TOROIDALLY ALTERNATING KNOTS AND LINKS

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1. INTRODUCTION

There have been various generalizations of the class of alternating links in $S^3$, specifically alternative, pseudo-alternating, homogeneous, adequate, augmented alternating and almost alternating links. These generalizations originated out of attempts to extend results known for alternating links to broader classes of links.

In this paper, we extend the class of alternating links to a new set, which we call toroidally alternating links. This set of links will be particularly broad, containing within it the set of alternating links, the set of almost alternating links, the subset of augmented alternating links with a single augmenting component, and a sub-class of the set of arborescent links, including all Montesinos links.

Surprisingly, if the tables of prime knots through eleven crossings and prime non-splittable links through ten crossings appearing in [6] are examined, all but three of the knots and two of the links can be shown to be toroidally alternating.

Let $T$ be a torus embedded in an orientable 3-manifold $M$. Let $L$ be a link in $M$ that can be isotoped into a neighborhood $T \times I$ of $T$. Suppose that if $T \times I$ is retracted onto $T$, $L$ projects to a connected 4-valent graph on $T$ such that if one keeps track of the crossings, they alternate between over and under as the components of the link are traversed, when viewed from one side of $T$. In addition, assume that every nontrivial curve on $T$ intersects the projection of $L$ onto $T$. Then $L$ is said to be toroidally alternating with respect to $T$.

In the case of a manifold with a genus one Heegaard splitting, there is a unique torus up to isotopy that splits the manifold into two solid tori. (See [3].) Hence, we can define a toroidally alternating link in these manifolds to be a link that is toroidally alternating with respect to this particular torus.

We will prove that a toroidally alternating knot in $S^3$ contains no closed incompressible meridionally incompressible surfaces in its complement. In particular, this will mean that a prime nontrivial toroidally alternating knot in $S^3$ is either a torus knot or it is hyperbolic. This generalizes a result that was proved for alternating links in [11] and almost alternating knots in [2].

In fact, we will prove that this is also the case when $S^3$ is replaced by a lens space $L(p, q)$ where $p$ is odd. In the case $p$ is even, toroidally alternating knots can have incompressible meridionally incompressible surfaces in their complement. However, we will show that if $K$ is a nontrivial prime non-torus toroidally alternating knot in $L(p, q)$, then $L(p, q) - K$ is hyperbolic, except for the lens spaces homeomorphic to $L(2, 1)$ and $L(4k, 2k - 1)$, where hyperbolicity will depend on the particular choice of a toroidally alternating knot. We will

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include $S^3 = L(0, 1)$ and $S^2 \times S^1 = L(1, 0)$ as lens spaces in our discussion. We always limit the lens spaces under discussion to $L(p, q)$ where $(p, q) = 1$, $p \geq 0$ and $0 \leq q \leq p/2$, as all lens spaces are homeomorphic to the ones in this set.

We also include extensions to toroidally alternating knots in more general 3-manifolds. We obtain limitations on the incompressible meridionally incompressible surfaces in the complement of a toroidally alternating knot in certain once-surgered Seifert fibered spaces and on the incompressible meridionally incompressible surfaces in the complement of a link that is toroidally alternating with respect to an incompressible torus boundary component of a manifold.

By a torus link in $L(p, q)$, we will mean a link $L$ that has a projection to the unique torus $T$ that decomposes $L(p, q)$ into two solid tori such that $L$ has no crossings and such that no component of $L$ is trivial in $L(p, q)$. We say that a knot or link $L$ in a 3-manifold $M$ is prime if there does not exist an incompressible boundary incompressible annulus properly embedded in $M - N(L)$ such that its boundary components are meridional on $L$.

We utilize the convention that a sphere is incompressible in a 3-manifold if it does not bound a 3-ball. See [8] and [9] for the standard definitions of incompressibility, boundary incompressibility, irreducibility, boundary irreducibility, sufficiently large manifolds and Haken manifolds.

A surface $S$ in the complement of a link $L$ in a 3-manifold $M$ is said to be meridionally compressible if there exists a disk $D$ in $M$ such that $D$ intersects the link once transversely in the interior of $D$ and $D \cap S = \partial D$. If no such disk exists, we say that the surface is meridionally incompressible. (Note that this property is called pairwise incompressibility in [11] and [12].)

By a $(p, q)$-curve on a solid torus, we mean a simple closed curve that wraps $p$ times longitudinally and $q$ times meridionally around the torus. In the figures that follow, a strand of a link that is marked with an "o" is a strand that goes over the next crossing in the link projection. A strand of a link marked with a "u" is a strand that goes under the next crossing in the link projection.

2. SUBCLASSES OF LINKS

In this section we investigate the extent of the class of toroidally alternating links in $S^3$. We first note that the trivial knot is toroidally alternating. A toroidally alternating projection of the trivial knot appears in Fig. 1(a). This projection also suggests how to make any connected alternating projection into a toroidally alternating projection as in Fig. 1(b).

In [2], an almost alternating link was defined to be a non-alternating link with a projection such that one crossing change in the projection yields an alternating projection. All such links are toroidally alternating, since we can find a toroidally alternating projection as in Fig. 2.
In [2], it was shown that of the knots and links appearing in the table of [6], all but at most three of the prime eleven or fewer crossing knots and all but two of the prime non-splittable ten or fewer crossing links were almost alternating links. Hence, they are all toroidally alternating links. We have not been able to determine whether the remaining five knots and links are or are not toroidally alternating.

We state the definition of an augmented alternating link from [1]. Let $L$ be a nontrivial non-splittable prime alternating link that is not a $(2, q)$-torus link. Let $P$ be a regular reduced alternating projection of $L$. Let $J_1, \ldots, J_n$ be $n$ nonisotopic embedded circles in $S^3 - L$, such that each $J_i$ intersects the projection plane in two points, the points lying in distinct regions of the projection plane for $L$, and such that each $J_i$ bounds a disk that lies in a plane perpendicular to the projection plane. Assume that the disks are all pairwise disjoint and each disk intersects $L$ transversely in two points. Then $L \cup (\bigcup J_i)$ is an augmented alternating link. Theorem 4.1 of [1] proved that an augmented alternating link in $S^3$ has hyperbolic complement.

