On the Characterization of Stability Concepts of Volterra Integro-Differential Equations

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Submitted by R. P. Boas

Received February 4, 1988

1. INTRODUCTION

We consider the systems of Volterra integro-differential equations

\[ x'(t) = A(t)x(t) + \int_0^t B(t, s)x(s)\,ds \quad \text{(L)} \]

and

\[ x'(t) = Ax(t) + \int_0^t C(t-s)x(s)\,ds, \quad \text{(CL)} \]

where \( A(t) \) and \( C(t) \) are \( n \times n \) continuous matrices for \( 0 \leq t < \infty \), \( B(t, s) \) is an \( n \times n \) continuous matrix for \( 0 \leq s \leq t < \infty \), and \( A \) is an \( n \times n \) constant matrix.

Let \( Z(t) \) be the unique solution of the matrix equation

\[ Z'(t) = AZ(t) + \int_0^t C(t-s)Z(s)\,ds, \quad Z(0) = I, \]

where \( I \) is the identity matrix. \( Z(t) \) is called the resolvent associated with (CL).

Miller [9] has shown the equivalence of uniform asymptotic stability of (CL) and \( Z(t) \in L^1(\mathbb{R}^+) \) under the condition that \( C(t) \in L^1(\mathbb{R}^+) \). In [2, Theorem 7] Burton and Mahfoud showed a set of equivalent conditions of uniform asymptotic stability for (CL). In Theorem 3.1 of this paper we generalize their results with respect to (CL). Our result is summarized in the following diagram:

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For the definitions above, see Section 2. Especially, we insist on the equivalence between UAS and $|Z(t)| \leq K e^{-\lambda t}$ for some $K \geq 1$ and $\lambda > 0$. By this relation we notice that the solutions of (CL) decay exponentially when the zero solution is UAS.

It is well known [12] that for the linear system

$$x' = A(t)x,$$

the following diagram of equivalences holds:

$$ Ub \leftrightarrow UUB \leftrightarrow UAS \leftrightarrow \text{ExAS} \rightarrow |X(t)X^{-1}(s)| \leq Ke^{-\lambda(t-s)} \rightarrow \exists V(t, x)$$

where $X(t)$ is a fundamental matrix of (1.1). The diagram for (CL) is a good analogy to that for (1.1).

In Section 4, as an application of Theorem 3.1, we study the perturbed systems of (L),

$$y'(t) = A(t)y(t) + \int_0^t B(t, s) y(s) \, ds + g(t, y(\cdot)) \quad (PL)$$

$$y'(t) = A(t)y(t) + \int_0^t B(t, s) y(s) \, ds + h(t), \quad (PL_h)$$

where $g(t, \phi)$ is a continuous functional on $\mathbb{R}^+ \times C(\mathbb{R}^+)$ into $\mathbb{R}^n$ and $h(t)$ is continuous on $\mathbb{R}^+$. In case $A(t)$ is a constant stable matrix, there are several works on the asymptotic behavior of the solutions of (PL) [1, 2, 5, 9, 11]. We will investigate conditions on $g(t, x(\cdot))$ under which uniform boundedness and uniform ultimate boundedness of (L) imply those of (PL) (Theorems 4.1 and 4.2). In Theorem 4.3, we prove that under some conditions on $A(t)$ and $B(t, s)$, if the solutions of (L) are UB and
UUB, then the solutions of \((PL_k)\) are UB and UUB if and only if 
\[ \sup_{r \geq 0} |e^{-r} \int_0^r e^s h(s) \, ds| < \infty. \]
This is an extension of \([7, \text{Theorem 5.2}]\) for \((1.1)\) to Volterra integro-differential equation. We also consider the perturbation problem with respect to uniform asymptotic stability from \((L)\) to \((PL)\) (Theorems 4.4–4.6).

2. Definitions and Lemmas

Let \(\mathbb{R}^n\) denote the Euclidean \(n\)-space. For \(x \in \mathbb{R}^n\), let 
\[ |x| = \left( \sum_{j=1}^n x_j^2 \right)^{1/2}. \]
For an \(n \times n\) matrix \(A\), define the norm \(|A|\) of \(A\) by 
\[ |A| = \sup_{|x| = 1} |Ax|. \]
Let \(\mathbb{R}^+\) be the half line \(0 \leq t < \infty\). For \(\phi \in C(\mathbb{R}^+)\), define 
\[ \|\phi\| = \max \{|\phi(s)|: 0 \leq s \leq t\} \] and 
\[ B_H = \{\phi: \sup_{t \geq 0} |\phi(t)| < H\}. \]

**Definition 2.1.** Let \(P(t, \phi)\) be a continuous functional on \(\mathbb{R}^+ \times C(\mathbb{R}^+)\) into \(\mathbb{R}^n\). Then \(P(t, \phi)\) is **locally Lipschitz with respect to \(\phi\)** if and only if given any pair of positive constants \(a\) and \(b\) there exists a constant \(K > 0\) such that 
\[ |P(t, \phi) - P(t, \psi)| \leq K \|\phi - \psi\|_t, \]
whenever \(0 \leq t \leq a\) and both \(\|\phi\|_t\) and \(\|\psi\|_t \leq b\).

