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On monotone method for first and second order periodic boundary value problems and periodic solutions of functional differential equations

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Abstract

In this paper, we show that the method of monotone iterative technique is valid to obtain two monotone sequences that converge uniformly to extremal solutions of first and second order periodic boundary value problems and periodic solutions of functional differential equations. We obtain some new results relative to the lower solution α and upper solution β with $\alpha \leqslant \beta$.

Keywords: Periodic boundary value problem; Periodic solution; Existence; Upper and lower solutions; Monotone iterative technique

1. Introduction

The method of upper and lower solutions coupled with the monotone iterative has been applied successfully to obtain results of existence and approximation of solutions for periodic boundary value problems for first and second order ordinary differential equations (see [1] and references therein).

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Some attempts have been made to extend these techniques to study periodic boundary value problems of functional differential equations (FDEs). In [2,3], the periodic problem

$$y'(t) = f(t, y, y_t), \qquad y(0) = y(T),$$

is considered, but in both papers it is required that $f(t, u, \phi)$ be monotone in the third variable.

In this paper, we apply monotone iterative technique to study first and second order periodic boundary value problems and periodic solutions of functional differential equations.

We consider the following periodic boundary value problems (PBVPs):

$$\begin{cases} y'(t) = f(t, y(t), y(\omega(t))), & t \in I = [0,\ T], \\ y(0) = y(T), \end{cases} \tag{1.1}$$

$$\begin{cases} -y''(t) = f(t, y(t), y(\omega(t))), & t \in I, \\ y(0) = y(T), \quad y'(0) = y'(T), \end{cases} \tag{1.2}$$

where $f \in C(I \times R^2, R)$, $\omega \in C(I, [a, b])$, here a, b are constants such that $[0, T] \subset [a, b]$, $T > 0$.

In a similar way to deal with (1.1)–(1.2), we consider the T-periodic solutions of the following functional differential equations (FDEs):

$$y'(t) = f\big(t, y(t), y\big(\omega(t)\big)\big), \quad t \in R, \tag{1.3}$$

$$-y''(t) = f\big(t, y(t), y\big(\omega(t)\big)\big), \quad t \in R, \tag{1.4}$$

where $f \in C(R^3, R)$, $f(t, u, v) = f(t + T, u, v)$, $T > 0$, $\omega(t) = t - \tau(t)$, $\tau \in C(R, R)$, $\tau(t) = \tau(t + T)$.

The definitions of solutions of (1.1)–(1.4) will be given in Section 3.

Note that (1.1)–(1.4) include ordinary, retarded and advanced differential equations.

PBVP (1.1) is considered by Liz and Nieto [4,5] without assuming f be monotone in the third variable. Recently, (1.1)–(1.4) are considered by Jiang and Wei [6] without assuming f to be monotone in the third variable by applying monotone method. However, in [4–6], the results are only valid for delay differential equations.

The main purpose of this paper is to improve and generalize the work of [2–6]. The new method we use for dealing with the maximum principle is different from [4–6].

In this paper, Section 2 is devoted to the maximum principle, which is the key to developing the monotone technique. Section 3 is devoted to develop the monotone method for (1.1)–(1.4).

2. Maximum principle

To prove the validity of the monotone iterative technique, we shall use the following maximum principle.

In the proof of our maximum principle below, we will use the lemma of Cabada [7].

Lemma 2.1 [7]. *If the linear problem*

$$z^{(n)}(t) + Mz(t) = 0,$$
$$z^{(i)}(a) - z^{(i)}(b) = 0, \quad i = 0, \dots, n-2,$$
$$z^{(n-1)}(a) - z^{(n-1)}(b) = 1$$

has a unique solution $r \in C^\infty[a,b]$, the problem

$$u^{(n)}(t) + Mu(t) = \sigma(t) \in L^1(I),$$
$$u^{(i)}(a) - u^{(i)}(b) = \lambda_i, \quad i = 0, \dots, n-1,$$

has a unique solution u given by the expression

$$u(t) = \int_a^b G_n(t,s)\sigma(s)\,ds + \sum_{i=0}^{n-1} r^{(i)}(t)\lambda_{n-1-i},$$

where

$$G_n(t,s) = \begin{cases} r(a+t-s), & a \leqslant s \leqslant t \leqslant b, \\ r(b+t-s), & a \leqslant t \leqslant s \leqslant b. \end{cases}$$

Theorem 2.1. *Let $y \in E_1 = C([a,b], R) \cap C^1([0,T], R)$ and $M > 0$, $N > 0$ such that*

(i) $y'(t) + My(t) + Ny\big(w(t)\big) \geqslant 0, \quad t \in I,$

(ii) $y(0) \geqslant y(T),$

(iii) $y(0) = y(t), \quad t \in [a, 0],$
$y(T) = y(t), \quad t \in [T, b],$

(iv) $\dfrac{e^{MT} - 1}{M} N < 1,$

where $w \in C(I, [a,b])$. Then $y(t) \geqslant 0$, $\forall t \in [a,b]$.