Note that if we augment an alternating link with a single additional component, the result is an almost alternating link, as in Fig. 3, and hence a toroidally alternating link. Moreover, after a $(p, q)$-surgery on the additional component, we obtain a toroidally alternating knot in the lens space $L(p, q)$. To see this, we again utilize the fact that a meridional disk in one of the two solid tori has boundary that intersects the link projection on the torus twice, allowing us to perform the trick portrayed in Fig. 1, ensuring that every nontrivial curve on the torus intersects the projection.

Let $L$ be a link with a projection given as a sequence of tangles connected as in Fig. 4, such that each tangle is individually alternating. Then, if any pair of adjacent tangles are
together alternating, absorb them into a single tangle. The result will either be a single alternating tangle or a sequence of an even number of alternating tangles, each of which is connected to the other tangles by non-alternating strands. Placing the link longitudinally around a torus, we can then push every other tangle through the torus as in Fig. 4(b) in order to obtain a toroidally alternating projection.

It is immediate that Montesinos links (also known as star links (see [14]), are all toroidally alternating, since they are obtained by placing rational tangles in available slots in Fig. 4(a), and rational tangles are well-known to have alternating projections. The set of Montesinos links forms a sub-class of the set of arborescent links. (See [7] for a definition.) Arborescent links can be represented by labelled trees. Each Montesinos link corresponds to a connected labelled tree with at most one vertex of degree greater than 2. In fact, it is easy to see that any arborescent link represented by a labelled tree that has the property that the removal of some vertex and its incident edges leaves a set of labelled trees, each of which represents an alternating arborescent link, will in fact be a toroidally alternating link.

The theory of polynomial invariants should prove fruitful in the study of toroidally alternating links. For instance, we note that the main result of [10] can be applied to these links. Kidwell proves that if \( Q \) is the Brandt Lickorish Millet Ho polynomial and \( r \) is the number of crossings in a projection that has maximal bridge length \( b \), then \( \deg(Q_L) \leq r - b \). But note that a toroidally alternating link projection can always be depicted as in Fig. 5.

Note also that both \( m \) and \( n \) must be even, since the strands leaving the tangle alternate between over and under crossings as we travel around the boundary of the tangle, and since a strand leaving the tangle after passing under a crossing must travel around the torus either meridianally or longitudinally and then meet up with a strand that passed over a crossing as it left the tangle.

Let \( s \) be the number of crossings in the alternating tangle. Then the maximal bridge length in the corresponding planar projection of the link is \( \max\{m, n\} + 1 \), and an upper bound on the crossing number is \( mn + s \). Hence, we have that \( \deg(Q_L) \leq mn + s - (\max\{m, n\} + 1) \).

However, there are results that were obtained for alternating links using the polynomials that will not generalize directly. For instance, although it is known that the number of crossings in a reduced alternating projection of a link is an invariant of the link, the same is not true for the number of crossings in a reduced toroidally alternating link projection.

We also note that we can generalize the concept of a toroidally alternating link and look at projections of links onto higher genus surfaces. For each of \( S^3 \) and the lens spaces, there is a unique genus \( n \) surface up to isotopy that decomposes the manifold into two handle-bodies. (See [4]). Therefore, we can speak of a genus \( n \) alternating knot or link in these...
manifolds. In [2], we defined an $m$-almost alternating link to be a link in $S^3$, such that it has a projection where $m$ crossing changes will make the projection alternating and such that there does not exist a projection where fewer crossing changes will make the projection alternating. An $m$-almost alternating link will then be a genus $m$ alternating link. However, the results that follow do not seem to generalize to the genus $m$ alternating knots and links for $m \geq 2$. In recent work, Chuichiro Hayashi has investigated such knots and links.

3. MAIN THEOREMS

Theorem 3.1. Let $K$ be a toroidally alternating knot in a genus one manifold $M = L(p, q)$. If there exists a closed orientable incompressible meridianally incompressible surface in $M - K$, then it bounds a twisted I-bundle in $M - K$.

Proof. Suppose that $M - K$ contained a closed incompressible meridianally incompressible surface $F$ where $M = L(p, q)$, and $K$ is a toroidally alternating knot in $M$. We will replace the surface with a surface that is in “Menasco” form.

Let $T$ be the torus that $K$ has been projected to. This torus splits the lens space into two solid tori. As in [11], we will put a bubble at each crossing on the torus so that the overstrand at the crossing runs over the outer hemisphere while the understrand runs over the inner hemisphere. Define $T_+$ to be the torus obtained by replacing each of the equatorial disks in the bubbles on $T$ by the outer hemispheres of the bubbles. Similarly, define $T_-$ to be $T$ with the equatorial disks replaced by the inner hemispheres. Define $V_+$ to be the solid torus that is bounded by $T_+$ and that does not intersect the interiors of the bubbles. Similarly, $V$ is the solid torus that is bounded by $T_-$ and that does not intersect the interiors of the bubbles.

The surface $F$ can then be isotoped to intersect each bubble in a set of saddles. We will assume that we have isotoped the surface to minimize the total number of saddles and curves, ordered lexicographically.

The curves in $F \cap T_+$ will then have a particular form. We will work on $F \cap T_+$, but everything that we say also applies to $F \cap T_-$. We first note that the projection of $K$ cuts the torus $T$ up into regions, each of which must be a disk by our requirement that nontrivial curves on $T$ intersect the projection and that the projection be connected. Hence, as pointed out in [11], a curve of $F \cap T_+$ that enters a region on the left side of a bubble must leave the region on the right side of a bubble. In particular, this means that as we travel around the entirety of a curve in $F \cap T_+$, we cross an even number of bubbles and they alternate between being crossed on the left and being crossed on the right.

Suppose that there exists a curve in $F \cap T_+$ that bounds a disk on $T_+$. Let $a$ be an innermost such curve.

