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Oscillation of harmonic functions for subordinate Brownian motion and its applications

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Abstract

In this paper, we establish an oscillation estimate of nonnegative harmonic functions for a pure-jump subordinate Brownian motion. The infinitesimal generator of such subordinate Brownian motion is an integro-differential operator. As an application, we give a probabilistic proof of the following form of relative Fatou theorem for such subordinate Brownian motion X in a bounded κ-fat open set; if u is a positive harmonic function with respect to X in a bounded κ-fat open set D and h is a positive harmonic function in D vanishing on D^c, then the non-tangential limit of u/h exists almost everywhere with respect to the Martin-representing measure of h.

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1. Introduction

Nowadays Lévy processes have been receiving intensive study due to their importance both in theories and applications. They are widely used in various fields, such as mathematical

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finance, actuarial mathematics and mathematical physics. Typically, the infinitesimal generators of general Lévy processes in $\mathbb{R}^d$ are not differential operators but integro-differential operators. Even though integro-differential operators are very important in the theory of partial differential equations, general Lévy processes and corresponding integro-differential operators are not easy to deal with. For a summary of some of these recent results from the probability literature, one can see [8] and the references therein. We refer readers to [11,12] for samples of recent progresses in the PDE literature.

Let $W = (W_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ and $S = (S_t : t \geq 0)$ be a subordinator independent of W. The process $X = (X_t : t \geq 0)$ defined by $X_t = W_{S_t}$ is a rotationally invariant Lévy process in $\mathbb{R}^d$ and is called a subordinate Brownian motion. Subordinate Brownian motions form a very large class of Lévy processes. Nonetheless, compared with general Lévy processes, subordinate Brownian motions are much more tractable. If we take the Brownian motion W as given, then X is completely determined by the Laplace exponent of subordinator S. Hence one can deduce the properties of X from the subordinator S, or equivalently the Laplace exponent of it.

The purpose of this paper is to give an oscillation estimate for (unbounded) harmonic functions (see Section 2 for the definition of harmonicity) for a large class of subordinate Brownian motions. Then using our estimates, we discuss non-tangential limits of the ratio of two harmonic functions with respect to such subordinate Brownian motions.

Now we state the first main result of this paper.

Theorem 1.1. *Suppose that $X = (X_t : t \geq 0)$ is a Lévy process whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where $\phi : (0, \infty) \to [0, \infty)$ is a complete Bernstein function such that $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$, $\alpha \in (0, 2)$ and $\ell : (0, \infty) \to (0, \infty)$ is slowly varying at ∞. Then for every $\eta > 0$, there exists $a = a(\eta, \alpha, d, \ell) \in (0, 1)$ such that for every $x_0 \in \mathbb{R}^d$ and $r \in (0, 1]$,*

$$\sup_{x \in B(x_0, ar)} u(x) \leq (1 + \eta) \inf_{x \in B(x_0, ar)} u(x)$$

for every nonnegative function u in $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ with respect to X.

Note that, for unlike a local operator, Theorem 1.1 cannot be obtained from the Harnack inequality and Moser's iteration method because harmonic functions in Theorem 1.1 are nonnegative in the whole space $\mathbb{R}^d$. On the other hand, if one just assumes that a harmonic function is nonnegative in $B(x_0, 2r)$, then even the Harnack inequality does not hold (see [21]).

Recently many results are obtained under the weaker assumption that ϕ is comparable to a regularly varying function at ∞ (see [24,26–28]). But our technical Lemmas 3.2–3.4 cannot be obtained under such assumptions.

Doob proved the relative Fatou theorem in the classical sense [17]. That is, the ratio u/h of two positive harmonic functions with respect to Brownian motion on a unit open ball has non-tangential limits almost everywhere with respect to the Martin measure of h. Later, the relative Fatou theorem in the classical sense has been extended to some general open sets (see [35] and references therein). But the relative Fatou theorem stated above and the Fatou theorem are not true for harmonic functions for the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta^{\alpha/2})$ when $\alpha \in (0, 2)$ (see [5] for some counterexamples). Correct formulation of the relative Fatou theorem for the integro-differential operator is the existence of non-tangential limits of the ratio u/h, where u is positive harmonic in an open set D and h is a positive harmonic function in D vanishing on D^c (see [9,23,25,30]).

In this paper, through a probabilistic method and Theorem 1.1, we show in Theorem 4.11 that the relative Fatou theorem holds for subordinate Brownian motion in very general open sets, namely, bounded κ-fat open sets, the family that includes bounded Lipschitz open sets.

This paper is organized as follows. In Section 2, we recall the definition of subordinate Brownian motion and its basic properties under our assumptions. In Section 3, we give the proof of Theorem 1.1. In these sections, the influence of [10] in our results will be apparent. Section 4 contains the proof of the relative Fatou theorem in bounded κ-fat open sets. The main idea of our proof is similar to [23], which is inspired by Doob's approach (see also [1]). We use the Harnack and the boundary Harnack principle obtained in [29] and our Theorem 1.1. If the open set is the unit ball in $\mathbb{R}^2$, we show that our result is the best possible one.

In the sequel, we will use the following convention: the value of the constant C_* will remain the same throughout this paper, while the constants $c_0, c_1, c_2, \ldots$ signify constants whose values are unimportant and which may change from location to location. The labeling of the constants $c_0, c_1, c_2, \ldots$ starts anew in the statement of each result. We use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$ and $f(t) \sim g(t)$, $t \to 0$ ($f(t) \sim g(t)$, $t \to \infty$, respectively) means $\lim_{t\to 0} f(t)/g(t) = 1$ ($\lim_{t\to\infty} f(t)/g(t) = 1$, respectively). For any open set U, we denote $\delta_U(x) = \mathrm{dist}(x, U^c)$. Let $A(x, a, b) := \{y \in \mathbb{R}^d : a \le |x - y| < b\}$ and $B(x_0, r)$ be a ball in $\mathbb{R}^d$ centered at x_0 whose radius is r. When x_0 is the origin, we simply denote $B_r := B(0, r)$.

2. Preliminaries

Suppose that $S = (S_t : t \ge 0)$ is a subordinator, that is, an increasing Lévy process taking values in $[0, \infty)$ with $S_0 = 0$. A subordinator S is completely characterized by its Laplace exponent ϕ via

$$\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)), \quad \lambda > 0.$$

A smooth function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \le 0$ for every natural number n. Every Bernstein function has a representation

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$$

where $a, b \ge 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$. a is called the killing coefficient, b is the drift and μ is the Lévy measure of the Bernstein function. A nonnegative function ϕ on $(0, \infty)$ is the Laplace exponent of a subordinator if and only if it is a Bernstein function with $\phi(0+) = 0$. We also call μ the Lévy measure of the subordinator S. A Bernstein function ϕ is called a complete Bernstein function if μ has a completely monotone density $t \mapsto \mu(t)$, i.e., $\mu(t)dt = \mu(dt)$ and $(-1)^n D^n \mu \ge 0$ for every non-negative integer n.

Throughout this paper we will assume the following.

(A1) *ϕ is a complete Bernstein function and regularly varying of index $\alpha/2$ at ∞ for some $\alpha \in (0, 2)$. That is,*

$$\phi(\lambda) = \lambda^{\alpha/2} \ell(\lambda) \tag{2.1}$$

for some $\alpha \in (0, 2)$ and some positive function ℓ which is slowly varying at ∞.

Note that, this is an assumption about ϕ at ∞ and nothing is assumed about the behavior near zero. Clearly (2.1) implies that $b = 0$ and $\lambda \to \ell(\lambda)$ is strictly positive and continuous on $(0, \infty)$.

We refer the reader to [29] for examples. From [8, Proposition 5.23], we get

$$\mu(t) \sim \frac{\alpha}{2\Gamma(1-\alpha/2)} t^{-1}\phi(t^{-1}) \quad \text{as } t \to 0 \tag{2.2}$$

where $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$.

Let $W := (W_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ be a Brownian motion on $\mathbb{R}^d$ with $\mathbb{P}_x(W_0 = x) = 1$ and $\mathbb{E}_x[e^{i\xi\cdot(W_t-W_0)}] = e^{-t|\xi|^2}$ for $\xi \in \mathbb{R}^d$, $t > 0$ and $x \in \mathbb{R}^d$. In the remainder of this paper we will use $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ to denote the subordinate Brownian motion defined by $X_t = W_{S_t}$, where $S = (S_t, t \geq 0)$ is a subordinator whose Laplace exponent is ϕ and S is independent of W.

Let

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\mu(t)dt \quad \text{for } r > 0 \tag{2.3}$$

where $\mu(t)$ is the Lévy density of S. Then $J(x) := j(|x|)$ is the Lévy density of X. Note that the function $r \mapsto j(r)$ is strictly positive, continuous and decreasing on $(0,\infty)$. Since $|\partial/\partial r(e^{-r^2/(4t)})| = 4r^{-1}\left(r^2/(8t)\, e^{-r^2/(8t)}\right) e^{-r^2/(8t)} \leq c\, r^{-1}e^{-r^2/(8t)}$ and $\int_0^\infty (4\pi t)^{-d/2} r^{-1} e^{-r^2/(8t)}\mu(t)dt = r^{-1}\, j(r/\sqrt{2})$, $j'(r)$ is well-defined and is continuous.

Applying [26, Lemma 13.3.1], we have the following.

Theorem 2.1.

$$j(r) \sim \frac{\alpha\Gamma((d+\alpha)/2)}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\alpha/2)} \frac{\phi(r^{-2})}{r^d} \quad \textit{as } r \to 0.$$

As an immediate consequence of Theorem 2.1 and the continuity of $r \mapsto j(r)$ on $(0,\infty)$, we have the following.

Corollary 2.2. *For every $R > 0$, there exists $c = c(R, \alpha, d, \ell) > 1$ such that for every positive y with $|y| \leq R$,*

$$c^{-1}|y|^{-d}\phi(|y|^{-2}) \leq J(y) \leq c\,|y|^{-d}\phi(|y|^{-2}).$$

By Kim et al. [26, Proposition 13.3.5], the function $r \mapsto j(r)$ enjoys the following properties.

Proposition 2.3. (1) *For any $M > 0$, there exists $c_1 = c_1(M) > 0$ such that $j(r) \leq c_1 j(2r)$ for every $r \in (0, M)$.*
(2) *There exists $c_2 > 0$ such that $j(r) \leq c_2 j(r+1)$ for every $r > 1$.*

For any open set D, we use τ_D to denote the first exit time of D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Given an open set $D \subset \mathbb{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a cemetery state. We now recall the definition of harmonic functions with respect to X.