Proof. There exists $\xi \in [0,T]$ such that

$$y(\xi) = \max_{t \in [0,T]} y(t) = \max_{t \in [a,b]} y(t),$$

thus

$$y'(t) + My(t) + Ny(\xi) \geqslant 0, \quad t \in I,$$
$$y(0) \geqslant y(T).$$

Let $\sigma(t) = y'(t) + My(t) + Ny(\xi) \geqslant 0$ and $\lambda = y(0) - y(T) \geqslant 0$.

Then we have

$$y'(t) + My(t) + Ny(\xi) = \sigma(t), \quad t \in I,$$
$$y(0) = y(T) + \lambda.$$

Let $u(t) = y(t) + (N/M)y(\xi)$; thus we obtain

$$u'(t) + Mu(t) = \sigma(t), \quad t \in I,$$
$$u(0) = u(T) + \lambda.$$

Let $r(t)$ be the unique solution $r \in C^\infty[0, T]$ to the problem

$$r'(t) + Mr(t) = 0, \qquad r(0) = r(T) + 1.$$

From Lemma 2.1, we obtain

$$u(t) = \begin{cases} \int_0^T G(0,s)\sigma(s)\,ds + \lambda r(0), & t \in [a, 0], \\ \int_0^T G(t,s)\sigma(s)\,ds + \lambda r(t), & t \in [0, T], \\ \int_0^T G(T,s)\sigma(s)\,ds + \lambda r(T), & t \in [T, b], \end{cases} \tag{2.1}$$

where

$$G(t,s) := \begin{cases} \frac{e^{M(T+s-t)}}{e^{MT}-1}, & 0 \leqslant s \leqslant t \leqslant T, \\ \frac{e^{M(s-t)}}{e^{MT}-1}, & 0 \leqslant t \leqslant s \leqslant T, \end{cases} \tag{2.2}$$

and

$$r(t) = \frac{e^{M(T-t)}}{e^{MT} - 1}.$$

A direct calculation shows that

$$0 < \frac{1}{e^{MT} - 1} \leqslant r(T) \leqslant G(t,s) \leqslant r(0) = \frac{e^{MT}}{e^{MT} - 1}. \tag{2.3}$$

By (2.1) and (2.3),

$$r(T)\left(\int_0^T \sigma(s)\,ds + \lambda\right) \leqslant u(t) \leqslant r(0)\left(\int_0^T \sigma(s)\,ds + \lambda\right), \quad \forall t \in [a, b]. \tag{2.4}$$

Since $u(t) = y(t) + (N/M)y(\xi)$, by (2.4) we have

$$y(\xi) = \frac{M}{M+N}u(\xi) \leqslant \frac{M}{M+N}r(0)\left(\int_0^T \sigma(s)\,ds + \lambda\right).$$

On the other hand, by (2.4) we also have

$$\begin{aligned} y(t) = u(t) - \frac{N}{M}y(\xi) &\geqslant r(T)\left(\lambda + \int_0^T \sigma(s)\,ds\right) - \frac{N}{M}y(\xi) \\ &\geqslant \left(\lambda + \int_0^T \sigma(s)\,ds\right)\left(r(T) - \frac{N}{M+N}r(0)\right) \\ &= \left(\lambda + \int_0^T \sigma(s)\,ds\right)r(0)\left(\delta - \frac{N}{M+N}\right), \end{aligned}$$

where $\delta = r(T)/r(0) = 1/e^{MT}$.

From the assumption (iv), we obtain $N/(M+N) < \delta$. So we get $y(t) \geqslant 0$, $\forall t \in [a,b]$. $\square$

Remark 2.1. Condition (iv) in Theorem 2.1 is the same as in [5,6], but we extend the range of $\omega(t)$.

