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Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control

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Abstract

We consider a linear Schrödinger equation, on a bounded interval, with bilinear control, that represents a quantum particle in an electric field (the control). We prove the exact controllability of this system, in any positive time, locally around the ground state.

Similar results were proved for particular models (Beauchard, 2005, 2008, 2006) [14,15,17], in non-optimal spaces, in long time and the proof relied on the Nash–Moser implicit function theorem in order to deal with an a priori loss of regularity.

In this article, the model is more general, the spaces are optimal, there is no restriction on the time and the proof relies on the classical inverse mapping theorem. A hidden regularizing effect is emphasized, showing there is actually no loss of regularity.

Then, the same strategy is applied to nonlinear Schrödinger equations and nonlinear wave equations, showing that the method works for a wide range of bilinear control systems.

Résumé

On considère une équation de Schrödinger linéaire, sur un intervalle borné, avec contrôle bilinéaire, représentant une particule quantique dans un champ électrique (le contrôle). On démontre la contrôlabilité exacte locale de ce système, en tout temps positif, localement au voisinage de l'état fondamental.

Des résultats similaires ont déjá été établis (Beauchard, 2005, 2008, 2006) [14,15,17], mais dans des espaces non optimaux, en temps long et leur démonstration reposait sur le théorème de Nash–Moser, pour gérer une apparente perte de régularité.

Dans cet article, le modèle étudié est plus général, les espaces sont optimaux, il n'y a pas de restriction sur le temps et la démonstration repose sur le théorème d'inversion locale classique. Un effet régularisant est exhibé, montrant qu'il n'y a finalement pas de perte de régularité.

La même stratégie est ensuite utilisée sur des équations de Schrödinger non linéaires et des équations des ondes non linéaires, montrant qu'elle s'applique de façon assez générale aux systèmes de contrôle bilinéaires.

Keywords: Control of partial differential equations; Bilinear control; Schrödinger equation; Wave equation

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1. Introduction

1.1. Main result

Following [57], we consider a quantum particle, in a 1D infinite square potential well, subjected to an electric field. It is represented by the following Schrödinger equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t}(t,x) = -\dfrac{\partial^2\psi}{\partial x^2}(t,x) - u(t)\mu(x)\psi(t,x), \quad x\in(0,1),\ t\in(0,T), \\ \psi(t,0)=\psi(t,1)=0, \end{cases} \tag{1}$$

where ψ is the wave function of the particle, u is the amplitude of the electric field and $\mu \in H^3((0,1),\mathbb{R})$ is the dipolar moment of the particle. The system (1) is a bilinear control system, in which

- the state is ψ, with $\|\psi(t)\|_{L^2(0,1)} = 1$, $\forall t \in (0,T)$,
- the control is the real valued function $u : [0,T] \to \mathbb{R}$.

Let us introduce some notations. The operator A is defined by:

$$D(A) := H^2 \cap H^1_0\big((0,1),\mathbb{C}\big), \qquad A\varphi := -\frac{d^2\varphi}{dx^2}. \tag{2}$$

Its eigenvalues and eigenvectors are:

$$\lambda_k := (k\pi)^2, \qquad \varphi_k(x) := \sqrt{2}\sin(k\pi x), \quad \forall k \in \mathbb{N}^*. \tag{3}$$

The family $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2((0,1),\mathbb{C})$, and

$$\psi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}, \quad \forall k \in \mathbb{N}^*$$

is a solution of (1) with $u \equiv 0$ called eigenstate, or ground state, when $k=1$. We define the spaces,

$$H^s_{(0)}\big((0,1),\mathbb{C}\big) := D\big(A^{s/2}\big), \quad \forall s > 0, \tag{4}$$

equipped with the norm

$$\|\varphi\|_{H^s_{(0)}} := \left(\sum_{k=1}^{\infty} |k^s\langle\varphi,\varphi_k\rangle|^2\right)^{1/2}.$$

We denote by $\langle .,. \rangle$ the $L^2((0,1),\mathbb{C})$ scalar product

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx,$$

and by $\mathcal{S}$ the unit $L^2((0,1),\mathbb{C})$-sphere. The first goal of this article is the proof of the following result

Theorem 1. *Let $T>0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\exists c > 0 \quad \textit{such that} \quad \frac{c}{k^3} \leqslant \big|\langle\mu\varphi_1,\varphi_k\rangle\big|, \quad \forall k \in \mathbb{N}^*. \tag{5}$$

There exists $\delta > 0$ and a C^1 map

$$\Gamma : \mathcal{V}_T \to L^2\big((0,T),\mathbb{R}\big),$$

where

$$\mathcal{V}_T := \big\{\psi_f \in \mathcal{S} \cap H^3_{(0)}\big((0,1),\mathbb{C}\big);\ \big\|\psi_f - \psi_1(T)\big\|_{H^3} < \delta\big\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (1) *with initial condition*

$$\psi(0) = \varphi_1 \tag{6}$$

and control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

Remark 1. Thanks to the time reversibility of the system, Theorem 1 ensures the local controllability of the system (1) around the ground state: for every $T > 0$, there exists $\delta > 0$ such that, for every $\psi_0, \psi_f \in \mathcal{S} \cap H^3_{(0)}((0,1),\mathbb{C})$ with $\|\psi_0 - \psi_1(0)\|_{H^3} + \|\psi_f - \psi_1(T)\|_{H^3} < \delta$, there exists a control $u \in L^2(0,T)$ such that the solution of (1) with initial condition $\psi(0) = \psi_0$ satisfies $\psi(T) = \psi_f$.

Remark 2. The assumption (5) holds, for example, with $\mu(x) = x^2$, because

$$\langle x^2\varphi_1, \varphi_k\rangle = \int_0^1 2x^2 \sin(k\pi x)\sin(\pi x)\,dx = \begin{cases} \frac{(-1)^{k+1}8k}{\pi^2(k^2-1)^2} & \text{if } k \geqslant 2, \\ \frac{-3+2\pi^2}{6\pi^2} & \text{if } k = 1. \end{cases} \tag{7}$$

But it does not hold when $\langle \mu\varphi_1, \varphi_k\rangle = 0$, for some $k \in \mathbb{N}^*$, or when μ has a symmetry with respect to $x = 1/2$. However, the assumption (5) holds generically with respect to $\mu \in H^3((0,1),\mathbb{R})$, because

$$\langle \mu\varphi_1, \varphi_k\rangle = \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu\varphi_1)'''(x)\cos(k\pi x)\,dx, \quad \forall k \in \mathbb{N}^* \tag{8}$$

(see Appendix A for a proof). Thus, Theorem 1 is very general.

1.2. A simpler proof

The local exact controllability of 1D Schrödinger equations, with bilinear control, has already been investigated in [14,15,17] (see also [16] for a similar result on a 1D beam equation). In these articles, three different models are studied. The local controllability of the nonlinear system is proved thanks to the linearization principle:

- first, we prove the controllability of a linearized system,
- then, we prove the local controllability of the nonlinear system, by applying an inverse mapping theorem.

This strategy is coupled with the return method and quasi-static deformations in [14,17] and with power series expansions in [15,17] (see [30,32] by Coron for a presentation of these technics). In these articles, the most difficult part of the proof is the application of the inverse mapping theorem. Indeed, because of an a priori loss of regularity, we were led to apply the Nash–Moser implicit function theorem (see, for instance [6] by Alinhac, Gérard and [38] by Hörmander), instead of the classical inverse mapping theorem. The Nash–Moser theorem requires, in particular, the controllability of an infinite number of linearized systems, and tame estimates on the corresponding controls. These two points are difficult to prove and lead to long technical developments in [14,15,17].

In this article, we propose a simpler proof, that uses only the classical inverse mapping theorem (needing the controllability of only one linearized system), because we emphasize a hidden regularizing effect (see Proposition 2).

Therefore, the controllability result of Theorem 1 enters the classical framework of local controllability results for nonlinear systems, proved with fixed point arguments (see, for instance, [55] by Rosier, [28] by Cerpa and Crépeau, [58] by Russell and Zhang, [63] by Zhang, [64] by Zuazua; this list is not exhaustive).

1.3. Additional results

The proof we developed for Theorem 1 is quite robust, thus we could apply it to other situations: other linear PDEs and also nonlinear PDEs, that are presented in the next subsections. This shows that the strategy proposed in this article works for a wide range of bilinear systems.

1.3.1. Generalization to higher regularities

The first situation is the analogue result of Theorem 1, but with higher regularities: we prove the local exact controllability of (1) in smoother spaces and with smoother controls. Namely, we prove the following result:

Theorem 2. *Let $T > 0$ and $\mu \in H^5((0,1),\mathbb{R})$ be such that* (5) *holds. There exists $\delta > 0$ and a C^1 map*

$$\Gamma : \mathcal{V}_T \to H^1_0\big((0,T),\mathbb{R}\big),$$
$$\psi_f \mapsto \Gamma(\psi_f),$$

where

$$\mathcal{V}_T := \big\{\psi_f \in \mathcal{S} \cap H^5_{(0)}\big((0,1),\mathbb{C}\big);\ \big\|\psi_f - \psi_1(T)\big\|_{H^5} < \delta\big\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (1), (6) *with control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.*

Of course, the strategy may be used to go further and prove the local exact controllability of (1) around the ground state:

- in $H^7_{(0)}(0,1)$ with controls in $H^2_0((0,T),\mathbb{R})$,
- in $H^9_{(0)}(0,1)$ with controls in $H^3_0((0,T),\mathbb{R})$, etc.

1.3.2. On the 3D ball with radial data

The second situation is the analogue result of Theorem 1, but for the Schrödinger equation posed on the three-dimensional unit ball B^3 for radial data. In polar coordinates, the Laplacian for radial data can be written:

$$\Delta u(r) = \partial_r^2 u(r) + \frac{2}{r}\partial_r u(r).$$

In particular, we have $\Delta(\frac{g(r)}{r}) = \frac{\partial_r^2 u(r)}{r}$. The eigenfunctions of the Dirichlet operator $A = -\Delta$ with domain $D(A) := H^2_{radial} \cap H^1_0(B^3)$ are $\varphi_k = \frac{\sin(k\pi r)}{r\sqrt{2\pi}}$ with eigenvalues $\lambda_k = (k\pi)^2$. Thus, we study the Schrödinger equation,

$$\begin{cases} i\dfrac{\partial\psi}{\partial t}(t,r) = -\Delta\psi(t,r) - u(t)\mu(r)\psi(t,r), \quad r \in (0,1), \\ \psi(t,1) = 0. \end{cases} \tag{9}$$

The theorem we obtain is very similar to Theorem 1.

Theorem 3. *Let $T > 0$ and $\mu \in H^3(B^3,\mathbb{R})$ radial be such that*

$$\exists c > 0 \quad \text{such that} \quad \frac{c}{k^3} \leqslant \big|\langle \mu\varphi_1, \varphi_k\rangle\big|, \quad \forall k \in \mathbb{N}^*. \tag{10}$$

There exists $\delta > 0$ and a C^1 map,

$$\Gamma : \mathcal{V}_T \to L^2\big((0,T),\mathbb{R}\big),$$

where

$$\mathcal{V}_T := \big\{\psi_f \in \mathcal{S} \cap H^3_{(0),rad}\big(B^3,\mathbb{C}\big);\ \big\|\psi_f - \psi_1(T)\big\|_{H^3} < \delta\big\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (9) *with initial condition*

$$\psi(0) = \varphi_1 \tag{11}$$

and control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

The analysis is very close to the 1D case since for this particular data, the Laplacian behaves as in dimension 1. We refer to Appendix A for the proof of the genericity of the assumption (10). Note that this simpler situation has also been used by Anton for proving global existence for the nonlinear Schrödinger equation [8].

1.3.3. Nonlinear Schrödinger equations

The third situation concerns nonlinear Schrödinger equations. More precisely we study the following nonlinear Schrödinger equation with Neumann boundary conditions:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t}(t,x) = -\dfrac{\partial^2\psi}{\partial x^2}(t,x) + |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x), & x\in(0,1),\ t\in(0,T), \\ \dfrac{\partial\psi}{\partial x}(t,0) = \dfrac{\partial\psi}{\partial x}(t,1) = 0. \end{cases} \tag{12}$$

It is a nonlinear control system, where

- the state is ψ, with $\|\psi(t)\|_{L^2(0,1)} = 1$, $\forall t \in [0,T]$,
- the control is the real valued function $u : [0,T] \to \mathbb{R}$.

We study its local controllability around the reference trajectory,

$$\big(\psi_{ref}(t,x) := e^{-it},\ u_{ref}(t) = 0\big).$$

More precisely, we prove the following result

Theorem 4. *Let $T > 0$ and $\mu \in H^2(0,1)$ be such that*

$$\exists c > 0 \quad \textit{such that} \quad \left|\int_0^1 \mu(x)\cos(k\pi x)\,dx\right| \geqslant \frac{c}{\max\{1,k\}^2}, \quad \forall k \in \mathbb{N}. \tag{13}$$

There exists $\eta > 0$ and a C^1 map,

$$\Gamma : \mathcal{V}_T \to L^2\big((0,T),\mathbb{R}\big),$$

where

$$\mathcal{V}_T := \big\{\psi_f \in \mathcal{S}\cap H^2(0,1);\ \psi_f'(0) = \psi_f'(1) = 0 \text{ and } \big\|\psi_f - e^{-iT}\big\|_{H^2} < \eta\big\},$$

such that, for every $\psi_f \in \mathcal{V}_T$, the solution of (12) *with initial condition,*

$$\psi(0,x) = 1, \quad \forall x \in (0,1), \tag{14}$$

and control $u := \Gamma(\psi_f)$ is defined on $[0,T]$ and satisfies $\psi(T) = \psi_f$.

Remark 3. The assumption (13) holds generically in $H^2(0,1)$. Indeed, integrations by part give,

$$\int_0^1 \mu(x)\cos(k\pi x)\,dx = \frac{1}{(k\pi)^2}\left((-1)^{k+1}\mu'(1) + \mu'(0) + \int_0^1 \mu''(x)\cos(k\pi x)\,dx\right), \quad \forall k \in \mathbb{N}^*.$$

Other versions of this result, with higher regularities may be proved: the system is exactly controllable, locally around the reference trajectory:

- in $H^4(0,1)$ with controls in $H_0^1(0,T)$,
- in $H^6(0,1)$ with controls in $H_0^2(0,T)$, etc.

Focusing nonlinearities may also be considered.

1.3.4. Nonlinear wave equations

The third situation concerns nonlinear wave equations. More precisely we study the following wave equation with Neumann boundary conditions:

$$\begin{cases} w_{tt} = w_{xx} + f(w, w_t) + u(t)\mu(x)(w + w_t), & x \in (0,1),\ t \in (0,T), \\ w_x(t,0) = w_x(t,1) = 0, \end{cases} \tag{15}$$

where f is an appropriate nonlinearity, that satisfies, in particular, $f(1,0) = 0$. It is a nonlinear control system, where

- the state is (w, w_t),
- the control is the real valued function $u : [0,T] \to \mathbb{R}$.

We study its exact controllability, locally around the reference trajectory,

$$\big(w_{ref}(t,x) = 1,\ u_{ref}(t) = 0\big).$$

More precisely, we prove the following result.

Theorem 5. *Let $T > 2$, $\mu \in H^2((0,1),\mathbb{R})$ be such that* (13) *holds and $f \in C^3(\mathbb{R}^2,\mathbb{R})$ be such that $f(1,0) = 0$ and $\nabla f(1,0) = 0$. There exists $\eta > 0$ and a C^1 map,*

$$\Gamma : \mathcal{V}_T \to L^2\big((0,T),\mathbb{R}\big),$$

where

$$\mathcal{V}_T := \big\{(w_f, \dot{w}_f) \in H^3 \times H^2\big((0,1),\mathbb{R}\big);\ w'_f(0) = w'_f(1) = \dot{w}'_f(0) = \dot{w}'_f(1) = 0,\ \text{and}\ \|w_f - 1\|_{H^3} + \|\dot{w}_f\|_{H^2} < \eta\big\}$$

such that $\Gamma(1,0) = 0$ and for every $(w_f, \dot{w}_f) \in \mathcal{V}_T$, the solution of (15) *with initial condition*

$$(w, w_t)(0,x) = (1,0), \quad \forall x \in (0,1) \tag{16}$$

and control $u := \Gamma(w_f, \dot{w}_f)$ is defined on $[0,T]$ and satisfies $(w, w_t)(T) = (w_f, \dot{w}_f)$.