Definition 2.4. Let D be an open subset in $\mathbb{R}^d$. A function u defined on $\mathbb{R}^d$ is said to be

(1) harmonic in D with respect to X if $\mathbb{E}_x\left[|u(X_{\tau_B})|\right] < \infty$ and $u(x) = \mathbb{E}_x[u(X_{\tau_B})]$ for every $x \in B$ and open set B whose closure is a compact subset of D;

(2) regular harmonic in D with respect to X if it is harmonic in D with respect to X and for each $x \in D$, $u(x) = \mathbb{E}_x\left[u(X_{\tau_D})\right]$;
(3) harmonic with respect to X^D if it is harmonic with respect to X in D and vanishes outside D.

By Kim et al. [26, Corollary 13.4.8], we have the following Harnack inequality.

Theorem 2.5 (*Harnack Inequality*)**.** *There exists a constant $C_0 > 0$ such that for every $r \in (0, 1)$, $x_0 \in \mathbb{R}^d$ and function $f \geq 0$ in $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ with respect to X, we have*

$$\sup_{y \in B(x_0, r/2)} f(y) \leq C_0 \inf_{y \in B(x_0, r/2)} f(y).$$

It follows from [8, Chapter 5] that the process X has a transition density $p(t, x, y)$ which is jointly continuous. By the joint continuity and the strong Markov property, one can easily check that

$$p_D(t, x, y) := p(t, x, y) - \mathbb{E}_x[p(t - \tau_D, X_{\tau_D}, y); t > \tau_D] \quad \text{for } x, y \in D$$

is the transition density of X^D, which is jointly continuous (for example, see [24, Lemma 5.5]). For any bounded open set $D \subset \mathbb{R}^d$, we will use G_D to denote the Green function of X^D, i.e.,

$$G_D(x, y) := \int_0^\infty p_D(t, x, y) dt \quad \text{for } x, y \in D.$$

Note that G_D is continuous in $(D \times D) \setminus \{(x, x) : x \in D\}$.

We define the Poisson kernel $P_D(x, y)$ as

$$P_D(x, y) := \int_D G_D(x, z) J(z - y)\, dz \quad \text{for } (x, y) \in \mathbb{R}^d \times \overline{D}^c.$$

Thus we have for every bounded open subset D, function $f \geq 0$ and $x \in D$,

$$\mathbb{E}_x\left[f(X_{\tau_D});\ X_{\tau_D -} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} P_D(x, y) f(y) dy. \tag{2.4}$$

Using the continuities of G_D and J, one can easily check that P_D is continuous on $D \times \overline{D}^c$. Moreover, from [33, Theorem 1] we know $\mathbb{P}_x(X_{\tau_{B_r}} \in \partial B_r) = 0$ for $x \in B_r$. Thus every harmonic function u in D is written as

$$u(x) = \int_{B_r^c} P_{B_r}(x, y) u(y) dy \quad \text{for } x \in B_r \subset \overline{B_r} \subset D. \tag{2.5}$$

When $r \leq 1$, by the continuity of $P_{B(x_0,r)}$ and the Harnack inequality (Theorem 2.5), we get

$$P_{B(x_0,r)}(x, y) \leq C_0\, P_{B(x_0,r)}(x_0, y) \quad \text{for every } (x, y) \in B(x_0, r/2) \times \overline{B(x_0, r)}^{\,c}.$$

Since $P_{B(x_0,r)}(x_0, y)|u(y)| \in L^1(D)$ for $y \in \overline{B(x_0, r)}^{\,c}$ by the definition of the harmonicity, applying the Lebesgue dominated convergence theorem to (2.5) we see that every harmonic function in D with respect to X is continuous.

3. Oscillation of harmonic functions

Recall that S_t is a subordinator with Laplace exponent ϕ, W is a Brownian motion independent of S_t and $X_t = W_{S_t}$. First we show that ϕ being a complete Bernstein function implies that its Lévy density of X cannot decrease too fast in the following sense.

Lemma 3.1.

$$\limsup_{\delta \downarrow 0} \sup_{t>1} \frac{\mu(t)}{\mu(t+\delta)} = 1.$$

Proof. Let $\eta > 0$ be given. Since μ is a completely monotone function, by Bernstein's theorem [31, Theorem 1.4] there exists a measure m on $[0, \infty)$ such that $\mu(t) = \int_{[0,\infty)} e^{-tx} m(dx)$. Choose $r = r(\eta) > 0$ such that

$$\eta \int_{[0,r]} e^{-x}\, m(dx) \geq \int_{(r,\infty)} e^{-x}\, m(dx).$$

Then for any $t > 1$, we have

$$\begin{aligned}\eta \int_{[0,r]} e^{-tx}\, m(dx) &= \eta \int_{[0,r]} e^{-(t-1)x} e^{-x}\, m(dx) \geq e^{-(t-1)r} \eta \int_{[0,r]} e^{-x}\, m(dx) \\ &\geq e^{-(t-1)r} \int_{(r,\infty)} e^{-x}\, m(dx) = \int_{(r,\infty)} e^{-(t-1)r} e^{-x}\, m(dx) \\ &\geq \int_{(r,\infty)} e^{-tx}\, m(dx).\end{aligned}$$

Thus for any $t > 1$ and $\delta > 0$,

$$\begin{aligned}\mu(t+\delta) &\geq \int_{[0,r]} e^{-(t+\delta)x}\, m(dx) \geq e^{-r\delta} \int_{[0,r]} e^{-tx}\, m(dx) \\ &= e^{-r\delta}(1+\eta)^{-1} \left(\int_{[0,r]} e^{-tx}\, m(dx) + \eta \int_{[0,r]} e^{-tx}\, m(dx) \right) \\ &\geq e^{-r\delta}(1+\eta)^{-1} \left(\int_{[0,r]} e^{-tx}\, m(dx) + \int_{(r,\infty)} e^{-tx}\, m(dx) \right) \\ &= e^{-r\delta}(1+\eta)^{-1} \int_{[0,\infty)} e^{-tx}\, m(dx) = e^{-r\delta}(1+\eta)^{-1} \mu(t).\end{aligned}$$

Therefore,

$$\limsup_{\delta \downarrow 0} \left(\sup_{t>1} \frac{\mu(t)}{\mu(t+\delta)} \right) \leq 1 + \eta.$$

Since $\eta > 0$ is arbitrary and $\frac{\mu(t)}{\mu(t+\delta)} \geq 1$, we conclude that this lemma holds. □

Lemma 3.2.

$$\limsup_{\delta \downarrow 0} \sup_{r>2} \frac{j(r)}{j(r+\delta)} = 1.$$

Proof. Fix $\varepsilon \in (0, 1)$ and let $L := \frac{\alpha}{2\Gamma(1-\alpha/2)}$. Using (2.1), (2.2) and the fact that ℓ is slowly varying, we choose $t_* = t_*(\varepsilon) \in (0, 1/2)$ such that for every $t \le 2t_*$,

$$\begin{aligned} &(1+\varepsilon)^{-1} L \frac{\phi(t^{-1})}{t} \le \mu(t) \le (1+\varepsilon)\, L \frac{\phi(t^{-1})}{t} \quad \text{and} \\ &1 \le \frac{\phi\left((1+\varepsilon)t^{-1}\right)}{\phi(t^{-1})} \le (1+\varepsilon)^{1+\alpha/2}. \end{aligned} \tag{3.1}$$

By (3.1) we get

$$\begin{aligned} \mu\left((1+\varepsilon)t\right) &\ge (1+\varepsilon)^{-1} L \frac{\phi\left((1+\varepsilon)^{-1}t^{-1}\right)}{(1+\varepsilon)t} \ge (1+\varepsilon)^{-3-\alpha/2} L \frac{\phi(t^{-1})}{t} \\ &\ge (1+\varepsilon)^{-4-\alpha/2} \mu(t) \quad \text{for every } t \le 2t_*. \end{aligned} \tag{3.2}$$

Now using Lemma 3.1, we choose $\delta_1 \in (0, \varepsilon(1+\varepsilon)^{-1}]$ such that for every $t \ge 1$,

$$\mu(t+\delta_1) \le \mu(t) \le (1+\varepsilon)\mu(t+\delta_1). \tag{3.3}$$

Since

$$\frac{\mu(t) - \mu\left((1-\delta)^{-1}t\right)}{\mu\left((1-\delta)^{-1}t\right)} \le \frac{\mu(t) - \mu\left((1-\delta)^{-1}t\right)}{\mu(4)}$$

and

$$\frac{\mu(t) - \mu(\delta+t)}{\mu(\delta+t)} \le \frac{\mu(t) - \mu(\delta+t)}{\mu(4)}$$

for every $\delta \in (0, 1/2)$ and $t \in [t_*, 2]$, by using the continuity of μ, we choose $\delta_2 \in (0, \delta_1]$ such that

$$\begin{aligned} &\mu(t) \le (1+\varepsilon)\, \mu\left(t(1-\delta_2)^{-1}\right) \quad \text{and} \\ &\mu(t) \le (1+\varepsilon)\, \mu(t+\delta_2) \text{ for every } t \in [t_*, 2]. \end{aligned} \tag{3.4}$$

Combining (3.2)–(3.4), we have that for every $\delta \le \delta_2$,

$$\mu(t) \le (1+\varepsilon)^{4+\alpha/2} \times \begin{cases} \mu\left(t(1-\delta)^{-1}\right) & \text{when } t < 2 \\ \mu(t+\delta) & \text{when } t \ge 1/2. \end{cases} \tag{3.5}$$

Let $r > 2$. Using (2.3), we put

$$j(r+\delta) = \left(\int_0^1 + \int_1^\infty\right) (4\pi t)^{-d/2} \exp\left(-\frac{(r+\delta)^2}{4t}\right) \mu(t)\, dt =: I + II.$$

Since $(1-\delta)(r+\delta)^2 \le r^2 + \delta(r+\delta)\left(2 - (r+\delta)\right) \le r^2$, by (3.5) and a change of variables,

$$\begin{aligned} I &\ge \int_0^1 (4\pi t)^{-d/2} \exp\left(-\frac{(1-\delta)^{-1}r^2}{4t}\right) \mu(t)\, dt \\ &= (1-\delta)^{-1+d/2} \int_0^{1-\delta} (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu\left(t(1-\delta)^{-1}\right) dt \\ &\ge (1-\delta)^{-1+d/2} (1+\varepsilon)^{-4-\alpha/2} \int_0^{1-\delta} (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(t)\, dt \end{aligned}$$

for every $\delta \le \delta_2$.