The following definitions are stated for the system of functional differential equations

\[ x'(t) = F(t, x(\cdot)), \quad (2.1) \]

where \(F: \mathbb{R}^+ \times C(\mathbb{R}^+) \to \mathbb{R}^n\) is continuous and \(x(\cdot)\) represents the function \(x\) on \([0, t]\), with the value of \(t\) always determined by the first coordinate of \(F\) in \((2.1)\). The solution of \((2.1)\) with initial values \((t_0, \phi)\) will be denoted by \(x(t; t_0, \phi)\), where \(t_0 \geq 0\) and \(\phi: [0, t_0] \to \mathbb{R}^n\) is a continuous function. In Definitions 2.2–2.4, we assume \(F(t, 0) \equiv 0\), so that \(x(t) \equiv 0\) is a solution of \((2.1)\).

**Definition 2.2.** The zero solution of \((2.1)\) is **uniformly stable** (US), if for any \(\varepsilon > 0\) there exists \(\delta(\varepsilon) > 0\) such that \(t_0 \geq 0\) and \(\|\phi\|_{t_0} < \delta(\varepsilon)\) imply 
\[ |x(t; t_0, \phi)| < \varepsilon \] for all \(t \geq t_0\).

**Definition 2.3.** The zero solution of \((2.1)\) is **uniformly attracting** (UA), if there exists \(\delta_0 > 0\) such that for any \(\varepsilon > 0\) there is a \(T(\varepsilon) > 0\) such that \(t_0 \geq 0\) and \(\|\phi\|_{t_0} < \delta_0\) imply 
\[ |x(t; t_0, \phi)| < \varepsilon \] for all \(t \geq t_0 + T(\varepsilon)\). The zero solution of \((2.1)\) is **uniformly asymptotically stable** (UAS), if it is US and UA.
**Definition 2.4.** The zero solution of (2.1) is **exponentially asymptotically stable** (ExAS), if there exists $\lambda > 0$ and for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\phi\|_{t_0} < \delta(\varepsilon)$ imply $|x(t; t_0, \phi)| < \varepsilon \exp\{ -\lambda(t - t_0) \}$ for all $t \geq t_0$.

**Definition 2.5.** The solutions of (2.1) are **uniformly bounded** (UB), if for any $\alpha > 0$ there exists $\beta(\alpha) > 0$ such that $t_0 \geq 0$ and $\|\phi\|_{t_0} < \alpha$ imply $|x(t; t_0, \phi)| < \beta(\alpha)$ for all $t \geq t_0$.

**Definition 2.6.** The solutions of (2.1) are **uniformly ultimately bounded** (UUB) for bound $B$, if there exists $B > 0$ and for any $\alpha > 0$ there exists $T(\alpha) > 0$ such that $t_0 \geq 0$ and $\|\phi\|_{t_0} < \alpha$ imply $|x(t; t_0, \phi)| < B$ for all $t \geq t_0 + T(\alpha)$.

Let $V(t, \phi) : R^+ \times C(R^+) \to R^1$ be a continuous functional satisfying a local Lipschitz condition with respect to $\phi$.

**Definition 2.7.** The derivative of $V(t, \phi)$ with respect to (2.1) is defined by

$$V_{(2.1)}(t, \phi) = \lim_{h \to 0^+} \sup \{ V(t + h, \phi^*) - V(t, \phi) \} / h,$$

where

$$\phi^*(s) = \begin{cases} \phi(s) & \text{on } 0 \leq s \leq t \\ \phi(s) + F(t, \phi)(s - t) & \text{on } t \leq s \leq t + h. \end{cases}$$

It is well known that

$$V_{(2.1)}(t_0, \phi) = \lim_{h \to 0^+} \sup \{ V(t_0 + h, x(\cdot; t_0, \phi)) - V(t_0, \phi) \} / h,$$

where $x(t; t_0, \phi)$ is the unique solution of (2.1) with initial values $(t_0, \phi) \in R^+ \times C(R^+)$ (see [4]).