If $a$ intersects any bubbles, then because the bubbles alternate left and right, there exists a bubble with its overcrossing to the inside of the disk bounded by $a$ on $T_+$. Since each curve crossing a side of a bubble corresponds to a saddle in the bubble, there must be a corresponding curve on the other side of the bubble. By our choice of $a$ as innermost, it must be that $a$ intersects the other side of the bubble. We can then choose an outermost saddle intersected by $a$ in this bubble. It must be the case that $a$ intersects both sides of this saddle. We can then construct a loop on $F$ that crosses through the saddle and then follows $a$ around until it returns to the other side of the saddle. There is a disk with boundary this loop, that is punctured once by $K$ and that intersects $F$ only in its boundary, contradicting
meridional incompressibility. (See [11] for more details.) Hence, such an intersection curve cannot exist.

If \( z \) intersects no bubbles, then since it bounds a disk in \( T_+ \) and \( F \) is incompressible, it bounds a disk on \( F \). By our minimality of intersection curves assumption, the disk on \( F \) cannot be isotopic to the disk on \( T_+ \). In particular, this means that the disk on \( F \) must contain some additional intersection curves on \( T_+ \). Moreover, the sphere given by the union of the disk on \( F \) with the disk on \( T_+ \) must be incompressible and meridionally incompressible.

Note that if there is such an intersection curve \( z \), \( F \) cannot be a sphere. For if \( F \) were a sphere, surgery along the disk in \( T_+ \) bounded by \( z \) would yield two spheres. If either sphere were compressible, we could have isotoped \( F \) to eliminate the intersection curve \( z \). However, for a sphere in \( M - K \) to be incompressible, it must bound a punctured lens space to one side and a ball minus a knot to the other. In order that both of these spheres be incompressible, they would have to be parallel in \( M - K \), contradicting the fact that we should not be able to isotope the one to the other.

We will form a new surface from \( F \) by replacing the disk on \( F \) bounded by \( z \) with the disk on \( T_+ \) bounded by \( z \). The resulting surface remains incompressible and meridionally incompressible and after a slight isotopy, has fewer intersection curves than the original surface did. Moreover, the sphere obtained from the disk on \( T_+ \) together with the disk on \( F \) is also incompressible and meridionally incompressible. We repeat this surgery process until no intersection curves that are trivial on either \( T_+ \) or \( T_- \) remain in either the surface or the spheres that have been cut off it. We denote the resulting surface and the set of spheres that have been cut off it by \( F' \). After a small isotopy, all of the components of \( F' \) intersect \( T_+ \) only in nontrivial parallel curves on \( T_+ \), and each component must intersect \( T_+ \) in at least one such curve. Note that each intersection curve must intersect at least one bubble, by the definition of a toroidally alternating knot.

We will show that a curve that intersects the same bubble more than once must travel around the torus in the interim. Suppose that there is a curve \( \beta \) that intersects a particular bubble \( B \) more than once and an arc in \( \beta - (\beta \cap B) \) that cuts a disk from \( T_+ - (T_+ \cap B) \). Taking one such arc so that the disk it cuts off \( T_+ - (T_+ \cap B) \) contains no other such arcs, the ends of the arc must either lie on distinct sides of the bubble or they lie on the same side. If they lie on distinct sides of the bubble, they must lie on distinct sides of the same saddle. As above, we may then construct a loop in \( F' \) that violates meridional incompressibility.

If the two ends of the arc lie on the same side of the bubble, then they must meet a pair of adjacent saddles in \( B \). As in Fig. 6, we could then isotope the surface \( F' \) to eliminate the pair of saddles by pulling a regular neighborhood of an arc \( \mu \) on the surface through the bubble, pulling the two saddles out with it and contradicting our choice of a surface in the isotopy class with a minimal number of saddles.

Each component of \( F' \) is incompressible. Suppose that a component of \( F' \cap V_+ \) compressed in \( V_+ \). Then, the boundary of the compression disk must form the boundary of a disk in \( F' \). If that disk is isotopic to the compression disk, we could lower the number of intersection curves. Hence, the two disks are not isotopic. As we did before, we will replace the disk on \( F' \) with the compression disk and add the sphere obtained from the disk on \( F' \) and the compression disk, after a small isotopy, to our set of spheres in \( F' \).

Thus, we can now assume that each component of \( F' \cap V_+ \) is incompressible in \( V_+ \). Since the fundamental group of a two-sided incompressible surface must inject, \( F' \cap V_+ \) must consist of a set of disks and annuli. However, a disk in \( V_+ \) with nontrivial boundary in \( T_+ \) must be meridional. If annuli are also present, their boundaries must then also be meridional.
Suppose that $F' \cap V_+$ contains at least one annulus. By taking an annulus in $F' \cap V_+$ that is outermost in $V_+$, we obtain a pair of adjacent curves on $T_+$ that form its boundary. Both curves must cross bubbles. However, by the left-right rule, they must both cross the same bubble, as adjacent curves. If they cross distinct sides of the same bubble, we can again form a loop in the surface, composed of an arc in the saddle and an arc on the annulus. This loop will bound a once-punctured disk, contradicting meridional incompressibility. If the two curves cross the same side of the bubble, then a neighborhood of an essential arc on the annulus can be isotoped to the surface of the bubble. We can then isotope $F'$ through the bubble to lower the number of saddles by two, contradicting our assumption that the number of saddles has already been minimized.

Hence, we can now assume that both $F' \cap V_+$ and $F' \cap V_-$ consist entirely of meridional disks in $V_+$ and $V_-$. We will temporarily exclude the case of $M = S^2 \times S^1$. Since $T_+, T_-$ and the components of $F'$ are then all separating surfaces, the number of intersection curves in each of $F' \cap T_+$ and $F' \cap T_-$ is even.

Thus, $F' \cap T_+$ cuts $T_+$ into an even number of annuli. We will color them alternately black and white, choosing the first black annulus to contain a part of the knot. Since each strand of the knot that passes under a bubble will also be passing under an even number of sub-arcs of curves of $F' \cap T_+$, two for each saddle, we know that the knot strand will re-surface in a black annulus on $T_+$. By traversing the knot all the way around, we see that the entire knot must be black, that is to say, it appears only in the black annuli. Moving to the intersection curves on $T_-$, we can color the annuli between them on $T_-$ correspondingly, so that the knot is again black. Therefore, the knot never appears in the white regions.
We can extend the white annuli on \( T_+ \) and \( T_- \) to white 2-handles in \( V_+ \) and \( V_- \), each of which is bounded by a white annulus and a pair of the meridional disks. This \( I \)-bundle structure can be extended through the bubbles coherently, by gluing on white 2-handles sandwiched between pairs of adjacent saddles. The resulting \( I \)-bundle has \( F' \) for its boundary.