On the other hand, from $0 \le (r+\delta-t)^2 = (r+\delta)^2 - 2tr + t(t-\delta) - \delta t$, we see that $t(t-\delta) \ge 2tr + \delta t - (r+\delta)^2$. Thus we get

$$\frac{(r+\delta)^2}{4t} - \frac{r^2}{4(t-\delta)} = \frac{(r+\delta)^2(t-\delta) - r^2 t}{4t(t-\delta)} = \frac{\delta(2tr+\delta t-(r+\delta)^2)}{4t(t-\delta)} \le \frac{\delta}{4}.$$

Therefore by using this, a change of variables, (3.5) and the inequality $t+\delta \le t(1-\delta)^{-1}$ for $1-\delta \le t < \infty$, we get

$$\begin{aligned} II &\ge e^{-\delta/4} \int_1^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4(t-\delta)}\right) \mu(t)\,dt \\ &= e^{-\delta/4} \int_{1-\delta}^\infty (4\pi(t+\delta))^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(t+\delta)\,dt \\ &\ge e^{-\delta/4}(1+\varepsilon)^{-4-\alpha/2}(1-\delta)^{d/2} \int_{1-\delta}^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(t)\,dt \end{aligned}$$

for every $\delta \le \delta_2$.

Consequently for every $\delta \le \delta_2$ and $r > 2$,

$$j(r+\delta) \ge \left((1-\delta)^{-1+d/2} \wedge e^{-\delta/4}(1-\delta)^{d/2}\right)(1+\varepsilon)^{-4-\alpha/2} j(r)$$

and so

$$\limsup_{\delta\downarrow 0} \left(\sup_{r>2} \frac{j(r)}{j(r+\delta)}\right) \le (1+\varepsilon)^{4+\alpha/2}.$$

Since $\varepsilon > 0$ is arbitrary and $\frac{j(r)}{j(r+\delta)} \ge 1$, the proof is completed. $\square$

Lemma 3.3.

$$\lim_{\delta\downarrow 0} \sup_{r\in(0,4]} \frac{j(r)}{j(r(1+\delta))} = 1.$$

Proof. Fix $\varepsilon > 0$ and let $\mathcal{A} := \alpha\Gamma((d+\alpha)/2)2^{-1+\alpha}\pi^{-d/2}(\Gamma(1-\alpha/2))^{-1}$. By Potter's Theorem [6, Theorem 1.5.6(i)], there exists $r_1 = r_1(\varepsilon) > 0$ such that

$$\frac{\ell(t^{-2})}{\ell(s^{-2})} \ge (1+\varepsilon)^{-1} \min\left\{\frac{t}{s}, \frac{s}{t}\right\} \quad \text{for } s, t \le 2r_1.$$

Moreover by Theorem 2.1, there exists $r_2 = r_2(\varepsilon) > 0$ such that

$$1+\varepsilon \ge \frac{\mathcal{A}\ell(s^{-2})}{s^{d+\alpha} j(s)} \ge (1+\varepsilon)^{-1} \quad \text{for } s \le 2r_2.$$

Thus for $r \le r_3 := r_1 \wedge r_2$ and $\delta \in (0,1)$

$$\begin{aligned} \frac{j(r(1+\delta))}{j(r)} &= \left(\frac{j(r(1+\delta))\, r^{d+\alpha}(1+\delta)^{d+\alpha}}{\mathcal{A}\,\ell\left(r^{-2}(1+\delta)^{-2}\right)}\right)\left(\frac{\mathcal{A}\,\ell(r^{-2})}{r^{d+\alpha} j(r)}\right) \frac{\ell\left(r^{-2}(1+\delta)^{-2}\right)}{\ell(r^{-2})} \\ &\quad \times (1+\delta)^{-d-\alpha} \\ &\ge (1+\varepsilon)^{-3}(1+\delta)^{-d-\alpha-1}. \end{aligned}$$

On the other hand for every $\delta \in (0,1)$ and $r \in [r_3, 4]$,

$$\frac{j(r) - j((1+\delta)r)}{j((1+\delta)r)} \leq \frac{j(r) - j((1+\delta)r)}{j(8)} \leq j(8)^{-1}\delta r|j'(r_3)| \leq 4j(8)^{-1}\delta|j'(r_3)|$$

and so $\left(1 + 4j(8)^{-1}\delta|j'(r_3)|\right) j(r(1+\delta)) \geq j(r)$. Therefore

$$\limsup_{\delta\downarrow 0}\left(\sup_{r\in(0,4]}\frac{j(r)}{j(r(1+\delta))}\right) \leq (1+\varepsilon)^3.$$

Since $\varepsilon > 0$ is arbitrary and $\frac{j(r)}{j(r(1+\delta))} \geq 1$, we complete the proof. □

In this section, for the notational convention we define

$$\Lambda_{a,b}(u) := \int_{A(0,a,b)} j(|y|)u(y)dy \quad \text{and} \quad \Lambda_a(u) := \int_{B_a^c} j(|y|)u(y)dy$$

for every nonnegative function u on $\mathbb{R}^d$ and constants a and b with $b > a > 0$. By Lemmas 3.2 and 3.3, there exists an increasing continuous function $\delta(\varepsilon) : (0, 1/2] \to (0, 1/2]$ such that $\lim_{\varepsilon\downarrow 0}\delta(\varepsilon) = 0$ and

$$\left(\sup_{r>2}\frac{j(r)}{j(r+\delta(\varepsilon))}\right) \vee \left(\sup_{r\in(0,4]}\frac{j(r)}{j(r(1+\delta(\varepsilon)))}\right) \leq 1+\varepsilon. \tag{3.6}$$

Lemma 3.4. *For every $0 < \varepsilon \leq 1/2$, $0 < p \leq 1/2$, $r \leq 2$ and any nonnegative function u in $\mathbb{R}^d$, we have for every $x \in B_{\delta pr/3}$*

$$(1+\varepsilon)^{-1}\Lambda_{pr}(u)\mathbb{E}_x[\tau_{B_{\delta pr/3}}] \leq \int_{B_{pr}^c} P_{B_{\delta pr/3}}(x,y)u(y)dy \leq (1+\varepsilon)\Lambda_{pr}(u)\mathbb{E}_x[\tau_{B_{\delta pr/3}}]$$

where $\delta = \delta(\varepsilon) \in (0, 1/2]$ is in (3.6).

Proof. If $z \in B_{\delta pr/3}$ and $y \in A(0, pr, 1)$, then we have

$$|y - z| \leq |y| + |z| \leq |y| + \delta pr/3 \leq (1+\delta/3)|y| \leq (1+\delta)|y|$$

and

$$|y - z| \geq |y| - |z| \geq |y| - \delta pr/3 \geq (1-\delta/3)|y| \geq (1+\delta)^{-1}|y|.$$

Thus by (3.6) and the fact that $r \mapsto j(r)$ is decreasing,

$$1+\varepsilon \geq \frac{j((1+\delta)^{-1}|y|)}{j(|y|)} \geq \frac{j(|y-z|)}{j(|y|)} \geq \frac{j((1+\delta)|y|)}{j(|y|)} \geq (1+\varepsilon)^{-1}$$

for $y \in A(0, pr, 1)$.

On the other hand, since the assumptions $r \leq 2$ and $p \leq 1/2$ imply $\delta pr/3 \leq \delta$, we have

$$|y - z| \leq |y| + |z| \leq |y| + \delta pr/3 \leq |y| + \delta$$

and

$$|y - z| \geq |y| - |z| \geq |y| - \delta pr/3 \geq |y| - \delta.$$

Thus by (3.6) and the fact that j is decreasing,

$$1+\varepsilon \geq \frac{j(|y|-\delta)}{j(|y|)} \geq \frac{j(|y-z|)}{j(|y|)} \geq \frac{j(|y|+\delta)}{j(|y|)} \geq (1+\varepsilon)^{-1} \quad \text{for } |y| \geq 1.$$

So we have for $x \in B_{\delta pr/3}$,

$$\begin{aligned}\int_{B_{pr}^c} P_{B_{\delta pr/3}}(x,y)u(y)dy &= \int_{B_{pr}^c}\int_{B_{\delta pr/3}} G_{B_{\delta pr/3}}(x,z)j(|z-y|)dz\,u(y)dy \\ &\leq (1+\varepsilon)\int_{B_{\delta pr/3}} G_{B_{\delta pr/3}}(x,z)dz \int_{B_{pr}^c} j(|y|)u(y)dy \\ &= (1+\varepsilon)\mathbb{E}_x[\tau_{B_{\delta pr/3}}]\Lambda_{pr}(u)\end{aligned}$$

and

$$\begin{aligned}\int_{B_{pr}^c} P_{B_{\delta pr/3}}(x,y)u(y)dy &\geq (1+\varepsilon)^{-1}\int_{B_{\delta pr/3}} G_{B_{\delta pr/3}}(x,z)dz \int_{B_{pr}^c} j(|y|)u(y)dy \\ &= (1+\varepsilon)^{-1}\mathbb{E}_x[\tau_{B_{\delta pr/3}}]\Lambda_{pr}(u). \quad \square\end{aligned}$$

The next two results were proved in [28] in a more general setting.

Lemma 3.5 (*[28, Lemma 5.2]*). *For every $p \in (0,1)$, there exists $c = c(\alpha, d, \ell, p) > 0$ such that for every $r \in (0,1)$ and $(x,y) \in B_{pr} \times B_r^c$,*

$$P_{B_r}(x,y) \leq \frac{c}{\phi(r^{-2})}\left(\int_{A(0,(1+p)r/2,r)} j(|z|)P_{B_r}(z,y)dz + j(|y|)\right).$$

Lemma 3.6 (*[28, Lemma 5.4]*). *There exists $c = c(\alpha, d, \ell) > 1$ such that for every $r \in (0,1)$ and $(x,y) \in B_{r/2} \times B_r^c$,*

$$P_{B_r}(x,y) \geq \frac{c}{\phi(r^{-2})}\left(\int_{A(0,r/2,r)} j(|z|)P_{B_r}(z,y)dz + j(|y|)\right).$$

Note that since ℓ is slowly varying at ∞ and ℓ is strictly positive and continuous on $(0,\infty)$, there exists a constant $c = c(\alpha,\ell) > 1$ such that for every $r \in (0,1)$,

$$c^{-1} \leq \frac{\ell\left((2r/3)^{-2}\right)}{\ell(r^{-2})} \leq \left(\frac{\ell\left((2r/3)^{-2}\right)}{\ell(r^{-2})} \vee \frac{\ell\left((r/2)^{-2}\right)}{\ell(r^{-2})}\right) \leq c. \tag{3.7}$$

Recall that C_0 is the constant in Theorem 2.5.

