u \mid X_0 = x\right),$$

where $\tau = \inf\{k > r: X_k \leqslant a\}$. Define $u_i = a + i(u-a)/(r+2)$, $i = 0, 1, \ldots, r+2$, and

$$\alpha_i = \sup_{x\leqslant u_i} \boldsymbol{P}\left(\sup_{i<k\leqslant\tau} X_k > u \mid X_i = x\right), \tag{10}$$

$$\beta_i = \sup_{x\leqslant u_i} \boldsymbol{P}(X_{i+1} > u_{i+1} \mid X_i = x). \tag{11}$$

By intersecting with the event $\{X_{i+1} > u_{i+1}\}$ and its complement, it is easy to show that $\alpha_i \leqslant \alpha_{i+1} + \beta_i$, for $i = 0, 1, \ldots, r-1$. For $x \leqslant u_r$ we have, as $X_{r+1} < a$ implies that $\tau = r+1$:

$$\begin{aligned}\boldsymbol{P}\left(\sup_{r<k\leqslant\tau} X_k > u \mid X_r = x\right) &= \boldsymbol{P}\left(\sup_{r<k\leqslant\tau} X_k > u, X_{r+1} > u_{r+1} \mid X_r = x\right)\\ &\quad + \boldsymbol{P}\left(\sup_{r<k\leqslant\tau} X_k > u, a \leqslant X_{r+1} \leqslant u_{r+1} \mid X_r = x\right)\\ &\leqslant \beta_r + \boldsymbol{P}\left(\sup_{r<k\leqslant\tau} X_k > u \mid a \leqslant X_{r+1} \leqslant u_{r+1}, X_r = x\right)\\ &\leqslant \beta_r + \boldsymbol{P}\left(\sup_{r<k\leqslant\tau} X_{r+1} + S^a_{k-r-1} > u \mid a \right.\\ &\qquad \left. \leqslant X_{r+1} \leqslant u_{r+1}, X_r = x\right)\\ &\leqslant \beta_r + \boldsymbol{P}\left(\sup_{k\geqslant 0} S^a_k > u - u_{r+1}\right).\end{aligned}$$

The second to last step follows from (3) because, by coupling of the steps of the Markov chain and the random walk, one can construct for arbitrary $X_{r+1} > a$, for $r+1 \leqslant k \leqslant \tau$,

$$X_k \leqslant X_{r+1} + S^a_{k-r-1}, \text{ a.s.} \tag{12}$$

The existence of a probability space supporting such a construction follows from Propositions 1 and 2 in a paper by Kamae et al. (1977).
So we have

$$\alpha_0 \leqslant \beta_0 + \cdots + \beta_r + \boldsymbol{P}\left(\sup_{k \geqslant 0} S^a_k > (u-a)/(r+2) \right). \tag{13}$$

Further,

$$\beta_i \leqslant \max \left\{ \sup_{x \in (-\infty, a] \cap \mathscr{S}} \boldsymbol{P}(X_1 > u_{i+1} \mid X_0 = x), \boldsymbol{P} \mid \xi^a > (u-a)/(r+2)) \right\} \tag{14}$$

and so $\beta_i \to 0$ as $u \to \infty$, by the Markov inequality, which combined with the negative drift of S^a_k implies $\alpha_0 \to 0$.

Hence, for u sufficiently large, $\boldsymbol{P}(F_i \mid X_0 > u) \leqslant \varepsilon$, and

$$\boldsymbol{P}\left(\sup_{N_a \leqslant k \leqslant C_0} X_k > u \mid X_0 > u \right) \leqslant \boldsymbol{P}(N > n_0) + \varepsilon n_0 = (1-p)^{n_0} + \varepsilon n_0.$$

This expression can be made arbitrarily small, since p does not depend on u. □

Remark. In the original corollary, Rootzén assumes that $EC_1^{2+\delta} < \infty$ for some positive δ, to conclude that $EC_0^{1+\delta} < \infty$. As pointed out by Perfekt (1994), $EC_1^2 < \infty$, or equivalently, $EC_0 < \infty$ is sufficient.

We are ready to formulate our main theorem.