**Lemma 2.1.** The solution of the equation

$$x'(t) = Ax(t) + \int_0^t C(t - s)x(s) \, ds + h(t)$$

with initial values $(t_0, \phi)$ is given by

$$x(t; t_0, \phi) = Z(t - t_0) \phi(t_0) + \int_{t_0}^t Z(t - s) h(s) \, ds$$

$$+ \int_{t_0}^t Z(t - s) \int_{s}^{t_0} C(s - u) \phi(u) \, du \, ds.$$
Proof. We differentiate (2.3) to verify that (2.3) is the solution of (2.2).

The following Lemmas 2.2-2.4 are well known.

**Lemma 2.2.** For all \( t, s \geq 0 \), \( Z(t) = Z(t - s) Z(s) \).

**Lemma 2.3** [3]. For an \( n \times m \) matrix function \( A(t) \) on \( \mathbb{R}^+ \),

\[
\lim_{t \to \infty} \int_t^{t+1} |A(s)| \, ds = 0
\]

if and only if \( \lim_{t \to \infty} e^{-\lambda t} \int_0^t e^{\lambda s} |A(s)| \, ds = 0 \) for any \( \lambda > 0 \), and

\[
\sup_{t \geq 0} \int_t^{t+1} |A(s)| \, ds < \infty
\]

if and only if \( \sup_{t \geq 0} e^{-\lambda t} \int_0^t e^{-\lambda s} |A(s)| \, ds < \infty \) for any \( \lambda > 0 \).

**Lemma 2.4.** If \( A(t) \in L^1(\mathbb{R}^+) \) or \( |A(t)| \to 0 \) as \( t \to \infty \), then we have \( \lim_{t \to \infty} \int_t^{t+1} |A(s)| \, ds = 0 \).

**Lemma 2.5.** Let \( v(t) \) be a continuous function on \([0, \infty)\). Suppose that \( v(t) > 0 \) for all \( t \geq 0 \) and there exists \( \lambda > 0 \) such that

\[
v(t) \int_0^t \frac{ds}{v(s)} \leq \frac{1}{\lambda} \quad \text{for all} \quad t \geq 0.
\]

Then there exists \( K > 0 \) such that

\[
v(t) \leq Ke^{-\lambda t} \quad \text{for all} \quad t \geq 0.
\]

Proof. Let \( H(t) = \lambda \int_0^t ds/v(s) \). Since \( H'(t) \geq \lambda H(t) \) for all \( t \geq 0 \), then we have, for some \( \alpha > 0 \),

\[
\frac{1}{v(t)} \geq H(t) \geq H(\alpha) e^{\lambda(t-\alpha)} \quad \text{for all} \quad t \geq \alpha > 0.
\]

Thus, we have \( v(t) \leq e^{-\lambda(t-\alpha)/H(\alpha)} \) for all \( t \geq \alpha > 0 \). Let \( K^* = \max_{t \in [0, \alpha]} e^{\lambda t} v(t) \), \( K = \max(K^*, e^{\lambda \alpha}/H(\alpha)) \), then we have \( v(t) \leq Ke^{-\lambda t} \) for all \( t \geq 0 \). This completes the proof.

### 3. Equivalence Theorem for Stability Concepts

In this section we present a set of statements which are equivalent to uniform asymptotic stability of the zero solution of (CL).
Theorem 3.1. Suppose that \( C(t) \in L^1(\mathbb{R}^+) \), then the following statements are equivalent:

(I) The zero solution of (CL) is UAS.

(II) The zero solution of (CL) is ExAS.

(III) There exist \( K > 0 \) and \( \lambda > 0 \) such that

\[
|Z(t)| \leq Ke^{-\lambda t} \quad \text{for all} \quad t \geq 0.
\]

(IV) \( Z(t) \in L^1(\mathbb{R}^+) \).

(V) The solutions of (CL) are UB and UUB.

(VI) There exist a continuous functional \( V(t, \phi) \) defined for all \( t \geq 0 \) and \( \phi \in C([0, t], \mathbb{R}^n) \), \( \lambda > 0 \) and \( K > 0 \) such that

(i) \( |\phi(t)| \leq V(t, \phi) \leq K \|\phi\| \),

(ii) \( |V(t, \phi) - V(t, \psi)| \leq K \|\phi - \psi\| \), for \( \phi, \psi \in C([0, t], \mathbb{R}^n) \),

(iii) \( V'_{(CL)}(t, \phi) \leq -\lambda V(t, \phi) \).

Remark 3.1. The assumption \( C(t) \in L^1(\mathbb{R}^+) \) should be demanded only in the implication (III) \( \rightarrow \) (I).