Lemma 3.7. *There exists $C_* = C_*(\alpha, d, \ell) \geq C_0$ such that for every $r \in (0,1)$, any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in B_r with respect to X and for any $x \in B_{r/2}$,*

$$\begin{aligned} C_*^{-1}\mathbb{E}_x[\tau_{B_r}]\Lambda_{r/2}(u) \leq u(x) &\leq C_*\,\mathbb{E}_x[\tau_{B_{2r/3}}]\Lambda_{3r/4}(u) \quad (3.8)\\ &\leq C_*\,\mathbb{E}_x[\tau_{B_r}]\Lambda_{r/2}(u). \quad (3.9)\end{aligned}$$

Proof. Since u is regular harmonic in B_r with respect to X and $\mathbb{P}_z(X_{\tau_{B_r}} \in \partial B_r) = 0$ for $z \in B_r$, we have $u(z) = \int_{B_r^c} P_{B_r}(z,y)u(y)dy$ for every $z \in B_r$ (see (2.5)). Thus by using Lemma 3.5 in the first, and (3.7) in the second inequality, we get

$$\begin{aligned}
u(x) &\le \frac{c_1}{\phi(r^{-2})}\left(\int_{B_r^c}\int_{A(0,3r/4,r)} j(|z|)P_{B_r}(z,y)dzu(y)dy + \int_{B_r^c} j(|y|)u(y)dy\right)\\
&= \frac{c_1}{\phi(r^{-2})}\left(\int_{A(0,3r/4,r)} j(|z|)\left(\int_{B_r^c} P_{B_r}(z,y)u(y)dy\right)dz + \int_{B_r^c} j(|y|)u(y)dy\right)\\
&= \frac{c_1}{\phi(r^{-2})}\left(\int_{A(0,3r/4,r)} j(|z|)u(z)dz + \int_{B_r^c} j(|y|)u(y)dy\right)\\
&\le \frac{c_2}{\phi\left((2r/3)^{-2}\right)}\int_{B_{3r/4}^c} j(|y|)u(y)dy.
\end{aligned}$$

Similarly using Lemma 3.6, we also get $u(x) \ge \frac{c_3}{\phi(r^{-2})}\int_{B_{r/2}^c} j(|y|)u(y)dy$. Now applying [26, Lemmas 13.4.2 and 13.4.3], we have proved (3.8). (3.9) follows immediately from (3.8). □

For the remainder of the section, we fix C_* in Lemma 3.7.

Lemma 3.8. *Suppose that* $r \in (0,1)$. *For nonnegative functions* u_1, u_2 *in* $\mathbb{R}^d$ *which are harmonic in* B_r *with respect to* X, *we have for every* $0 < p < q/4 < 1/8$,

$$\left(\sup_{B_{pr}}\frac{g_1}{g_2} - \inf_{B_{pr}}\frac{g_1}{g_2}\right) \le \frac{C_*^2-1}{C_*^2+1}\left(\sup_{B_{qr}}\frac{u_1}{u_2} - \inf_{B_{qr}}\frac{u_1}{u_2}\right),$$

where $g_i(x) := \mathbb{E}_x[u_i(X_{\tau_{B_{2pr}}}) : X_{\tau_{B_{2pr}}} \in A(0,2pr,qr)]$.

Proof. For $a > 0$, we define $m_a = \inf_{B_a}(u_1/u_2)$ and $M_a = \sup_{B_a}(u_1/u_2)$. Let

$$f(x) := \mathbb{E}_x[(u_1 - m_{qr}u_2)(X_{\tau_{B_{2pr}}}) : X_{\tau_{B_{2pr}}} \in A(0,2pr,qr)] = g_1(x) - m_{qr}g_2(x)$$

and

$$h(x) := \mathbb{E}_x[(M_{qr}u_2 - u_1)(X_{\tau_{B_{2pr}}}) : X_{\tau_{B_{2pr}}} \in A(0,2pr,qr)] = M_{qr}g_2(x) - g_1(x),$$

then f and h are regular harmonic in B_{2pr} and nonnegative in $\mathbb{R}^d$. Thus by applying (3.9) to f and h, we get

$$\sup_{B_{pr}}\frac{g_1}{g_2} - m_{qr} = \sup_{B_{pr}}\frac{f}{g_2} \le C_*^2\inf_{B_{pr}}\frac{f}{g_2} = C_*^2\left(\inf_{B_{pr}}\frac{g_1}{g_2} - m_{qr}\right)$$

and

$$M_{qr} - \inf_{B_{pr}}\frac{g_1}{g_2} = \sup_{B_{pr}}\frac{h}{g_2} \le C_*^2\inf_{B_{pr}}\frac{h}{g_2} = C_*^2\left(M_{qr} - \sup_{B_{pr}}\frac{g_1}{g_2}\right).$$

By adding these inequalities, we proved the lemma. □

Now we are ready to prove the main result of this section. We prove the main result for the quotient of two harmonic functions in the next theorem. We closely follow the proof of [10, Lemma 8].

Theorem 3.9. *For every $\eta > 0$, there exists $a = a(\eta, \alpha, d, \ell) \in (0, 1)$ such that for every $x_0 \in \mathbb{R}^d$ and $r \in (0, 1]$,*

$$\sup_{B(x_0,ar)} \frac{u_1}{u_2} \le (1+\eta) \inf_{B(x_0,ar)} \frac{u_1}{u_2}$$

for nonnegative functions u_1 and u_2 in $\mathbb{R}^d$ which are harmonic in $B(x_0, r)$ with respect to X.

Proof. We assume $x_0 = 0$. We fix $r \in (0, 1]$ and nonnegative functions u_1, u_2 in $\mathbb{R}^d$ which are harmonic in B_r with respect to X. Fix $\eta > 0$ and let

$$\varphi(t) := 1 + \frac{\eta}{2(C_*^2+1)} + \frac{C_*^2}{C_*^2+1}(t-1)$$

for $t \ge 1$ and $\varphi^1 := \varphi$, $\varphi^{l+1} := \varphi(\varphi^l)$ for $l = 1, 2, \ldots$.

Then

$$\varphi^l(C_*^2) = 1 + \frac{\eta}{2(C_*^2+1)} \sum_{i=0}^{l-1} \left(\frac{C_*^2}{C_*^2+1}\right)^i + \left(\frac{C_*^2}{C_*^2+1}\right)^l (C_*^2 - 1)$$
$$\le 1 + \frac{\eta}{2} + \left(\frac{C_*^2}{C_*^2+1}\right)^l (C_*^2 - 1).$$

Choose $l = l(C_*, \eta)$ large such that

$$\left(\frac{C_*^2}{C_*^2+1}\right)^l (C_*^2 - 1) < \frac{\eta}{2} \quad \text{so that } \varphi^l(C_*^2) < 1 + \eta. \tag{3.10}$$

Also we choose $\varepsilon = \varepsilon(\eta)$ small enough so that

$$1 + \frac{\eta}{C_*^2+1} \ge \left(C_*^3\,\varepsilon + (1+\varepsilon)\right)^2 (1+\varepsilon)^2, \tag{3.11}$$

$$(1 + C_*^2\varepsilon)^2 \le 1 + \frac{\eta}{2(C_*^2+1)} \quad \text{and} \quad 1 + C_*^2\,\varepsilon \le \frac{C_*^2}{C_*^2 - 1}. \tag{3.12}$$

Let $k = k(\varepsilon) \ge 3$ be the smallest integer such that $k > 1 + 1/\varepsilon^2$. We recall that $\delta = \delta(\varepsilon) > 0$ is the constant from (3.6) and fix it. Let $p_i := (\delta/6)^i/2$ for $i = 0, \ldots, lk-1$. For simplicity, we put $m_a := \inf_{B_a} u_1/u_2$ and $M_a := \sup_{B_a} u_1/u_2$.

Case 1. Suppose that the following holds for both $i = 1$ and 2; for every $0 \le m < lk$,

$$\int_{A(0,rp_{m+1},rp_m)} j(|y|)u_i(y)\,dy = \Lambda_{rp_{m+1},rp_m}(u_i) > \varepsilon\Lambda_{rp_m}(u_i)$$
$$= \varepsilon \int_{B^c_{rp_m}} j(|y|)u_i(y)\,dy.$$

By the definition of k, for $0 \le j \le l-1$

$$\Lambda_{2rp_{(j+1)k},rp_{jk}}(u_i) \ge \Lambda_{rp_{(j+1)k-1},rp_{jk}}(u_i) = \sum_{m=0}^{k-2} \Lambda_{rp_{jk+m+1},rp_{jk+m}}(u_i)$$
$$\ge \varepsilon \sum_{m=0}^{k-2} \Lambda_{rp_{jk+m}}(u_i) \ge (k-1)\varepsilon\,\Lambda_{rp_{jk}}(u_i) \ge \varepsilon^{-1}\Lambda_{rp_{jk}}(u_i). \tag{3.13}$$

For $i=1,2$ and $j=1,\dots,l-1$, we let

$$f_i^j(x) := \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{(j+1)k}}}}) : X_{\tau_{B_{2rp_{(j+1)k}}}} \in B^c_{rp_{jk}}] = \int_{B^c_{rp_{jk}}} P_{B_{2rp_{(j+1)k}}}(x,y)\,u_i(y)\,dy$$

and

$$\begin{aligned} g_i^j(x) &:= \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{(j+1)k}}}}) : X_{\tau_{B_{2rp_{(j+1)k}}}} \in A(0,2rp_{(j+1)k},rp_{jk})] \\ &= \int_{A(0,2rp_{(j+1)k},rp_{jk})} P_{B_{2rp_{(j+1)k}}}(x,y)\,u_i(y)\,dy, \end{aligned}$$

which are regular harmonic in $B_{2rp_{(j+1)k}}$ and $u_i = f_i^j + g_i^j$.