Theorem 1. *Suppose $\{X_k\}$ satisfies condition (2) and the sandwich (3), and for each m the set $(-\infty, a_m] \cap \mathscr{S}$ is a regeneration set with aperiodic cycle length. Suppose that for some m_0 condition (6) with $a = a_{m_0}$ is fulfilled. Suppose further that for $u \to \infty$,*

$$\nu_u(x, \infty) = (1 - F(x+u))/(1 - F(u)) \xrightarrow{\mathrm{d}} \nu(x, \infty), \tag{15}$$

for some non-degenerate measure ν on $\mathbb{R}^+$; then the extremal index θ_1 of $\{X_k\}$ is equal to

$$\theta_1 = \int_0^\infty \boldsymbol{P}\left(\sup_{k \geqslant 1} S_k \leqslant -x \right) \nu(\mathrm{d}x). \tag{16}$$

Moreover, the compounding probabilities π_l, $l \geqslant 1$, of the limiting Poisson process of exceedances of level $u_n(\tau)$, are given by $\pi_l = (\theta_l - \theta_{l+1})/\theta_1$ with

$$\theta_l = \int_0^\infty \boldsymbol{P}(\mathrm{card}\{k \geqslant 1 \colon S_k > -x\} = l-1) \nu(\mathrm{d}x), \quad l \geqslant 2. \tag{17}$$

Proof. Condition (15) implies that F lies in the domain of attraction of the Gumbel distribution and $\nu(x, \infty) = \mathrm{e}^{-\alpha x}$ for some positive α; it also implies $EX_0^+ < \infty$. As in the lemma, the domination of $\{X_k\}$ in (a_m, ∞) by a random walk with negative drift implies that $EC_0 < \infty$, for any $m \geqslant m_0$.

The method of proof is that we establish, for each a_m and for $n \to \infty$,

$$P\left(\sup_{k \geqslant 1} X_0 + S_k^{a_m} \leqslant u_n \mid X_0 > u_n\right) + o(1) \leqslant P\left(\sup_{1 \leqslant k \leqslant C_0} X_k \leqslant u_n \mid X_0 > u_n\right)$$

$$\leqslant P\left(\sup_{k \geqslant 1} X_0 + S_k \leqslant u_n \mid X_0 > u_n\right) + o(1). \tag{18}$$

It then follows (with $v_n = v_{u_n}$) that

$$\limsup_{n \to \infty} P\left(\sup_{1 \leqslant k \leqslant C_0} X_k \leqslant u_n \mid X_0 > u_n\right) \leqslant \limsup_{n \to \infty} \int_0^\infty P\left(\sup_{k \geqslant 1} S_k \leqslant -x\right) v_n(\mathrm{d}x)$$

$$= \int_0^\infty P\left(\sup_{k \geqslant 1} S_k \leqslant -x\right) v(\mathrm{d}x), \tag{19}$$

where the equality follows from the weak convergence of v_n to the absolute continuous probability measure v. Similarly, we have

$$\liminf_{n \to \infty} P\left(\sup_{1 \leqslant k \leqslant C_0} X_k \leqslant u_n \mid X_0 > u_n\right) \geqslant \int_0^\infty P\left(\sup_{k \geqslant 1} S_k^{a_m} \leqslant -x\right) v(\mathrm{d}x). \tag{20}$$

Borovkov (1976, Section 21) proves that $\sup_{k \geqslant 0} S_k^{a_m} \xrightarrow{\mathrm{d}} \sup_{k \geqslant 0} S_k$, when $\xi^{a_m} \xrightarrow{\mathrm{d}} \xi$ and $E[\xi^{a_m}]^+ \to E\xi^+$, as $m \to \infty$. Clearly, both conditions hold: the first by hypothesis and the second condition by monotone convergence. From this one easily obtains

$$\sup_{k \geqslant 1} S_k^{a_m} \stackrel{\mathrm{d}}{=} \xi^{a_m} + \sup_{k \geqslant 0} S_k^{a_m} \xrightarrow{\mathrm{d}} \xi + \sup_{k \geqslant 0} S_k \stackrel{\mathrm{d}}{=} \sup_{k \geqslant 1} S_k. \tag{21}$$

So

$$\lim_{m \to \infty} \int_0^\infty P\left(\sup_{k \geqslant 1} S_k^{a_m} \leqslant -x\right) v(\mathrm{d}x) = \int_0^\infty P\left(\sup_{k \geqslant 1} S_k \leqslant -x\right) v(\mathrm{d}x).$$

Hence, (16) follows when we prove that for each a_m and for $n \to \infty$ the sandwich (18) holds.