There is at most one non-spherical component of \( F' \). Hence it must bound a twisted \( I \)-bundle over a non-orientable surface.

Since each of the spherical components of \( F' \) could individually have been treated as the entire incompressible meridionally incompressible surface, they must each bound a twisted \( I \)-bundle over a projective plane. As the only manifold from among the lens spaces that contains a projective plane is \( L(2, 1) = RP^3 \) (see [5]), we know that except for this case and possibly \( S^2 \times S^1 \), which we will deal with below, \( F' = F \) and there are no spherical components to \( F' \). Hence \( F \) bounds a twisted \( I \)-bundle.

In the case of \( L(2, 1) \), a given spherical component of \( F' \) must bound a twisted \( I \)-bundle over a projective plane. However, the complement of the interior of a twisted \( I \)-bundle over \( P^2 \) in \( RP^3 \) is a ball. Hence, this sphere must bound a ball minus the knot to its other side. The non-spherical component \( G \) of \( F' \) is incompressible and since a lens space (punctured or not) cannot contain a closed incompressible orientable surface of positive genus, \( G \) must be contained in the ball minus the knot. However, \( G \) must also bound a twisted \( I \)-bundle over a non-orientable surface and since a ball cannot contain a closed non-orientable surface, it must be the case that \( F' \) consists of only a single component which bounds an \( I \)-bundle.

We will now deal with the one remaining case when \( M = S^2 \times S^1 \). Define a bubble disk on \( T_+ \) to be the intersection of a bubble with \( T_+ \). Isotope the intersection curves on \( T_+ \) to a set of parallel meridional curves on \( V_+ \), stretching the bubble disks out and around the torus in the process, as in Fig. 7.

Then no bubble disk intersects the same curve in \( F' \cap T_+ \) more than once, unless it wraps all the way around the torus in the process. In particular, this means that if we order the intersection curves cyclically as we traverse a longitude on \( V_+ \), each bubble disk on \( T_+ \) must intersect the meridional curves on \( T_+ \) in this cyclic order.

However, when curve surgery is performed to determine the curves in \( F' \cap T_- \) from the curves in \( F' \cap T_+ \), each of the resulting curves is forced to be a \((p, q)\)-curve on \( T_- \), where \( p \) is nonzero. But in the case of \( S^2 \times S^1 \), the intersection curves on \( T_- \) must bound meridional disks in \( V_- \) and hence be \((0, 1)\)-curves, a contradiction.

We note that in the end of the proof above, we proved the following lemma.

**Lemma 3.2.** Let \( K \) be a toroidally alternating knot in \( S^2 \times S^1 \). Then \( S^2 \times S^1 \setminus K \) contains no incompressible meridionally incompressible surfaces.
Corollary 3.3. Let $K$ be a toroidally alternating knot in $S^3$ or $L(p, q)$ where $p$ is odd. Then $L(p, q) - K$ contains no incompressible meridionally incompressible surfaces.

Proof. If the surface is orientable, it bounds a twisted $I$-bundle. But a twisted $I$-bundle in an orientable manifold must contain a closed non-orientable surface. For $p$ odd, the lens spaces $L(p, q)$ contains no closed non-orientable surfaces, by considering homology with $\mathbb{Z}_2$ coefficients.

Lemma 3.4. Let $L$ be a toroidally alternating link in $L(p, q)$ where $p$ is odd. If $F$ is an incompressible meridionally incompressible surface in $L(p, q) - L$, then $F$ must separate components of $L$.

Proof. We can repeat the proof of Theorem 3.1 until we reach the point where we color the annuli on $T$, alternately black and white. If the surface $F'$ consists of a single component, and if that component did not separate components of the link, all of the link would be colored black and we could again form a twisted $I$-bundle in the complement of the link using the annuli that are colored white. As in the proof of Corollary 3.2, such a twisted $I$-bundle cannot exist.

If the surface $F'$ consists of more than one component, all but at most one of those components are spheres. Since each component individually is incompressible and meridianally incompressible, the argument in the paragraph above implies that each component must individually separate components of $L$ from one another.

Suppose that $F'$ has a component $G$ that is not a sphere. Then, $G$ is a boundary component of two components of $M - F'$. If either of those two components does not contain a component of $L$, we can drop all of the surfaces of $F'$ that do not bound this component, and apply the above argument to show that this component must be an $I$-bundle. But since an $I$-bundle cannot have different boundary components of distinct genus, and we have already seen that these lens spaces cannot contain twisted $I$-bundles, this is a contradiction. Hence, there are components of $L$ to either side of $G$ that are not separated from $G$ by spherical components of $F'$. This implies that the original surface $F$ must separate components of $L$.

Finally, suppose that $F'$ has only spherical components, meaning that $F$ is a sphere. If $F$ does not separate components of $L$, then there exists a component of $M - F'$ that contains no components of $L$. As above, it must be an $I$-bundle. Because there are no projective planes in these lens spaces, it must be homeomorphic to $S^2 \times I$. However, in re-constructing the original surface $F$, we must glue a disk on one boundary component of the $S^2 \times I$ to a disk on the other boundary component. The resulting sphere is compressible and hence could not have occurred in the process of creating $F'$. Hence, $F$ separates the components of $L$.

Corollary 3.5. If $K$ is a toroidally alternating knot in $M$ and $S$ is an incompressible sphere in $M - K$, then $M = L(2, 1) = \mathbb{RP}^3$.