By (3.8) applied to $B_{rp_{(j+1)k}}$ in the first, and the facts that $f_i^j(x)=0$ on $A(0,2rp_{(j+1)k},rp_{jk})$ and $f_i^j(x)=u_i(x)$ on $B^c_{rp_{jk}}$ in the second inequality, we have for $x\in B_{rp_{(j+1)k}}$,

$$f_i^j(x) \le C_*\,\mathbb{E}_x[\tau_{B_{\frac{4}{3}rp_{(j+1)k}}}]\,\Lambda_{\frac{3}{2}rp_{(j+1)k}}(f_i^j) \le C_*\,\mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\,\Lambda_{rp_{jk}}(u_i)$$

for $j=1,\dots,l-1$.

Hence by (3.13), the fact that $g_i^j(x)=u_i(x)$ on $A(0,2p_{(j+1)k}r,p_{jk}r)$ and (3.9) applied to $B_{rp_{(j+1)k}}$,

$$\begin{aligned} f_i^j(x) &\le C_*\,\varepsilon\,\mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\,\Lambda_{2rp_{(j+1)k},rp_{jk}}(u_i) = C_*\,\varepsilon\,\mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\,\Lambda_{2rp_{(j+1)k}}(g_i^j) \\ &\le C_*\,\varepsilon\,\mathbb{E}_x[\tau_{B_{2rp_{(j+1)k}}}]\,\Lambda_{rp_{(j+1)k}}(g_i^j) \le C_*^2\,\varepsilon\,g_i^j(x) \end{aligned}$$

for $x\in B_{rp_{(j+1)k}}$ and $j=1,\dots,l-1$.

Since $u_i(x)=f_i^j(x)+g_i^j(x)$ and $\frac{g_1^j}{f_2^j+g_2^j}\le\frac{u_1}{u_2}\le\frac{f_1^j+g_1^j}{g_2^j}$, we have

$$(1+C_*^2\,\varepsilon)^{-1}\inf_{B_{rp_{(j+1)k}}}\frac{g_1^j}{g_2^j} \le m_{rp_{(j+1)k}} \le M_{rp_{(j+1)k}} \le (1+C_*^2\,\varepsilon)\sup_{B_{rp_{(j+1)k}}}\frac{g_1^j}{g_2^j}$$

for $j=1,\dots,l-1$.

Thus by Lemma 3.8,

$$\begin{aligned} &(C_*^2+1)\left((1+C_*^2\varepsilon)^{-1}M_{rp_{(j+1)k}} - (1+C_*^2\varepsilon)m_{rp_{(j+1)k}}\right) \\ &\quad\le (C_*^2+1)\left(\sup_{B_{rp_{(j+1)k}}}\frac{g_1^j}{g_2^j} - \inf_{B_{rp_{(j+1)k}}}\frac{g_1^j}{g_2^j}\right) \le (C_*^2-1)(M_{rp_{jk}}-m_{rp_{jk}}) \end{aligned}$$

for $j=1,\dots,l-1$.

Multiplying by $(1+C_*^2\,\varepsilon)/(m_{rp_{(j+1)k}}(C_*^2+1))$ and using the obvious fact $m_{rp_{(j+1)k}}\ge m_{rp_{jk}}$, we obtain

$$\frac{M_{rp_{(j+1)k}}}{m_{rp_{(j+1)k}}} \le (1+C_*^2\varepsilon)^2 + (1+C_*^2\varepsilon)\frac{C_*^2-1}{C_*^2+1}\left(\frac{M_{rp_{jk}}}{m_{rp_{jk}}}-1\right).$$

By the definition of φ and (3.12), $\frac{M_{rp_{(j+1)k}}}{m_{rp_{(j+1)k}}} \le \varphi\left(\frac{M_{rp_{jk}}}{m_{rp_{jk}}}\right)$. We already know that $\frac{M_{r/2}}{m_{r/2}} \le C_*^2$ by (3.9). And also by the monotonicity of φ and (3.10), we get

$$\frac{M_{rp_{lk}}}{m_{rp_{lk}}} \le \varphi\left(\frac{M_{rp_{(l-1)k}}}{m_{rp_{(l-1)k}}}\right) \le \cdots \le \varphi^l\left(\frac{M_{r/2}}{m_{r/2}}\right) \le \varphi^l(C_*^2) < 1+\eta.$$

Case 2. Suppose that there exists $m < lk$ such that for either $i = 1$ or 2,

$$\begin{aligned}\int_{A(0,rp_{m+1},rp_m)} j(|y|)u_i(y)\,dy &= \Lambda_{rp_{m+1},rp_m}(u_i) \le \varepsilon\,\Lambda_{rp_m}(u_i)\\ &= \varepsilon\int_{B^c_{rp_m}} j(|y|)u_i(y)\,dy.\end{aligned}$$

Note that by (3.9),

$$C_*^{-1}\frac{u_{3-i}(y)}{\Lambda_{rp_m}(u_{3-i})} \le \mathbb{E}_y[\tau_{B_{2rp_m}}] \le C_*\frac{u_i(y)}{\Lambda_{rp_m}(u_i)} \quad \text{for } y \in A(0, rp_{m+1}, rp_m).$$

Hence by integrating on $A(0, rp_{m+1}, rp_m)$, we get

$$\frac{\Lambda_{rp_{m+1},rp_m}(u_{3-i})}{\Lambda_{rp_m}(u_{3-i})} \le C_*^2\frac{\Lambda_{rp_{m+1},rp_m}(u_i)}{\Lambda_{rp_m}(u_i)} \le C_*^2\varepsilon.$$

Thus

$$\Lambda_{rp_{m+1},rp_m}(u_i) \le C_*^2\varepsilon\,\Lambda_{rp_m}(u_i) \quad \text{for both } i = 1 \text{ and } 2. \tag{3.14}$$

Let

$$\begin{aligned}f_i^m(x) = f_i(x) &:= \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{m+1}}}}) : X_{\tau_{B_{2rp_{m+1}}}} \in B^c_{rp_m}]\\ &= \int_{B^c_{rp_m}} P_{B_{2rp_{m+1}}}(x,y)\,u_i(y)\,dy\end{aligned}$$

and

$$\begin{aligned}g_i^m(x) = g_i(x) &:= \mathbb{E}_x[u_i(X_{\tau_{B_{2rp_{m+1}}}}) : X_{\tau_{B_{2rp_{m+1}}}} \in A(0, 2rp_{m+1}, rp_m)]\\ &= \int_{A(0,2rp_{m+1},rp_m)} P_{B_{2rp_{m+1}}}(x,y)\,u_i(y)\,dy,\end{aligned}$$

so that $u_i = f_i + g_i$. Since g_i is regular harmonic in $B_{2rp_{m+1}}$, by (3.8) we obtain for $x \in B_{rp_{m+1}}$,

$$g_i(x) \le C_*\,\mathbb{E}_x[\tau_{B_{\frac{4}{3}rp_{m+1}}}]\Lambda_{\frac{3}{2}rp_{m+1}}(g_i) \le C_*\,\mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_{m+1}}(g_i).$$

Also since $g_i = 0$ on $\overline{B_{rp_m}}^c$ and $g_i = u_i$ on $A(0, 2rp_{m+1}, rp_m)$, we get

$$\begin{aligned}g_i(x) &\le C_*\,\mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_{m+1},rp_m}(g_i) \le C_*\,\mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_{m+1},rp_m}(u_i)\\ &\le \varepsilon\,C_*^3\,\mathbb{E}_x[\tau_{B_{2rp_{m+1}}}]\Lambda_{rp_m}(u_i) \quad \text{for } x \in B_{rp_{m+1}}.\end{aligned}$$

The last inequality comes from (3.14).

Then by (3.14), applying Lemma 3.4 to $f_i(x)$ and the fact that $\frac{f_1}{f_2+g_2} \le \frac{u_1}{u_2} \le \frac{f_1+g_1}{f_2}$, we have

$$\frac{(1+\varepsilon)^{-1}\Lambda_{rp_m}(u_1)}{\left((1+\varepsilon)+\varepsilon\, C_*^3\right)\Lambda_{rp_m}(u_2)} \le \frac{u_1(x)}{u_2(x)} \le \frac{\left((1+\varepsilon)+\varepsilon\, C_*^3\right)\Lambda_{rp_m}(u_1)}{(1+\varepsilon)^{-1}\Lambda_{rp_m}(u_2)} \quad \text{for } x \in B_{rp_{m+1}}.$$

So by (3.11), $\frac{M_{rp_{lk}}}{m_{rp_{lk}}} \le \frac{M_{rp_{m+1}}}{m_{rp_{m+1}}} \le \left(\varepsilon\, C_*^3 + (1+\varepsilon)\right)^2 (1+\varepsilon)^2 \le 1 + \frac{\eta}{C_*^2+1} < 1+\eta$.

In these two cases, we prove the theorem with $a = p_{lk}$. □

Proof of Theorem 1.1. Take $u_1 = u$ and $u_2 \equiv 1$ in Theorem 3.9. □

As a corollary of Theorem 1.1, we get the following.

Corollary 3.10. *There exists an increasing continuous function* $\theta : (0,1) \to (0,\infty)$ *with* $\lim_{t\to 0} \theta(t) = 0$ *such that for every* $x_0 \in \mathbb{R}^d$, $R \in (0,1]$ *and* $r < R/2$,

$$\sup_{x,y\in B(x_0,R/2),|x-y|<r} |u(x)-u(y)| \le \theta(|x-y|/r) \sup_{w\in B(x_0,R)} |u(w)|$$

for nonnegative function u *in* $\mathbb{R}^d$ *which is harmonic in* $B(x_0,R)$ *with respect to* X.

Proof. Without loss of generality, we assume $x_0 = 0$. For fixed $R \in (0,1]$ and r with $r < R/2$, let $x, y \in B_{R/2}$ be such that $|x-y| < r$ and $x, y \in B(z, |x-y|) \subset B_R$ for some $z \in B_{R/2}$. For a nonnegative integer k, by Theorem 1.1 we can choose $a_{k+1} < a_k$ recurrently such that

$$\sup_{B(z,ra_k)} u \le (1+2^{-k-1}) \inf_{B(z,ra_k)} u \quad \text{for } z \in B_{R/2}. \tag{3.15}$$

Define $a(\eta)$ using the linear interpolation as

$$a(\eta) = \begin{cases} a_k & \text{if } \eta = 2^{-k} \\ \dfrac{a_k - a_{k+1}}{2^{-k}-2^{-k-1}}\,\eta + 2a_{k+1} - a_k & \text{if } 2^{-k-1} < \eta < 2^{-k}. \end{cases}$$

Then $a(\eta)$ is continuous and strictly increasing, so there exists an inverse function $\theta := a^{-1} : (0,1) \to (0,\infty)$, which is increasing and continuous.