For brevity, replace a_m by a. It follows from (3) and results by Kamae et al. (1977) that a coupling of the Markov chain and the random walks exists such that for $X_0 > a$ and $k \leqslant N_a$,

$$X_0 + S_k \leqslant X_k \leqslant X_0 + S_k^a, \text{ a.s.} \tag{22}$$

Further, we have $T_a = \inf\{k\colon X_0 + S_k \leqslant a\} \leqslant N_a \leqslant R_a = \inf\{k\colon X_0 + S_k^a \leqslant a\}$, and $N_a \leqslant C_0$. From these inequalities, for arbitrary n,

$$P\left(\sup_{1 \leqslant k \leqslant C_0} X_k \leqslant u_n \mid X_0 > u_n\right)$$

$$\geqslant P\left(\sup_{1 \leqslant k \leqslant N_a} X_k \leqslant u_n \mid X_0 > u_n\right) - P\left(\sup_{N_a \leqslant k \leqslant C_0} X_k > u_n \mid X_0 > u_n\right)$$

$$\geqslant \boldsymbol{P}\left(\sup_{1 \leqslant k \leqslant N_a} X_0 + S_k^a \leqslant u_n \mid X_0 > u_n\right) - \boldsymbol{P}\left(\sup_{N_a \leqslant k \leqslant C_0} X_k > u_n \mid X_0 > u_n\right)$$

$$\geqslant \boldsymbol{P}\left(\sup_{1 \leqslant k} X_0 + S_k^a \leqslant u_n \mid X_0 > u_n\right) - \boldsymbol{P}\left(\sup_{N_a \leqslant k \leqslant C_0} X_k > u_n \mid X_0 > u_n\right). \tag{23}$$

Since condition (6) is fulfilled for some a_{m_0}, we have for $a_{m_0} < x \leqslant a_m$ because of (3): $E[x + g(x, Y)]^+ \leqslant E[x - a_{m_0} + a_{m_0} + \xi^{a_{m_0}}]^+ \leqslant a_m - a_{m_0} + E[a_{m_0} + g(a_{m_0}, Y)]^+ < \infty$, so (6) is fulfilled for each $a_m > a_{m_0}$. Hence, from the proof of Lemma 1 follows that for all but a finite number of a's.

$$\lim_{n \to \infty} \boldsymbol{P}\left(\sup_{N_a \leqslant k \leqslant C_0} X_k > u_n \mid X_0 > u_n\right) = 0. \tag{24}$$

This establishes the left-hand side of (18). Now use the left-hand side of (22) to conclude

$$\boldsymbol{P}\left(\sup_{1 \leqslant k \leqslant C_0} X_k \leqslant u_n \mid X_0 > u_n\right) \leqslant \boldsymbol{P}\left(\sup_{1 \leqslant k \leqslant T_a} X_0 + S_k \leqslant u_n \mid X_0 > u_n\right). \tag{25}$$

Because T_a is the hitting time of the set $(-\infty, a]$ by the random walk $\{S_k\}$, which has negative drift, we have for $n \to \infty$,

$$\boldsymbol{P}\left(\sup_{1 \leqslant k \leqslant T_a} X_0 + S_k \leqslant u_n \mid X_0 > u_n\right) = \boldsymbol{P}\left(\sup_{k \geqslant 1} X_0 + S_k \leqslant u_n \mid X_0 > u_n\right) + \mathrm{o}(1). \tag{26}$$

The sandwich (18) is now immediate from (23)–(26). This finishes the proof of (16). The proof of (17) follows the same lines and is therefore omitted. □

Remark. Note that without the convergence $\xi^{a_m} \xrightarrow{\mathrm{d}} \xi$ one would still obtain bounds on θ_1 (if it exists):

$$\int_0^\infty \boldsymbol{P}\left(\sup_{k \geqslant 1} S_k^{a_m} \leqslant -x\right) \nu(\mathrm{d}x) \leqslant \theta_1 \leqslant \int_0^\infty \boldsymbol{P}\left(\sup_{k \geqslant 1} S_k \leqslant -x\right) \nu(\mathrm{d}x). \tag{27}$$