The following proofs of Theorem 3.1 are carried out for (CL), but the same proofs on (I), (II), (V), and (VI) are valid for (L).

Corollary 3.1. The equivalence relations among (I), (II), (V), and (VI) hold for (L).

We shall sketch the proof of Theorem 3.1 according to the arrows in the diagram in Section 1.

Proof of (I) \( \rightarrow \) (V). Since the equation (CL) is linear, the proof is similar to that of ordinary differential equations (cf. [12, Theorem 11.2]).

Proof of (I) \( \rightarrow \) (II). Define a linear operator \( X(t, t_0) : C[0, t_0] \rightarrow C[0, t] \) by \( (X(t, t_0)\phi)(t) = x(t; t_0, \phi) \). The uniqueness of solutions of (CL) yields the relation

\[
X(t, t_0) = X(t, s) X(s, t_0) \quad \text{for} \quad t \geq s \geq t_0.
\]

From uniform stability, it follows that for \( \varepsilon_0 > 0 \), there exists \( \delta(\varepsilon_0) > 0 \) such that \( t_0 \geq 0, \|\phi\|_{t_0} < \delta(\varepsilon_0) \), and \( t \geq t_0 \) imply \( |(X(t, t_0)\phi)(t)| = |x(t; t_0, \phi)| < \varepsilon_0 \). Hence, for all \( t \geq t_0 \),

\[
\|X(t, t_0)\| = \sup_{\|\phi\|_{t_0} \leq 1} |(X(t, t_0)\phi)(t)| \leq \frac{\varepsilon_0}{\delta(\varepsilon_0)}.
\]
The remainder of the proof is similar to that of ordinary differential equations (cf. [12, Theorem 7.2]).

**Proof of (II) \(\rightarrow\) (III).** Lemma 2.1 implies that \(Z(t) \phi(0) = x(t; 0, \phi(0))\), where \(x(t; 0, \phi(0))\) is a solution of (CL), and hence the proof is obvious.

**Proof of (III) \(\rightarrow\) (I).** From Lemma 2.1 and (III) we have

\[
|x(t; t_0, \phi)| \leq |Z(t - t_0)| \cdot |\phi(t_0)|
+ \int_{t_0}^{t} |Z(t - s)| \int_{0}^{s} |C(s - u)| \cdot |\phi(u)| \, du \, ds
\leq K \exp\{-\lambda(t - t_0)\} \cdot ||\phi||_{L^1}
+ ||\phi||_{L^1} K \exp\{-\lambda(t - t_0)\} \int_{0}^{t - t_0} \exp(\lambda t) \int_{t}^{\infty} |C(s)| \, ds \, dt.
\]

Since \(C(t) \in L^1(\mathbb{R}^+)\), we have

\[
\int_{t}^{\infty} |C(s)| \, ds \to 0 \quad (t \to \infty).
\]

Therefore it follows from Lemmas 2.3 and 2.4 that the last term of the above inequality tends to zero as \(t \to \infty\) independently of \(t_0\). Thus \(x(t; t_0, \phi) \to 0 \quad (t \to \infty)\) independently of \(t_0\). The proof is complete.

**Proof of (III) \(\not\rightarrow\) (IV).** The necessity is trivial. We shall show the sufficiency. Suppose that (IV) holds, then there exists \(\lambda > 0\) such that

\[
\int_{0}^{t} |Z(s)| \, ds \leq \frac{1}{\lambda} \quad \text{for all} \quad t \geq 0.
\]

By Lemma 2.2, we have

\[
|Z(t)| \int_{0}^{t} \frac{ds}{|Z(s)|} \leq \int_{0}^{t} |Z(t - s)| \cdot |Z(s)| \cdot \frac{ds}{|Z(s)|}
\leq \frac{1}{\lambda} \quad \text{for all} \quad t \geq 0.
\]

Therefore it follows from Lemma 2.5 that there exists \(K > 0\) such that

\[
|Z(t)| \leq Ke^{-\lambda t} \quad \text{for all} \quad t \geq 0.
\]

**Proof of (II) \(\not\rightarrow\) (VI).** The sufficiency is carried out by the standard
arguments for the functional $V(t, \phi)$. We shall show the necessity. Since the zero solution of (CL) is ExAS, then there exist $K > 0$ and $\lambda > 0$ such that

$$|x(t; t_0, \phi)| \leq K \exp\{-\lambda(t - t_0)\} \|\phi\|_{t_0} \quad \text{for all } t_0 \geq 0 \text{ and } t \geq t_0.$$  

Let $V(t, \phi) = \sup_{t_0 \geq 0} |x(t + t_0; t, \phi)| e^{2t_0}$, then in a way similar to [12, Theorem 33.4], we can show that $V(t, \phi)$ satisfies (i)-(iii). Thus the proof of Theorem 3.1 is now completed.