Proof. If there exists a sphere that doesn't bound a ball to either side in $M - K$, the manifold in question cannot be $S^3$. By Lemma 3.2, the manifold cannot be $S^1 \times S^1$ either. Since the lens spaces with nonzero $p$ are irreducible, the sphere must bound the complement of a knot in a ball to one side and a punctured lens space to the other side. In particular, the sphere cannot be contained completely in either $V_+$ or $V_-$. By Theorem 3.1, such a sphere must then form the boundary of a twisted $I$-bundle over a projective plane. However, the only lens space that contains a projective plane is $L(2, 1)$, as proved in [5].
At the end of this section, we give an example of an incompressible sphere in the complement of a toroidally alternating knot in $RP^3$.

**Corollary 3.6.** A trivial knot in $M = L(p,q)$ is not toroidally alternating unless $M = S^3$ or $RP^3$.

*Proof.* Assume that $M$ is not $S^3$. If a trivial knot were toroidally alternating, we could let $D$ be a disk bounded by the knot. Let $S$ be a sphere with boundary $N(D)$. Then $S$ is incompressible, as it bounds a ball missing a knot to one side and a nontrivial punctured lens space to the other side. By Corollary 3.4, $M$ must be $RP^3$.

In the case that we have a toroidally alternating knot in a lens space $L(p,q)$ where $p$ is even, results from [5] can again be applied. The authors define an integer-valued function $N$ recursively by $N(2,1) = 1$ and $N(2k,q) = N(2(k-q),q') + 1$ where $q' \leq q$ and $q' = \pm q \mod (2(k-q))$. They prove that the lens space $L(7k,q)$ contains a nonorientable closed surface consisting of the connected sum of $h$ copies of the projective plane if and only if $h = N(2k,q) + 2i$ for $i = 0, 1, 2, \ldots$

**Corollary 3.7.** Let $K$ be a toroidally alternating knot in $L(p,q)$ with $p$ even, $p \geq 2$. Let $S$ be a closed orientable incompressible meridionally incompressible surface in $L(p,q) - K$ of genus $g$. Then $g = N(2k,q) + 2i$ for some $i = 0, 1, 2, \ldots$

*Proof.* By Theorem 3.1, we know that $S$ forms the boundary of a twisted $I$-bundle. Hence, there must be a closed non-orientable surface $S'$ inside the lens space that is double-covered by $S$. The result then follows immediately from [5].

Note that this result from [5] implies that the lens spaces $L(2k,q)$ contain only the connected sum of an odd number of copies of the projective plane when $k$ is odd and an even number of copies of the projective plane when $k$ is even. Moreover, the only lens spaces that contain a Klein bottle are those of the form $L(4k, 2k - 1)$. Thus, we have the following corollary.

**Corollary 3.8.** Let $K$ be a toroidally alternating knot in $L(p,q)$ where $p$ is even and $0 \leq p, 0 \leq q \leq p/2$.

(i) If $p = 4k$, then $L(p,q) - K$ contains no closed orientable incompressible meridianally incompressible surfaces of even genus.

(ii) If $p = 4k + 2$, then $L(p,q) - K$ contains no closed orientable incompressible meridionally incompressible surfaces of odd genus.

(iii) If $(p,q) \neq (4k,2k - 1)$, then $L(p,q) - K$ contains no incompressible meridionally incompressible tori.

This leads to the following result.

**Corollary 3.9.** Let $K$ be a nontrivial prime non-torus toroidally alternating knot in $L(p,q)$ for $0 \leq p, 0 \leq q \leq p/2$. If $(p,q) \neq (2,1), (4k, 2k - 1)$, then $L(p,q) - K$ is hyperbolic.

*Proof.* By the work of Thurston (see [16]), it is enough to show that $L(p,q) - K$ is a Haken manifold that contains no essential tori or annuli. To ensure irreducibility, we need only exclude $L(2,1)$ by Corollary 3.3.

To see boundary-irreducibility, suppose that the boundary of $L(p,q) - N(K)$ compressed. Then there is an $(r,s)$-curve on $\partial N(K)$ that bounds a disk $D$ in $L(p,q) - N(K)$. If
$r = 0$, the disk can be glued along its boundary to the boundary of a meridional disk in $N(K)$, forming a sphere in $M$ that doesn't separate. Hence, $M = S^2 \times S^1$. The knot $K$ passes through the sphere once. Utilizing the light bulb trick (see p. 257 of [15]), the knot must then be isotopic to a core curve of either of the pair of solid tori that give us $S^2 \times S^1$. In particular, $K$ can be isotoped onto $T$, and is therefore, a torus knot, contradicting our hypotheses.

If $|r| = 1$, then the disk $D$ can be extended so that its boundary is the knot. Hence the knot is trivial, contradicting our hypotheses. In fact, Corollary 3.6 showed that with the exceptions of $S^3$ and $RP^3$, a trivial knot cannot be toroidally alternating. Hence, the inclusion of the hypothesis that the knot be nontrivial is necessary only for these two manifolds.

If $|r| \geq 2$, the manifold $M' = N(K) \cup N(D)$ is a lens space with the interior of a ball removed. Since it is a submanifold of $L(p, q)$, $M'$ must be the complement in $M$ of an open ball. By the irreducibility of $M - K$, we know that the knot is not contained in the ball. However, $M'$ consists of a solid torus $N(K)$ and a 2-handle $N(D)$. Gluing the missing ball to the 2-handle $N(D)$ along a pair of disks in their boundaries gives us a second solid torus, which, together with $N(K)$, forms a genus one splitting of $M$. Therefore, by the uniqueness of genus one splittings, $K$ is isotopic to one of the core curves of $V_+$ or $V_-$. Such a curve is a torus knot in $I(p, q)$, contradicting our hypothesis.

Corollary 3.8 says that if an incompressible non-boundary parallel torus existed, it would be meridianally compressible. After a meridianal compression, we would have created an incompressible boundary incompressible annulus with meridional boundaries, contradicting primeness of the knot.

Hence, we have eliminated all essential tori. Suppose that there is an essential annulus. Then $M - \text{int}(N(K))$ is Seifert fibered with $\partial A$ consisting of two of the fibers. If the fibers on $\partial N(K)$ are not meridional with respect to $K$, we can extend the Seifert fibration to all of $M = L(p, q)$. However, the knot then appears as a fiber in the Seifert fibration of $L(p, q)$. But any fiber in a Seifert fibration of a lens space is isotopic to a nontrivial curve on the splitting torus, contradicting our assumption that the knot is not a torus knot.