Other versions of this result, with higher regularities may be proved: the system is exactly controllable, locally around the reference trajectory:

- in $H^4 \times H^3(0,1)$ with controls in $H^1_0(0,T)$,
- in $H^5 \times H^4(0,1)$ with controls in $H^2_0(0,T)$, etc.

1.4. A brief bibliography

1.4.1. A previous negative result

First, let us recall an important negative controllability result, for Eq. (1), proved by Turinici [61]. It is a corollary of a more general result due to Ball, Marsden and Slemrod [10].

Proposition 1. *Let $\psi_0 \in \mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$ and $U[T; u, \psi_0]$ be the value at time T of the solution of* (1) *with initial condition $\psi(0) = \psi_0$. The set of attainable states from ψ_0,*

$$\big\{U[T; u, \psi_0];\ T > 0,\ u \in L^2\big((0,T),\mathbb{R}\big)\big\},$$

has an empty interior in $\mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$. Thus (1) *is not controllable in $\mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$ with controls in $L^2_{loc}([0,+\infty),\mathbb{R})$.*

Proposition 1 is a rather weak negative controllability result, because it does not prevent from positive controllability results, in different spaces. This had already been emphasized for the particular cases studied in [14,15,17], in which the reachable set is proved to contain $H^7_{(0)}$ or $H^{5+}_{(0)}$. In this article, we prove that the reachable set (at least locally, with small controls in $L^2((0,T),\mathbb{R})$), coincides with $\mathcal{S}\cap H^3_{(0)}$ (which has, indeed, an empty interior in $\mathcal{S}\cap H^2_{(0)}$). Therefore, sometimes, Ball, Marsden and Slemrod's negative result is only due to an 'unfortunate' choice of functional spaces, that does not allow the controllability. It may not be due to a deep non-controllability (such as, for example, when a subsystem evolves independently of the control).

1.4.2. Iterated Lie brackets

Now, let us quote some articles about the controllability of quantum systems.

First, the controllability of *finite-dimensional* quantum systems (i.e. modeled by an ordinary differential equation) is well understood. Let us consider the quantum system

$$i\frac{dX}{dt} = H_0X + u(t)H_1X, \tag{17}$$

where $X \in \mathbb{C}^n$ is the state, H_0, H_1 are $n*n$ hermitian matrices, and $t\mapsto u(t)\in\mathbb{R}$ is the control. The controllability of (17) is linked to the rank of the Lie algebra spanned by H_0 and H_1 (see for instance [5] by Albertini and D'Alessandro, [7] by Altafini, [26] by Brockett, see also [3] by Agrachev and Sachkov, [32] by Coron for a more general discussion).

In *infinite dimension*, there are cases where the iterated Lie brackets provide the right intuition. For instance, it holds for the non-controllability of the harmonic oscillator (see [49] by Mirrahimi and Rouchon). However, the Lie brackets are often less powerful in infinite dimension than in finite dimension. It is precisely the case of our system. Indeed, let us define the operators:

$$D(f_0) := H^2\cap H^1_0(0,1), \quad f_0(\psi) := -\psi'',$$
$$D(f_1) := L^2(0,1), \quad f_1(\psi) := x^2\psi,$$

which correspond to $\mu(x) = x^2$. Let us compute the iterated Lie brackets at the point $\varphi_1(x) = \sqrt{2}\sin(\pi x)$. Since $\varphi_1 \in D(f_0)$, we can compute,

$$[f_0, f_1](\varphi_1) = -4x\varphi_1' - 2\varphi_1,$$
$$\big[f_1,[f_0,f_1]\big](\psi) = 8x^2\varphi_1 = 8f_1(\varphi_1).$$

Notice that $[f_0,f_1](\varphi_1)$ does not belong to $D(f_0)$ because $[f_0,f_1](\varphi_1)(1) = 4\sqrt{2}\pi \neq 0$. Thus, in order to give a sense to the Lie bracket $[f_0,[f_0,f_1]]$, one needs to extend the definition of f_0 to functions that do not vanish at $x=0,1$. A natural choice is

$$f_0(\psi) := -\psi'' + \psi(0)\delta_0' - \psi(1)\delta_1' \tag{18}$$

because, with this choice, we have:

$$\langle f_0(\psi), \tilde{\psi}\rangle = \langle \psi, f_0(\tilde{\psi})\rangle, \quad \forall\psi\in D(f_0), \forall\tilde{\psi}\in H^2(0,1),$$

in the sense

$$-\int_0^1 \psi''(x)\tilde{\psi}(x)\,dx = -\int_0^1 \psi(x)\tilde{\psi}''(x)\,dx - \psi'(1)\tilde{\psi}(1) + \psi'(0)\tilde{\psi}(0).$$

With the definition (18), we get:

$$\big[f_0,[f_0,f_1]\big](\psi) = -8f_0(\psi) + 4\psi'(1)\delta_1'.$$

But then, again, $[f_0,[f_0,[f_0,f_1]]]$ is not well defined. Moreover, even if we could give a sense to any iterated Lie bracket, because of the presence of Dirac masses, it would not be clear which space the Lie algebra should generate in case of local controllability. Therefore, the way the Lie algebra rank condition could be used directly in infinite dimension is not clear (see also [32] for the same discussion on other examples). This is why we develop completely analytic methods in this article.

Finally, let us quote important articles about the controllability of PDEs, in which positive results are proved by applying geometric control methods to the (finite-dimensional) Galerkin approximations of the equation. In [4] by Sarychev and Agrachev and [59] by Shirikyan, the authors prove exact controllability results for dissipative equations. In [29], by Boscain, Chambrion, Mason and Sigalotti, the authors prove the approximate controllability in L^2, for bilinear Schrödinger equations such as (1).

We also refer to the following works about the controllability of finite-dimensional quantum systems [2,20–25], by Agrachev, Boscain, Chambrion, Charlot, Gauthier, Guérin, Jauslin and Mason, [40] by Khaneja, Glaser and Brockett, [53] by Ramakrishna, Salapaka, Dahleh, Rabitz, [60] by Sussmann and Jurdjevic, [62] by Turinici and Rabitz. Let us also mention [50] by Mirrahimi, Rouchon, Turinici and [18] for explicit feedback controls, inspired by Lyapunov technics.

1.4.3. Controllability results for Schrödinger and wave equations

The controllability of Schrödinger equations with distributed and boundary controls, that act linearly on the state, is studied since a long time.

For linear equations, the controllability is equivalent to an observability inequality that may be proved with different technics: multiplier methods (see [36] by Fabre, [47] by Machtyngier), microlocal analysis (see [46] by Lebeau, [27] by Burq), Carleman estimates (see [42,43] by Lasiecka, Triggiani, Zhang), or number theory (see [54] by Ramdani, Takahashi, Tenenbaum and Tucsnak).

For nonlinear equations, we refer to [33] by Dehman, Gérard, Lebeau, [41] by Lange Teismann, [45,44] by Laurent, [56] by Rosier, Zhang.

1.4.4. Other results about bilinear quantum systems

The study of the controllability of Schrödinger PDEs with bilinear controls started later.

The first result is negative and it is due to Turinici (see [61] and Proposition 1). It is a corollary of a more general result by Ball, Marsden and Slemrod [10]. Because of this noncontrollability result, such equations have been considered as non-controllable for a long time. However, important progress have been made in the last years and this question is now better understood (see Section 1.4.1). Let us also mention that this negative result has been adapted to nonlinear Schrödinger equations in [39] by Ilner, Lange and Teismann.

Concerning exact controllability issues, local results for 1D models have been proved in [14,15] by Beauchard; almost global results have been proved in [17], by Coron and Beauchard. In [31], Coron proved that a positive minimal time was required for the local controllability of the 1D model (1) with $\mu(x) = x - 1/2$.

Now, let us quote some approximate controllability results. In [19] Mirrahimi and Beauchard proved the global approximate controllability, in infinite time, for a 1D model and in [48] Mirrahimi proved a similar result for equations involving a continuous spectrum. Approximate controllability, in finite time, has been proved for particular models by Boscain and Adami in [1], by using adiabatic theory and intersection of the eigenvalues in the space of controls. Approximate controllability, in finite time, for more general models, have been studied by 3 teams, with different tools: by Boscain, Chambrion, Mason, Sigalotti [29], with geometric control methods; by Nersesyan [52,51] with feedback controls and variational methods; and by Ervedoza and Puel [35] thanks to a simplified model.

Let us emphasize that the local exact controllability result of this article and the global approximate controllability of [52,51] can be put together in order to get the global exact controllability of 1D models (see [51]).

Optimal control techniques have also been investigated for Schrödinger equations with a non-linearity of Hartree type in [11,12] by Baudouin, Kavian, Puel and in [34] by Cances, Le Bris, Pilot. An algorithm for the computation of such optimal controls is studied in [13] by Baudouin and Salomon.

1.5. Structure of this article

This article is organized as follows.

Section 2 aims at proving the controllability for the linear Schrödinger equations. Sections 2.1, 2.2, 2.3 and 2.4 are dedicated to the different steps of the proof of Theorem 1, where the equation is posed on a bounded interval. Section 2.5 is dedicated to the proof of the same result with higher regularities, i.e. Theorem 2. Section 2.6 is dedicated to the Schrödinger equation for radial data on the three-dimensional ball, i.e. the proof of Theorem 3.

In Section 3, we prove Theorem 4 concerning the nonlinear Schrödinger equation (12).

In Section 4, we prove Theorem 5 concerning the nonlinear wave equation (15).

Finally, in Section 5, we state some conclusions, open problems and perspectives.

1.6. Notations

Let us introduce some conventions and notations that are valid in all this article. Unless otherwise specified, the functions considered are complex valued and, for example, we write $H_0^1(0,1)$ for $H_0^1((0,1),\mathbb{C})$. When the functions considered are real valued, we specify it and we write, for example, $L^2((0,T),\mathbb{R})$. We use the spaces,

$$h^s(\mathbb{N}^*,\mathbb{C}) := \left\{ a = (a_k)_{k\in\mathbb{N}^*} \in \mathbb{C}^{\mathbb{N}^*};\ \sum_{k=1}^{\infty} |k^s a_k|^2 < +\infty \right\},$$

equipped with the norm:

$$\|a\|_{h^s} := \left(\sum_{k=1}^{\infty} |k^s a_k|^2 \right)^{1/2}.$$

The same letter C denotes a positive constant, that can change from one line to another one. If $(X, \|.\|)$ is a normed vector space and $R > 0$, $B_R[X]$ denotes the open ball $\{x \in X;\ \|x\| < R\}$ and $\overline{B}_R[X]$ denotes the closed ball $\{x \in X;\ \|x\| \leqslant R\}$.

2. Linear Schrödinger equations

The goal of this section is the proof of controllability results for linear Schrödinger equations, with bilinear controls.

Sections 2.1–2.4 are dedicated to the different steps of the proof of Theorem 1, where the equation is posed on a bounded interval. In Section 2.1, we prove existence, uniqueness, regularity results and bounds on the solution of the Cauchy problem (1), (6). In Section 2.2, we prove the C^1-regularity of the end-point map associated to our control problem. In Section 2.3, we prove the controllability of the linearized system around the ground state. Finally, in Section 2.4, we deduce Theorem 1 by applying the inverse mapping theorem.

Section 2.5 is dedicated to the proof of the same result with higher regularities, i.e. Theorem 2.

Section 2.6 is dedicated to the Schrödinger equation for radial data on the three-dimensional ball, i.e. the proof of Theorem 3.

In all this sections (except in Section 2.6), the operator A is defined by (2), the spaces $H^s_{(0)}(0,1)$ are defined by (4) and e^{-iAt} denotes the group of isometries of $H^s_{(0)}(0,1)$, $\forall s \geqslant 0$ generated by $-iA$,

$$e^{-iAt}\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle e^{-i\lambda_k t} \varphi_k, \quad \forall \varphi \in L^2(0,1). \tag{19}$$

We use few classical results concerning trigonometric moment problems that are recalled in Appendix B.

2.1. Well posedness of the Cauchy problem

This subsection is dedicated to the statement of existence, uniqueness, regularity results, and bounds for the weak solutions of the Cauchy problem:

$$\begin{cases} i\dfrac{\partial \psi}{\partial t} = -\dfrac{\partial^2 \psi}{\partial x^2} - u(t)\mu(x)\psi - f(t,x), & x \in (0,1),\ t \in \mathbb{R}_+, \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{20}$$

Proposition 2. *Let $\mu \in H^3((0,1),\mathbb{R})$, $T > 0$, $\psi_0 \in H^3_{(0)}(0,1)$, $f \in L^2((0,T), H^3 \cap H^1_0)$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique weak solution of* (20)*, i.e. a function $\psi \in C^0([0,T], H^3_{(0)})$ such that the following equality holds in $H^3_{(0)}(0,1)$ for every $t \in [0,T]$,*

$$\psi(t) = e^{-iAt}\psi_0 + i\int_0^t e^{-iA(t-\tau)}\big[u(\tau)\mu\psi(\tau) + f(\tau)\big]\,d\tau. \tag{21}$$

Moreover, for every $R > 0$, there exists $C = C(T, \mu, R) > 0$ such that, if $\|u\|_{L^2(0,T)} < R$, then this weak solution satisfies:

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leqslant C\big(\|\psi_0\|_{H^3_{(0)}} + \|f\|_{L^2((0,T),H^3\cap H^1_0)}\big). \tag{22}$$

If $f \equiv 0$, then

$$\big\|\psi(t)\big\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \quad \forall t \in [0, T]. \tag{23}$$

The main difficulty of the proof of this result is that $f(s)$ is not assumed to belong to $H^3_{(0)}(0,1)$ (i.e. $f''(s,.)$ may not vanish at $x = 0$ and $x = 1$), and μ is not assumed to satisfy $\mu'(0) = \mu'(1) = 0$ (and thus the operator $\varphi \mapsto \mu\varphi$ does not preserve $H^3_{(0)}(0,1)$ because for $\varphi \in H^3_{(0)}(0,1)$, we have $(\mu\varphi)'' = 2\mu'\varphi'$ at $x = 0$ and $x = 1$). The argument for proving Proposition 2 comes from the following lemma.

Lemma 1. *Let $T > 0$ and $f \in L^2((0,T), H^3 \cap H^1_0)$. The function $G : t \mapsto \int_0^t e^{iAs} f(s)\,ds$ belongs to $C^0([0,T], H^3_{(0)})$, moreover*

$$\|G\|_{L^\infty((0,T),H^3_{(0)})} \leqslant c_1(T)\|f\|_{L^2((0,T),H^3\cap H^1_0)}, \tag{24}$$

where the constants $c_1(T)$ are uniformly bounded for T lying in bounded intervals.