### 4. Perturbation Theorems

In this section we consider the systems of perturbed Volterra integro-differential equations

$$y'(t) = A(t) y(t) + \int_0^t B(t, s) y(s) \, ds + g(t, y(\cdot)) \quad \text{(PL)}$$

$$y'(t) = A(t) y(t) + \int_0^t B(t, s) y(s) \, ds + h(t). \quad \text{(PL}_h\text{)}$$

We shall study the perturbation problems on boundedness and asymptotic stability from (L) to (PL) and (PL$_h$). We denote the solution of (PL) or (PL$_h$) with initial values $(t_0, \phi)$ by $y(t) = y(t; t_0, \phi)$. We now give a lemma which is similar to Theorem 3.1.

Let $t \geq 0$, $\phi \in C([0, t], \mathbb{R}^n)$, and $p(t, \phi) = \sup_{-r \leq s \leq t} |\phi(s)|$, where $r \geq 0$ and $\phi(s) = \phi(0)$ for $s \leq 0$.

**Lemma 4.1.** The zero solution of (L) is ExAS if and only if there exist a continuous functional $W(t, \phi)$ defined for $t \geq 0$ and $\phi \in C([0, t], \mathbb{R}^n)$, $\lambda > 0$ and $K > 0$ such that

(i) $p(t, \phi) \leq W(t, \phi) \leq K \|\phi\|_t$,

(ii) $|W(t, \phi) - W(t, \psi)| \leq K \|\phi - \psi\|$ for $\phi, \psi \in C([0, t], \mathbb{R}^n)$,

(iii) $W(t, \phi) \leq -\lambda W(t, \phi)$.

**Proof.** Let $W(t, \phi) = \sup_{t_0 \geq 0} p(t + t_0, x(\cdot; t, \phi)) e^{2t_0}$, then the proof is similar to that of (II) \(\Rightarrow\) (VI) in Theorem 3.1.

**Theorem 4.1.** Suppose that the solutions of (L) are UB and UUB and

$$|g(t, \phi(\cdot))| \leq \gamma(t) \sup_{t - r \leq s \leq t} |\phi(s)| \quad \text{for } t \geq 0 \text{ and } \phi \in C(\mathbb{R}^+),$$

where $r \geq 0$ and $\limsup_{t \to \infty} \int_0^t \gamma(s) \, ds$ is sufficiently small. Then the solutions of (PL) are UB and UUB.
Proof. Since the solutions of (1) are UB and UUB, the zero solution of (L) is ExAS by the use of Corollary 3.1. Thus there exists \( W(t, \phi) \) satisfying the conditions of Lemma 4.1. Let \( \varepsilon < \lambda K^{-1} \), then there exists \( T > 0 \) such that

\[
\int_{t}^{t+1} \gamma(s) \, ds < \varepsilon \quad \text{for all} \quad t \geq T.
\]  

(4.1)

By Lemma 4.1, we have

\[
W_{(PL)}(t, y(\cdot)) \leq W_{(L)}(t, y(\cdot)) + K |g(t, y(\cdot))| \\
\leq -\lambda W(t, y(\cdot)) + K \gamma(t) p(t, y(\cdot)) \\
\leq (K \gamma(t) - \lambda) W(t, y(\cdot)),
\]

for all \( t \geq t_0 \). Hence, we have

\[
W(t, y(\cdot)) \leq W(t_0, \phi) \exp \left\{ K \int_{t_0}^{t} \gamma(s) \, ds - \lambda (t - t_0) \right\}.
\]

Thus, for all \( t \geq t_0 \),

\[
|y(t; t_0, \phi)| \leq p(t, y(\cdot)) \\
\leq K \|\phi\|_{t_0} \exp \left\{ K \int_{t_0}^{t} \gamma(s) \, ds - \lambda (t - t_0) \right\}.
\]  

(4.2)

In case \( t_0 \geq T \), from (4.1) and (4.2) we have

\[
|y(t; t_0, \phi)| \leq Ke^{K\varepsilon} \|\phi\|_{t_0} e^{-(\lambda - K\varepsilon)(t - t_0)}
\]

for \( t \geq t_0 \). In case \( t_0 < T \), from (4.1) and (4.2) we have

\[
|y(t; t_0, \phi)| \leq K \exp \left\{ K \left( \varepsilon + \int_{t_0}^{t} \gamma(s) \, ds \right) \right\} \|\phi\|_{t_0} e^{-(\lambda - K\varepsilon)(t - t_0)}
\]

for \( t \geq t_0 \). Thus the solutions of (PL) are UB and UUB. This completes the proof.