If the fibers on $\partial N(K)$ are meridional with respect to $K$, then the essential annulus in $M - N(K)$ can be capped off to form a sphere in $M$. That sphere bounds a ball in $M$, containing an arc of the knot. If that arc is nontrivial, the knot is not prime. If the arc is trivial, the original annulus is not essential.

Since a boundary-parallel torus ensures that the manifold is sufficiently large, $L(p, q) - K$ is hyperbolic.

**Corollary 3.10.** If $K$ is a nontrivial prime non-torus toroidally alternating knot in $L(p, q)$ for $p$ odd, then the hyperbolic manifold $L(p, q) - K$ contains no closed totally geodesic surfaces.

**Proof.** As was pointed out in [13], a totally geodesic surface in a hyperbolic manifold must be an incompressible surface with no accidental parabolics. An accidental parabolic is a curve on the surface that is homotopic to a nontrivial curve on the boundary of the manifold. For these manifolds, we have shown that any incompressible surface is meridionally compressible and hence contains an accidental parabolic.

We end this section with a few examples. In each of the following figures, the shaded disks represent any connected alternating tangles that respect the "u" and "o" labels and generate a knot. Figure 8 depicts a toroidally alternating knot on the boundary of a solid torus. If we glue a second solid torus to the first via a $(2, 1)$ surgery, the knot becomes...
a toroidally alternating knot in $RP^3$. We also show a meridional disk and saddle which, together with a meridional disk in the solid torus that is not shown form an incompressible meridionally incompressible projective plane in the complement of the knot. The boundary of a regular neighborhood of the projective plane is an essential sphere in the complement of the knot.

In Fig. 9, we give examples of toroidally alternating knots in $L(4, 3)$ that have incompressible meridionally incompressible tori in their complements. Specifically, this means the knot complement is not hyperbolic. We display that part of the corresponding Klein bottle that sits in the solid torus shown.

In Fig. 10, we give an example of a toroidally alternating link in $S^3$ that does contain an incompressible meridionally incompressible torus in its complement, demonstrating that the hypothesis that $K$ be a knot rather than a link is necessary in the statements of Theorem 3.1 and Corollaries 3.3, 3.7, 3.8 and 3.9.

4. EXTENSIONS

In this section, we extend Theorem 3.1 to apply to toroidally alternating knots and links in a broader class of manifolds. We are particularly interested in manifolds where the
incompressible surfaces with boundary to one side of the torus are parallel to one another, allowing the construction of an \( I \)-bundle bounded by the surface.

**THEOREM 4.1.** Let \( M \) be an orientable Seifert fibered space with one boundary component \( T \) and let \( M' \) be the manifold obtained by gluing a solid torus to \( M \) along its boundary so that a meridian of the solid torus is glued to a fiber in the Seifert fibration on \( M \). Let \( K \) be a knot in \( M' \) that is toroidally alternating with respect to \( T \). Then a closed orientable separating incompressible meridionally incompressible surface in \( M' - K \) is either the boundary of a twisted \( I \)-bundle or it is a torus in \( M \) that is saturated in the Seifert fibration of \( M \).

**Proof.** Let \( V \) be the solid torus bounded by \( T \), considered as a submanifold of \( M' \). Let \( T_+ \) be the torus \( T \) with the equatorial disks of the crossing bubbles removed and replaced by the hemispheres of the bubbles to the \( V \) side. Let \( V_+ \) be the solid torus within \( V \) that is bounded by \( T_+ \). Let \( T_- \) be the torus with the equatorial disks of the crossing bubbles removed and replaced by the hemispheres of the bubbles to the \( M \) side. Let \( M_- \) be the submanifold of \( M \) bounded by \( T_- \).

Just as we did in Theorem 3.1, we can replace the surface \( F \) with a surface \( F' \) so that it is in Menasco form. In particular, we can be assured that \( V_+ \cap F' \) consists only of meridional disks.

On the other hand, we can apply Theorem VI.34 of [9] to determine the possibilities for \( M_- \cap F' \). Each component must be incompressible in \( M_- \). There are four possibilities for incompressible surfaces in a Seifert fibered space. The first possibility is that a component is either a disk or annulus parallel to the boundary. Since the intersections of \( T_- \cap F' \) are all nontrivial curves on \( T_- \), we cannot have a disk parallel to the boundary. If there is an annulus parallel to the boundary, we can take an outermost such. Just as we argued previously, if such an annulus exists, its boundary components are two adjacent curves on \( T_- \). They must share a bubble. If they share the same side of a bubble, \( F' \) can be isotoped to lower the number of saddles. If they lie on opposite sides of a bubble, this contradicts meridional incompressibility.

The second possibility is that a component of \( M_- \cap F' \) is a fiber in a surface bundle over \( S^1 \) structure for \( M \). Then every other surface in \( M_- \cap F' \) must also be a fiber in this fiber bundle, as once we cut open along the first surface, we are left with the product of this surface and an interval. Hence the set of surface components cut \( M_- \) into a set of \( I \)-bundles, as we desire.

The third possibility is that this component of \( M_- \cap F' \) separates \( M_- \) into two twisted \( I \)-bundles. Any additional components in \( M_- \cap F' \) will be parallel to this component and will bound an \( I \)-bundle to each side.

The fourth and last possibility is that a component of \( M_- \cap F' \) is an annulus or torus that is saturated in some Seifert fibration of \( M_- \). In fact, the only Seifert fibered 3-manifolds with one boundary component that have more than one Seifert fibration are the solid torus and the twisted \( I \)-bundle over the Klein bottle. We have proved this theorem already for the case where \( M_- \) is a solid torus. If \( M_- \) is a twisted \( I \)-bundle over a Klein bottle, the only orientable incompressible surfaces with boundary are annuli. As we have already eliminated boundary-parallel annuli, the only such annuli cut \( M_- \) into \( I \)-bundles, as we desire.