Proof of Lemma 1. By definition, we have:

$$G(t) = \sum_{k=1}^{\infty}\left(\int_0^t \langle f(s), \varphi_k\rangle e^{i\lambda_k s}\,ds\right)\varphi_k.$$

For almost every $s \in (0,T)$, $f(s) \in H^3 \cap H^1_0$, and, we have:

$$\begin{aligned}\langle f(s), \varphi_k\rangle &= \frac{1}{\lambda_k}\langle Af(s), \varphi_k\rangle \\ &= -\frac{\sqrt{2}}{\lambda_k}\int_0^1 f''(s,x)\sin(k\pi x)\,dx \\ &= \frac{\sqrt{2}}{(k\pi)^3}\big((-1)^k f''(s,1) - f''(s,0)\big) - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 f'''(s,x)\cos(k\pi x)\,dx.\end{aligned}$$

Thus, we have:

$$\begin{aligned}\big\|G(t)\big\|_{H^3_{(0)}} &= \left\|\int_0^t \langle f(s), \varphi_k\rangle e^{i\lambda_k s}\,ds\right\|_{h^3} \\ &\leqslant \frac{\sqrt{2}}{\pi^3}\left(\left\|\int_0^t f''(s,1)e^{i\lambda_k s}\,ds\right\|_{l^2} + \left\|\int_0^t f''(s,0)e^{i\lambda_k s}\,ds\right\|_{l^2}\right) \\ &\quad + \frac{1}{\pi^3}\left\|\int_0^t \big\langle f'''(s), \sqrt{2}\cos(k\pi x)\big\rangle e^{i\lambda_k s}\,ds\right\|_{l^2}.\end{aligned}$$

The family $(\sqrt{2}\cos(k\pi x))_{k\in\mathbb{N}^*}$ is orthonormal in $L^2(0,1)$, thus

$$\begin{aligned}\left\| \int_0^t \langle f'''(s), \sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k s}\,ds \right\|_{l^2} &= \left(\sum_{k=1}^{\infty} \left| \int_0^t \langle f'''(s), \sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k s}\,ds \right|^2 \right)^{1/2} \\ &\leqslant \left(\sum_{k=1}^{\infty} t \int_0^t \left|\langle f'''(s), \sqrt{2}\cos(k\pi x)\rangle\right|^2 ds \right)^{1/2} \\ &\leqslant \sqrt{t} \left(\int_0^t \left\| f'''(s) \right\|_{L^2}^2 ds \right)^{1/2} \\ &\leqslant \sqrt{t} \| f \|_{L^2((0,t),H^3)}.\end{aligned}$$

Thanks to Corollary 4 (in Appendix B), we get:

$$\begin{aligned}\left\| G(t) \right\|_{H^3_{(0)}} &\leqslant \frac{\sqrt{2}C(t)}{\pi^3} \left(\left\| f''(.\,,0) \right\|_{L^2(0,t)} + \left\| f''(.\,,1) \right\|_{L^2(0,t)} \right) + \frac{\sqrt{t}}{\pi^3} \| f \|_{L^2((0,t),H^3)} \\ &\leqslant c_1(t) \| f \|_{L^2((0,t),H^3 \cap H^1_0)},\end{aligned}$$

where $c_1(t)$ is uniformly bounded for t lying in bounded intervals. This bound shows that $G(t)$ belongs to $H^3_{(0)}(0,1)$ for every $t \in [0,T]$ and that the map $t \in [0,T] \mapsto G(t) \in H^3_{(0)}$ is continuous at $t = 0$ (because $c_1(t)$ is uniformly bounded when $t \to 0$ and $\| f \|_{L^2((0,t),H^3 \cap H^1_0)} \to 0$ when $t \to 0$, thanks to the dominated convergence theorem). The continuity of G at any $t \in (0,T)$ can be proved similarly. □

Proof of Proposition 2. Let $\mu \in H^3((0,1),\mathbb{R})$, $T > 0$, $\psi_0 \in H^3_{(0)}(0,1)$, $f \in L^2((0,T), H^3 \cap H^1_0)$ and $u \in L^2((0,T),\mathbb{R})$. We consider the map,

$$\begin{aligned} F : C^0\big([0,T], H^3_{(0)}\big) &\to C^0\big([0,T], H^3_{(0)}\big), \\ \psi &\mapsto \xi, \end{aligned}$$

where $\xi := F(\psi)$ is defined by

$$\xi(t) := e^{-iAt}\psi_0 + i \int_0^t e^{-iA(t-s)}\big(u(s)\mu\psi(s) + f(s)\big)\,ds, \quad \forall t \in [0,T]. \tag{25}$$

We have assumed that $f \in L^2((0,T), H^3 \cap H^1_0)$ and $u \in L^2(0,T)$, thus, for every $\psi \in C^0([0,T], H^3_{(0)})$, the map $(u\mu\psi + f)$ belongs to $L^2((0,T), H^3 \cap H^1_0)$ and Lemma 1 ensures that F takes values in $C^0([0,T], H^3_{(0)})$. We have also used that in dimension 1, H^3 is an algebra.

Thanks to (24), we get, for every $t \in [0,T]$,

$$\begin{aligned}\left\| F(\psi_1)(t) - F(\psi_2)(t) \right\|_{H^3_{(0)}} &= \left\| \int_0^t e^{iAs} u(s)\mu(\psi_1 - \psi_2)(s)\,ds \right\|_{H^3_{(0)}} \\ &\leqslant c_1(t) \left\| u\mu(\psi_1 - \psi_2) \right\|_{L^2((0,t),H^3 \cap H^1_0)} \\ &\leqslant c_1(t) \|u\|_{L^2(0,t)} \left\| \mu(\psi_1 - \psi_2) \right\|_{L^\infty((0,t),H^3 \cap H^1_0)} \\ &\leqslant c_1(t) \|u\|_{L^2(0,t)} C(\mu) \|\psi_1 - \psi_2\|_{L^\infty((0,t),H^3_{(0)})}\end{aligned}$$

thus

$$\left\| F(\psi_1) - F(\psi_2) \right\|_{L^\infty((0,T),H^3_{(0)})} \leqslant c_2(T,\mu) \|u\|_{L^2(0,T)} \|\psi_1 - \psi_2\|_{L^\infty((0,T),H^3_{(0)})}. \tag{26}$$

If $\|u\|_{L^2(0,T)}$ is small enough, then F is a contraction. Thanks to the Banach fixed point theorem, there exists $\psi \in C^0([0,T], H^3_{(0)})$ such that $F(\psi) = \psi$. The previous arguments show that, for this fixed point, we have:

$$\|\psi\|_{L^\infty((0,T),H^3_{(0)})} \leqslant \|\psi_0\|_{H^3_{(0)}} + c_2(T,\mu)\|u\|_{L^2(0,T)}\|\psi\|_{L^\infty((0,T),H^3_{(0)})} + c_1(T)\|f\|_{L^2((0,T),H^3\cap H^1_0)}.$$

Thus, if $c_2(T,\mu)\|u\|_{L^2(0,T)} \leqslant 1/2$, then, we get (22).

We have proved Proposition 2 when $\|u\|_{L^2(0,T)}$ is small enough. If it is not the case, one may consider $0 = T_0 < T_1 < \cdots < T_N = T$ such that $\|u\|_{L^2(T_j,T_{j+1})}$ is small and apply the previous result on $[T_0,T_1],\ldots,[T_{N-1},T_N]$ in order to get the conclusion. Since our constant $c_1(t)$ is uniform on bounded sets, we easily get that N only depends on R, so that the constant in Proposition 2 does only depend on T, μ and R as claimed.

Now, let us prove that (23) holds when $f = 0$. Classical arguments allow to prove that, when $u \in C^0([0,T],\mathbb{R})$, then $\psi \in C^1([0,T],L^2)$ and the first equality of (1) holds in L^2 for every $t \in [0,T]$. Thus, when $u \in C^0([0,T],\mathbb{R})$, we can take the L^2-scalar product of this equation with ψ; and the imaginary part of the resulting equality gives:

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2} = 0.$$

Thus, we have (23) when $u \in C^0([0,T],\mathbb{R})$. A density argument allows to prove (23) when u only belongs to $L^2((0,T),\mathbb{R})$. □

2.2. C^1-regularity of the end-point map

For $T > 0$ we introduce the tangent space of $\mathcal{S}$ at $\psi_1(T)$,

$$V_T := \{\xi \in L^2(0,1);\ \Re\langle \xi, \psi_1(T)\rangle = 0\},$$

and the orthogonal projection

$$P_T : L^2(0,1) \to V_T.$$

Proposition 2 allows to consider the map

$$\begin{aligned} \Theta_T : L^2\big((0,T),\mathbb{R}\big) &\to V_T \cap H^3_{(0)}(0,1), \\ u &\mapsto P_T\big[\psi(T)\big], \end{aligned} \tag{27}$$

where ψ is the solution of (1), (6). The goal of this section is the proof of the following result:

Proposition 3. *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$. The map Θ_T defined by* (27) *is C^1. Moreover, for every $u, v \in L^2((0,T),\mathbb{R})$, we have*:

$$d\Theta_T(u).v = P_T\big[\Psi(T)\big] \tag{28}$$

where Ψ is the weak solution of the linearized system

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = -\Psi'' - u(t)\mu(x)\Psi - v(t)\mu(x)\psi, & x \in (0,1),\ t \in (0,T), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0, \end{cases} \tag{29}$$

and ψ is the solution of (1), (6).

Proof of Proposition 3. Let $T > 0$, $\mu \in H^3((0,1),\mathbb{R})$ and $u \in L^2((0,T),\mathbb{R})$. First, let us emphasize that the linear map $v \mapsto \Psi(T)$ is continuous from $L^2((0,T),\mathbb{R})$ to $H^3_{(0)}(0,1)$ thanks to Proposition 2.

First step: We prove that Θ_T is differentiable and that (28) *holds*. Let ψ be the weak solution of (1), (6), Ψ solution of (29) and $\tilde{\psi}$ solution of:

$$\begin{cases} i\dfrac{\partial \tilde{\psi}}{\partial t} = -\tilde{\psi}'' - (u+v)(t)\mu(x)\tilde{\psi}, & x \in (0,1),\ t \in (0,T), \\ \tilde{\psi}(t,0) = \tilde{\psi}(t,1) = 0, \\ \tilde{\psi}(0,x) = \varphi_1. \end{cases} \tag{30}$$

Then $\Delta := \tilde{\psi} - \psi - \Psi$ is the weak solution of:

$$\begin{cases} i\dfrac{\partial \Delta}{\partial t} = -\Delta'' - (u+v)(t)\mu(x)\Delta - v(t)\mu\Psi, \quad x \in (0,1),\ t \in (0,T), \\ \Delta(t,0) = \Delta(t,1) = 0, \\ \Delta(0,x) = 0. \end{cases} \tag{31}$$

Let us prove that

$$\|\Delta\|_{C^0([0,T],H^3_{(0)})} = o\big(\|v\|_{L^2}\big) \quad \text{when } \|v\|_{L^2} \to 0, \tag{32}$$

which gives the conclusion. Let $R > 0$ be such that $\|u\|_{L^2(0,T)} < R$ and $\|u+v\|_{L^2(0,T)} < R$. Thanks to Proposition 2, there exists $C_j = C_j(T,\mu,R) > 0$ for $j = 0, 1$ such that

$$\|\Delta\|_{C^0([0,T],H^3_{(0)})} \leqslant C_0\|v\mu\Psi\|_{L^2((0,T),H^3\cap H^1_0)} \leqslant C_1\|v\|_{L^2}\|\Psi\|_{C^0([0,T],H^3_{(0)})},$$
$$\|\Psi\|_{C^0([0,T],H^3_{(0)})} \leqslant C_0\|v\mu\psi\|_{L^2((0,T),H^3\cap H^1_0)} \leqslant C_1\|v\|_{L^2}\|\psi\|_{C^0([0,T],H^3_{(0)})} \leqslant C_0C_1\|v\|_{L^2}\|\varphi_1\|_{H^3_{(0)}},$$

which proves (32).

Second step: We prove that $d\Theta_T$ is continuous. Actually, we prove that this map is locally Lipschitz. Let $u, \tilde{u} \in L^2((0,T),\mathbb{R})$ and $v \in L^2((0,T),\mathbb{R})$. Let ψ be the solution of (1), (6), Ψ solution of (29) and $\tilde{\psi}$, $\tilde{\Psi}$ solution of:

$$\begin{cases} i\dfrac{\partial \tilde{\psi}}{\partial t} = -\tilde{\psi}'' - \tilde{u}(t)\mu(x)\tilde{\psi}, \\ \tilde{\psi}(t,0) = \tilde{\psi}(t,1) = 0, \\ \tilde{\psi}(0,x) = \varphi_1, \end{cases} \qquad \begin{cases} i\dfrac{\partial \tilde{\Psi}}{\partial t} = -\tilde{\Psi}'' - \tilde{u}(t)\mu(x)\tilde{\Psi} - v(t)\mu(x)\tilde{\psi}, \\ \tilde{\Psi}(t,0) = \tilde{\Psi}(t,1) = 0, \\ \tilde{\Psi}(0,x) = 0. \end{cases}$$

We have:

$$\big[d\Theta_T(u) - d\Theta_T(\tilde{u})\big].v = P_T\big[\Psi(T) - \tilde{\Psi}(T)\big] = P_T\big[\Xi(T)\big],$$

where Ξ is the weak solution of

$$\begin{cases} i\dfrac{\partial \Xi}{\partial t} = -\dfrac{\partial^2 \Xi}{\partial x^2} - u(t)\mu\Xi - (u-\tilde{u})\mu\tilde{\Psi} - v\mu(\psi - \tilde{\psi}), \\ \Xi(t,0) = \Xi(t,1) = 0, \\ \Xi(0) = 0. \end{cases}$$

Let $R > 0$ be such that $\|u\|_{L^2(0,T)} < R$, $\|\tilde{u}\|_{L^2(0,T)} < R$. Let us prove that

$$\|\Xi\|_{C^0([0,T],H^3_{(0)})} \leqslant \mathcal{C}\|v\|_{L^2}\|u - \tilde{u}\|_{L^2},$$

where $\mathcal{C} = \mathcal{C}(T,\mu,R) > 0$, which gives the conclusion. Thanks to Proposition 2, we have:

$$\begin{aligned} \|\Xi\|_{C^0([0,T],H^3_{(0)})} &\leqslant C_2\big\|(u-\tilde{u})\mu\tilde{\Psi} + v\mu(\psi - \tilde{\psi})\big\|_{L^2((0,T),H^3\cap H^1_0)} \\ &\leqslant C_3\big(\|u-\tilde{u}\|_{L^2}\|\tilde{\Psi}\|_{C^0([0,T],H^3_{(0)})} + \|v\|_{L^2}\|\psi - \tilde{\psi}\|_{C^0([0,T],H^3_{(0)})}\big) \\ &\leqslant C_4\big(\|u-\tilde{u}\|_{L^2}\|v\mu\tilde{\psi}\|_{L^2((0,T),H^3\cap H^1_0)} + \|v\|_{L^2}\|(\tilde{u}-u)\mu\tilde{\psi}\|_{L^2((0,T),H^3\cap H^1_0)}\big) \\ &\leqslant C_5\big(\|u-\tilde{u}\|_{L^2}\|v\|_{L^2}\|\tilde{\psi}\|_{C^0([0,T],H^3_{(0)})} + \|v\|_{L^2}\|\tilde{u}-u\|_{L^2}\|\tilde{\psi}\|_{C^0([0,T],H^3_{(0)})}\big) \\ &\leqslant C_6\|u-\tilde{u}\|_{L^2}\|v\|_{L^2}, \end{aligned}$$

where $C_j = C_j(T,\mu,R) > 0$ for $j = 2,\dots,6$. $\square$

2.3. Controllability of the linearized system

The goal of this section is the proof of the following result.

Proposition 4. *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that* (5) *holds. The linear map*

$$d\Theta_T(0) : L^2\big((0,T),\mathbb{R}\big) \to V_T \cap H^3_{(0)}(0,1)$$

has a continuous right inverse $d\Theta_T(0)^{-1} : V_T \cap H^3_{(0)}(0,1) \to L^2((0,T),\mathbb{R})$.

The proof of Proposition 4 relies on an Ingham inequality, due to Haraux (see [37] and Appendix B).

Proof of Proposition 4. We have $d\Theta_T(0).v = \Psi(T)$, where

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = -\Psi'' - v(t)\mu\psi_1, \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0) = 0, \end{cases} \tag{33}$$

thus

$$\Psi(T) = \sum_{k=1}^{\infty} i\langle \mu\varphi_1, \varphi_k\rangle \left(\int_0^T v(t) e^{i(\lambda_k - \lambda_1)t}\, dt \right) e^{-i\lambda_k T}\varphi_k.$$

Let $\Psi_f \in V_T \cap H^3_{(0)}(0,1)$. If Ψ is the solution of (33) for some $v \in L^2((0,T),\mathbb{R})$, then, the equality $\Psi(T) = \Psi_f$ is equivalent to the trigonometric moment problem:

$$\int_0^T v(t) e^{i(\lambda_k - \lambda_1)t}\, dt = d_{k-1}(\Psi_f) := \frac{\langle \Psi_f, \varphi_k\rangle e^{i\lambda_k T}}{i\langle \mu\varphi_1, \varphi_k\rangle}, \quad \forall k \in \mathbb{N}^*. \tag{34}$$

Now, we apply Corollary 1 (see Appendix B) with $\omega_k := \lambda_{k+1} - \lambda_1$, $\forall k \in \mathbb{N}$, and we get the conclusion with

$$d\Theta_T(0)^{-1}(\Psi_f) := L\big[d(\Psi_f)\big],$$

where $d(\Psi_f) := (d_k(\Psi_f))_{k \in \mathbb{N}}$. Indeed, for $\Psi_f \in V_T \cap H^3_{(0)}(0,1)$, the sequence $d(\Psi_f)$ belongs to $l^2_r(\mathbb{N},\mathbb{C})$ thanks to the assumption (5). □

2.4. Proof of Theorem 1

Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that (5) holds. Let $R_1 > 0$ and $\delta_1 > 0$ be such that

$$\forall u \in B_{R_1}\big[L^2\big((0,T),\mathbb{R}\big)\big], \quad \text{the solution of (1), (6) satisfies } \Re\big\langle \psi(T), \psi_1(T)\big\rangle > 0$$

(see Proposition 2), and

$$\forall \psi_f \in \mathcal{S} \cap H^3_{(0)}(0,1) \quad \text{with } \big\|\psi_f - \psi_1(T)\big\|_{H^3_{(0)}} < \delta_1, \text{ we have } \Re\big\langle \psi_f, \psi_1(T)\big\rangle > 0.$$

The spaces $\overline{B}_{R_1}[L^2((0,T),\mathbb{R})]$ and $V_T \cap H^3_{(0)}(0,1)$ are Banach spaces. The map

$$\Theta_T : \overline{B}_{R_1}\big[L^2\big((0,T),\mathbb{R}\big)\big] \to V_T \cap H^3_{(0)}(0,1)$$

is C^1 (see Proposition 3), its differential at 0 has a continuous right inverse

$$d\Theta_T(0)^{-1} : V_T \cap H^3_{(0)}(0,1) \to L^2\big((0,T),\mathbb{R}\big)$$

(see Proposition 4). Thanks to the inverse mapping theorem, there exists $\delta \in (0,\delta_1)$ and a C^1 map,

$$\Theta_T^{-1} : B_\delta\big[V_T \cap H^3_{(0)}(0,1)\big] \to \overline{B}_{R_1}\big[L^2\big((0,T),\mathbb{R}\big)\big],$$

such that $\Theta_T(\Theta_T^{-1}(\tilde{\psi}_f)) = \tilde{\psi}_f$ for every $\tilde{\psi}_f \in B_\delta[V_T \cap H^3_{(0)}(0,1)]$.