3. Computation

Suppose that for a stationary Markov chain $\{X_k\}$ the conditional increment $X_{k+1} - X_k$, given $X_k = x$, converges in distribution to a random variable ξ, and that the distribution of X_0, given $X_0 > x$, converges to an exponential distribution with parameter α, as $x \to \infty$. This one has, under suitable conditions (cf. Perfekt, 1994; Smith, 1992; or Theorem 1 of Section 2) that

$$\theta_1 = \int_0^\infty \boldsymbol{P}\left(\inf_{k \geqslant 1} S_k' \geqslant x\right) \alpha \mathrm{e}^{-\alpha x} \, \mathrm{d}x \tag{28}$$

and

$$\theta_l = \int_0^\infty \boldsymbol{P}(\text{card}\{k \geqslant 1\colon S'_k < x\} = l-1)\alpha e^{-\alpha x}\,dx, \quad l \geqslant 2, \tag{29}$$

where $S'_k = -S_k$, $k \geqslant 0$, and S_k is the random walk with stepsize ξ. We prefer to work with the random walk S'_k, which has positive drift, to facilitate the use of Grübel's algorithm. Denote by G the distribution function of $-\xi$; $G(t) = \boldsymbol{P}(\xi \geqslant -t)$, $t \in \mathbb{R}$. Define $T_l = S'_{(l)}$ the lth smallest of $S'_1, S'_2, \ldots$ and $P_l(x) = \boldsymbol{P}(T_l \leqslant x)$, all for $l \geqslant 1$. (To use $\boldsymbol{P}(T_l < x)$ instead would seem more natural, but P_l is convenient, and without consequence as it is integrated with respect to an exponential density).

Using the Wiener–Hopf factorization algorithm by Grübel (1991), a discrete approximation to the distribution of $M = \inf\{S'_k\colon k \geqslant 0\}$ can be obtained. For independent M, ξ, and $E \overset{d}{=} \exp(\alpha)$ one obtains $T_1 \overset{d}{=} M - \xi$ and so $\theta_1 = \boldsymbol{P}(M - \xi \geqslant E)$. Thus, θ_1 is easy to compute.

Below we shall derive a recursive scheme to compute the $\theta_l, l \geqslant 2$, building on this result. We have for $l \geqslant 2$,

$$\theta_l = \int_0^\infty (P_{l-1}(x) - P_l(x))\alpha e^{-\alpha x}\,dx \tag{30}$$

and

$$\begin{aligned} P_l(x) &= \boldsymbol{P}(T_l \leqslant x, \xi < -x) + \boldsymbol{P}(T_l \leqslant x, \xi \geqslant -x) \\ &= \int_{(x,\infty)} P_l(x-y)\,dG(y) + \int_{(-\infty,x]} P_{l-1}(x-y)\,dG(y). \end{aligned} \tag{31}$$

The second term on the right-hand side can be obtained from recursion, the first requires $P_l(z)$ for $z < 0$.

Let V be the point of first entry of the random walk $\{S'_k\}$ into $(-\infty, z]$; formally $\tau = \inf\{k\colon S'_k \leqslant z\}$, $V = S'_\tau$, and $V = \infty$ if $\{k\colon S'_k \leqslant z\}$ is empty. Let L be the (defective) distribution of the negative ladder heights and $m_L = \sum_{n \geqslant 0} L^{n*}$ the renewal function of the corresponding (defective) renewal process on $(-\infty, 0]$. Then

$$F_V(v) = \boldsymbol{P}(V \leqslant v) = \int_{(z,0]} L(v-x)\,dm_L(x),$$

for $v \leqslant z$, and so for $z < 0$, by conditioning on V,

$$\begin{aligned} P_l(z) &= \boldsymbol{P}(T_l \leqslant z, V \leqslant z) \\ &= \int_{(-\infty,z]} \boldsymbol{P}(T_{l-1} \leqslant z - v)\,dF_V(v) \\ &= \int_{(z,0]} \int_{(-\infty,z]} P_{l-1}(z-v)\,dL(v-x)\,dm_L(x) \\ &= ((P^+_{l-1} * L)^- * m_L)(z), \end{aligned} \tag{32}$$

where for any (defective) distribution function H, H^+ and H^- are defined by

$$H^+(x) = H(x)1_{x \geqslant 0}$$

and

$$H^-(x) = H(x)1_{x<0}.$$

Combined with (31), which reads in this notation:

$$P_l = P_l^- * G + P_{l-1}^+ * G, \tag{33}$$

this yields the recursion.