Hence, we now suppose that \( M_- \) does have a unique Seifert fibration. Any annulus in \( M_- \cap F' \) must have boundary a fiber in the original Seifert fibration of \( M \). However, the solid torus has been glued to \( M \) in such a way that the boundary of a meridional disk went to this fiber. Hence the curves in \( T_+ \cap F' \) are isotopic to these curves. In the process of doing curve surgery to transform the curves of \( T_+ \cap F' \) to the curves of \( T_- \cap F' \), we must change
the homotopy type of these curves with respect to \(T\) (using the fact that bubble disks travel monotonically around the torus). Hence, the intersection curves in \(T_\pm \cap F\) are distinct from the fibers in the Seifert fibration and hence cannot be the boundaries of the annuli.

Therefore, if \(F\) intersects \(T\) at all, it must intersect \(M_-\) in a set of components that cut \(M_-\) into \(I\)-bundles. Since \(F\) and \(T\) separate, there are an even number of intersection curves in \(T_\pm \cap F\) and \(T_\pm \cap F\). We can again colorize the annuli in \(T\) black and white, such that the annuli containing the knot are all black. By extending the \(I\)-bundle structure between components of \(M_- \cap F\) in \(M\) through the white annuli, across pairs of saddles and then across pairs of disks in \(V_+\), we obtain an \(I\)-bundle bounded by \(F\).

Suppose that there is more than one component in \(F\). In particular, this means that there is a spherical component \(S\). By the definition of \(F\) and the fact that a Seifert fibered space is irreducible, \(S\) must intersect \(T_+\) and \(T_-\). The components of \(S \cap V_\pm\) must all be meridional disks. Since \(M-\text{int}(V_+)\) is Seifert fibered, \(S \cap M-\text{int}(V_+)\) must either be a saturated annulus or a fiber in a surface bundle over \(S^1\). In the second case, the boundary components of the fibering surface would not be fibers in the Seifert fibration of \(M-\text{int}(V_+)\), contradicting the fact that \(V\) was glued on so that meridional curves went to fibers. Hence \(S \cap M-\text{int}(V_+)\) is a saturated annulus.

This annulus must contain saddles within it. However, then \(M_- \cap S\) has a disk as one of its components. Again, by Theorem VI.34 of [9], this disk must either be boundary-parallel or a fiber in a surface bundle over \(S^1\). If the disk were boundary-parallel, we would have a curve in \(T_\pm \cap S\) that was trivial on \(T_\pm\), contradicting the fact that \(F\) is in Menasco form.

Hence, the disk is a fiber in a surface bundle over \(S^1\), implying that \(M_-\) is a solid torus, and \(M'\) is a lens space. As we have already shown in the proof of Theorem 3.1, it must then be the case that \(M'\) is \(L(2,1)\), and \(S\) is the only component in \(F\).

Thus, we have shown that \(F\) is isotopic to \(F\), and therefore that \(F\) bounds an \(I\)-bundle, assuming that it intersects \(T\). If \(F\) does not intersect \(T\) at all, then it must lie entirely to the \(M_-\) side of \(T\). Utilizing Theorem VI.34 of [9] again, this implies that it must be a saturated torus in \(M_-\).

For the purpose of discussing a more specific set of examples, we include the following Lemma.

**Lemma 4.2.** Let \(M\) be a 3-manifold and let \(T\) be a torus boundary component of \(M\). Let \(L\) be a link in \(M\) that is toroidally alternating with respect to \(T\). Then \(T\) is incompressible and meridionally incompressible in \(M-\text{int}(L)\).

**Proof.** Let \(T'\) be a torus in the interior of \(M\) that is parallel to \(T\). Then \(L\) is also toroidally alternating with respect to \(T'\). We isotope \(L\) to a toroidally alternating projection on \(T'\). We can then form \(T_+\) and \(T_-\) as before, choosing \(T_+\) to be the torus that is closer to \(T\). Let \(W\) be the submanifold of \(M\) that is homeomorphic to \(T^2 \times I\) and that is bounded by \(T_+\) and \(T_-\).

Suppose that \(T\) is either compressed or meridionally compressed. Then there is a disk \(D\) in \(M\) such that \(D \cap T = \partial D\), where \(\partial D\) is a nontrivial curve on \(T\), and such that \(D\) either misses \(L\) or is punctured once by \(L\). As in [12], we can put \(D\) in Menasco form relative to \(L\). Note that in the process, we may have replaced the original disk with a new disk, by exchanging a disk on its surface for a disk on \(T_+\) or \(T_-\) and then isotoping to eliminate a trivial curve of intersection on \(T_+\) or \(T_-\). Assume that we have isotoped the disk while fixing its boundary, to minimize the number of saddles and the number of intersection curves, ordered lexicographically. Since \(T\) does not intersect \(T_\pm\), each of \(D \cap T_+\) and \(D \cap T_-\) must consist of simple closed curves on \(D\), one of which passes through the puncture on \(D\), in the case \(D\) is
punctured. As the projection is toroidally alternating, each of the intersection curves in $D \cap T^+_n$ and $D \cap T^-_n$ must either cross an even number of saddles or an odd number of saddles and one puncture.

Since $W$ is homeomorphic to $T^2 \times I$, $D \cap W$ must consist of annuli and disks. However, if there were a disk in $D \cap W$ with boundary that avoids the puncture, its boundary would bound a disk on $T^+_n$, contradicting the fact that there are no trivial intersection curves that do not cross a puncture for a surface in Menasco form.

It must be the case that $D \cap W$ contains an annulus, one boundary component of which is the boundary of $D$. If there is an annulus in $D \cap W$ that has both boundary components on $T^+_n$ and both boundary components avoiding the puncture, we can argue as we did in the proof of Theorem 3.1 that there is an outermost such, causing both of its boundary components to cross the same side of the same bubble, allowing us to isotope a nontrivial arc on the annulus through the bubble and lower the number of saddles by two. Thus, either $D \cap W$ consists of a single annulus, or two annuli, one of which has a puncture on its boundary, or an annulus and a disk, where the disk has a puncture on its boundary.