For $\psi_f \in \mathcal{S} \cap H^3_{(0)}(0,1)$ with $\|\psi_f - \psi_1(T)\|_{H^3_{(0)}} < \delta$, we have $\|P_T\psi_f\|_{H^3_{(0)}} < \delta$, thus we can define:

$$\Gamma(\psi_f) =: \Theta_T^{-1}[P_T\psi_f].$$

Thanks to the choice of R_1 and δ_1 we know that the solution of (1), (6) with $u = \Gamma(\psi_f)$ satisfies:

$$\psi(T) = P_T\big(\psi(T)\big) + \sqrt{1 - \big\|P_T\psi(T)\big\|^2_{L^2}}\,\psi_1(T) = P_T(\psi_f) + \sqrt{1 - \|P_T\psi_f\|^2_{L^2}}\,\psi_1(T) = \psi_f.$$

2.5. Generalization to higher regularities

The goal of this section is the proof of Theorem 2. The first step of the proof consists in adapting Proposition 2.

Proposition 5. *Let $\mu \in H^5((0,1),\mathbb{R})$, $T > 0$, $\psi_0 \in H^5_{(0)}(0,1)$, $f \in H^1_0((0,T), H^3 \cap H^1_0)$ and $u \in H^1_0((0,T),\mathbb{R})$. There exists a unique function $\psi \in C^1([0,T], H^3_{(0)})$ such that the equality (21) holds in $C^1([0,T], H^3_{(0)})$. Moreover, for every $R > 0$ there exists $C = C(T,\mu,R) > 0$ such that, if $\|u\|_{H^1_0(0,T)} < R$, then, this weak solution satisfies*:

$$\|\psi\|_{C^1([0,T],H^3_{(0)})} \leqslant C\big(\|\psi_0\|_{H^5_{(0)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)}\big). \tag{35}$$

The proof of Proposition 5 is the same as the one of Proposition 2, except that we use the following lemma, instead of Lemma 1:

Lemma 2. *Let $T > 0$, $u_0 \in H^5 \cap H^3_{(0)}$ and $f \in H^1((0,T), H^3 \cap H^1_0)$ be such that $-iAu_0 + f(0) \in H^3_{(0)}$. The function $G : t \mapsto e^{-iAt}u_0 + \int_0^t e^{-iA(t-s)} f(s)\,ds$ belongs to $C^1([0,T], H^3_{(0)})$, moreover*

$$\|G\|_{C^1([0,T],H^3_{(0)})} \leqslant c_1(T)\big(\|u_0\|_{H^3_{(0)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)} + \big\|-iAu_0 + f(0)\big\|_{H^3_{(0)}}\big),$$

where the constants $c_1(T)$ are uniformly bounded for T lying in bounded intervals. We also have:

$$\big\|-iAG(T) + f(T)\big\|_{H^3_{(0)}} \leqslant c_1(T)\big(\|u_0\|_{H^3_{(0)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)} + \big\|-iAu_0 + f(0)\big\|_{H^3_{(0)}}\big).$$

Proof of Lemma 2. We already know that $G \in C^0([0,T], H^3_{(0)})$. First let us write:

$$G(t) = e^{-iAt}u_0 + \int_0^t e^{-iA\tau} f(t-\tau)\,d\tau.$$

Since $u_0 \in H^4_{(0)}$ and $f \in H^1((0,T), H^2_{(0)})$, we know that $G \in C^1([0,T], H^2_{(0)})$ and the following equality holds in $H^2_{(0)}$ for every $t \in [0,T]$,

$$\begin{aligned}\frac{\partial G}{\partial t}(t) &= -iAe^{-iAt}u_0 + e^{-iAt}f(0) + \int_0^t e^{-iA\tau}\frac{\partial f}{\partial t}(t-\tau)\,d\tau \\ &= e^{-iAt}\big[-iAu_0 + f(0)\big] + \int_0^t e^{-iA(t-s)}\frac{\partial f}{\partial t}(s)\,ds\end{aligned}$$

(the proof of this result involves classical technics). Thanks to this expression and Lemma 1, we get:

$$\frac{\partial G}{\partial t} \in C^0\big([0,T], H^3_{(0)}\big).$$

Now, let us prove that $G \in C^1([0,T], H^3_{(0)})$, i.e. for every $t \in [0,T]$,

$$\left\| \frac{G(t+h)-G(t)}{h} - \frac{\partial G}{\partial t}(t) \right\|_{H^3_{(0)}} \to 0 \quad \text{when } h \to 0.$$

We have:

$$\begin{aligned}\frac{G(t+h)-G(t)}{h} - \frac{\partial G}{\partial t}(t) &= e^{-iAt}\left[\frac{e^{-iAh}-Id}{h}u_0 + iAu_0 - f(0)\right] + \frac{1}{h}\int_t^{t+h} e^{-iA\tau} f(t+h-\tau)\,d\tau \\ &\quad + \int_0^t e^{-iA\tau}\left[\frac{f(t+h-\tau)-f(t-\tau)}{h} - \frac{\partial f}{\partial t}(t-\tau)\right] d\tau. \qquad (36)\end{aligned}$$

By applying Lemma 1, we see that the $H^3_{(0)}(0,1)$-norm of the term on the second line of the right-hand side of (36) tends to zero when $h \to 0$ because $f \in H^1((0,T), H^3 \cap H^1_0)$. Thanks to several changes of variables, the term on the first line of the right-hand side of (36) may be decomposed in the following way:

$$e^{-iAt}\left[\frac{e^{-iAh}-Id}{h}\big(u_0 + iA^{-1}f(0)\big) + iA\big(u_0 + iA^{-1}f(0)\big)\right] + e^{-iAt}\frac{1}{h}\int_0^h e^{-iAs}\big(f(h-s)-f(0)\big)\,ds. \qquad (37)$$

The $H^3_{(0)}(0,1)$-norm of the first term of (37) tends to zero when $h \to 0$ because $u_0 + iA^{-1}f(0) \in H^5_{(0)}(0,1)$. The $H^3_{(0)}(0,1)$-norm of the second term of (37) also tends to zero when $h \to 0$ because, thanks to Lemma 1 and Cauchy–Schwarz inequality, it is bounded by:

$$\begin{aligned}\left\| \int_0^h e^{iAs}\left(\frac{f(s)-f(0)}{h}\right) ds \right\|_{H^3_{(0)}} &\leqslant c_1(h) \left\| \frac{f(.)-f(0)}{h} \right\|_{L^2((0,h),H^3\cap H^1_0)} \leqslant \frac{c_1(h)}{h} \left\| \int_0^{\cdot} \frac{\partial f}{\partial t}(\tau)\,d\tau \right\|_{L^2((0,h),H^3\cap H^1_0)} \\ &\leqslant \frac{c_1(h)\sqrt{h}}{h} \left\| \int_0^{\cdot} \frac{\partial f}{\partial t}(\tau)\,d\tau \right\|_{L^\infty((0,h),H^3\cap H^1_0)} \leqslant c_1(h) \left\| \frac{\partial f}{\partial t} \right\|_{L^2((0,h),H^3\cap H^1_0)}.\end{aligned}$$

The estimate (24) of Lemma 1 gives the first inequality of Lemma 2. Moreover, by integration by part in time, we get:

$$\begin{aligned}-iAG(t) &= -iAe^{-iAt}u_0 - \int_0^t iAe^{-iA\tau} f(t-\tau)\,d\tau \\ &= -iAe^{-iAt}u_0 + e^{-iAt}f(0) - f(t) + \int_0^t e^{-iA\tau}\frac{\partial f}{\partial t}(t-\tau)\,d\tau.\end{aligned}$$

We get the second estimate thanks to the identity:

$$-iAG(t) + f(t) = e^{-iAt}\big[-iAu_0 + f(0)\big] + \int_0^t e^{-iA(t-\tau)}\frac{\partial f}{\partial t}(\tau)\,d\tau. \qquad \square$$

The following statement is the appropriate adaptation of Proposition 3.

Proposition 6. *Let $T > 0$ and $\mu \in H^5((0,1),\mathbb{R})$. The map Θ_T defined by (27) is C^1 from $H^1_0((0,T),\mathbb{R})$ to $V_T \cap H^5_{(0)}(0,1)$.*

Proof of Proposition 6. *First step: We prove that Θ_T maps $H_0^1((0,T),\mathbb{R})$ into $V_T \cap H^5_{(0)}(0,1)$.* Let $u \in H_0^1((0,T),\mathbb{R})$ and ψ be the weak solution of (1), (6). Then $\psi \in C^1([0,T], H^2_{(0)}) \cap C^0([0,T], H^4_{(0)})$ and the first equality of (1) holds in $H^2_{(0)}$ for every $t \in [0,T]$ (the proof of this result involves classical technics). In particular, we have:

$$\|\psi(T)\|_{H^5_{(0)}} = \|\psi''(T)\|_{H^3_{(0)}} = \left\|\frac{\partial \psi}{\partial t}(T)\right\|_{H^3_{(0)}} \quad \text{because } u(T)=0$$

which is finite, thanks to Proposition 5.

Second step: We prove that $\Theta_T : H_0^1((0,T),\mathbb{R}) \to V_T \cap H^5_{(0)}$ is differentiable. Let $u, v \in H_0^1((0,T),\mathbb{R})$, ψ, Ψ, $\tilde{\psi}$ be the weak solutions of (1), (6), (29), (30). Then, $\Delta := \tilde{\psi} - \psi - \Psi$ is the weak solution of (31). Let us prove that

$$\|\Delta(T)\|_{H^5_{(0)}} = o\big(\|v\|_{H^1_0}\big) \quad \text{when } \|v\|_{H^1_0} \to 0,$$

which gives the conclusion. Let $R > 0$ be such that $\|u\|_{H^1_0} < R$ and $\|u+v\|_{H^1_0} < R$. Thanks to Proposition 5, there exists $C = C(T,\mu,R) > 0$, $C_1 = C_1(\mu) > 0$ such that

$$\begin{aligned}
\|\Delta(T)\|_{H^5_{(0)}} &= \|\Delta''(T)\|_{H^3_{(0)}} = \left\|\frac{\partial \Delta}{\partial t}(T)\right\|_{H^3_{(0)}} \quad \text{because } u(T)=v(T)=0 \\
&\leqslant C\|v\mu\Psi\|_{H^1_0((0,T),H^3\cap H^1_0)} \leqslant CC_1\|v\|_{H^1_0}\|\Psi\|_{C^1([0,T],H^3_{(0)})} \\
&\leqslant C^2C_1\|v\|_{H^1_0}\|v\mu\psi\|_{H^1_0((0,T),H^3\cap H^1_0)} \leqslant C^2C_1^2\|v\|^2_{H^1_0}\|\psi\|_{C^1([0,T],H^3_{(0)})}.
\end{aligned}$$

The proof of the continuity of the map $d\Theta_T : H_0^1((0,T),\mathbb{R}) \to \mathcal{L}(H_0^1, V_T \cap H^5_{(0)})$ involves similar arguments. □

Remark 4. With the same kind of arguments, we could get that $A\psi(t) - u(t)\mu\psi(t) \in C^0([0,T], H^3_{(0)})$. Therefore, $\psi(t)$ does not, in general, belong to $H^5_{(0)}(0,1)$ for $t \in (0,T)$.

The following statement is the appropriate generalization of Proposition 4.

Proposition 7. *Let $T > 0$, $\mu \in H^5((0,1),\mathbb{R})$ be such that (5) holds and Θ_T be defined by* (27). *The linear map $d\Theta_T(0) : H_0^1((0,T),\mathbb{R}) \to V_T \cap H^5_{(0)}(0,1)$ has a continuous right inverse*

$$d\Theta_T(0)^{-1} : V_T \cap H^5_{(0)}(0,1) \to H_0^1\big((0,T),\mathbb{R}\big).$$

Proof of Proposition 7. Let $\Psi_f \in V_T \cap H^5_{(0)}(0,1)$. If Ψ is the solution of (33) for some $v \in H_0^1((0,T),\mathbb{R}$, then, the equality $\Psi(T) = \Psi_f$ is equivalent to the trigonometric moment problem (34), or equivalently:

$$\begin{gathered}
\int_0^T \dot{v}(t)\,dt = 0, \\
\int_0^T (T-t)\dot{v}(t)\,dt = \frac{1}{i\langle\mu\varphi_1,\varphi_1\rangle}\langle\Psi_f,\varphi_1\rangle e^{i\lambda_1 T}, \\
\int_0^T \dot{v}(t)e^{i(\lambda_k-\lambda_1)t}\,dt = \frac{\lambda_1-\lambda_k}{\langle\mu\varphi_1,\varphi_k\rangle}\langle\Psi_f,\varphi_k\rangle e^{i\lambda_k T}, \quad \forall k \geqslant 2.
\end{gathered} \tag{38}$$

The conclusion comes from Corollary 2 (in Appendix B). □

Now, Theorem 2 may be proved exactly as Theorem 1.

2.6. Case of the three-dimensional ball with radial data

The goal of this section is the proof of Theorem 3. This proof is very similar to the case of the interval and we only give the necessary modifications. The equivalent of Lemma 1 is proved with a similar computation for $f \in L^2((0,T), H^3_{rad} \cap H^1_{(0)})$. More precisely, for almost every $s \in (0,T)$, we have:

$$\begin{aligned}\langle f(s), \varphi_k\rangle &= \int_{B^3} f(s)\varphi_k = \frac{1}{\lambda_k^2}\int_{B^3} f(s)\Delta^2\varphi_k = \frac{1}{\lambda_k^2}\int_{B^3} \Delta f(s)\Delta\varphi_k \\ &= -\frac{1}{\lambda_k^2}\int_{B^3} \nabla\Delta f(s)\cdot\nabla\varphi_k + \frac{1}{\lambda_k^2}\int_{S^2} \Delta f(s)\frac{\partial\varphi_k}{\partial n}\,d\sigma.\end{aligned}$$

To bound the first term, we use $\nabla\Delta f \in L^2((0,T), L^2(B^3)^3)$ and the fact that the functions $(\nabla\varphi_k/\sqrt{\lambda_k})_{k\in\mathbb{N}^*}$ form an orthonormal family of $L^2(B^3)^3$, because

$$\int_{B^3} \nabla\varphi_i\cdot\nabla\varphi_j = -\int_{B^3} \varphi_i\Delta\varphi_j = \lambda_j\delta_{i,j}.$$

For the second term, since f and φ_k are radial, we have:

$$\frac{1}{\lambda_k^2}\int_{S^2} \Delta f(s)\frac{\partial\varphi_k}{\partial n}\,d\sigma = \frac{2^{3/2}\sqrt{\pi}(-1)^k}{\lambda_k^{3/2}}\Delta f(s, r=1).$$

We conclude as in Lemma 1 for this term since the eigenvalues are the same and Corollary 4 still applies. The genericity of assumption (10) is detailed in Appendix A, Proposition 17.

Remark 5. It is very likely that the same analysis would work in any dimension $n \leqslant 5$, provided that H^3 remains an algebra. However, this would require the analysis of the zeros of the Bessel functions and we have chosen to present the simplest result.

3. Nonlinear Schrödinger equations

In this section, we study the nonlinear Schrödinger equation with Neumann boundary conditions (12). The goal is the proof of Theorem 4.