The actual calculation scheme is as follows. As in Grübel's algorithm, it is based on Fourier transforms (FT) of discrete approximations to relevant distributions. Grübel's algorithm yields the FTs of M, L and m_L. The product of the FTs of M and $-\xi$ yields the FT of P_1. The $+$ and $-$ operators are easily implemented, so by applying (32) we get P_2^- from P_1, and then P_2 from (33). P_3 is derived from P_2 in the same manner. Thus, recursively, one obtains the FTs of the P_l. Formula (30) is evaluated easily: by convolution with the exponential distribution. All this has been implemented in Matlab; the code is available from the authors. Matters related to the accuracy of results obtained this way are discussed in the next section.

4. Applications

4.1. Validation of the algorithm and the G/G/1 queue

Rootzén (1988) showed that for waiting times in the G/G/1 queue the general expressions for $\theta_1, \theta_2, \ldots$ simplify to those as in Theorem 1; it is straightforward to show that the conditions of this theorem are satisfied. For this case the parameter $\alpha > 0$ in Theorem 1 is the unique positive solution to

$$E\exp(\alpha(B - A)) = 1. \tag{34}$$

where B and A denote a service time and an interarrival time, respectively (cf. Iglehart, 1972).

The reason for mentioning the G/G/1 queue here is that some exact results are available which can be used to validate the numerical procedure. For the M/G/1 queue extremal index θ_1 can be expressed in terms of familiar quantities: the Poisson arrival rate λ, the traffic intensity $\rho = \lambda EB$, and $\alpha > 0$ which is the unique positive solution of

$$E\exp(\alpha B) = 1 + \frac{\alpha}{\lambda} \tag{35}$$

(cf. Cohen, 1969, III.7). A straightforward, but lengthy, calculation then shows that

$$\theta_1 = \frac{\alpha(1-\rho)}{\alpha + \lambda}; \tag{36}$$

so if (35) can be solved, the extremal index is known. For exponential, uniform, Erlang-k and deterministic service times, among others, this can be done either explicitly or through a simple iteration procedure. For these distributions and for two values of ρ (0.5 and 0.9) we shall below compare exact values for the extremal index θ_1 to the values obtained with the numerical procedure.

Some error considerations are in order, however, as several factors affect the accuracy of the numerical procedure. In order to use the fast Fourier transform, probability distributions have to be replaced by discretized, truncation versions on grid points kh, $k = -m, \ -m+1, \dots, m-1$. The values of the discretization parameters m and h are of considerable influence on the accuracy of the results.

Here, as in the case described by Grübel (1991), the discretization error is almost eliminated by applying Richardson extrapolation. (Here this means: because the error in $\theta^{(h)}$ is $ch + o(h)$, the error in the extrapolated value $\theta_R = 2\theta^{(h)} - \theta^{(2h)}$ is $o(h)$, in practice $O(h^2)$.) Our experience indicates that the effect of the truncation is best assessed through the amount of neglected probability mass, i.e., the total mass outside the interval that supports the discrete approximations (both for ξ and the exponential distribution with parameter α). If δ is the maximum of these missing masses, for given m and h, then the following rule of thumb holds:

the error in the value of the extremal index θ_1 *extrapolated from the cases* (m, h) *and* $(m/2, 2h)$ *is smaller than* $\max(10\delta, h^2)$.

There are few hard results on these issues, fewer of practical use; see Embrechts et al. (1993) for a more elaborate discussion, and Grübel and Pitts (1992) for some related theoretical material. Table 1 lists true values for θ_1, computed using (35) and (36) and the errors in the numerical approximation, based on $m = 2^{13}$.