In the case that $D$ is unpunctured, it must then be the case that $D \cap W$ consists of a single annulus $A$, one boundary component of which is the boundary of $D$. Let $D' = D - \text{int}(A)$. Let $W'$ be the closure of the component of $M - T'_n$ that does not contain $\text{int}(W)$. Then $D'$ must consist of components of $D \cap W'$ and saddles. However, the boundary of each saddle must consist of four arcs, two in $T^+_n$ and two in $T^-_n$. Choosing an outermost saddle on $D'$, it will cut a disk from $D'$ that must lie in $W'$. However, the boundary of that disk intersects only one saddle, contradicting the fact that all of the intersection curves must hit an even number of saddles.

In the case that $D$ is punctured, the three possibilities for $D \cap W$ imply that one of the components in $D - \text{int}(D \cap W)$ must be either a disk with a puncture on its boundary or an annulus with a single puncture on one of its boundaries. Let $E$ be such a component. Then $E$ must be made up of saddles and components from $D \cap W'$.

If $E$ is a disk, there exists an outermost saddle with no puncture on its boundary. This implies that there is a disk in $D \cap W'$, the boundary of which intersects only one saddle and no punctures, a contradiction.

If $E$ is an annulus, then no saddle on $E$ can begin and end on the same boundary component of $E$ unless it cuts a disk off from $E$ that contains the puncture on its boundary. All such saddles on $E$ must be concentric. If there are no such saddles, then the intersection curve in $D \cap T'^-_n$ that contains the puncture will also cross an even number of saddles, a contradiction.

If there are such saddles, choose an innermost such saddle $S$ on $E$ and let $E'$ be the annulus obtained by removing from $E$ both $S$ and the disk $S$ cuts from $E$. All saddles in $E'$ must begin and end on distinct boundary components of $E'$. However, the intersection curve in $D \cap W''$ that is on $E'$ and that intersects $S$ will intersect no punctures and an odd number of saddles, a contradiction.

As a corollary to the Lemma, we have the following.

**Corollary 4.3.** Let $M$ be a 3-manifold with an incompressible torus boundary component $T$. Let $L$ be a link in $M$ that is toroidally alternating with respect to $T$, in a toroidally alternating projection within $N(T)$. Let $T'$ be the other boundary component of $N(T)$. Then the closed incompressible meridionally incompressible surfaces in $M - L$ are exactly the closed incompressible surfaces in $M$ together with the tori isotopic to $T$ and $T'$.

**Proof.** The fact that $T$ and $T'$ are incompressible and meridionally incompressible in $M - L$ follows immediately from Lemma 4.2. Let $F$ be an incompressible surface in $M$. We
assume that $F$ has been isotoped to avoid intersecting $T'$. Then $F$ will remain incompressible in $M - L$. Suppose that $F$ meridianally compressed in $M - L$. Let $D$ be the meridional compression disk. Then $D$ must intersect $T'$. Since $T'$ is incompressible, one of the components of $D - (D \cap T')$ must be a once-punctured disk. However, this contradicts the meridional incompressibility of $T'$.

Let $T''$ be a torus that is between $T$ and $T'$ and put $L$ in a toroidally alternating projection with respect to $T''$. Let $T''_+$ and $T''_-$ be the resulting tori, where $T''_+$ is closer to $T$. Let $W$ be the $T^2 \times I$ bounded by $T''_+$ and $T$.

Given a surface $F$ that is incompressible and meridianally incompressible in $M - L$, we replace it with a surface $F'$ in Menasco form, with a minimal number of saddles and intersection curves, ordered lexicographically. Since the components of $W \cap F'$ are incompressible in $W$, they must be disks and annuli parallel to $T''_+$ in $W$. The boundaries of the disks must be trivial curves on $T''_+$, a contradiction, as we have seen before. The annuli can again be isotoped to eliminate two saddles, a contradiction. Therefore $F'$ cannot intersect $W$ and all of the components of $F'$ are contained within $M - N(T)$. This implies that $F$ is contained within $M - N(T)$. If $F$ compressed in $M$, $F$ would compress in $M - N(T)$, implying that $F$ compressed in $M - L$, a contradiction. Hence, $F$ must be an incompressible surface in $M$.

In the following corollary to Theorem 4.1 and Lemma 4.2, instead of gluing two solid tori together along their boundary as in Section 3, we glue a solid torus to a torus knot exterior. However, the potential gluing maps are limited if we desire a strong conclusion.

**Corollary 4.4.** Let $M'$ be the manifold obtained by gluing a solid torus $V$ with boundary $T$ to a $(p, q)$-torus knot exterior in $S^3$ by $(pq, 1)$-surgery. Let $K$ be a knot in $M'$ that is toroidally alternating with respect to $T$, in a toroidally alternating projection in $N(T)$. If both $p$ and $q$ are odd, then any closed incompressible meridianally incompressible surface in $M' - K$ is isotopic to the torus that is a boundary component of $N(T)$ and that bounds a solid torus in $M$ containing the knot $K$.

**Proof.** First note that the manifold $M'$ does not contain any closed nonorientable surfaces. This follows from the fact that relative homology with $Z$ coefficients shows that such a surface cannot separate. However, since $H_1(M; Z) = Z_{pq}$ and $H_2(M; Z) = 0$, relative homology with $Z_2$ coefficients shows that the surface must separate. It is also the case that any orientable surface must separate the manifold.

Let $K'$ be the torus knot in $S^3$, and let $M$ be the submanifold of $M'$ corresponding to $S^3 - N(K)$. Note that $S^3 - N(K)$ is Seifert fibered with two exceptional fibers of index $p$ and $q$ and orbit manifold a disk with two exceptional points. A fiber on the boundary of the manifold is given by a $(pq, 1)$-curve. The only incompressible saturated torus in the Seifert fibration of $M$ is boundary-parallel in $M$. This torus is incompressible and meridionally incompressible in $M' - K$ by Lemma 4.2 and the fact that the boundary of a torus knot exterior is incompressible. This is the torus referred to in the conclusion of the statement of this corollary.

By Theorem 4.1, any other orientable incompressible meridionally incompressible surface in $M' - K$ must bound a twisted $I$-bundle in $M' - K$. However, the existence of such a twisted $I$-bundle in $M'$ would imply the existence of a closed nonorientable surface, a contradiction.

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