First, let us introduce the following notations, that will be valid in all Section 3. The operator A is defined by:

$$D(A) = H^2_{(0)}(0,1) := \{\varphi \in H^2(0,1);\ \varphi'(0) = \varphi'(1) = 0\}, \qquad A\varphi = -\varphi''. \tag{39}$$

Its eigenvectors $(\varphi_k)_{k\in\mathbb{N}}$ and eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ are,

$$\begin{aligned}\varphi_0 &:= 1, & \lambda_0 &:= 0,\\ \varphi_k(x) &:= \sqrt{2}\cos(k\pi x), & \lambda_k &:= (k\pi)^2, \quad \forall k \in \mathbb{N}^*.\end{aligned} \tag{40}$$

We introduce the spaces:

$$H^s_{(0)}(0,1) := D(A^{s/2}), \quad \forall s > 0, \tag{41}$$

and the notation

$$k_* := \max\{k, 1\}, \quad \forall k \in \mathbb{N}. \tag{42}$$

3.1. Well posedness of the Cauchy problem

The goal of this subsection is the proof of the following result:

Proposition 8. *Let $\mu \in H^2((0,1),\mathbb{R})$ and $T > 0$. There exists $\delta > 0$ such that, for every $u \in B_\delta[L^2(0,T)]$, there exists a unique weak solution $\psi \in C^0([0,T], H^2_{(0)})$ of* (12), (14)*. Moreover, we have*:

$$\left\|\psi(t)\right\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \quad \forall t \in [0,T].$$

We search ψ in the form $\psi(t,x) = e^{-it}(1+\zeta(t,x))$, where ζ is a weak solution of:

$$\begin{cases} i\dfrac{\partial \zeta}{\partial t} = -\zeta'' + \left(|1+\zeta|^2 - 1\right)(1+\zeta) - u\mu(1+\zeta), \\ \zeta'(t,0) = \zeta'(t,1) = 0, \\ \zeta(0,x) = 0. \end{cases} \tag{43}$$

Proposition 8 will be the consequence of the existence and uniqueness of a weak solution ζ for (43) (the conservation of the L^2-norm may be proved as in the linear case). In order to precise the definition of such a weak solution, let us introduce the operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) := H^2_{(0)}(0,1), \qquad \mathcal{A}\zeta := -\zeta'' + 2\Re(\zeta).$$

Then for every $\zeta \in H^2_{(0)}(0,1)$ and every $t \in \mathbb{R}$, we have:

$$e^{-i\mathcal{A}t}\zeta = \sum_{k=0}^{\infty}\big(a_k(t) + ib_k(t)\big)\varphi_k,$$

where

$$a_0(t) := \Re\big(\langle\zeta,\varphi_0\rangle\big), \qquad b_0(t) := \Im\big(\langle\zeta,\varphi_0\rangle\big) - 2t\Re\big(\langle\zeta,\varphi_0\rangle\big),$$

$$a_k(t) := \Re\big(\langle\zeta,\varphi_k\rangle\big)\cos\big[\sqrt{\lambda_k(\lambda_k+2)}t\big] + \sqrt{\frac{\lambda_k}{\lambda_k+2}}\Im\big(\langle\zeta,\varphi_k\rangle\big)\sin\big[\sqrt{\lambda_k(\lambda_k+2)}t\big], \quad \forall k \in \mathbb{N}^*,$$

$$b_k(t) := -\sqrt{\frac{\lambda_k+2}{\lambda_k}}\Re\big(\langle\zeta,\varphi_k\rangle\big)\sin\big[\sqrt{\lambda_k(\lambda_k+2)}t\big] + \Im\big(\langle\zeta,\varphi_k\rangle\big)\cos\big[\sqrt{\lambda_k(\lambda_k+2)}t\big], \quad \forall k \in \mathbb{N}^*.$$

Remark that these formulae are only the result of the diagonalization of the matrix $\left(\begin{smallmatrix} 0 & \Delta \\ -\Delta+2 & 0\end{smallmatrix}\right)$ obtained by the decomposition in real and imaginary part. Then Proposition 8 is equivalent to the following statement.

Proposition 9. *Let $\mu \in H^2((0,1),\mathbb{R})$ and $T > 0$. There exists $\delta > 0$ such that, for every $u \in B_\delta[L^2((0,T),\mathbb{R})]$, there exists a unique weak solution of* (43)*, i.e. a function $\zeta \in C^0([0,T], H^2_{(0)})$ such that the following equality holds in $H^2_{(0)}$ for every $t \in [0,T]$*

$$\zeta(t) = \int_0^t e^{-i\mathcal{A}(t-s)}\Big(\big[\left|1+\zeta(s)\right|^2 - 1\big]\big[1+\zeta(s)\big] - 2\Re\big[\zeta(s)\big] - u(s)\mu\big[1+\zeta(s)\big]\Big)\,ds. \tag{44}$$

The proof of Proposition 9 relies on the following lemma.

Lemma 3. *Let $T > 0$ and $f \in L^2((0,T), H^2)$. The function $G : t \mapsto \int_0^t e^{-i\mathcal{A}(t-s)} f(s)\,ds$ belongs to $C^0([0,T], H^2_{(0)})$, moreover*

$$\|G\|_{L^\infty((0,T),H^2_{(0)})} \leqslant c_0(T)\|f\|_{L^2((0,T),H^2)},$$

where the constants $c_0(T)$ are uniformly bounded for T lying in bounded intervals.

Proof of Lemma 3. The proof of this lemma is similar to the one of Lemma 1. By definition, we have:

$$G(t)=\sum_{k=0}^{\infty}\sum_{a=1}^{4}\left(\int_0^t y_k^a(t,s)\,ds\right)\varphi_k,$$

where

$$\begin{gathered}
y_k^1(t,s):=\Re\big(\langle f(s),\varphi_k\rangle\big)\cos\big[\sqrt{\lambda_k(\lambda_k+2)}(t-s)\big],\quad \forall k\in\mathbb{N},\\
y_k^2(t,s):=\sqrt{\frac{\lambda_k}{\lambda_k+2}}\Im\big(\langle f(s),\varphi_k\rangle\big)\sin\big[\sqrt{\lambda_k(\lambda_k+2)}(t-s)\big],\quad \forall k\in\mathbb{N}^*,\\
y_k^3(t,s):=-i\sqrt{\frac{\lambda_k+2}{\lambda_k}}\Re\big(\langle f(s),\varphi_k\rangle\big)\sin\big[\sqrt{\lambda_k(\lambda_k+2)}(t-s)\big],\quad \forall k\in\mathbb{N}^*,\\
y_k^4(t,s):=i\Im\big(\langle f(s),\varphi_k\rangle\big)\cos\big[\sqrt{\lambda_k(\lambda_k+2)}(t-s)\big],\quad \forall k\in\mathbb{N},\\
y_0^2(t,s):=0,\qquad y_0^3(t,s):=-2t\Re\big(\langle f(s),\varphi_k\rangle\big).
\end{gathered}$$

We have

$$\|G(t)\|_{H^2_{(0)}}\leqslant\sum_{a=1}^{4}\left(\sum_{k=1}^{\infty}\left|k_*^2\int_0^t y_k^a(t,s)\,ds\right|^2\right)^{1/2}.$$

Let us prove that there exists a constant $c=c(t)>0$ (uniformly bounded on bounded intervals of t) such that

$$\left(\sum_{k=1}^{\infty}\left|k_*^2\int_0^t y_k^1(t,s)\,ds\right|^2\right)^{1/2}\leqslant c(t)\|f\|_{L^2((0,t),H^2)}\tag{45}$$

(the other terms may be treated in the same way). Integrations by part give, for almost every $s\in(0,T)$,

$$\langle f(s),\varphi_k\rangle=\frac{\sqrt{2}}{(k\pi)^2}\left((-1)^k f'(s,1)-f'(s,0)-\int_0^1 f''(s,x)\cos(k\pi x)\,dx\right),\quad \forall k\in\mathbb{N}^*.$$

Thus, we have, for every $k\in\mathbb{N}^*$,

$$\begin{aligned}
k^2\int_0^t y_k^1(t,s)\,ds={}&\frac{\sqrt{2}(-1)^k}{(\pi)^2}\int_0^t f'(s,1)\cos\big[\sqrt{\lambda_k(\lambda_k+2)}(t-s)\big]\,ds\\
&+\frac{\sqrt{2}}{(\pi)^2}\int_0^t f'(s,0)\cos\big[\sqrt{\lambda_k(\lambda_k+2)}(t-s)\big]\,ds\\
&-\frac{\sqrt{2}}{(\pi)^2}\int_0^t \langle f''(s),\varphi_k\rangle\cos\big[\sqrt{\lambda_k(\lambda_k+2)}(t-s)\big]\,ds.
\end{aligned}$$

We get (45) thanks to Corollary 4, as in the proof of Lemma 1. □

Proof of Proposition 9. We introduce the function $g:\mathbb{C}\to\mathbb{C}$ defined by:

$$g(z):=\big[|1+z|^2-1\big][1+z].\tag{46}$$

We have $dg(0).\zeta=2\Re(\zeta)$. Let $c_0=c_0(T)$ be as in Lemma 3. Let $c_1,c_2,c_3>0$ be such that

$$\big\|g(\zeta)-dg(0).\zeta\big\|_{H^2}\leqslant c_1\big(\|\zeta\|^2_{H^2_{(0)}}+\|\zeta\|^3_{H^2_{(0)}}\big),\quad \forall\zeta\in H^2_{(0)},\tag{47}$$

$$\big\|g(\tilde\zeta)-g(\zeta)-dg(0).(\tilde\zeta-\zeta)\big\|_{H^2}\leqslant c_2\|\zeta-\tilde\zeta\|_{H^2_{(0)}}\max\big\{\|\xi\|_{H^2_{(0)}},\|\xi\|^2_{H^2_{(0)}};\ \xi\in\{\zeta,\tilde\zeta\}\big\},\quad\forall\zeta,\tilde\zeta\in H^2_{(0)},\tag{48}$$

$$\|\mu\zeta\|_{H^2}\leqslant c_3\|\zeta\|_{H^2_{(0)}},\quad\forall\zeta\in H^2_{(0)}.\tag{49}$$

Let $R>0$ be small enough so that

$$c_0c_1\sqrt{T}\big(R^2+R^3\big)<\frac{R}{2}\quad\text{and}\quad c_0c_2\sqrt{T}\max\big\{R,R^2\big\}<\frac{1}{4}.\tag{50}$$

Let $\delta>0$ be small enough so that

$$c_0\delta c_3(1+R)<\frac{R}{2}\quad\text{and}\quad c_0\delta c_3<\frac{1}{4}.\tag{51}$$

Let $u\in L^2((0,T),\mathbb{R})$ be such that $\|u\|_{L^2(0,T)}<\delta$. We consider the map,

$$F:\overline{B}_R\big[C^0\big([0,T],H^2_{(0)}\big)\big]\to\overline{B}_R\big[C^0\big([0,T],H^2_{(0)}\big)\big],$$
$$\zeta\mapsto\xi,$$

where $\xi:=F(\zeta)$ is defined by:

$$\xi(t)=-i\int_0^t e^{-i\mathcal{A}(t-s)}\big(\big[g\big(\zeta(s)\big)-dg(0).\zeta(s)-u(s)\mu\big[1+\zeta(s)\big]\big)\,ds.$$

For $\zeta\in\overline{B}_R[C^0([0,T],H^2_{(0)})]$, the function $g(\zeta)-dg(0).\zeta-u\mu[1+\zeta]$ belongs to $L^2((0,T),H^2)$, thus ξ belongs to $C^0([0,T],H^2_{(0)})$ thanks to Lemma 3. Moreover, using (47), (49)–(51), we get:

$$\begin{aligned}\|\xi\|_{L^\infty((0,T),H^2_{(0)})}&\leqslant c_0\big\|g(\zeta)-dg(0).\zeta-u\mu[1+\zeta]\big\|_{L^2((0,T),H^2)}\\&\leqslant c_0\big[\sqrt{T}\big\|g(\zeta)-dg(0).\zeta\big\|_{L^\infty((0,T),H^2)}+\|u\|_{L^2(0,T)}\big\|\mu[1+\zeta]\big\|_{L^\infty((0,T),H^2)}\big]\\&\leqslant c_0\big[\sqrt{T}c_1\big(R^2+R^3\big)+\delta c_3(1+R)\big]\\&\leqslant R.\end{aligned}$$

Thus, F takes values in $\overline{B}_R[C^0([0,T],H^2_{(0)})]$.

For $\zeta,\tilde\zeta\in\overline{B}_R[C^0([0,T],H^2_{(0)})]$, using (48)–(51), we get:

$$\begin{aligned}\|\xi-\tilde\xi\|_{L^\infty((0,T),H^2_{(0)})}&\leqslant c_0\big\|g(\zeta)-g(\tilde\zeta)-dg(0).(\zeta-\tilde\zeta)-u\mu(\zeta-\tilde\zeta)\big\|_{L^2((0,T),H^2)}\\&\leqslant c_0\big[\sqrt{T}c_2\|\zeta-\tilde\zeta\|_{L^\infty((0,T),H^2_{(0)})}\max\big\{R,R^2\big\}+\delta c_3\|\zeta-\tilde\zeta\|_{L^\infty((0,T),H^2_{(0)})}\big]\\&\leqslant\frac{1}{2}\|\zeta-\tilde\zeta\|_{L^\infty((0,T),H^2_{(0)})}.\end{aligned}$$

Thus F is a contraction. □

3.2. C^1-regularity of the end-point map

Let $T>0$ and $\delta>0$ be as in Proposition 8. Let

$$V_T:=\left\{\varphi\in L^2(0,1);\ \Re\left(e^{iT}\int_0^1\varphi(x)\,dx\right)=0\right\},$$

and $P_T:L^2(0,1)\to V_T$ be the associated orthogonal projection. Then, the following map is well defined:

$$\begin{aligned}\Theta_T : B_\delta\big[L^2\big((0,T),\mathbb{R}\big)\big] &\to H^2_{(0)}(0,1),\\ u &\mapsto P_T\big[\psi(T)\big],\end{aligned} \tag{52}$$

where ψ solves (12), (14). We want to prove that the map Θ_T is C^1 on a neighborhood of zero. We have seen that $\psi(t) = e^{-it}(1+\zeta(t))$, where ζ solves (43). Thus, it is sufficient to prove the following statement:

Proposition 10. *Let $\mu \in H^2((0,1),\mathbb{R})$, $T > 0$, δ be as in Proposition* 9, *and*

$$\begin{aligned}\tilde{\Theta}_T : B_\delta\big[L^2\big((0,T),\mathbb{R}\big)\big] &\to H^2_{(0)}(0,1),\\ u &\mapsto \zeta(T),\end{aligned}$$

where ζ solves (43). *There exists $\delta' \in (0,\delta)$ such that the map Θ_T is C^1 on $B_{\delta'}[L^2((0,T),\mathbb{R})]$. Moreover, for every $u \in B_{\delta'}[L^2((0,T),\mathbb{R})]$ and $v \in L^2((0,T),\mathbb{R})$ we have*:

$$d\tilde{\Theta}_T(u).v = \xi(T), \tag{53}$$

where ξ solves

$$\begin{cases} i\dfrac{\partial \xi}{\partial t} = -\xi'' + dg(\zeta).\xi - u\mu\xi - v\mu(1+\zeta),\\ \xi'(t,0) = \xi'(t,1) = 0,\\ \xi(0,x) = 0,\end{cases} \tag{54}$$

g is defined by (46) *and ζ solves* (43).

Proof of Proposition 10. We use the same notations $c_0, c_1, c_2, c_3, R, \delta$ as in the proof of Proposition 9, in particular, the relations (47)–(51) are satisfied. We introduce constants $c_4, c_5 > 0$ such that

$$\big\|\big[dg(\zeta) - dg(0)\big].h\big\|_{H^2} \leqslant c_4\|h\|_{H^2_{(0)}} \max\big\{\|\zeta\|_{H^2_{(0)}}, \|\zeta\|^2_{H^2_{(0)}}\big\}, \quad \forall \zeta, h \in H^2_{(0)}, \tag{55}$$

$$\big\|g(\tilde{\zeta}) - g(\zeta) - dg(0).(\tilde{\zeta} - \zeta)\big\|_{H^2} \leqslant c_5\|\tilde{\zeta} - \zeta\|_{H^2_{(0)}} \max\big\{\|\xi\|_{H^2_{(0)}}, \|\xi\|^2_{H^2_{(0)}};\ \xi \in \{\zeta, \tilde{\zeta}\}\big\}, \quad \forall \zeta, \tilde{\zeta} \in H^2_{(0)}. \tag{56}$$

Moreover, we assume that

$$c_0\sqrt{T}\max\{c_4, c_5\}\max\big\{R, R^2\big\} < \frac{1}{4} \tag{57}$$

(this additional assumption may change δ into a smaller value δ').