Example 4.1. We show some examples for \( g(t, \phi) \) satisfying the conditions in Theorem 4.1:

(i) \( g(t, x) \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n] \) satisfying \( |g(t, x)| \leq \gamma(t) |x| \),

(ii) \( g(t, \phi(\cdot)) = Q(t) \phi(t - r) \),

(iii) \( g(t, \phi(\cdot)) = \int_{t-r}^{t} Q(s) \phi(s) \, ds \).
In (ii) and (iii), $r \geq 0$ and $Q(t)$ is an $n \times n$ matrix function such that
\[ \limsup_{t \to \infty} \int_{t}^{t+1} |Q(s)| \, ds \text{ is sufficiently small.} \]

**Theorem 4.2.** Suppose that the solutions of (L) are UB and UUB and
\[ |g(t, \phi(\cdot))| \leq \gamma(t) \sup_{t-r \leq s \leq t} |\phi(s)|^\beta \quad \text{for } t \geq 0 \text{ and } \phi \in C(\mathbb{R}^+), \]
where $0 \leq \beta < 1$, $r \geq 0$, and $\sup_{t \geq 0} \int_{t}^{t+1} \gamma(s) \, ds < \infty$. Then the solutions of (PL) are UB and UUB.

**Proof.** Since the solutions of (L) are UB and UUB, there exists $W(t, \phi)$ satisfying the conditions of Lemma 4.1. Then,
\[ W_{(PL)}(t, y(\cdot)) = -\lambda W(t, y(\cdot)) + K\gamma(t) p(t, y(\cdot))^\beta \]
for $t \geq t_0$. Let $U(t, \phi(\cdot)) = \{ W(t, \phi(\cdot)) + e^{-\lambda t} \}^{1-\beta}$. Then $U(t, \phi(\cdot))$ satisfies a local Lipschitz condition with respect to $\phi$, because $W(t, \phi(\cdot)) + e^{-\lambda t} > 0$. Thus
\[ U_{(PL)}(t, y(\cdot)) = (1-\beta) \{ -\lambda e^{-\lambda t} + W_{(PL)}(t, y(\cdot)) \} \{ W(t, y(\cdot)) + e^{-\lambda t} \}^{-\beta} \]
\[ \leq -\lambda (1-\beta) U(t, y(\cdot)) + K(1-\beta) \gamma(t). \]

Let $\sigma = \lambda (1-\beta) > 0$, then
\[ U(t, y(\cdot)) \leq U(t_0, \phi) e^{-\sigma(t-t_0)} + K(1-\beta) \int_{t_0}^{t} e^{\sigma s} \gamma(s) \, ds. \] (4.3)

From Lemma 2.3, Lemma 4.1, and (4.3) there exists $L > 0$ such that
\[ |y(t; t_0, \phi)|^{1-\beta} \leq (K\|\phi\|_{t_0} + 1)^{1-\beta} e^{-\sigma(t-t_0)} + K(1-\beta)L \]
for $t \geq t_0$. Hence the solutions of (PL) are UB and UUB. This completes the proof.

**Theorem 4.3.** Suppose that
\[ \sup_{t \geq 0} \int_{t}^{t+1} |A(s)| \, ds < \infty, \quad \sup_{t \geq 0} \int_{t}^{t} |B(t, s)| \, ds < \infty, \]
and the solutions of (L) are UB and UUB. Then the solutions of (PL, h) are UB and UUB if and only if
\[ \sup_{t \geq 0} \left| e^{-t} \int_{0}^{t} e^{sh(s)} \, ds \right| < \infty. \]
Proof. We first show the sufficiency. Let \( y(t) \) be the solution of \((PL_h)\) with initial values \((t_0, \phi)\). Let \( H(t) = e^{-t} \int_0^t e^s h(s) \, ds \) and \( x(t) = y(t) - H(t) \), then \( H(t) \) is bounded for all \( t \geq 0 \) and

\[
x'(t) - A(t) x(t) + \int_0^t B(t, s) x(s) \, ds + \int_0^t B(t, s) H(s) \, ds + \{ A(t) + I \} H(t).
\]

(4.4)

Let \( \tilde{h}(t) = \int_0^t B(t, s) H(s) \, ds + \{ A(t) + I \} H(t) \), then by the use of Lemma 2.3, we have \( \sup_{t \geq 0} e^{-t} \int_0^t e^s |\tilde{h}(s)| \, ds < \infty \). Thus \( \tilde{h}(t) \) satisfies the condition of Theorem 4.2 for \( \beta = 0 \), hence the solutions of (4.4) are UB and UUB. Since \( y(t) = x(t) + H(t) \), the solutions of \((PL_h)\) are UB and UUB.