Now we choose a nonnegative integer k such that $a_{k+1} \le \frac{|x-y|}{r} < a_k$, so that $2^{-k-1} \le \theta\left(\frac{|x-y|}{r}\right)$. Using this and (3.15), we get

$$\begin{aligned}\sup_{B(z,|x-y|)} u &\le \sup_{B(z,ra_k)} u \le (1+2^{-k-1}) \inf_{B(z,ra_k)} u \le \left(1+\theta\left(\frac{|x-y|}{r}\right)\right) \inf_{B(z,ra_k)} u \\ &\le \left(1+\theta\left(\frac{|x-y|}{r}\right)\right) \inf_{B(z,|x-y|)} u.\end{aligned}$$

Therefore

$$\begin{aligned}|u(x)-u(y)| &\le \sup_{B(z,|x-y|)} u - \inf_{B(z,|x-y|)} u \le \theta\left(\frac{|x-y|}{r}\right) \inf_{B(z,|x-y|)} u \\ &\le \theta\left(\frac{|x-y|}{r}\right) \sup_{B_R} u. \quad \square\end{aligned}$$

Even though this corollary gives merely the continuity estimates, notice that the supremum is taken over the ball $B(x_0,R)$ and not the whole space $\mathbb{R}^d$ as in the existing literature (see [3,4,2,12,18,19,22,32,34]).

4. The relative Fatou theorem

In this section, we assume that $d \geq 2$. In the case $d = 2$, we will always assume the following.

(A2) *There exists* $\gamma \in (0, 1)$ *such that* $\liminf_{\lambda \to 0} \phi(\lambda)/\lambda^{\gamma} > 0$.

Then by the criterion of Chung–Fuchs type, the process X is transient under this assumption (see [26, (13.3.1)]).

In this section, using Theorem 1.1 we prove the relative Fatou theorem. The proofs of the results in this section are similar to the corresponding parts of [23]. For this reason, some proofs in this section will be omitted.

In this section, we assume that D is a bounded κ-fat open set. We recall the definition of κ-fat open set.

Definition 4.1. Let $\kappa \in (0, 1/2]$. We say that an open set D in $\mathbb{R}^d$ is κ-fat if there exists $R > 0$ such that for each $Q \in \partial D$ and $r \in (0, R)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (R, κ) is called the characteristics of the κ-fat open set D.

Note that all Lipschitz domains and all non-tangentially accessible domains (see [20] for the definition) are κ-fat. The boundary of a κ-fat open set may be not rectifiable, and in general, no regularity of its boundary can be inferred. A bounded κ-fat open set may be disconnected.

The following boundary Harnack principle is the main result in [29,26].

Theorem 4.2 ([29, Theorem 4.8] and [26, Theorem 13.4.22])**.** *Suppose that D is a κ-fat open set with the characteristics (R, κ). There exists a constant $c = c(\alpha, d, \ell, R, \kappa) > 1$ such that if $r \leq R \wedge \frac{1}{4}$ and $Q \in \partial D$, then for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have*

$$c^{-1} \frac{u(A_r(Q))}{v(A_r(Q))} \leq \frac{u(x)}{v(x)} \leq c \frac{u(A_r(Q))}{v(A_r(Q))} \quad \textit{for } x \in D \cap B\left(Q, \frac{r}{2}\right).$$

Let $x_0 \in D$ be fixed and set

$$M_D(x, y) := \frac{G_D(x, y)}{G_D(x_0, y)}, \quad \text{for } x, y \in D \quad \text{and} \quad y \neq x_0.$$

For each fixed $z \in \partial D$ and $x \in D$, let $M_D(x, z) := \lim_{D \ni y \to z} M_D(x, y)$, which exists by Kim et al. [29, Theorem 5.5]. For each $z \in \partial D$, set $M_D(x, z)$ to be zero for $x \in D^c$. M_D is called the Martin kernel of D with respect to X.

As a consequence of [29, Theorem 5.11], for every nonnegative harmonic function h for X^D, there exists a unique finite measure ν on ∂D such that

$$h(x) = \int_{\partial D} M_D(x, z) \nu(dz) \quad \text{for } x \in D.$$

ν is called the Martin measure of h.

We will use $G(x, y) = G(x - y) = \int_0^\infty p(t, x, y)\, dt$ to denote the Green function of X. G is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$.

The proof of the next result is similar to [15, Theorem 2.4] and [23, Lemma 3.2].

Lemma 4.3. *For each $z \in \partial D$, $M_D(\cdot, z)$ is bounded regular harmonic in $D \setminus B(z, \varepsilon)$ for every $\varepsilon > 0$.*

Proof. Fix $z \in \partial D$ and $\varepsilon > 0$, and let $h(x) := M_D(x, z)$ for $x \in \mathbb{R}^d$. Note that $G(x, y) \geq G_D(x, y)$. By Kim et al. [26, Theorem 13.3.2] and [27, Lemma 3.3] and Theorem 4.2, there exist $c_1, c_2 > 0$ which depend on $\alpha, d, \ell, \kappa, R$ and $\mathrm{diam}(D)$ such that for every $x \in D \setminus B(z, \varepsilon/2)$,

$$
\begin{aligned}
h(x) = M_D(x, z) &= \lim_{D \ni y \to z} \frac{G_D(x, y)}{G_D(x_0, y)} \leq c_1 \frac{G_D(x, A)}{G_D(x_0, A)} \\
&\leq c_1 \frac{G(x, A)}{G_D(x_0, A)} \leq c_2 \sup_{y \in D \setminus B(z, \varepsilon/2)} \frac{1}{|y - A|^d \phi(|y - A|^{-2}) G_D(x_0, A)} < \infty
\end{aligned}
$$

where $A := A_{\varepsilon/16}(z)$ (see Definition 4.1). Take an increasing sequence of smooth open sets $\{D_m\}_{m \geq 1}$ such that $\overline{D_m} \subset D_{m+1}$ and $\cup_{m=1}^{\infty} D_m = D \setminus B(z, \varepsilon)$. Set $\tau_m := \tau_{D_m}$ and $\tau_\infty := \tau_{D \setminus B(z, \varepsilon)}$. Then $\tau_m \uparrow \tau_\infty$ and $\lim_{m \to \infty} X_{\tau_m} = X_{\tau_\infty}$ by quasi-left continuity of X. Set $E = \{\tau_m = \tau_\infty$ for some $m \geq 1\}$ and N be the set of irregular boundary points of D. Since X is symmetric, by Blumenthal and Getoor [7, (VI.4.6), (VI.4.10)] we get

$$
\mathbb{P}_x(X_{\tau_\infty} \in N) = 0 \quad \text{for } x \in D. \tag{4.1}
$$

We also know from [29, Lemma 5.9(i)] that if $w \in \partial D$, $w \neq z$ and w is a regular boundary point, then $h(x) \to 0$ as $x \to w$ so that h is continuous on $\overline{D \setminus B(z, \varepsilon)} \setminus N$. Since h is bounded on $\mathbb{R}^d \setminus B(z, \varepsilon/2)$, by the bounded convergence theorem and (4.1), we have

$$
\begin{aligned}
\lim_{m \to \infty} \mathbb{E}_x \left[h(X_{\tau_m}); \tau_m < \tau_\infty \right] &= \lim_{m \to \infty} \mathbb{E}_x \left[h(X_{\tau_m}) 1_{\overline{D \setminus B(z, \varepsilon)} \setminus N}(X_{\tau_m}); \tau_m < \tau_\infty \right] \\
&= \mathbb{E}_x \left[h(X_{\tau_\infty}) 1_{\overline{D \setminus B(z, \varepsilon)} \setminus N}(X_{\tau_\infty}); E^c \right] \\
&= \mathbb{E}_x \left[h(X_{\tau_\infty}); E^c \right].
\end{aligned} \tag{4.2}
$$

Since $\tau_m \uparrow \tau_\infty$ and $\{\tau_m = \tau_\infty\} = \{\tau_n = \tau_\infty, n \geq m\} \uparrow E$ as $m \to \infty$, by (4.2) and the monotone convergence theorem,

$$
\begin{aligned}
h(x) &= \lim_{m \to \infty} \mathbb{E}_x[h(X_{\tau_m})] = \lim_{m \to \infty} \mathbb{E}_x[h(X_{\tau_m}); \tau_m < \tau_\infty] \\
&\quad + \lim_{m \to \infty} \mathbb{E}_x[h(X_{\tau_\infty}); \tau_m = \tau_\infty] \\
&= \mathbb{E}_x[h(X_{\tau_\infty}); E^c] + \mathbb{E}_x[h(X_{\tau_\infty}); E] = \mathbb{E}_x[h(X_{\tau_\infty})]. \quad \square
\end{aligned}
$$

Throughout this paper, $\mathcal{F}_t$ is the augmented right continuous σ-field generated by X_t^D. For a positive harmonic function h with respect to X^D, we let $(\mathbb{P}_x^h, X_t^h)$ be the h-transform of $(\mathbb{P}_x, X_t^D)$, that is,

$$
\mathbb{P}_x^h(A) := \mathbb{E}_x \left[\frac{h(X_t^D)}{h(x)}; A \right] \quad \text{if } A \in \mathcal{F}_t.
$$

When $h(\cdot) = M_D(\cdot, z)$, we use the notation $(\mathbb{P}_x^z, X_t^z) := (\mathbb{P}_x^h, X_t^h)$ so that $(\mathbb{P}_x^z, X_t^z)$ is $M_D(\cdot, z)$-transform of $(\mathbb{P}_x, X_t^D)$.

Let τ_D^z be the life time of X^z. Using [24, Theorem 3.10] and (A1), the proof of the next result is similar to [23, Theorem 3.3].

Theorem 4.4.

$$
\mathbb{P}_x^z \left(\lim_{t \uparrow \tau_D^z} X_t^z = z, \tau_D^z < \infty \right) = 1 \quad \textit{for every } x \in D, z \in \partial D.
$$

Proof. See [23, Theorem 3.3]. □

The following result is a simple consequence of Theorem 4.4.

Proposition 4.5. *Let h be a positive harmonic function with respect to X^D with Martin measure ν. Then*

$$\mathbb{P}_x^h\left(A \cap \left\{\lim_{t\uparrow\tau_D^h} X_t^h \in K\right\}\right) = \frac{1}{h(x)}\int_K M_D(x,z)\mathbb{P}_x^z(A)\nu(dz)$$

for every $x \in D$, $A \in \mathcal{F}_{\tau_D}$ and Borel subset K of ∂D.

