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Universal Realization

G. NAUDÉ

National Research Institute for Mathematical Sciences, CSIR, Pretoria 0001, Republic of South Africa

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A study is made of input-process machines, in the sense of Arbib and Manes, and their behavior. For a given input-process **X**: **K** → **K** the categories **Mach(X)** of machines and **Beh(X)** of behaviors are constructed, also a functor **E**: **Mach(X)** → **Beh(X)** which assigns to each machine its behavior. It is shown that **E** has a left adjoint and that abstract Nerode realization is universal. A consequence is a characterization of minimal realization functors: a result similar to those arrived at by Goguen for machines in closed categories. We then show that by restricting machine and behavior morphisms, realization is universal for the most general type of Nerode realization, i.e., reflexive Nerode realization.

Following the Arbib–Manes theory of machines in a category **K** [2–4], a category **Mach(X)** of machines is constructed where **X**: **K** → **K** is an input process. We can consider **Mach(X)** as a formalization of an internal or state-space description of systems. An external description of a system is usually given in terms of its input–output behavior and this can be formalized as a category **Beh(X)** of behaviors. These definitions are given in Section 1, which also includes some further background on the Arbib–Manes approach to machines in a category.

Various functorial relationships between the categories **Mach(X)** and **Beh(X)** are also investigated in Section 1. It is shown that we have a functor **E**: **Mach(X)** → **Beh(X)** which assigns to each machine its behavior and that **E** has a left adjoint which assigns to each behavior its free realization.

Minimal realizations are considered in Sections 2 and 3. Adámek showed in [1] that external Nerode realization is universal.[1] We now show in Section 2 that the more general abstract Nerode realization is universal. This result leads to a characterization of minimal realization functors, which are similar to earlier ones of Goguen [8–10] and Ehrig *et al.* [7] for machines in closed categories. Results in [13] on state–behavior machines are also extended to input-process machines. The relationship between our results and those of Adámek is discussed in Section 2.

Adámek also showed that realization is seldom universal, for instance, that it is not universal for tree automata. In Section 3 we show that by requiring our categories of

[1] The author is grateful to a referee for bringing this paper to his attention.

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machines and behaviors to have morphisms with split epi first component, we obtain universal realization for the most general type of Nerode realization, i.e., reflexive Nerode realization. For the first time this result can be applied to tree automata.

1. Free Realization

In this section the notation used in the rest of the paper will be established, and a brief description given of some of the basic notions of the Arbib–Manes theory. Various categories are defined and the results of the present paper are related to those arrived at in our previous work [13].

Let $\mathbf{K}$ be an arbitrary category and $\mathbf{X}\colon \mathbf{K} \to \mathbf{K}$ any functor. The category $\mathbf{Dyn}(\mathbf{X})$ has as objects pairs (Q, δ), where Q is a $\mathbf{K}$-object and $\delta\colon \mathbf{X}Q \to Q$ is a $\mathbf{K}$-morphism. (Q, δ) is called an $\mathbf{X}$-*dynamics*. A $\mathbf{Dyn}(\mathbf{X})$-*morphism* $f\colon (Q, \delta) \to (Q', \delta')$ is a $\mathbf{K}$-morphism $f\colon Q \to Q'$ for which $f\delta = \delta'\mathbf{X}f$. A functor $\mathbf{X}\colon \mathbf{K} \to \mathbf{K}$ is called an *input process* if the forgetful functor $\mathbf{U}\colon \mathbf{Dyn}(\mathbf{X}) \to \mathbf{K}\colon (Q, \delta) \mapsto Q$ has a left adjoint. Thus $\mathbf{X}\colon \mathbf{K} \to \mathbf{K}$ is an input process if for each $\mathbf{K}$-object A there exists an $\mathbf{X}$-dynamics $(\mathbf{X}^{@}A, \mu A)$ together with a $\mathbf{K}$-morphism $\eta A\colon A \to \mathbf{X}^{@}A$ such that given any $\mathbf{X}$-dynamics (Q, δ) and any $\mathbf{K}$-morphism $f\colon A \to Q$, there exists a unique $\mathbf{Dyn}(\mathbf{X})$-morphism $f^{\#}\colon (\mathbf{X}^{@}A, \mu A) \to (Q, \delta)$ such that $f^{\#}\eta A = f$. $f^{\#}$ is called the $\mathbf{Dyn}(\mathbf{X})$-*extension* of f.

A functor $\mathbf{X}\colon \mathbf{K} \to \mathbf{K}$ is called a *state–behavior process* if $\mathbf{U}$ has both a left and a right adjoint.