We next show the necessity. From Lemma 2.3, there exist \( L_1 > 0 \) and \( L_2 > 0 \) such that

\[
e^{-t} \int_0^t e^s \left| A(s) + I \right| \, ds \leq L_1
\]

and

\[
e^{-t} \int_0^t e^s \int_0^s |B(s, u)| \, du \, ds \leq L_2,
\]

for all \( t \geq 0 \). Let \( y(t) = y(t; 0, 0) \) be a solution of \((PL_h)\). Then there exists \( M > 0 \) such that \( |y(t)| \leq M \) for all \( t \geq 0 \). Note that \( h(t) = y'(t) - A(t) y(t) - \int_0^t B(t, s) y(s) \, ds \). Then we have, by an integration by parts,

\[
\int_0^t e^s h(s) \, ds = e^t y(t) - \int_0^t \{ A(s) + I \} e^s y(s) \, ds - \int_0^t e^s \int_0^s B(s, u) y(u) \, du \, ds.
\]

Hence \( h(t) \) satisfies

\[
\left| e^{-t} \int_0^t e^s h(s) \, ds \right| \leq (1 + L_1 + L_2) M \quad \text{for all} \quad t \geq 0.
\]

This completes the proof.

Theorem 4.4. Suppose that the zero solution of \((L)\) is UAS and

\[
|g(t, \phi(\cdot))| \leq \gamma(t) \sup_{t-r \leq s \leq t} |\phi(s)| \quad \text{for} \quad t \geq 0,
\]
and \( \phi \in B_H \) for some \( H > 0 \), where \( r \geq 0 \) and \( \limsup_{t \to \infty} \int_0^t \gamma(s) \, ds \) is sufficiently small. Then the zero solution of (PL) is UAS.

**Proof.** By the same argument as in the proof of Theorem 4.1, there exists \( M > 0 \) such that

\[
|y(t; t_0, \phi)| \leq M \|\phi\|_{\infty} e^{-(\lambda - \kappa_0)(t - t_0)} \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \|\phi\|_{\infty} < HM^{-1}.
\]

Hence the zero solution of (PL) is UAS. This completes the proof.

**Theorem 4.5.** Suppose that the zero solution of (L) is UAS and there exist \( A > 0 \) and \( B > 1 \) such that

\[
|g(t, \phi(\cdot))| \leq M \sup_{t - r \leq s \leq t} |\phi(s)|^\beta \quad \text{for} \quad t \geq 0,
\]

and \( \phi \in B_H \) for some \( H > 0 \). Then the zero solution of (PL) is UAS.

**Proof.** Since the zero solution of (L) is UAS, there exists \( W(t, \phi) \) satisfying the conditions of Lemma 4.1. Let \( \varepsilon_0 < K^{-1} \), \( \delta(\varepsilon_0) = \min\{(\varepsilon_0/M)^{1/(\beta - 1)}, H\} \), then if \( |\phi(t)| < \delta(\varepsilon_0) \), we have \( |g(t, \phi(\cdot))| \leq \varepsilon_0 |\phi(t)| \).

We consider the solution \( y(t) \) of (PL) such that \( \|\phi\|_{\infty} < \delta(\varepsilon_0) \). We assume that there exists \( t_1 \geq t_0 \) such that \( |y(t_1; t_0, \phi)| = \delta(\varepsilon_0) \) and \( |y(t; t_0, \phi)| < \delta(\varepsilon_0) \) on \([t_0, t_1)\). Then we have

\[
W'(PL)(t, y(\cdot)) \leq W'(L)(t, y(\cdot)) + K|g(t, y(\cdot))| \leq -(\lambda - \kappa_0) W(t, y(\cdot)),
\]

for all \( t \in [t_0, t_1] \). Hence,

\[
|y(t_1; t_0, \phi)| \leq K \|\phi\|_{\infty} e^{-(\lambda - \kappa_0)(t_1 - t_0)} < \delta(\varepsilon_0).
\]

This is a contradiction. Thus

\[
|y(t; t_0, \phi)| \leq K \|\phi\|_{\infty} e^{-(\lambda - \kappa_0)(t - t_0)} \quad \text{for all} \quad t \geq t_0.
\]

Therefore, the zero solution of (PL) is UAS. This completes the proof.