Let $u, v \in B_\delta[L^2((0,T),\mathbb{R})]$ be such that $(u+v) \in B_\delta[L^2(0,T)]$. Let ζ, ξ and $\tilde{\zeta}$ be the solutions of (43), (54), and

$$\begin{cases} i\dfrac{\partial \tilde{\zeta}}{\partial t} = -\tilde{\zeta}'' + \big(|1+\tilde{\zeta}|^2 - 1\big)(1+\tilde{\zeta}) - (u+v)\mu(1+\tilde{\zeta}),\\ \tilde{\zeta}'(t,0) = \tilde{\zeta}'(t,1) = 0,\\ \tilde{\zeta}(0,x) = 0.\end{cases}$$

The existence of ξ may be proved in a similar way as the existence of ζ.

First step: Let us prove that

$$\|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})} \leqslant 2c_0c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}\|v\|_{L^2}. \tag{58}$$

Thanks to Lemma 3, (56), (49), (57) and (51), we have:

$$\begin{aligned}\|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})} &\leqslant c_0\big\|g(\tilde{\zeta}) - g(\zeta) - dg(0).(\tilde{\zeta} - \zeta) - (u+v)\mu(\tilde{\zeta} - \zeta) - v\mu(1+\zeta)\big\|_{L^2((0,T),H^2)}\\ &\leqslant c_0\big[\sqrt{T}c_5\|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})}\max\big\{R, R^2\big\} + \delta c_3\|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})}\\ &\quad + \|v\|_{L^2}c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}\big]\\ &\leqslant \frac{1}{2}\|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})} + c_0\|v\|_{L^2}c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}\end{aligned}$$

which gives (58).

Second step: Let us prove that the linear map

$$\left| \begin{array}{rcl} L^2(0,T) & \to & H^2_{(0)}(0,1), \\ v & \mapsto & \xi(T), \end{array} \right.$$

is continuous. Thanks to Lemma 3, (55), (49), (57) and (51), we have,

$$\begin{aligned} \|\xi\|_{L^\infty((0,T),H^2_{(0)})} &\leqslant c_0 \big\| \big[dg(\zeta) - dg(0)\big].\xi - u\mu\xi - v\mu(1+\zeta) \big\|_{L^2((0,T),H^2)} \\ &\leqslant c_0 \big[\sqrt{T} c_4 \|\xi\|_{L^\infty((0,T),H^2_{(0)})} \max\{R, R^2\} + \delta c_3 \|\xi\|_{L^\infty((0,T),H^2_{(0)})} \\ &\quad + \|v\|_{L^2} c_3 \|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})} \big] \\ &\leqslant \frac{1}{2} \|\xi\|_{L^\infty((0,T),H^2_{(0)})} + c_0 \|v\|_{L^2} c_3 \|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}, \end{aligned}$$

which gives

$$\|\xi\|_{L^\infty((0,T),H^2_{(0)})} \leqslant 2c_0 c_3 \|v\|_{L^2} \|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}. \tag{59}$$

Third step: Let us prove that $\tilde{\Theta}_T$ is differentiable and that (53) *holds*. Let $\Delta := \tilde{\zeta} - \zeta - \xi$. We want to prove that

$$\big\|\Delta(T)\big\|_{H^2_{(0)}} = o\big(\|v\|_{L^2}\big) \quad \text{when } \|v\|_{L^2} \to 0.$$

Let $\epsilon > 0$. There exists $\eta > 0$ such that, for every $f \in L^\infty((0,T), H^2_{(0)})$ with $\|f - \zeta\|_{L^\infty((0,T),H^2_{(0)})} < \eta$, we have:

$$\big\|g(f) - g(\zeta) - dg(\zeta).(f-\zeta)\big\|_{L^\infty((0,T),H^2_{(0)})} < \epsilon \|f - \zeta\|_{L^\infty((0,T),H^2_{(0)})}.$$

Let us assume that v is small enough so that

$$2c_0 c_3 \|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})} \|v\|_{L^2} < \eta.$$

Then, thanks to Lemma 3 and (58), (55) and (49), we have:

$$\begin{aligned} \|\Delta\|_{L^\infty((0,T),H^2_{(0)})} &\leqslant c_0 \big\| g(\tilde{\zeta}) - g(\zeta) - dg(\zeta).(\tilde{\zeta} - \zeta) + \big[dg(\zeta) - dg(0)\big].\Delta - (u+v)\mu\Delta - v\mu\xi \big\|_{L^2((0,T),H^2)} \\ &\leqslant c_0 \big[\sqrt{T}\epsilon \|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})} + \sqrt{T} c_4 \big(R + R^2\big) \|\Delta\|_{L^\infty((0,T),H^2_{(0)})} \\ &\quad + \delta c_3 \|\Delta\|_{L^\infty((0,T),H^2_{(0)})} + \|v\|_{L^2} c_3 \|\xi\|_{L^\infty((0,T),H^2_{(0)})} \big]. \end{aligned}$$

Thanks to (57) and (49), we get:

$$\|\Delta\|_{L^\infty((0,T),H^2_{(0)})} \leqslant 2c_0 \big[\sqrt{T}\epsilon \|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})} + \|v\|_{L^2} c_3 \|\xi\|_{L^\infty((0,T),H^2_{(0)})} \big],$$

which gives the conclusion, thanks to (58) and (59).

The continuity of the map $d\tilde{\Theta}_T$ may be proved with similar arguments. □

3.3. Controllability of the linearized system

The goal of this section is the proof of the following result.

Proposition 11. *Let $T > 0$ and $\mu \in H^2((0,1),\mathbb{R})$ be such that* (13) *holds. Let $\delta > 0$ be as in Proposition* 8 *and Θ_T be defined by* (52). *The linear map $d\Theta_T(0) : L^2((0,T),\mathbb{R}) \to V_T \cap H^2_{(0)}(0,1)$ has a continuous right inverse $d\Theta_T(0)^{-1} : V_T \cap H^2_{(0)}(0,1) \to L^2((0,T),\mathbb{R})$.*

Proof of Proposition 11. It is equivalent to prove that the continuous linear map

$$d\tilde{\Theta}_T(0) : L^2\big((0,T),\mathbb{R}\big) \to \tilde{V} \cap H^2_{(0)}(0,1)$$

has a continuous right inverse, where

$$\tilde{V} := \left\{ \varphi \in L^2(0,1);\ \Re \int_0^1 \varphi(x)\,dx = 0 \right\}.$$

We have $d\tilde{\Theta}_T(0).v = \xi(T)$ where ξ is the weak solution of:

$$\begin{cases} i\dfrac{\partial \xi}{\partial t} = -\xi'' + 2\Re(\xi) - v(t)\mu(x), & x \in (0,1),\ t \in (0,T), \\ \xi'(t,0) = \xi'(t,1) = 0, \\ \xi(0,x) = 0. \end{cases}$$

In particular, we have:

$$\xi(T) = i\int_0^T e^{-i\mathcal{A}(T-s)} v(s)\mu\,ds = i\sum_{k=0}^{\infty}\big[a_k(T) + ib_k(T)\big]\varphi_k,$$

where

$$a_0(T) = \langle \mu, \varphi_0\rangle \int_0^T v(s)\,ds, \qquad b_0(T) = -2\langle \mu, \varphi_0\rangle \int_0^T (T-t)v(s)\,ds,$$

$$a_k(T) = \langle \mu, \varphi_k\rangle \int_0^T v(s)\cos\big[\sqrt{\lambda_k(\lambda_k+2)}(T-s)\big]\,ds, \quad \forall k \in \mathbb{N}^*,$$

$$b_k(T) = -\sqrt{\frac{\lambda_k+2}{\lambda_k}}\langle \mu, \varphi_k\rangle \int_0^T v(s)\sin\big[\sqrt{\lambda_k(\lambda_k+2)}(T-s)\big]\,ds, \quad \forall k \in \mathbb{N}^*.$$

For $\xi_f \in \tilde{V} \cap H^2_{(0)}(0,1)$, the equality $\xi(T) = \xi_f$ is equivalent to the following trigonometric moment problem:

$$\begin{cases} \displaystyle\int_0^T v(s)\,ds = d_0(\xi_f) := \Im\frac{\langle \xi_f, \varphi_0\rangle}{\langle \mu, \varphi_0\rangle}, \\ \displaystyle\int_0^T v(s)e^{-i\sqrt{\lambda_k(\lambda_k+2)}s}\,ds = d_k(\xi_f) := \frac{e^{-i\sqrt{\lambda_k(\lambda_k+2)}T}}{\langle \mu, \varphi_k\rangle}\left(\Im\langle \xi_f, \varphi_k\rangle + i\sqrt{\frac{\lambda_k}{\lambda_k+2}}\Re\langle \xi_f, \varphi_k\rangle\right), \quad \forall k \in \mathbb{N}^*, \\ \displaystyle\int_0^T s v(s)\,ds = \tilde{d}(\xi_f) := T d_0(\xi_f). \end{cases}$$

We conclude thanks to Corollary 2 (in Appendix B). □

The proof of Theorem 4 is completed using the same arguments as in Section 2.4 using the inverse mapping theorem and the conservation of the L^2-norm.

Remark 6. With the same method, one may prove the local exact controllability of the focusing nonlinear Schrödinger equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t}(t,x) = -\dfrac{\partial^2\psi}{\partial x^2}(t,x) - |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x), & x\in(0,1),\ t\in(0,T),\\ \dfrac{\partial\psi}{\partial x}(t,0) = \dfrac{\partial\psi}{\partial x}(t,1) = 0, \end{cases}$$

around the reference trajectory ($\psi_{ref}(t,x) = e^{it}, u_{ref}(t) = 0$). The only difference in the proof is that we get the frequencies $\sqrt{\lambda_k(\lambda_k - 2)}$ (instead of $\sqrt{\lambda_k(\lambda_k + 2)}$) in the moment problem. When the space domain is the interval $(0,1)$, then all the quantities $\lambda_k(\lambda_k - 2)$, for $k \in \mathbb{N}^*$, are positive (because $\lambda_k = (k\pi)^2$), thus there is no additional difficulty. When the space domain is different, for instance $(0,a)$ with a large, then $\lambda_k = (k\pi/a)^2$, thus a finite number of the quantities $\lambda_k(\lambda_k - 2)$ are negative: we get a new moment problem with a finite number of moments with real valued exponentials, and an infinite number of trigonometric moments, that can be easily solved by adapting the tools used in this article.

4. Nonlinear wave equations

In this section, we study the nonlinear wave equation with Neumann boundary conditions (15). The goal is the proof of Theorem 5. In all this section, we use the notations defined in (39)–(42) and all the functions are real valued.

First, let us check that the Cauchy problem is well posed in $H^3_{(0)} \times H^2_{(0)}(0,1)$, when $u \in L^2(0,T)$. In order to write the system (15) in first order form, let us introduce:

$$D(\mathcal{A}) := H^2_{(0)} \times H^1(0,1), \qquad \mathcal{A} := \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix},$$
$$D(\mathcal{B}) := L^2 \times L^2(0,1), \qquad \mathcal{B} := \mu(x)\begin{pmatrix} 0 & 0 \\ Id & Id \end{pmatrix}, \tag{60}$$

and $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(y_1, y_2) := (0, f(y_1, y_2))$. The operator $\mathcal{A}$ generates a C^0-group of bounded operators of $H^2_{(0)} \times H^1(0,1)$ defined by:

$$e^{\mathcal{A}t}\begin{pmatrix} w_0 \\ \dot w_0 \end{pmatrix} = \begin{pmatrix} w(t) \\ \dot w(t) \end{pmatrix},$$

where

$$w(t) = \big(\langle w_0, \varphi_0\rangle + \langle \dot w_0, \varphi_0\rangle t\big)\varphi_0 + \sum_{k=1}^{\infty}\bigg(\langle w_0, \varphi_k\rangle \cos(\sqrt{\lambda_k}t) + \frac{1}{\sqrt{\lambda_k}}\langle \dot w_0, \varphi_k\rangle \sin(\sqrt{\lambda_k}t)\bigg)\varphi_k,$$
$$\dot w(t) = \langle \dot w_0, \varphi_0\rangle\varphi_0 + \sum_{k=1}^{\infty}\big(-\sqrt{\lambda_k}\langle w_0, \varphi_k\rangle \sin(\sqrt{\lambda_k}t) + \langle \dot w_0, \varphi_k\rangle \cos(\sqrt{\lambda_k}t)\big)\varphi_k.$$

With the notation

$$\mathcal{W} := \begin{pmatrix} w \\ \frac{\partial w}{\partial t} \end{pmatrix}, \qquad \mathcal{W}_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

Eq. (15) may be written:

$$\frac{\partial \mathcal{W}}{\partial t}(t,x) = \mathcal{A}\mathcal{W}(t,x) + F(\mathcal{W}) + u(t)\mathcal{B}\mathcal{W}(t,x), \quad x \in (0,1). \tag{61}$$

Proposition 12. *Let $\mu \in H^2(0,1)$, $T > 0$, $f \in C^3(\mathbb{R}^2, \mathbb{R})$ be such that $f(1,0) = 0$ and $\nabla f(1,0) = 0$. There exists $\delta > 0$ such that, for every $u \in B_\delta[L^2(0,T)]$, there exists a unique weak solution of* (61), (16)*, i.e. a function $\mathcal{W} \in C^0([0,T], H^3_{(0)} \times H^2_{(0)})$ such that the following equality holds in $H^3_{(0)} \times H^2_{(0)}(0,1)$, for every $t \in [0,T]$,*

$$\mathcal{W}(t) = e^{\mathcal{A}t}\mathcal{W}_0 + \int_0^t e^{\mathcal{A}(t-\tau)}\big(F\big(\mathcal{W}(\tau)\big) + u(\tau)\mathcal{B}\mathcal{W}(\tau) + \mathcal{F}(\tau)\big)\, d\tau. \tag{62}$$

The proof of this proposition relies on the following lemma.