Given any input process $\mathbf{X}\colon \mathbf{K} \to \mathbf{K}$ we have a functor $\mathbf{X}^{@}\colon \mathbf{K} \to \mathbf{K}$ which assigns to each $\mathbf{K}$-object A the $\mathbf{K}$-object $\mathbf{X}^{@}A$ and to each $\mathbf{K}$-morphism $f\colon A \to B$ the $\mathbf{K}$-morphism $(\eta Bf)^{\#}$. We now have natural transformations $\eta\colon 1_{\mathbf{K}} \dot{\to} \mathbf{X}^{@}$, $\mu\colon \mathbf{X}\mathbf{X}^{@} \dot{\to} \mathbf{X}^{@}$, and $\eta_1\colon \mathbf{X} \dot{\to} \mathbf{X}^{@}$, where $\eta_1 Q = \mu \mathbf{X} \eta Q$.

Given an input process $\mathbf{X}\colon \mathbf{K} \to \mathbf{K}$ *the run map* $\delta^{@}\colon \mathbf{X}^{@}Q \to Q$ of a dynamics (Q, δ) is the $\mathbf{Dyn}(\mathbf{X})$-extension of $1_Q\colon Q \to Q$.

Lemma 1.1 [2, pp. 25, 26]. *Let* $\mathbf{X}\colon \mathbf{K} \to \mathbf{K}$ *be an input process. Then* $\delta^{@}\eta_1 Q = \delta$. *Furthermore, if* $\psi\colon (Q, \delta) \to (Q', \delta')$ *is a* $\mathbf{Dyn}(\mathbf{X})$-*morphism, then* $\psi\delta^{@} = (\delta')^{@}\,\mathbf{X}^{@}\psi$.

A *machine* in $\mathbf{K}$ is a septuple $M = (\mathbf{X}, Q, \delta, I, \tau, Y, \beta)$, where $\mathbf{X}$ is an input process, (Q, δ) is an $\mathbf{X}$-dynamics, and $\tau\colon I \to Q$, $\beta\colon Q \to Y$ are $\mathbf{K}$-morphisms. The *reachability morphism* $r\colon (\mathbf{X}^{@}I, \mu I) \to (Q, \delta)$ of M is defined to be the $\mathbf{Dyn}(\mathbf{X})$-extension of $\tau\colon I \to Q$. Reachability of M can now be defined relative to a class $\mathscr{E}$ of $\mathbf{K}$-epimorphisms: M is called $\mathscr{E}$-*reachable* if $r \in \mathscr{E}$. The *behavior* of M is the $\mathbf{K}$-morphism $\mathbf{E}M = \beta r\colon \mathbf{X}^{@}I \to Y$. A machine M is called a *realization* of a $\mathbf{K}$-morphism $f\colon \mathbf{X}^{@}I \to Y$ if f is the behavior of M.

By an appropriate definition of a machine morphism we obtain a category $\mathbf{Mach}(\mathbf{X})$ of machines with $\mathbf{X}$ as input process. Given machines $M_i = (\mathbf{X}, Q_i, \delta_i, I_i, \tau_i, Y_i, \beta_i)$, $i = 1, 2$, a *machine morphism* $M_1 \to M_2$ is a triple (a, b, c), where $a\colon I_1 \to I_2$, $b\colon Q_1 \to Q_2$, and $c\colon Y_1 \to Y_2$ are $\mathbf{K}$-morphisms such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{X}Q_1 \arrow[rr, "\delta_1"] \arrow[ddd, "\mathbf{X}b"'] & & Q_1 \arrow[dr, "\beta_1"] \arrow[ddd, "b"] & \\
& I_1 \arrow[ur, "\tau_1"] \arrow[d, "a"] & & Y_1 \arrow[d, "c"] \\
& I_2 \arrow[dr, "\tau_2"'] & & Y_2 \\
\mathbf{X}Q_2 \arrow[rr, "\delta_2"] & & Q_2 \arrow[ur, "\beta_2"'] &
\end{tikzcd}$$

We note that b is a $\mathbf{Dyn(X)}$-morphism. Composition of machine morphisms is defined componentwise.