Proof. See [23, Proposition 3.5]. □

Definition 4.6. $A \in \mathcal{F}_{\tau_D}$ is shift-invariant if whenever $T < \tau_D$ is a stopping time, $1_A \circ \theta_T = 1_A \mathbb{P}_x$-a.s. for every $x \in D$.

Using [29, Theorem 5.11], the proof of the next proposition is the same as the one in [23, Proposition 3.7] (see also [1, p. 196]).

Proposition 4.7 (*0–1 Law*)**.** *If A is shift-invariant, then $x \to \mathbb{P}_x^z(A)$ is a constant function which is either 0 or 1.*

Using (2.1), [6, Theorem 1.5.3] and the 0-version of [6, Theorem 1.5.11], we have the following inequalities; there exists $c = c(\alpha, d, \ell) > 0$ such that

$$s^d\phi(s^{-2}) \le c\, r^d\phi(r^{-2}) \quad \text{for } 0 < s < r \le 4 \tag{4.3}$$

and

$$\int_0^r \frac{1}{s\,\phi(s^{-2})}ds \le c\,\frac{1}{\phi(r^{-2})} \quad \text{for } 0 < r \le 4. \tag{4.4}$$

From now on, we use notations $T_B := \inf\{t > 0 : X_t \in B\}$, $T_B^z := \inf\{t > 0 : X_t^z \in B\}$ and $B_y^\lambda := B(y, \lambda\delta_D(y))$ for the convenience.

Proposition 4.8. *There exists $c = c(\alpha, \ell, D) > 1$ such that if $0 < \lambda < 1/2$ and $x, y \in D$ with $|y - x| > 2\delta_D(y)$, then*

$$\mathbb{P}_x\left(T_{B_y^\lambda} < \tau_D\right) \ge c\, G_D(x,y)\lambda^d\delta_D(y)^d\phi\left((2\lambda\delta_D(y))^{-2}\right).$$

Proof. Fix $\lambda \in (0, 1/2)$ and $x, y \in D$ with $|y - x| > 2\delta_D(y)$. Since $x \notin B(y, \delta_D(y))$, by Kim et al. [27, Theorem 2.14] we get

$$\mathbb{E}_x\left[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\right] = \int_{B_y^\lambda} G_D(x,z)dz \ge c_1 G_D(x,y)\lambda^d\delta_D(y)^d. \tag{4.5}$$

On the other hand, by the strong Markov property,

$$\begin{aligned}\mathbb{E}_x\left[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\right] &= \mathbb{E}_x\left[\mathbb{E}_{X_{T_{B_y^\lambda}}}\left[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\right] : T_{B_y^\lambda} < \tau_D\right]\\ &\le \mathbb{P}_x\left(T_{B_y^\lambda} < \tau_D\right)\sup_{w\in\overline{B_y^\lambda}}\mathbb{E}_w\left[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\right]. \end{aligned}\tag{4.6}$$

Note that since $0 < \lambda\,\delta_D(y) \le \mathrm{diam}(D)$, by (4.4) and [26, Theorem 13.3.2], we obtain for every $w \in \overline{B_y^\lambda}$

$$
\begin{aligned}
\mathbb{E}_w\left[\int_0^{\tau_D} 1_{B_y^\lambda}(X_s)ds\right] &\le \int_{B_y^\lambda} G(w-v)dv \le c_2 \int_{B_y^\lambda} \frac{dv}{|w-v|^d \phi(|w-v|^{-2})} \\
&\le c_2 \int_{\{|w-v|\le 2\lambda\delta_D(y)\}} \frac{dv}{|w-v|^d \phi(|w-v|^{-2})} \\
&= c_3 \int_0^{2\lambda\delta_D(y)} \frac{1}{s\phi(s^{-2})} ds \le c_4 \frac{1}{\phi\left((2\lambda\delta_D(y))^{-2}\right)}.
\end{aligned}
$$

Combining this with (4.5)–(4.6), we finish the proof. □

Now we define the Stolz open set for κ-fat open set D with the characteristics (R, κ).

Definition 4.9. For $z \in \partial D$ and $\beta > (1-\kappa)/\kappa$, let $A_z^\beta := \{y \in D; \delta_D(y) < R \wedge (\delta_D(x_0)/3)$ and $|y-z| < \beta\,\delta_D(y)\}$. We call A_z^β the Stolz open set for D at z with the angle β.

Since $\beta > (1-\kappa)/\kappa$, there exists a sequence $\{y_k\}_{k\ge1} \subset A_z^\beta$ such that $\lim_{k\to\infty} y_k = z$ (see [23, Lemma 3.9]).

Proposition 4.10. *Given $\beta > (1-\kappa)/\kappa$ and $x \in D$, there exists $c = c(\alpha, \beta, D, x) > 0$ such that for every $z \in \partial D$, $\lambda \in (0, 1/2)$ and $y \in A_z^\beta$ with $\delta_D(y) \le \frac{1}{2}|x-y| \wedge \delta_D(x)$, we have*

$$
\mathbb{P}_x^z\left(T^z_{B_y^\lambda} < \tau_D^z\right) > c\,\lambda^d \frac{\phi\left((2\lambda\delta_D(y))^{-2}\right)}{\phi\left((\delta_D(y)/8)^{-2}\right)}.
$$

Proof. Fix $\beta > (1-\kappa)/\kappa$, $z \in \partial D$, $x \in D$, $\lambda \in (0, 1/2)$ and $y \in A_z^\beta$ with $\delta_D(y) \le \frac{1}{2}|x-y| \wedge \delta_D(x)$. Let $z_1 := A_{\delta_D(y)/8}(z)$ so that $B(z_1, \kappa\,\delta_D(y)/8) \subset B(z, \delta_D(y)/8) \cap D$ and fix $z_2 \in \partial B(y, \delta_D(y)/8)$. Since $M_D(\cdot, z)$ is a harmonic function with respect to X in D (Lemma 4.3), by the Harnack principle [27, Theorem 2.14] and Proposition 4.8 we have

$$
\begin{aligned}
\mathbb{P}_x^z\left(T^z_{B_y^\lambda} < \tau_D^z\right) &= \mathbb{E}_x\left[\frac{M_D(X_{T_{B_y^\lambda}}, z)}{M_D(x,z)}; T_{B_y^\lambda} < \tau_D\right] \ge c_1\,\mathbb{P}_x\left(T_{B_y^\lambda} < \tau_D\right)\frac{M_D(y,z)}{M_D(x,z)} \\
&\ge c_2\,G_D(x,y)\lambda^d \delta_D(y)^d \phi\left((2\lambda\delta_D(y))^{-2}\right) \lim_{D\ni w\to z} \frac{G_D(y,w)}{G_D(x,w)} \\
&\ge c_3\,G_D(x,y)\lambda^d \delta_D(y)^d \phi\left((2\lambda\delta_D(y))^{-2}\right) \frac{G_D(y,z_1)}{G_D(x,z_1)}.
\end{aligned}
$$

The last inequality comes from Theorem 4.2 because $|y-z| \wedge |x-z| > \delta_D(y)/2$. We see that $\delta_D(z_1) \ge \kappa\delta_D(y)/8 > \delta_D(y)/\left(8(\beta+1)\right)$, $\delta_D(z_2) > \delta_D(y)/2$ and $|z_2 - y| = \delta_D(y)/8$. Moreover using our assumptions that $\delta_D(y) \le \delta_D(x)$ and $|x-y| \ge 2\delta_D(y)$, we have

$$
\begin{aligned}
|z_2 - x| &\ge |x-y| - |y-z_2| \ge 2\delta_D(y) - \frac{\delta_D(y)}{8} > \delta_D(y), \\
|z_1 - x| &\ge |x-z| - |z-z_1| \ge \delta_D(x) - \frac{\delta_D(y)}{8} > \frac{\delta_D(y)}{2}
\end{aligned}
$$

and

$$
|z_1 - y| \ge |y-z| - |z_1 - z| \ge \delta_D(y) - \frac{\delta_D(y)}{8} > \frac{\delta_D(y)}{2}.
$$

Thus $G_D(y,\cdot)$ and $G_D(x,\cdot)$ are harmonic functions in $B(z_1, 8^{-1}(\beta+1)^{-1}\delta_D(y)) \cup B(z_2, 8^{-1}(\beta+1)^{-1}\delta_D(y))$. Since $|z_1 - z_2| \le |z_1 - z| + |z - y| + |y - z_2| < (4^{-1}+\beta)\,\delta_D(y)$, by Kim et al. [27, Theorem 2.14] we have $G_D(y, z_1) \ge c_4 G_D(y, z_2)$ and $G_D(x, z_1) \le c_5 G_D(x, z_2) \le c_6 G_D(x, y)$. On the other hand, by Kim et al. [27, Lemma 3.3] and (4.3), we get

$$G_D(y, z_2) \ge c_7 \frac{1}{|y - z_2|^d \phi(|y - z_2|^{-2})} \ge c_8 \frac{1}{\delta_D(y)^d \phi\left((\delta_D(y)/8)^{-2}\right)}.$$

Combining these observations, we prove the proposition. □

Now we are ready to show the relative Fatou theorem for the harmonic function with respect to X in D. The proof is similar to the proof of [23, Theorem 3.13]. But, since we state a slightly more general version, we spell out detail for the reader's convenience.