Theorem 2.2. *Let* $y \in E_2 = C([a,b], R) \cap C^2([0,T], R)$ *and* $M > 0$, $N > 0$ *such that*

$$\begin{aligned}
&\text{(i)} \quad -y''(t) + My(t) + Ny\big(w(t)\big) \geqslant 0, \quad t \in I,\\
&\text{(ii)} \quad y(0) = y(T), \qquad y'(0) \leqslant y'(T),\\
&\text{(iii)} \quad y(0) = y(t), \quad t \in [a,0] \cup [T,b],\\
&\text{(iv)} \quad \frac{2N}{M}\left(\operatorname{sh}\frac{T\sqrt{M}}{4}\right)^2 < 1,
\end{aligned}$$

where $\operatorname{sh} x = (e^x - e^{-x})/2$, $w \in C(I, [a,b])$. *Then* $y(t) \geqslant 0$, $\forall t \in [a,b]$.

Proof. Let $z(t) = -y(t)$, thus

$$\begin{aligned}
&z''(t) - Mz(t) - Nz\big(w(t)\big) \geqslant 0, \quad t \in I,\\
&z(0) = z(T), \qquad z'(0) \geqslant z'(T),\\
&z(0) = z(t), \quad t \in [a,0] \cup [T,b].
\end{aligned}$$

There exists $\xi \in [0,T]$ such that

$$z(\xi) = \min_{t \in [0,T]} z(t) = \min_{t \in [a,b]} z(t),$$

thus

$$\begin{aligned}
&z''(t) - Mz(t) - Nz(\xi) \geqslant 0, \quad t \in I,\\
&z(0) = z(T), \qquad z'(0) \geqslant z'(T).
\end{aligned}$$

Let $\sigma(t) = z''(t) - Mz(t) - Nz(\xi) \geqslant 0$ and $\lambda = z'(0) - z'(T) \geqslant 0$. Then we have

$$\begin{aligned}
&z''(t) - Mz(t) - Nz(\xi) = \sigma(t), \quad t \in I,\\
&z(0) = z(T), \qquad z'(0) = z'(T) + \lambda.
\end{aligned}$$

Let $u(t) = z(t) + (N/M)z(\xi)$, thus

$$\begin{aligned}
&u''(t) - Mu(t) = \sigma(t), \quad t \in I,\\
&u(0) = u(T), \qquad u'(0) = u'(T) + \lambda.
\end{aligned}$$

Let $r(t)$ be the unique solution $r \in C^\infty[0,T]$ to the problem

$$\begin{aligned}
&r''(t) - Mr(t) = 0,\\
&r(0) = r(T), \qquad r'(0) = r'(T) + 1.
\end{aligned}$$

From Lemma 2.1, we obtain

$$u(t) = \begin{cases} \int_0^T G(t,s)\sigma(s)\,ds + \lambda r(t), & t \in [0,T], \\ \int_0^T G(0,s)\sigma(s)\,ds + \lambda r(0), & t \in [a,0] \cup [T,b], \end{cases} \tag{2.5}$$

where $m = \sqrt{M}$,

$$G(t,s) := \begin{cases} \frac{e^{m(t-s)} + e^{m(T-t+s)}}{2m(1-e^{mT})}, & 0 \leqslant s \leqslant t \leqslant T, \\ \frac{e^{m(T+t-s)} + e^{m(s-t)}}{2m(1-e^{mT})}, & 0 \leqslant t \leqslant s \leqslant T, \end{cases} \tag{2.6}$$

and

$$r(t) = \frac{e^{mt} + e^{m(T-t)}}{2m(1-e^{mT})} < 0.$$

A direct calculation shows that

$$\frac{1+e^{mT}}{2m(1-e^{mT})} = r(0) \leqslant G(t,s) \leqslant r\left(\frac{T}{2}\right) = \frac{2e^{mT/2}}{2m(1-e^{mT})} < 0. \tag{2.7}$$

By (2.5) and (2.7) we have that

$$r(0)\left(\int_0^T \sigma(s)\,ds + \lambda\right) \leqslant u(t) \leqslant r\left(\frac{T}{2}\right)\left(\int_0^T \sigma(s)\,ds + \lambda\right), \quad \forall t \in [a,b]. \tag{2.8}$$

Since $u(t) = z(t) + (N/M)z(\xi)$, so by (2.8) we have that

$$z(\xi) = \frac{M}{M+N}u(\xi) \geqslant \frac{M}{M+N}r(0)\left(\int_0^T \sigma(s)\,ds + \lambda\right).$$