We can also define a category $\mathbf{Beh(X)}$ of behaviors. As objects we take quadruples $(\mathbf{X}, I, f, Y)$ where $\mathbf{X}$ is an input-process. I and Y are $\mathbf{K}$-objects and $f\colon \mathbf{X}^{@}I \to Y$ is a $\mathbf{K}$-morphism. Given behaviors $B_i = (\mathbf{X}, I_i, f_i, Y_i)$, $i = 1, 2$, a *behavior morphism* $B_1 \to B_2$ is a pair (a, c), where $a\colon I_1 \to I_2$ and $c\colon Y_1 \to Y_2$ are $\mathbf{K}$-morphisms such that $cf_1 = f_2\mathbf{X}^{@}a$.

Composition of behavior morphisms is again defined componentwise. A behavior $(\mathbf{X}, I, f, Y)$ is sometimes simply denoted by the $\mathbf{K}$-morphism f.

If $\mathbf{X}$ is a state–behavior process we can also construct a category $\mathbf{TBeh(X)}$ of total behaviors; cf. [13].

We now have the following relationship between the categories $\mathbf{Beh(X)}$ and $\mathbf{TBeh(X)}$. The proof is straightforward and is left to the reader.

THEOREM 1.2. *For a state–behavior process* $\mathbf{X}$ *we have* $\mathbf{TBeh(X)} \simeq \mathbf{Beh(X)}$.

In a previous paper we investigated various adjoint situations between the categories $\mathbf{Mach(X)}$ and $\mathbf{TBeh(X)}$ [13]. Similar results on the categories $\mathbf{Mach(X)}$ and $\mathbf{Beh(X)}$ are obtained in Sections 2 and 3. The above result shows that if $\mathbf{X}$ is a state–behavior process some of the present results reduce to those in [13].

We introduced the categories $\mathbf{Mach(X)}$ of machines and $\mathbf{Beh(X)}$ of behaviors in an arbitrary category $\mathbf{K}$ and it was seen that with every machine we can associate its behavior. We now show that this defines a functor $\mathbf{E}\colon \mathbf{Mach(X)} \to \mathbf{Beh(X)}$ and that $\mathbf{E}$ has a left adjoint $\mathbf{F}\colon \mathbf{Beh(X)} \to \mathbf{Mach(X)}$ which assigns to each behavior a reachable realization.

We need the following elementary result, which is easily proved by using the uniqueness of $\mathbf{Dyn(X)}$-extensions.

LEMMA 1.3. *Let M and M' be machines with reachability morphisms $\tau^{\#} = r\colon \mathbf{X}^{@}I \to Q$ and $(\tau')^{\#} = r'\colon \mathbf{X}^{@}I' \to Q'$, respectively. If $(a, b, c)\colon M \to M'$ is a* $\mathbf{Mach(X)}$*-morphism, then $br = r'\mathbf{X}^{@}a$.*

By using the above lemma it is a straightforward matter to prove the following result.

THEOREM 1.4. *There is a functor* $\mathbf{E}: \mathbf{Mach(X)} \to \mathbf{Beh(X)}$ *which assigns to every machine* M *its behavior* $\mathbf{E}M = \beta r: \mathbf{X}^{@}I \to Y$, *and to each* $\mathbf{Mach(X)}$ *morphism* (a, b, c): $M \to M'$ *the* $\mathbf{Beh(X)}$*-morphism* $(a, c): \mathbf{E}M \to \mathbf{E}M'$.

It is well known that every behavior has a free realization which is reachable. The general result is expressed in the following theorem. Again the proof is easy and is left to the reader.

THEOREM 1.5. *There is a functor* $\mathbf{F}: \mathbf{Beh(X)} \to \mathbf{Mach(X)}$ *which assigns to each behavior* $f: \mathbf{X}^{@}I \to Y$ *the reachable machine*

$$\mathbf{F}M = (\mathbf{X}, \mathbf{X}^{@}I, \mu I, I, \eta I, Y, f),$$

and to each $\mathbf{Beh(X)}$*-morphism* $(a, c): f \to f'$ *the* $\mathbf{Mach(X)}$*-morphism*

$$\mathbf{F}(a, c) = (a, \mathbf{X}^{@}a, c): \mathbf{F}f \to \mathbf{F}f'.$$

The relationship between $\mathbf{E}$ and $\mathbf{F}$ is expressed in our next result.

THEOREM 1.6. $\mathbf{E}$ *is a right adjoint, left inverse of* $\mathbf{F}$.

Proof. We note that for any $f: \mathbf{X}^{@}I \to Y$ the reachability morphism of $\mathbf{F}f$ is $1_{\mathbf{X}^{@}I}$. Thus $\mathbf{EF}f = f$. Similarly we have $\mathbf{EF}(a, c) = (a, c)$ for each $\mathbf{Beh(X)}$-morphism (a, c), so that $\mathbf{E}$ is a left inverse of $\mathbf{F}$.