Theorem 4.11. *Let h be a positive harmonic function with respect to X^D with the Martin measure ν. If u is a nonnegative function which is harmonic in D with respect to X and $x \in D$, then for ν-a.e. $z \in \partial D$, $\lim_{t\uparrow\tau^z_D} u(X^z_t)/h(X^z_t)$ exists and is finite $\mathbb{P}^z_x$-a.s. Moreover, for every $x \in D$ and every $\beta > \frac{1-\kappa}{\kappa}$,*

$$\lim_{t\uparrow\tau^z_D} \frac{u(X^z_t)}{h(X^z_t)} = \lim_{A^\beta_z \ni y \to z} \frac{u(y)}{h(y)} \quad \mathbb{P}^z_x\text{-a.s.} \tag{4.7}$$

In particular, for ν-a.e. $z \in \partial D$,

$$\lim_{A^\beta_z \ni y \to z} \frac{u(y)}{h(y)} \quad \text{exists for every } \beta > \frac{1-\kappa}{\kappa}. \tag{4.8}$$

Proof. Without loss of generality, we assume $\nu(\partial D) = 1$ and fix $x \in D$. Note that u is a non-negative and continuous superharmonic function with respect to X^D, i.e., for $x \in B$, $u(x) \ge \mathbb{E}_x\left[u(X^D_{\tau_B})\right]$ for every open set B whose closure is a compact subset of D. Since X^D is a Hunt process and u is non-negative and continuous superharmonic with respect to X^D, u is excessive with respect to X^D (see [7, Corollary II.5.3] and the second part of the proof of [1, Proposition II.6.7]). In particular, $\mathbb{E}_w[u(X^D_t)] \le u(w)$ for every $w \in D$. So by the Markov property for the conditional process (for example, see [16, Chapter 11]), we have for every $t, s > 0$

$$\mathbb{E}^h_x\left[\left.\frac{u(X^h_{t+s})}{h(X^h_{t+s})}\right| \mathcal{F}_s\right] = \mathbb{E}^h_{X^h_s}\left[\frac{u(X^h_t)}{h(X^h_t)}\right] = \frac{1}{h(X^h_s)}\,\mathbb{E}_{X^h_s}\left[u(X^D_t)\right] \le \frac{u(X^h_s)}{h(X^h_s)}.$$

Therefore we see that $u(X^h_t)/h(X^h_t)$ is a non-negative supermartingale with respect to $\mathbb{P}^h_x$, and so the martingale convergence theorem gives $\lim_{t\uparrow\tau^h_D} u(X^h_t)/h(X^h_t)$ exists and is finite $\mathbb{P}^h_x$-a.s.. Thus by Proposition 4.5, for ν-a.e. $z \in \partial D$,

$$\mathbb{P}^z_x\left(\lim_{t\uparrow\tau^z_D} \frac{u(X^z_t)}{h(X^z_t)} \text{ exists and is finite}\right) = 1. \tag{4.9}$$

Fix $z \in \partial D$ satisfying (4.9) and $\beta > (1-\kappa)/\kappa$. By (2.1) and Proposition 4.10, for every sequence $\{y_k\}_{k=1}^\infty \subset A^\beta_z$ converging to z, $\mathbb{P}^z_x\left(T^z_{B^\lambda_{y_k}} < \tau^z_D \text{ i.o.}\right) \ge \liminf_{k\to\infty} \mathbb{P}^z_x\left(T^z_{B^\lambda_{y_k}} < \tau^z_D\right) > 0$ for every $\lambda \in (0, 1/2)$. Since $\{T^z_{B^\lambda_{y_k}} < \tau^z_D \text{ i.o.}\}$ is shift-invariant, by Proposition 4.7,

$$\mathbb{P}^z_x \left(X^z_t \text{ hits infinitely many } B^\lambda_{y_k} \right) = \mathbb{P}^z_x \left(T^z_{B^\lambda_{y_k}} < \tau^z_D \text{ i.o.} \right) = 1 \quad \text{for every } \lambda \in (0, 1/2). \tag{4.10}$$

Now let

$$m := \liminf_{A^\beta_z \ni y \to z} \frac{u(y)}{h(y)} \quad \text{and} \quad l := \limsup_{A^\beta_z \ni y \to z} \frac{u(y)}{h(y)}.$$

First we note that $l < \infty$. If not, for any $M > 1$, there exists a sequence $\{x_k\}_{k=1}^\infty \subset A^\beta_z$ such that $u(x_k)/h(x_k) > 4M$ and $x_k \to z$. By Theorem 1.1, there exists $\lambda_1 = \lambda_1(M, \alpha, d, \ell) > 0$ such that $u(w)/h(w) \geq M^2(M+1)^{-2} u(x_k)/h(x_k) > M$ for every $w \in B^{\lambda_1}_{x_k}$. Thus by (4.10) we have $\lim_{t \uparrow \tau^z_D} u(X^z_t)/h(X^z_t) > M$, $\mathbb{P}^z_x$-a.s. for every $M > 1$, which is a contradiction to (4.9). Also if $l = 0$, then $0 \leq m \leq l = 0$ so the theorem is clear. So we assume $0 < l < \infty$.

For given $\varepsilon > 0$, choose sequences $\{y_k\}_{k=1}^\infty \cup \{z_k\}_{k=1}^\infty \subset A^\beta_z$ such that $u(y_k)/h(y_k) > (1+\varepsilon)^{-1} l$, $u(z_k)/h(z_k) < m + \varepsilon$ and $y_k, z_k \to z$. By Theorem 1.1, there is $\lambda_2 = \lambda_2(\varepsilon, \alpha, d, \ell) > 0$ such that

$$\frac{u(w)}{h(w)} \geq \frac{u(y_k)}{(1+\varepsilon)^2\, h(y_k)} > \frac{l}{(1+\varepsilon)^3} \quad \text{for every } w \in B^{\lambda_2}_{y_k} \tag{4.11}$$

and

$$\frac{u(w)}{h(w)} \leq (1+\varepsilon)^2 \frac{u(z_k)}{h(z_k)} < (1+\varepsilon)^2 (m+\varepsilon) \quad \text{for every } w \in B^{\lambda_2}_{z_k}. \tag{4.12}$$

Applying (4.9)–(4.12) and letting $\varepsilon \downarrow 0$, we obtain both (4.7) and (4.8). $\square$

If u and h are harmonic functions in D and u/h is bounded, then u can be recovered from non-tangential boundary limit values of u/h.

Theorem 4.12. *If u is a harmonic function in D with respect to X and u/h is bounded for a positive harmonic function h in D with respect to X^D with the Martin measure ν, then for every $x \in D$*

$$u(x) = h(x)\, \mathbb{E}^h_x \left[\varphi_u \left(\lim_{t \uparrow \tau^h_D} X^h_t \right) \right]$$

where $\varphi_u(z) := \lim_{A^\beta_z \ni x \to z} u(x)/h(x)$, $\beta > (1-\kappa)/\kappa$, which is well-defined for ν-a.e. $z \in \partial D$. If we further assume that u is positive in D, then $\varphi_u(z)$ is the Radon–Nikodym derivative of the (unique) Martin measure μ_u with respect to ν.

Proof. Using our Propositions 4.5 and 4.7, the proof is the same as [23, Theorem 3.18] (There are typos in the proof of [23, Theorem 3.18]; v should be replaced by h). $\square$

When the boundary of D is sufficiently smooth, by Kim et al. [27, Theorem 1.1] the Martin kernel enjoys the following estimate:

$$c^{-1} \left(\phi(\delta_D(x)^{-2}) \right)^{-1/2} |x-z|^{-d} \leq M_D(x,z) \leq c \left(\phi(\delta_D(x)^{-2}) \right)^{-1/2} |x-z|^{-d}. \tag{4.13}$$

Now suppose that $d = 2$, $D = B := B(0, 1)$, $x_0 = 0$ and σ_1 is the normalized surface measure on ∂B. It is showed in [23] that the Stolz domain is the best possible one for the Fatou theorem in B for the $(-\Delta)^{\alpha/2}$-harmonic function. Similarly, using (4.13), we can show that our Stolz open set is also the best possible one here.

A curve $\mathcal{C}_0$ is called a tangential curve in B which ends on ∂B if $\mathcal{C}_0 \cap \partial B = \{w_0\} \in \partial B$, $\mathcal{C}_0 \setminus \{w_0\} \subset B$ and there are no $r > 0$ and $\beta > 1$ such that $\mathcal{C}_0 \cap B(w_0, r) \subset A^{\beta}_{w_0} \cap B(w_0, r)$.

Theorem 4.13. *Let $h(x) := \int_{\partial B} M_B(x, w)\sigma_1(dw)$, $\mathcal{C}_0$ be a tangential curve in B which ends on ∂B and $\mathcal{C}_\theta$ be the rotation of $\mathcal{C}_0$ about x_0 through an angle θ. Then there exists a positive harmonic function u with respect to X in $B := B(x_0, 1)$ such that for a.e. $\theta \in [0, 2\pi]$ with respect to Lebesgue measure,*

$$\lim_{|x|\to 1, x\in\mathcal{C}_\theta} \frac{u(x)}{h(x)} \quad \textit{does not exist.}$$

Proof. See [23, Lemma 3.22 and Theorem 3.23]. □

With the relative Fatou theorem given in Theorem 4.11, the proof of Theorem 4.14 is almost identical to the corresponding parts of [23]. For this reason, the proof of Theorem 4.14 will be omitted. We refer [13,14,23] for the definitions of $\mathbf{S}_\infty(X^D)$ and $\mathbf{A}_\infty(X^D)$.

For a smooth measure μ associated with a continuous additive functional A^μ and a Borel measurable function F on $D \times D$ that vanishes along the diagonal, define

$$e_{A^\mu+F}(t) := \exp\left(A^\mu_t + \sum_{0<s\le t} F(X^D_{s-}, X^D_s)\right) \quad \text{for } t \ge 0.$$

Let $\mu \in \mathbf{S}_\infty(X^D)$ and $F \in \mathbf{A}_\infty(X^D)$ such that the gauge function $x \mapsto \mathbb{E}_x\left[e_{A^\mu+F}(\tau_D)\right]$ is bounded. A Borel measurable function k defined on D is said to be a positive (μ, F)-harmonic function if $k > 0$ and $\mathbb{E}_x\left[e_{A^\mu+F}(\tau_B)k(X^D_{\tau_B})\right] = k(x)$ for every open set B whose closure is a compact subset of D and $x \in B$. By Chen and Kim [14, Theorem 5.16 and Section 6], there is a unique finite measure ν on ∂D such that $k(x) = \int_{\partial D} K_D(x, z)\,\nu(dz)$, where $K_D(x, z)$ is the Martin kernel for the semigroup $Q_t f(x) := \mathbb{E}_x[e_{A^\mu+F}(t) f(X^D_t)]$. We call ν the Martin-representing measure of k.

Theorem 4.14. *Let D be a bounded κ-fat open set and k be a positive (μ, F)-harmonic function with the Martin-representing measure ν. If u is a nonnegative (μ, F)-harmonic function, then for ν-a.e. $z \in \partial D$, $\lim_{A^\beta_z \ni x\to z} \frac{u(x)}{k(x)}$ exists for every $\beta > (1-\kappa)/\kappa$.*

Proof. See the proof of [23, Theorem 4.7]. □

Using the same argument as the one in [23, Lemma 4.9 and Theorem 4.10], one can see that the Stolz open set is the best possible one like Theorem 4.13.

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