Proposition 13. *Let $T > 0$ and $g \in L^2((0,T), H^2)$. The function G defined by,*

$$G(t) := \int_0^t e^{\mathcal{A}s} \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds,$$

belongs to $C^0([0,T], H^3_{(0)} \times H^2_{(0)})$. Moreover, there exists a constant $c_0(T) > 0$, uniformly bounded for T lying in bounded intervals, such that, for every $g \in L^2((0,T), H^2)$,

$$\|G\|_{L^\infty((0,T), H^3_{(0)} \times H^2_{(0)})} \leqslant c_0(T) \|g\|_{L^2((0,T),H^2)}. \tag{63}$$

Proof of Proposition 13. We have, for every $t \in [0,T]$,

$$G(t) = \int_0^t \begin{pmatrix} \langle g(s), \varphi_0\rangle s\varphi_0 + \sum_{k=1}^\infty \frac{\langle g(s),\varphi_k\rangle}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k}s)\varphi_k \\ \langle g(s), \varphi_0\rangle\varphi_0 + \sum_{k=1}^\infty \langle g(s), \varphi_k\rangle \cos(\sqrt{\lambda_k}s)\varphi_k \end{pmatrix} ds.$$

Thus, there exists $C > 0$ such that

$$\|G(t)\|_{H^3_{(0)} \times H^2_{(0)}} \leqslant C\left[\sum_{k=0}^\infty \left| k_*^2 \int_0^t \langle g(s), \varphi_k\rangle e^{ik\pi s}\, ds\right|^2 + \left|\int_0^t s\langle g(s), \varphi_0\rangle\, ds\right|^2\right].$$

We get the conclusion as in the previous sections. □

Proof of Proposition 12. Let us introduce the constants c_1, c_2, c_3 such that

$$\|f(w,w_t)\|_{H^2} \leqslant c_1 \|(w-1,w_t)\|^2_{H^3_{(0)} \times H^2_{(0)}}, \quad \forall (w,w_t) \in (1,0) + B_1\big[H^3_{(0)} \times H^2_{(0)}\big], \tag{64}$$

$$\begin{aligned} \|f(w,w_t) - f(\tilde{w},\tilde{w}_t)\|_{H^2} \leqslant c_2 &\|(w-\tilde{w}, w_t - \tilde{w}_t)\|_{H^3_{(0)} \times H^2_{(0)}} \\ &\times \max\big\{\|(w-1,w_t)\|_{H^3_{(0)} \times H^2_{(0)}}, \|(\tilde{w}-1,\tilde{w}_t)\|_{H^3_{(0)} \times H^2_{(0)}}\big\}, \\ &\forall (w,w_t), (\tilde{w},\tilde{w}_t) \in (1,0) + B_1\big[H^3_{(0)} \times H^2_{(0)}\big], \end{aligned} \tag{65}$$

and (49) holds. Let $R \in (0,1)$ be small enough so that

$$\sqrt{T} c_0 c_1 R^2 \leqslant \frac{R}{2}, \qquad \sqrt{T} c_0 c_2 R \leqslant \frac{1}{4}. \tag{66}$$

Let $\delta > 0$ be small enough so that

$$\delta c_0 c_3 < \frac{1}{4}, \qquad \delta c_0 c_3 (1+R) < \frac{R}{2}. \tag{67}$$

Let $u \in B_\delta[L^2(0,T)]$. We consider the map:

$$\begin{aligned} \mathcal{F} : (1,0) + \overline{B}_R\big[C^0\big([0,T], H^3_{(0)} \times H^2_{(0)}\big)\big] &\to (1,0) + \overline{B}_R\big[C^0\big([0,T], H^3_{(0)} \times H^2_{(0)}\big)\big], \\ \zeta &\mapsto \xi, \end{aligned}$$

where

$$\xi(t) = e^{\mathcal{A}t}\mathcal{W}_0 + \int_0^t e^{\mathcal{A}(t-\tau)}\big(F\big(\zeta(\tau)\big) + u(\tau)\mathcal{B}\zeta(\tau)\big)\, d\tau, \quad \forall t \in [0,T].$$

For $\zeta = (w, w_t) \in (1,0) + \overline{B}_R[C^0([0,T], H^3_{(0)} \times H^2_{(0)})]$, the second component of $F(\zeta) + u\mathcal{B}\zeta$ belongs to $L^2((0,T), H^2)$, thus ξ belongs to $C^0([0,T], H^3_{(0)} \times H^2_{(0)})$ thanks to Proposition 13. Moreover, thanks to (63), (64), (49), (66), and (67), we have, for every $t \in [0,T]$,

$$\begin{aligned}
\big\|\xi(t) - (1,0)\big\|_{H^3_{(0)} \times H^2_{(0)}} &\leqslant c_0 \big\| f(w, w_t) + u\mu[w + w_t]\big\|_{L^2((0,T),H^2)} \\
&\leqslant c_0\big(\sqrt{T}\big\| f(w, w_t)\big\|_{L^\infty((0,T),H^2)} + \|u\|_{L^2(0,T)}\big\|\mu[w + w_t]\big\|_{L^\infty((0,T),H^2)}\big) \\
&\leqslant c_0\big(\sqrt{T}c_1R^2 + \delta c_3(R+1)\big) \\
&\leqslant R.
\end{aligned}$$

Thus, $\mathcal{F}$ takes values in $(1,0) + \overline{B}_R[C^0([0,T], H^3_{(0)} \times H^2_{(0)})]$.

For $\zeta = (w, w_t)$, $\tilde{\zeta} = (\tilde{w}, \tilde{w}_t) \in (1,0) + \overline{B}_R[C^0([0,T], H^3_{(0)} \times H^2_{(0)})]$, thanks to (65), (49), (66) and (67), we have:

$$\begin{aligned}
\big\|\mathcal{F}(\zeta) - \mathcal{F}(\tilde{\zeta})\big\|_{L^\infty((0,T),H^3_{(0)} \times H^2_{(0)})} &\leqslant c_0\big\| f(w, w_t) - f(\tilde{w}, \tilde{w}_t) + u\mu[w - \tilde{w} + w_t - \tilde{w}_t]\big\|_{L^2((0,T),H^2)} \\
&\leqslant c_0\big[\sqrt{T}\big\| f(w, w_t) - f(\tilde{w}, \tilde{w}_t)\big\|_{L^\infty((0,T),H^2)} + \delta c_3\|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^3_{(0)} \times H^2_{(0)})}\big] \\
&\leqslant c_0\big[\sqrt{T}c_2R\|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^3_{(0)} \times H^2_{(0)})} + \delta c_3\|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^3_{(0)} \times H^2_{(0)})}\big] \\
&\leqslant \frac{1}{2}\|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})}.
\end{aligned}$$

Thus $\mathcal{F}$ is a contraction. □

Let $T > 0$, $\mu \in H^2(0,1)$, $f \in C^3(\mathbb{R}^2, \mathbb{R})$ be such that $f(1,0) = 0$, $\nabla f(1,0) = 0$ and $\delta > 0$ be as in Proposition 12. Then, the following map is well defined,

$$\begin{aligned}
\Theta_T : B_\delta\big[L^2(0,T)\big] &\to H^3_{(0)} \times H^2_{(0)}, \\
u &\mapsto (w, w_t)(T),
\end{aligned} \tag{68}$$

where (w, w_t) is the weak solution of (15), (16). Working as in the previous section, one may prove the following statements.

Proposition 14. *Let $\mu \in H^2(0,1)$, $T > 0$, $f \in C^3(\mathbb{R}^2, \mathbb{R})$ be such that $f(1,0) = 0$, $\nabla f(1,0) = 0$ and $\delta > 0$ be as in Proposition* 12. *The map Θ_T defined by* (68) *is C^1. Moreover, for every $u \in B_\delta[L^2(0,T)]$ and $v \in L^2(0,T)$, we have $d\Theta_T(u).v = (W, W_t)(T)$, where (W, W_t) is the weak solution of*:

$$\begin{cases}
W_{tt} = W_{xx} + \dfrac{\partial f}{\partial y_1}(w, w_t).W + \dfrac{\partial f}{\partial y_2}(w, w_t).W_t + u(t)\mu[W + W_t] + v(t)\mu(x)[w + w_t], \\
W_x(t,0) = W_x(t,1) = 0, \\
(W, W_t)(0,x) = 0,
\end{cases} \tag{69}$$

and (w, w_t) is the weak solution of (15), (16).

Proposition 15. *Let $T > 2$, $\mu \in H^2(0,1)$ be such that* (13) *holds and $f \in C^3(\mathbb{R}^2, \mathbb{R})$ be such that $f(1,0) = 0$, $\nabla f(1,0) = 0$. The linear map $d\Theta_T(0) : L^2(0,T) \to H^3_{(0)} \times H^2_{(0)}$ has a continuous right inverse*

$$d\Theta_T(0)^{-1} : H^3_{(0)} \times H^2_{(0)} \to L^2(0,T).$$

The proof is the same except that the gap between the eigenvalues does not tend to infinity and we use Corollary 3.

5. Conclusion, open problems, perspectives

In this article, we have proposed a method for the proof of the local exact controllability for linear and nonlinear bilinear systems. We have applied it to Schrödinger and wave equations, showing it works for a wide range of problems. It also works on other equations (for instance it may prove an optimal version of the controllability result proved in [16] for a 1D Beam equation).

In this article, we have presented various examples of application of the method. However, they all have in common that the linearized system fulfills a gap condition on the eigenvalues of the operator. This condition is not necessarily realized for the Schrödinger equation in higher space dimensions. Even in two dimension, we do not know any example of domain where it is true. So, one challenging question is the extension (or the impossibility to do it) of these results to other dimensions.

Appendix A. Genericity of the assumption on μ

The goal of this section is the proof of the following result.

Proposition 16. *The set* $\{\mu \in H^3((0,1),\mathbb{R});\ (5)\ \textit{holds}\}$ *is dense in* $H^3((0,1),\mathbb{R})$.

Proof. First, let us notice that

$$\mathcal{V} := \big\{\mu \in H^3\big((0,1),\mathbb{R}\big);\ \mu'(1) \pm \mu'(0) \neq 0\big\}$$

is a dense open subset of $H^3((0,1),\mathbb{R})$. Now, let us prove that the set,

$$\mathcal{U} := \big\{\mu \in \mathcal{V};\ \langle \mu\varphi_1, \varphi_k\rangle \neq 0,\ \forall k \in \mathbb{N}^*\big\},$$

is dense in $H^3((0,1),\mathbb{R})$. It is sufficient to prove that this set is dense in $\mathcal{V}$. For $n \in \mathbb{N}$, we introduce the set:

$$\mathcal{U}_n := \big\{\mu \in \mathcal{V};\ \langle \mu\varphi_1, \varphi_k\rangle \neq 0,\ \forall k \in \{1,\dots,n\}\big\},$$

with the convention $\mathcal{U}_0 := \mathcal{V}$. Then the sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is decreasing and

$$\mathcal{U} = \bigcap_{n=0}^{\infty} \mathcal{U}_n.$$

Thanks to Baire lemma, it is sufficient to check that, for every $n \in \mathbb{N}$, $\mathcal{U}_{n+1}$ is dense in $\mathcal{U}_n$ for the $H^3((0,1),\mathbb{R})$-topology. Let $n \in \mathbb{N}$ and let $\mu \in \mathcal{U}_n - \mathcal{U}_{n+1}$. Then $\mu \in \mathcal{V}$, $\langle \mu\varphi_1, \varphi_k\rangle \neq 0$ for $k = 1,\dots,n$ and $\langle \mu\varphi_1, \varphi_{n+1}\rangle = 0$. Thanks to (7), $\mu + \epsilon x^2 \in \mathcal{U}_{n+1}$ for every $\epsilon \in \mathbb{R}$ such that

$$\epsilon \neq -\frac{\langle \mu\varphi_1, \varphi_j\rangle}{\langle x^2\varphi_1, \varphi_j\rangle}, \quad \forall j \in \{1,\dots,n\}.$$

Thus $\mathcal{U}_{n+1}$ is dense in $\mathcal{U}_n$.

Finally, thanks to (8), we have:

$$\mathcal{U} \subset \big\{\mu \in H^3\big((0,1),\mathbb{R}\big);\ (5)\ \text{holds}\big\},$$

which gives the conclusion. □

Proposition 17. *The set* $\{\mu \in H^3_{rad}(B^3,\mathbb{R});\ (10)\ \textit{holds}\}$ *is dense in* $H^3(B^3,\mathbb{R})$.

Proof. We make the same proof. We use the formula

$$\langle \mu\varphi_1, \varphi_k\rangle = \frac{4\pi(-1)^{k+1}}{\lambda_k^{3/2}}\partial_r\mu(1) - \frac{1}{\lambda_k^2}\int_{B^3} \nabla\Delta(\mu\varphi_1)\cdot\nabla\varphi_k$$

instead of (8). Moreover, we can find one $\mu(r) = r^2$ that fulfills (10). □

Appendix B. Moment problems

In this section, we recall classical results about moment problems (see, for instance [9]). The proofs are given for sake of completeness.

B.1. Families of vectors in Hilbert spaces

Let H be a separable Hilbert vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\Theta := (\xi_j)_{j\in\mathbb{Z}}$ be a family of vectors of H with $\xi_j \neq 0, \forall j \in \mathbb{Z}$.

Definition 1. The family Θ is minimal in H if, for every $j \in \mathbb{Z}$, $\xi_j \notin \overline{\mathrm{Span}\{\xi_i;\ i \in \mathbb{Z} - \{j\}\}}$.

Proposition 18. *The family Θ is minimal in H if and only if there exists a biorthogonal family $\Theta' = (\xi'_j)_{j\in\mathbb{Z}}$, i.e. Θ' is a family of vectors of H such that*

$$\langle \xi_i, \xi'_j \rangle = \delta_{i,j}, \quad \forall i, j \in \mathbb{Z}. \tag{70}$$

Proof of Proposition 18. We assume Θ is minimal. For $j \in \mathbb{Z}$, let v_j be the orthogonal projection of ξ_j over the closed vector space $\overline{\mathrm{Span}\{\xi_i,\ i \neq j\}}$, i.e.

$$v_j \in \overline{\mathrm{Span}\{\xi_i,\ i \neq j\}} \quad \text{and} \quad \langle \xi_j - v_j, \xi_i \rangle = 0, \quad \forall i \neq j.$$

Let

$$\xi'_j := \frac{\xi_j - v_j}{\|\xi_j - v_j\|^2}, \quad \forall j \in \mathbb{Z}.$$

Then, the families (ξ_j) and (ξ'_j) are biorthogonal.

Now, we assume that there exists a biorthogonal family $\Theta' = (\xi'_j)_{j\in\mathbb{Z}}$. Let us assume that there exists $j \in \mathbb{Z}$ such that $\xi_j \in \overline{\mathrm{Span}\{\xi_i;\ i \in \mathbb{Z} - \{j\}\}}$. Then (70) implies $\langle \xi_j, \xi'_j \rangle = 1$ which is a contradiction. $\square$

Remark 7. If Θ is minimal, then there exists a unique biorthogonal family Θ' such that $\Theta' \subset \overline{\mathrm{Span}\{\xi_i;\ i \in \mathbb{Z}\}}$. In the end of this appendix, the expression "the" biorthogonal family of Θ, refers to this unique biorthogonal family in $\overline{\mathrm{Span}\{\xi_i;\ i \in \mathbb{Z}\}}$.

Definition 2. The family Θ is a Riesz basis of $\overline{\mathrm{Span}\,\Theta}$ if Θ is the image of some orthonormal family by an isomorphism.

Remark 8. It is clear that, if Θ is a Riesz basis of $\overline{\mathrm{Span}\,\Theta}$, then Θ is minimal in H.

Proposition 19.

(1) *If Θ is a Riesz basis of $\overline{\mathrm{Span}\,\Theta}$, then its biorthogonal family Θ' is also a Riesz basis of $\overline{\mathrm{Span}\,\Theta}$.*
(2) *Θ is a Riesz basis of $\overline{\mathrm{Span}\,\Theta}$ if and only if there exists $C_1, C_2 \in (0, +\infty)$ such that, for every scalar sequence $(c_j)_{j\in\mathbb{Z}}$ with finite support,*

$$C_1\left(\sum_{j=-\infty}^{\infty} |c_j|^2\right)^{1/2} \leqslant \left\|\sum_{j=-\infty}^{\infty} c_j \xi_j\right\| \leqslant C_1\left(\sum_{j=-\infty}^{\infty} |c_j|^2\right)^{1/2}. \tag{71}$$

(3) *If Θ is a Riesz basis of $\overline{\mathrm{Span}\,\Theta}$ then there exists $C > 0$ such that, for every $f \in H$, we have:*

$$\left(\sum_{j\in\mathbb{Z}} \left|\langle f, \xi_j \rangle\right|^2\right)^{1/2} \leqslant C\|f\|.$$

Proof of Proposition 19. (1) We assume Θ is a Riesz basis of $\overline{\operatorname{Span}\Theta}$. Let $\mathcal{H}$ be a Hilbert space, $(\zeta_j)_{j\in\mathbb{Z}}$ be an orthonormal family of $\mathcal{H}$, $V:\mathcal{H}\to\overline{\operatorname{Span}\Theta}$ an isomorphism such that $\xi_j=V(\zeta_j)$, $\forall j\in\mathbb{Z}$. Then the adjoint operator $V^*:\overline{\operatorname{Span}\Theta}\to\mathcal{H}$ is also an isomorphism and we have $\xi'_j=(V^*)^{-1}(\zeta_j)$, $\forall j\in\mathbb{Z}$. Indeed, for every $j,k\in\mathbb{Z}$,

$$\delta_{j,k}=\langle\xi_j,\xi'_k\rangle_H=\langle V(\zeta_j),\xi'_k\rangle_H=\langle\zeta_j,V^*(\xi'_k)\rangle_{\mathcal{H}}.$$

Thus Θ' is a Riesz basis of $\overline{\operatorname{Span}\Theta}$.

(2) We assume Θ is a Riesz basis of $\overline{\operatorname{Span}\Theta}$. Let $\mathcal{H}$ be a Hilbert space, $(\zeta_j)_{j\in\mathbb{Z}}$ be an orthonormal family of $\mathcal{H}$, $V:\mathcal{H}\to\overline{\operatorname{Span}\Theta}$ an isomorphism such that $\xi_j=V(\zeta_j)$, $\forall j\in\mathbb{Z}$ and $(c_j)_{j\in\mathbb{Z}}$ a scalar sequence with finite support. We have:

$$\left\|\sum_{j=-\infty}^{\infty}c_j\xi_j\right\|=\left\|V\left[\sum_{j=-\infty}^{\infty}c_j\zeta_j\right]\right\|\leqslant\|V\|\left\|\sum_{j=-\infty}^{\infty}c_j\zeta_j\right\|=\|V\|\left(\sum_{j=-\infty}^{\infty}|c_j|^2\right)^{1/2},$$

and

$$\left(\sum_{j=-\infty}^{\infty}|c_j|^2\right)^{1/2}=\left\|\sum_{j=-\infty}^{\infty}c_j\zeta_j\right\|=\left\|V^{-1}\left[\sum_{j=-\infty}^{\infty}c_j\xi_j\right]\right\|\leqslant\|V^{-1}\|\left\|\sum_{j=-\infty}^{\infty}c_j\xi_j\right\|,$$

thus, we have (71) with $C_1=1/\|V^{-1}\|$ and $C_2=\|V\|$.