For the M/M/1 queue more analytical results can be obtained: each θ_l, $l \geqslant 1$, can be expressed in the traffic intensity ρ. The limiting random walk has a step distribution with two exponential tails, and this allows the explicit solution of $P_1, P_2, \dots$ from the recursive scheme in Section 3, and evaluation of $\theta_1, \theta_2, \dots$ through formula (28) and (30). These are exact formulas for θ_1–θ_5:

$$\theta_1 = (1-\rho)^2,$$

$$\theta_2 = \frac{2\rho(1-\rho)^2}{1+\rho},$$

Table 1

	$\rho = 0.5$			$\rho = 0.9$		
		$mh = 20$			$mh = 100$	
	θ_1	Error	δ	θ_1	Error	δ
Uniform	0.320999	$-5\ 10^{-6}$	$2\ 10^{-9}$	0.014614	$-4\ 10^{-6}$	$4\ 10^{-8}$
Erlang-3	0.313499	$-5\ 10^{-6}$	$2\ 10^{-9}$	0.014502	$-4\ 10^{-6}$	$4\ 10^{-8}$
Deterministic	0.357666	$-3\ 10^{-6}$	$2\ 10^{-9}$	0.018710	$-4\ 10^{-6}$	10^{-10}

$$\theta_3 = \frac{\rho^2(1-\rho)^2}{(1+\rho)^3}\{3\rho + 5\},$$

$$\theta_4 = \frac{\rho^3(1-\rho)^2}{(1+\rho)^5}\{4\rho^2 + 14\rho + 14\},$$

$$\theta_5 = \frac{\rho^4(1-\rho)^2}{(1+\rho)^7}\{5\rho^3 + 27\rho^2 + 54\rho + 42\}.$$

In Table 2 exact values are compared against the numerical answers, again with $m = 2^{13}$. The rule of thumb stated above seems very reliable for θ_1, and it is based on evidence from a different array of cases: it seems inadvisable, however, to extend this rule to the other thetas, without more evidence or a theoretical footing for this error estimate.

4.2. *The G/M/c queue*

Consider a c-server queue with (general) renewal arrivals, and exponential service times, where each server has rate μ. The extremal quantities $\theta_1, \theta_2, \ldots$ of the waiting time process depend only on the interarrival distribution and on $\rho = \lambda/c\mu$. In other words, the extremal behaviour of a $\mathrm{G/M}(\mu)/c$ queue is the same as that of a $\mathrm{G/M}(c\mu)/1$ queue with identical interarrival distribution.

This can be seen as follows. Let X_k^c and X_k^1 be the waiting time of the kth arriving customer, in the c-server and single server queue, respectively. Assume that the processes $\{X_k^c\}$ and $\{X_k^1\}$ are stationary. It is well-known (see, e.g. Wolff, 1989, Ch. 8) that, conditional upon the waiting times being non-zero, X_0^c and X_0^1 have the same distribution, and under this condition, the distributions of the number of customers in the *queue* are identical as well. This implies that, given that the zeroth customer has to wait, the two systems can be probabilistically coupled. This can also be done for subsequent transitions, as long as all the servers in the c-server queue are busy, as follows. That arrivals can be coupled is trivial. Service completions can be coupled because (as long as all servers are busy) time between subsequent service completions form an i.i.d. exponential($c\mu$) sequence that can be coupled to the service times in the single-server queue.

Table 2

	$\rho = 0.5$		$\rho = 0.9$	
l	θ_l	Error when $mh = 16$ ($\delta = 2\ 10^{-7}$)	θ_l	Error when $mh = 160$ ($\delta = 2\ 10^{-8}$)
1	0.250000	-1×10^{-6}	0.0100000	-7×10^{-6}
2	0.166667	4×10^{-6}	0.0094737	10^{-5}
3	0.120370	-4×10^{-6}	0.0090932	10^{-7}
4	0.090535	-2×10^{-5}	0.0087853	10^{-7}
5	0.069845	-4×10^{-5}	0.0085228	10^{-7}

As an example (with a heavy tail), consider the Pareto distribution

$$G(x) = 1 - \left(\frac{r}{x}\right)^3, \quad x \geqslant r. \tag{37}$$

For this case, Eq. (34) becomes

$$\phi(\alpha) + \alpha/c\mu - 1 = 0, \tag{38}$$

where $\phi(t) = \int_r^\infty e^{-ty}\,dG(y)$, the Laplace transform of G. To obtain the two values of ρ (0.5 and 0.9), we took $c\mu = 1$ and $r = \frac{4}{3}$, $r = \frac{27}{20}$, respectively. A simple Newton–Raphson iteration gives: $\alpha = 0.735257$ ($\rho = 0.5$) and $\alpha = 0.742667$ ($\rho = 0.9$). In Table 3 we compare the extremal index θ_1, obtained with the algorithm ($m = 2^{13}$, $mh = 40$, yielding $\delta = 4 \times 10^{-5}$ in both cases), with the exact value. This exact value follows from Cohen (1969, III.7, Theorem 7.5), and reads for $c\mu = 1$:

$$\theta_1 = \alpha(1 + \phi'(\alpha)). \tag{39}$$

Note that the results obtained with the algorithm are very satisfactory, considering the heavy tail of the Pareto distribution.