On the other hand, by (2.8) we also have

$$\begin{aligned} z(t) = u(t) - \frac{N}{M}z(\xi) &\leqslant r\left(\frac{T}{2}\right)\left(\lambda + \int_0^T \sigma(s)\,ds\right) - \frac{N}{M}z(\xi) \\ &\leqslant \left(\lambda + \int_0^T \sigma(s)\,ds\right)\left(r\left(\frac{T}{2}\right) - \frac{N}{M+N}r(0)\right) \\ &= \left(\lambda + \int_0^T \sigma(s)\,ds\right)r(0)\left(\delta - \frac{N}{M+N}\right), \end{aligned}$$

where

$$\delta = \frac{r(T/2)}{r(0)} = \frac{2e^{mT/2}}{1+e^{mT}}.$$

From the assumption (iv), we obtain $N/(M+N) < \delta$. So we get $z(t) \leqslant 0$, i.e., $y(t) = -z(t) \geqslant 0$, $\forall t \in [a,b]$. $\square$

Remark 2.2. Condition (iv) in Theorem 2.2 is weaker than condition (iv) in Theorem 2.2* of [6], and it holds provided N is suitably small. Condition (iv) in Theorem 2.2* of [6] is

$$\frac{2N}{M}\left(\operatorname{sh}\frac{T\sqrt{M}}{2}\right)^2 < 1.$$

In the same way as proof of Theorems 2.1 and 2.2, we have the following results.

Theorem 2.3. *Let* $y \in X_1 = \{y \in C^1(R,R)\colon\ y(t) = y(t+T)\}$ *and* $M > 0$, $N > 0$ *such that*

(i) $y'(t) + My(t) + Ny\big(w(t)\big) \geqslant 0, \quad t \in R,$

(ii) $\dfrac{e^{MT}-1}{M}N < 1,$

where $w(t) = t - \tau(t)$, $\tau \in C(R,R)$, $\tau(t) = \tau(t+T)$, $T > 0$. *Then* $y(t) \geqslant 0$, $\forall t \in R$.

Theorem 2.4. *Let* $y \in X_2 = \{y \in C^2(R,R)\colon\ y(t) = y(t+T)\}$ *and* $M > 0$, $N > 0$ *such that*

(i) $-y''(t) + My(t) + Ny\big(w(t)\big) \geqslant 0, \quad t \in R,$

(ii) $\dfrac{2N}{M}\left(\operatorname{sh}\dfrac{T\sqrt{M}}{4}\right)^2 < 1,$

where $w(t) = t - \tau(t)$, $\tau \in C(R,R)$, $\tau(t) = \tau(t+T)$, $T > 0$. *Then* $y(t) \geqslant 0$, $\forall t \in R$.

3. Monotone method for PBVPs and periodic solutions of FDEs

In this section, we are in a position to prove the validity of monotone method for (1.1)–(1.4).

First we consider PBVPs (1.1) and (1.2). We shall denote by

$$E_{01} = \big\{y \in E_1\colon\ y(t) = y(0),\ \forall t \in [a,0] \text{ and } y(t) = y(T),\ \forall t \in [T,b]\big\}$$

and

$$E_{02} = \big\{y \in E_2\colon\ y(t) = y(0),\ \forall t \in [a,0] \cup [T,b]\big\},$$

where E_i $(i = 1,2)$ are defined in Section 2.

For $\alpha, \beta \in E_{0i}$ $(i = 1,2)$, we shall write $\alpha \leqslant \beta$ if $\alpha(t) \leqslant \beta(t)$ for all $t \in [a,b]$. In such a case, we shall denote

$$[\alpha,\beta] = \big\{y \in E_{0i}\colon\ \alpha \leqslant y \leqslant \beta,\ i = 1,2\big\}.$$

A function $\alpha \in E_{01}$ is said to be a lower solution to (1.1), if it satisfies

$$\begin{cases} \alpha'(t) \leqslant f(t,\alpha(t),\alpha(w(t))), & t \in I, \\ \alpha(0) \leqslant \alpha(T). \end{cases} \tag{3.1}$$

An upper solution for (1.1) is defined analogously by reversing the inequalities of above.

A function $\alpha \in E_{02}$ is said to be a lower solution to (1.2), if it satisfies

$$\begin{cases} -\alpha''(t) \leqslant f(t, \alpha(t), \alpha(w(t))), & t \in I, \\ \alpha(0) = \alpha(T), \quad \alpha'(0) \geqslant \alpha'(T). \end{cases} \tag{3.2}$$

An upper solution for (1.2) is defined analogously by reversing the inequalities of above.

A function $y \in E_{01}$ (or E_{02}) is said to be a solution to (1.1) (or (1.2)) if it is both a lower and an upper solution to (1.1) (or (1.2)).

By similar arguments as in [6], we can prove Theorems 3.1 and 3.2 below.