Hereafter we assume that

\[
|g(t, \phi(\cdot))| \leq \gamma(t) \sup_{t - r \leq s \leq t} |\phi(s)|^\beta \quad \text{for} \quad t \geq 0, \quad \text{and} \quad \phi \in B_H \text{ for } (H-1)
\]

for some \( H > 0 \), where \( r \geq 0, 0 < \beta < 1 \), and \( \lim_{t \to \infty} \int_t^{t+1} \gamma(s) \, ds = 0 \).

We investigate whether UAS for (L) yields UAS for (PL) under the assumption (H-1). Unfortunately, this is not the case strictly, but the eventual uniform asymptotic stability is preserved. We now give the definition of eventual uniform stability.
DEFINITION 4.1. The zero solution of (2.1) is eventually uniformly stable (EvUS), if for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $\alpha(\varepsilon) > 0$ such that $t_0 \geq \alpha(\varepsilon)$ and $\|\phi\|_{t_0} < \delta(\varepsilon)$ imply $|x(t; t_0, \phi)| < \varepsilon$ for all $t \geq t_0$.

THEOREM 4.6. Suppose that the zero solution of (L) is UAS and $g(t, \phi(\cdot))$ satisfies the assumption (H-1). Then the zero solution of (PL) is EvUS and UA.

Proof. In a way similar to the proof of Theorem 4.2, we have

$$|y(t; t_0, \phi)|^{1-\beta} \leq (K\|\phi\|_{t_0} + e^{-\lambda t_0})^{1-\beta} e^{-\sigma(t-t_0)} + K(1-\beta) e^{-\sigma t} \int_{t_0}^{t} e^{\sigma r} y(s) \, ds.$$

From Lemma 2.3 for any $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that

$$e^{-\alpha t} \int_{t_0}^{t} e^{\alpha r} y(s) \, ds < \varepsilon \quad \text{and} \quad e^{-\lambda t_0} < \varepsilon \quad \text{for all} \quad t \geq t_0 \geq \alpha(\varepsilon).$$

Thus the zero solution of (PL) is EvUS. It is clear that the zero solution of (PL) is UA. This completes the proof.

Remark 4.1. We note that the zero solution of (PL) is not US under the conditions of Theorem 4.6. Consider the equation

$$y'(t) = -y(t) + y(t) y(t)^{1/2}, \quad (4.5)$$

where

$$\gamma(t) = \begin{cases} c > 0 & (0 \leq t < 1) \\ -ct + 2c & (1 \leq t < 2) \\ 0 & (t \geq 2). \end{cases}$$

Then (4.5) satisfies the conditions of Theorem 4.6 and we have as the solution of (4.5)

$$y(t; t_0, 0) = e^{-t\{c(e^{t/2} - e^{t_0/2})\}^2} \quad \text{for} \quad 0 \leq t_0 \leq t \leq 1.$$ 

Hence the zero solution of (4.5) is not unique. Thus the zero solution of (4.5) is not US.

Remark 4.2. Let $\phi \in C([0, t], \mathbb{R}^n)$ be given and define $\phi, \in B = C((-\infty, 0], \mathbb{R}^n)$ by

$$\phi_1(s) = \begin{cases} \phi(t + s) & \text{for} \quad -t \leq s \leq 0 \\ \phi(0) & \text{for} \quad s < -t, \end{cases}$$
then $C([0, t], \mathbb{R}^n)$ is naturally embedded in $B$. Let $|\cdot|_B$ be a norm which satisfies the axioms in [6, 8, 10], for example,

$$|\phi|_B = e^{-\gamma t} \sup_{0 \leq s \leq t} e^{\gamma s} |\phi(s)|,$$

then we can show that UB and UUB for (L) in $B$ imply those for (PL) in $B$ under the condition

$$|g(t, \phi(\cdot))| \leq \gamma(t) |\phi|_B.$$

However, the norm $\|\phi\|_t$ in the paper does not satisfy the axiom $(\gamma_2)$ in [6]. With respect to this norm, stability property in $B$ is not equivalent to that in $\mathbb{R}^n$.

Remark 4.3. Consider the equation due to Seifert [11]:

$$y'(t) = -2y(t) + y(0). \quad (4.6)$$

Then $g(t, \phi) = \phi(0)$ satisfies

$$|g(t, \phi)| \leq \|\phi\|_t,$$

but the solutions of (4.6) are not UUB, since $y(t) = (1 + e^{-2t}) y(0)/2$ is a solution of (4.6). Thus in the theorems of Section 4, the condition

$$|g(t, \phi(\cdot))| \leq \gamma(t) \sup_{t-r \leq s \leq t} |\phi(s)|^\beta$$

cannot be replaced by $|g(t, \phi(\cdot))| \leq \gamma(t) \|\phi\|_t^\beta$.

REFERENCES