4.3. ARCH-processes

As described in Section 2, the ARCH process satisfies the recursion

$$X_{k+1} = X_k + g(X_k, Y_{k+1}),$$

with

$$Y_k = \log(\lambda U_k^2),$$

$$g(x, Y) = Y + \log\left(1 + \frac{\beta}{\lambda} e^{-x}\right),$$

where $U_1, U_2, \ldots$ are i.i.d. standard normal. $\beta > 0$ and $0 < \lambda < 1$.

In this example, we can take $\xi = Y$. By Jensen's inequality the random walk $\{S_k\}$ has a negative drift

$$E\xi = EY = E\log(\lambda U_1^2) \leqslant \log(\lambda E U_1^2) = \log\lambda < 0.$$

Table 3

	$\rho = 0.5$		$\rho = 0.9$	
l	θ_l	Exact value	θ_l	Exact value
1	0.403948	0.403938	0.413874	0.413894
2	0.213546	–	0.215324	–
3	0.127611	–	0.126922	–
4	0.080895	–	0.079447	–
5	0.053158	–	0.051583	–

Table 4

λ	α	θ_1	π_1	π_2	π_3	π_4	π_5
0.3	4.17990	0.88556	0.89136	0.09181	0.01382	0.00243	0.00046
		0.887	0.892	0.094	0.011	0.003	0.003
0.95	1.07211	0.43308	0.50458	0.20993	0.10950	0.06309	0.03852
		0.438	0.524	0.184	0.118	0.063	0.037

It is immediate that $\xi^a = g(a, Y) \xrightarrow{\mathrm{d}} \xi$ and that (3) is satisfied. Condition (6) follows since

$$x + \log\left(1 + \frac{\beta}{\lambda}\mathrm{e}^{-x}\right) = \log(\mathrm{e}^x + \gamma),$$

where $\gamma = \beta/\lambda$ is bounded on $(-\infty, a]$.

In order to show that $(-\infty, a]$ is a regeneration set we put $r = 1$, $\delta(x) = \log(\mathrm{e}^x + \gamma)$ and $b = \sup_{x \leqslant a} |\delta(x)|$. With $m(A)$ the Lebesgue measure of a set A, μ equal to the law of $Y = \log(\lambda U_k^2)$, M an arbitrary positive number and

$$p = \min_{|x| < M + b} \frac{\mathrm{d}\mu}{\mathrm{d}m}(x) > 0,$$

we have for an arbitrary Borel set B, and arbitrary $x \leqslant a$,

$$\begin{aligned} \boldsymbol{P}(X_{n+1} \in B \mid X_n = x) &= \mu(B - \delta(x)) \\ &\geqslant \mu(B - \delta(x) \cap [-M - b, M + b]) \\ &\geqslant pm(B - \delta(x) \cap [-M - b, M + b]) \\ &\geqslant pm(B \cap [-M - b + \delta(x), M + b + \delta(x)]) \\ &\geqslant pm(B \cap [-M, M]). \end{aligned}$$

Finally, it is well-known from the literature (cf. Kesten, 1973) that (15) holds with $v(x, \infty) = \exp(-\alpha x)$, where α is the unique solution of $E(\zeta U_1^2)^\alpha = 1$; in particular, this implies that F has a finite first moment.

In Table 4, for each λ value, the top rows list the values of θ_l and $\pi_l = (\theta_l - \theta_{l+1})/\theta_1$, $l = 1, \ldots, 5$ computed with the algorithm. Our rule of thumb gives that the error in θ_1 is smaller than $h^2 < 3.10^{-5}$. The bottom rows show the values of θ_1 and π_l, $l = 1, \ldots, 5$ taken from de Haan et al. (1989), and based on 1000 simulations each of length 100.

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