Theorem 3.1. *Suppose that there exists a lower solution α and an upper solution β of* (1.1) *such that $\alpha \leqslant \beta$ on $[a, b]$. Assume that there exist two constants $M > 0$, $N > 0$ satisfying*

$$(\mathrm{H}_1) \quad f(t, u_2, v_2) - f(t, u_1, v_1) \geqslant -M(u_2 - u_1) - N(v_2 - v_1)$$

for $t \in I$, whenever $\alpha(t) \leqslant u_1 \leqslant u_2 \leqslant \beta(t)$ and $\alpha(w(t)) \leqslant v_1 \leqslant v_2 \leqslant \beta(w(t))$;

$$(\mathrm{H}_2) \quad \frac{e^{MT} - 1}{M} N < 1.$$

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal solution to the problem (1.1) *in the segment* $[\alpha, \beta]$.

Theorem 3.2. *Suppose that there exists a lower solution α and an upper solution β of* (1.2) *such that $\alpha \leqslant \beta$ on $[a, b]$. Assume that there exist two constants $M > 0$, $N > 0$ satisfying*

$$(\mathrm{B}_1) \quad f(t, u_2, v_2) - f(t, u_1, v_1) \geqslant -M(u_2 - u_1) - N(v_2 - v_1)$$

for $t \in I$, whenever $\alpha(t) \leqslant u_1 \leqslant u_2 \leqslant \beta(t)$ and $\alpha(w(t)) \leqslant v_1 \leqslant v_2 \leqslant \beta(w(t))$;

$$(\mathrm{B}_2) \quad \frac{2N}{M} \left(\operatorname{sh} \frac{T\sqrt{M}}{4} \right)^2 < 1.$$

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal solution to the problem (1.2) *in the segment* $[\alpha, \beta]$.

Next we consider periodic solutions of (1.3) and (1.4). Let X_1 and X_2 be as in Section 2.

A function $\alpha \in X_1$ is said to be a lower solution to (1.3) if it satisfies

$$\alpha'(t) \leqslant f\big(t, \alpha(t), \alpha\big(w(t)\big)\big), \quad t \in R. \tag{3.3}$$

An upper solution for (1.3) is defined analogously by reversing the inequalities of above.

A function $\alpha \in X_2$ is said to be a lower solution to (1.4) if it satisfies

$$-\alpha''(t) \leqslant f\big(t, \alpha(t), \alpha\big(w(t)\big)\big), \quad t \in R. \tag{3.4}$$

An upper solution for (1.4) is defined analogously by reversing the inequalities of above.

A function $y \in X_1$ (or X_2) is said to be a solution to (1.3) (or (1.4)) if it is both a lower and an upper solution to (1.3) (or (1.4)).

Also by similar arguments as in [6], we have the following results.

Theorem 3.3. *Suppose that there exists a lower solution α and an upper solution β of* (1.3) *such that $\alpha \leqslant \beta$ in R. Assume that there exist two constants $M > 0$, $N > 0$ satisfying*

$$(\mathrm{H}_1) \quad f(t, u_2, v_2) - f(t, u_1, v_1) \geqslant -M(u_2 - u_1) - N(v_2 - v_1)$$

for $t \in R$, whenever $\alpha(t) \leqslant u_1 \leqslant u_2 \leqslant \beta(t)$ and $\alpha(w(t)) \leqslant v_1 \leqslant v_2 \leqslant \beta(w(t))$;

$$(\mathrm{H}_2) \quad \frac{e^{MT} - 1}{M} N < 1.$$

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal T-periodic solution to (1.3) *in the segment $[\alpha, \beta]$.*

Theorem 3.4. *Suppose that there exists a lower solution α and an upper solution β of* (1.4) *such that $\alpha \leqslant \beta$ in R. Assume that there exist two constants $M > 0$, $N > 0$ satisfying*

$$(\mathrm{B}_1) \quad f(t, u_2, v_2) - f(t, u_1, v_1) \geqslant -M(u_2 - u_1) - N(v_2 - v_1)$$

for $t \in R$, whenever $\alpha(t) \leqslant u_1 \leqslant u_2 \leqslant \beta(t)$ and $\alpha(w(t)) \leqslant v_1 \leqslant v_2 \leqslant \beta(w(t))$;

$$(\mathrm{B}_2) \quad \frac{2N}{M}\left(\operatorname{sh}\frac{T\sqrt{M}}{4}\right)^2 < 1.$$

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal T-periodic solution to (1.4) *in the segment $[\alpha, \beta]$.*

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