Now, we assume that (71) holds. Then the linear map

$$V:l^2(\mathbb{Z},\mathbb{K})\to\overline{\operatorname{Span}\Theta}$$

defined by $V[(c_j)_{j\in\mathbb{Z}}]=\sum_{j=-\infty}^{\infty}c_j\xi_j$ is well defined and injective. Let $h\in\overline{\operatorname{Span}\Theta}$. There exists $(h_N)_{N\in\mathbb{N}}$ such that $h_N\to h$ in H when $N\to+\infty$ and for every $N\in\mathbb{N}$, there exists a sequence $c^{(N)}=(c_j^{(N)})_{j\in\mathbb{Z}}$ with finite support such that $h_N=\sum_{j=-\infty}^{\infty}c_j^{(N)}\xi_j$. Then $(h_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in H, thus, thanks to (71), $(c^{(N)})_{N\in\mathbb{N}}$ is a Cauchy sequence in $l^2(\mathbb{Z})$ and there exists $c=(c_j)_{j\in\mathbb{Z}}\in l^2(\mathbb{N})$ such that $c^N\to c$ in $l^2(\mathbb{Z})$. Then, (71) proves that $\sum_{j=-\infty}^{\infty}(c_j-c_j^{(N)})\xi_j\to 0$ in H, i.e. $h=\sum_{j=-\infty}^{\infty}c_j\xi_j$. We have proved that V is an isomorphism, thus Θ is a Riesz basis of $\overline{\operatorname{Span}\Theta}$.

(3) $\overline{\operatorname{Span}\Theta}$ is a close vector subspace of H thus we have the orthogonal decomposition $H=\overline{\operatorname{Span}\Theta}+\overline{\operatorname{Span}\Theta}^{\perp}$ and the associated orthogonal projection $P:H\to\overline{\operatorname{Span}\Theta}$. For $f\in H$, we have:

$$\left(\sum_{j\in\mathbb{Z}}\left|\langle f,\xi_j\rangle\right|^2\right)^{1/2}=\left(\sum_{j\in\mathbb{Z}}\left|\langle Pf,\xi_j\rangle\right|^2\right)^{1/2}\leqslant\frac{1}{C_1}\left\|\sum_{j\in\mathbb{Z}}\langle Pf,\xi_j\rangle\xi'_j\right\|=\frac{1}{C_1}\|Pf\|_H\leqslant\frac{1}{C_1}\|f\|.\qquad\square$$

Remark 9. We have proved that, if Θ is a Riesz basis of $\overline{\operatorname{Span}\Theta}$, then, for every $h\in\overline{\operatorname{Span}\Theta}$ there exists $c=(c_j)_{j\in\mathbb{Z}}\in l^2(\mathbb{Z},\mathbb{K})$ such that $h=\sum_{j=-\infty}^{\infty}c_j\xi_j$. Moreover, if Θ' and Θ are biorthogonal families, then necessarily $c_j=\langle h,\xi'_j\rangle$, $\forall j\in\mathbb{Z}$. Thus, every $h\in\overline{\operatorname{Span}\Theta}$ can be decomposed in the following way,

$$h=\sum_{j=-\infty}^{\infty}\langle h,\xi'_j\rangle\xi_j=\sum_{j=-\infty}^{\infty}\langle h,\xi_j\rangle\xi'_j,\tag{72}$$

where the series converge in H and the coefficients $(\langle h,\xi'_j\rangle)_{j\in\mathbb{Z}}$, $(\langle h,\xi_j\rangle)_{j\in\mathbb{Z}}$, belong to $l^2(\mathbb{Z},\mathbb{K})$.

B.2. Abstract moment problems

Now, we move to the investigation of abstract moment problems: given a scalar sequence $(d_j)_{j\in\mathbb{Z}}$ is it possible to find $f\in H$ such that

$$\langle f,\xi_j\rangle=d_j,\quad\forall j\in\mathbb{Z}.$$

Let us introduce the operator:

$$\begin{aligned} J_\Theta : H &\to l^2(\mathbb{Z}, \mathbb{K}), \\ f &\mapsto \big(\langle f, \xi_j \rangle\big)_{j\in\mathbb{Z}}, \end{aligned}$$

with domain $D_\Theta := \{f \in H;\ J_\Theta(f) \in l^2(\mathbb{Z})\}$. It is clear that, if the family Θ is not complete in H, then the operator J_Θ has a non-trivial null space $\overline{\operatorname{Span}\Theta}^\perp$. This motivates the introduction of the operator $J_\Theta^0 := J_\Theta|_{\overline{\operatorname{Span}\Theta}}$.

Proposition 20. *The operator $J_\Theta^0 : \overline{\operatorname{Span}\Theta} \to l^2(\mathbb{Z}, \mathbb{K})$ is an isomorphism if and only if Θ is a Riesz basis of $\overline{\operatorname{Span}\Theta}$.*

Proof of Proposition 20. We assume $J_\Theta^0 : \overline{\operatorname{Span}\Theta} \to l^2(\mathbb{Z}, \mathbb{K})$ is an isomorphism. Let $(\zeta_j)_{j\in\mathbb{Z}}$ be the canonical orthonormal basis of $l^2(\mathbb{Z})$. Then, the family,

$$\big(\big(J_\Theta^0\big)^{-1}(\zeta_j)\big)_{j\in\mathbb{Z}},$$

is a Riesz basis of $\overline{\operatorname{Span}\Theta}$. Moreover, it is the biorthogonal family to Θ in $\overline{\operatorname{Span}\Theta}$. Thanks to Proposition 19(1), Θ is also a Riesz basis of $\overline{\operatorname{Span}\Theta}$.

We assume Θ is a Riesz basis of $\overline{\operatorname{Span}\Theta}$. Thanks to Remark 9, it is clear that $J_\Theta^0 : \overline{\operatorname{Span}\Theta} \to l^2(\mathbb{Z}, \mathbb{K})$ is an isomorphism. □

B.3. Trigonometric moment problems

In this section, we recall important results on trigonometric moment problems. The following Ingham inequality is due to Haraux [37].

Theorem 6. *Let $N \in \mathbb{N}$, $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers such that*

$$\begin{aligned} \omega_{k+1} - \omega_k &\geqslant \gamma > 0, \quad \forall k \in \mathbb{Z},\ |k| \geqslant N, \\ \omega_{k+1} - \omega_k &\geqslant \rho > 0, \quad \forall k \in \mathbb{Z}, \end{aligned}$$

and $T > 2\pi/\gamma$. There exists $C_1 = C_1(\gamma, \rho, N, T)$, $C_2 = C_2(\gamma, \rho, N, T) \in (0, +\infty)$ such that, for every sequence $(c_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ with finite support, we have

$$C_1 \sum_{k\in\mathbb{Z}} |c_k|^2 \leqslant \int_0^T \left| \sum_{k=-\infty}^{+\infty} c_k e^{-i\omega_k t} \right|^2 dt \leqslant C_2 \sum_{k\in\mathbb{Z}} |c_k|^2.$$

Let us introduce the space:

$$l_r^2(\mathbb{N}, \mathbb{C}) := \big\{(d_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N}, \mathbb{C});\ d_0 \in \mathbb{R}\big\}.$$

Thanks to Proposition 19 and Theorem 6, we have the following statement, which is used in the proof of Proposition 4.

Corollary 1. *Let $T > 0$ and $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0, +\infty)$ such that $\omega_0 = 0$, and*

$$\omega_{k+1} - \omega_k \to +\infty \quad \text{when } k \to +\infty.$$

There exists a continuous linear map,

$$\begin{aligned} L : l_r^2(\mathbb{N}, \mathbb{C}) &\to L^2\big((0, T), \mathbb{R}\big), \\ d &\mapsto L(d), \end{aligned}$$

such that, for every $d = (d_k)_{k\in\mathbb{N}} \in l_r^2(\mathbb{N}, \mathbb{C})$, the function $v := L(d)$ solves

$$\int_0^T v(t) e^{i\omega_k t}\, dt = d_k, \quad \forall k \in \mathbb{N}.$$

Proof of Corollary 1. We define $\omega_{-k} := -\omega_k$, $\forall k \in \mathbb{N}^*$. Theorem 6 ensures that the family $(e^{i\omega_k t})_{k\in\mathbb{Z}}$ is a Riesz basis of $F := \mathrm{Adh}_{L^2(0,T)}(\mathrm{Span}\{e^{i\omega_k t};\ k \in \mathbb{Z}\})$. Thanks to Proposition 20, the map,

$$\begin{aligned} J : F &\to l^2(\mathbb{Z},\mathbb{C}),\\ v &\mapsto \left(\int_0^T v(t)e^{i\omega_k t}\,dt\right)_{k\in\mathbb{Z}}, \end{aligned}$$

is an isomorphism. For $d = (d_k)_{k\in\mathbb{N}} \in l^2_r(\mathbb{N},\mathbb{C})$, we define $\tilde{d} := (\tilde{d}_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z},\mathbb{C})$ by $\tilde{d}_k := d_k$ if $k \geqslant 0$ and $\bar{d}_{-k}$ if $k < 0$. Now, we define $L : l^2_r(\mathbb{N},\mathbb{C}) \to L^2((0,T),\mathbb{R})$ by $L(d) = J^{-1}(\tilde{d})$. The map L takes values in real valued functions because $\tilde{d}_{-k} = \overline{\tilde{d}_k}$, $\forall k \in \mathbb{N}$ for every $d \in l^2_r(\mathbb{N},\mathbb{C})$. $\square$

Theorem 6 is also crucial in the proof of the following statement, used in the proof of Proposition 7.

Corollary 2. *Let $T > 0$ and $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0,+\infty)$ such that $\omega_0 = 0$, and*

$$\omega_{k+1} - \omega_k \to +\infty \quad \text{when } k \to +\infty. \tag{73}$$

There exists a continuous linear map,

$$\begin{aligned} L : \mathbb{R} \times l^2_r(\mathbb{N},\mathbb{C}) &\to L^2\big((0,T),\mathbb{R}\big),\\ (\tilde{d}, d) &\mapsto L(\tilde{d}, d), \end{aligned}$$

such that, for every $\tilde{d} \in \mathbb{R}$, $d = (d_k)_{k\in\mathbb{N}} \in l^2_r(\mathbb{N},\mathbb{C})$, the function $v := L(\tilde{d}, d)$ solves:

$$\int_0^T v(t)e^{i\omega_k t}\,dt = d_k, \quad \forall k \in \mathbb{N},$$

$$\int_0^T t v(t)\,dt = \tilde{d}. \tag{74}$$

Proof of Corollary 2. Let $\omega_k := -\omega_{-k}$, for every $k \in \mathbb{Z}$ with $k < 0$. From Proposition 6, $\Theta := (e^{i\omega_k t})_{k\in\mathbb{Z}}$ is a Riesz basis of $\mathrm{Adh}_{L^2(0,T)}(\mathrm{Span}\,\Theta)$.

First step: We prove that the family $\tilde{\Theta} := \{t, e^{i\omega_k t}; k \in \mathbb{Z}\}$ is minimal in $L^2(0,T)$.

Working by contradiction, we assume that $\tilde{\Theta}$ is not minimal in $L^2(0,T)$. Then, necessarily

$$t \in \mathrm{Adh}_{L^2(0,T)}\,\mathrm{Span}\,\Theta. \tag{75}$$

With successive integrations, we get:

$$t^j \in \mathrm{Adh}_{C^0[0,T]}(\mathrm{Span}\,\tilde{\Theta}), \quad \forall j \in \mathbb{N} \text{ with } j \geqslant 2.$$

The Stone Weierstrass theorem ensures that $\{1, t^j;\ j \in \mathbb{N},\ j \geqslant 2\}$ is dense in $C^0([0,T],\mathbb{C})$, thus, it is also dense in $L^2(0,T)$. From (75), we deduce that $\mathrm{Span}\,\Theta$ is dense in $L^2(0,T)$. This is a contradiction, because, thanks to Theorem 6, for every $\omega \in \mathbb{R} - \{\omega_k,\ k \in \mathbb{Z}\}$, the family $\{e^{i\omega t}, e^{i\omega_k t};\ k \in \mathbb{Z}\}$ is minimal, i.e.

$$e^{i\omega t} \notin \mathrm{Adh}_{L^2(0,T)}(\mathrm{Span}\,\Theta).$$

Second step: We conclude.

For $k < 0$, we define $d_k := \overline{d_{-k}}$. Let $\{\tilde{\xi}, \xi_k;\ k \in \mathbb{Z}\}$ be the biorthogonal family to $\{t, e^{i\omega_k t};\ k \in \mathbb{Z}\}$. From Theorem 6, there exists $C > 0$ and a unique solution $v \in \mathrm{Adh}_{L^2(0,T)}(\mathrm{Span}\,\Theta)$ of,

$$\int_0^T v(t)e^{i\omega_k t}\,dt = d_k, \quad \forall k \in \mathbb{Z}$$

and it satisfies

$$\|v\|_{L^2(0,T)} \leqslant C\left(\sum_{k\in\mathbb{Z}} |d_k|^2\right)^{1/2}.$$

The uniqueness guarantees that v is real valued. Let us define:

$$L(\tilde{d}, d) := u := v + \left(\tilde{d} - \int_0^T t v(t)\, dt\right)\tilde{\xi}.$$

Then, u is real valued (because v and $\tilde{\xi}$ are), u solves (74), and

$$\begin{aligned}
\|u\|_{L^2} &\leqslant \|v\|_{L^2} + \left(|\tilde{d}| + \left|\int_0^T t v(t)\, dt\right|\right)\|\tilde{\xi}\|_{L^2} \\
&\leqslant \|v\|_{L^2}\left(1 + \sqrt{\frac{T^3}{3}}\|\tilde{\xi}\|_{L^2}\right) + |\tilde{d}|\|\tilde{\xi}\|_{L^2} \\
&\leqslant \left(C\left(1 + \sqrt{\frac{T^3}{3}}\|\tilde{\xi}\|_{L^2}\right) + \|\tilde{\xi}\|_{L^2}\right)\left(|\tilde{d}|^2 + \sum_{k\in\mathbb{Z}} |d_k|^2\right)^{1/2}. \qquad \square
\end{aligned}$$

For the wave equation, the gap between two successive frequencies does not tend to infinity, so we will need the following corollary which is proved similarly.

Corollary 3. *Let $T > 2$. We make the same assumptions as in Corollary* 2 *except that we assume,*

$$\omega_{k+1} - \omega_k \geqslant \pi,$$

instead of (73). *Then, we have the same conclusion as Corollary* 2.

Corollary 4. *Let $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0, +\infty)$ such that $\omega_0 = 0$, and*

$$\omega_{k+1} - \omega_k > \gamma > 0.$$

There exists a nondecreasing function,

$$\begin{aligned}
C : [0, +\infty) &\to \mathbb{R}_+^*, \\
T &\mapsto C(T),
\end{aligned}$$

such that, for every $T > 0$ and for every $g \in L^2(0, T)$, we have:

$$\left(\sum_{k=0}^{\infty}\left|\int_0^T g(t) e^{i\omega_k t}\, dt\right|^2\right)^{1/2} \leqslant C(T)\|g\|_{L^2(0,T)}.$$

Proof of Corollary 4. The existence of $C(T)$, for large $T \geqslant 2\pi/\gamma + 1$, is a consequence of Theorem 6 and Proposition 19(3). Let us choose for $C(T)$ the smallest value possible for this constant. For $T \leqslant 2\pi/\gamma + 1$, we choose $C(T) = C(2\pi/\gamma + 1)$. Let $0 < T_1 < T_2 < +\infty$, $g \in L^2(0, T_1)$ and $\tilde{g} \in L^2(0, T_2)$ be defined by $\tilde{g} = g$ on $(0, T_1)$ and 0 on (T_1, T_2). By applying the inequality on $\tilde{g}$, we get $C(T_1) \leqslant C(T_2)$. $\square$

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