For any machine M we define a $\mathbf{Mach(X)}$-morphism

$$\theta_M = (1_I, r, 1_Y): \mathbf{FE}M \to M.$$

The rest of the proof is routine. ∎

2. ABSTRACT NERODE REALIZATION

In this section we consider minimal realizations of behaviors. We say that M is a *minimal realization* of f if M is a reachable realization of f with the property that, given any other reachable realization M' of f, there exists a unique $\mathbf{Mach(X)}$-morphism $\psi: M' \to M$. Such an M is therefore a terminal object in the full subcategory of $\mathbf{Mach(X)}$ with objects reachable realizations of f, and M is thus unique up to $\mathbf{Mach(X)}$-isomorphism. We need a few more definitions from [2].

Given a behavior $f: \mathbf{X}^{@}I \to Y$, we say that a pair of $\mathbf{K}$-morphisms $\alpha, \gamma: E_f \to \mathbf{X}^{@}I$ is the *abstract Nerode equivalence* of f if $f\alpha^{\#} = f\gamma^{\#}$ and if, whenever $f(\alpha')^{\#} = f(\gamma')^{\#}$ for $\alpha', \gamma': E' \to \mathbf{X}^{@}I$, there exists a unique $\psi: E' \to E_f$ such that $\alpha\psi = \alpha'$ and $\gamma\psi = \gamma'$ ($\alpha^{\#}$ is the unique $\mathbf{Dyn(X)}$-extension of α, etc.).

Before considering minimal realizations we prove the following result.

LEMMA 2.1. *Suppose that we have the following data*:

(i) *a functor* $\mathbf{G}\colon \mathbf{K} \to \mathbf{K}$;

(ii) $\mathbf{Dyn}(\mathbf{G})$-*morphisms* $f_i\colon (A_i, \delta_{A_i}) \to (B_i, \delta_{B_i})$, $i = 1, 2$, *such that* $\mathbf{G}f_1$ *is an epimorphism*;

(iii) *a commutative square*, $g_2 f_1 = f_2 g_1$, *in* $\mathbf{K}$.

Then if $\delta_{A_2}\mathbf{G}g_1 = g_1\delta_{A_1}$, *we have* $\delta_{B_2}\mathbf{G}g_2 = g_2\delta_{B_1}$.

Proof. In the following diagram we have to show that (z) commutes, while all other squares are commutative. This is easily done by using the fact that $\mathbf{G}f_1$ is an epimorphism. ∎

$$\begin{tikzcd}
\mathbf{G}A_1 \arrow[rrr, "\mathbf{G}f_1"] \arrow[ddd, "\mathbf{G}g_1"'] \arrow[dr, "\delta_{A_1}"] & & & \mathbf{G}B_1 \arrow[ddd, "\mathbf{G}g_2"] \arrow[dl, "\delta_{B_1}"'] \\
& A_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] & B_1 \arrow[d, "g_2\ (z)"] & \\
& A_2 \arrow[r, "f_2"] & B_2 & \\
\mathbf{G}A_2 \arrow[rrr, "\mathbf{G}f_2"'] \arrow[ur, "\delta_{A_2}"] & & & \mathbf{G}B_2 \arrow[ul, "\delta_{B_2}"']
\end{tikzcd}$$

We now state the following result, which is proved in [2].

THEOREM 2.2. *Let the behavior* $f\colon \mathbf{X}^{@}I \to Y$ *satisfy the following postulates*:

(1) f *has an abstract Nerode equivalence* $\alpha, \gamma\colon E_f \to \mathbf{X}^{@}I$;

(2) $r_f = \operatorname{coeq}(\alpha, \gamma)\colon \mathbf{X}^{@}I \to Q_f$ *exists*;

(3) *there exists a dynamics* (Q_f, δ_f) *such that* $r_f\colon (\mathbf{X}^{@}I, \mu I) \to (Q_f, \delta_f)$ *is a* $\mathbf{Dyn}(\mathbf{X})$-*morphism*;

(4) $\mathbf{X}$ *is such that if* (Q, δ) *has reachability morphism* $r\colon \mathbf{X}^{@}I \to Q$ *with* r *a coequalizer, then either* $\mathbf{X}r$ *or* $\mathbf{X}^{@}r$ *is an epimorphism.*

Then f *has a minimal coequalizer-reachable realization* $\mathbf{R}f$.

$\mathbf{R}f$ is defined by

$$\mathbf{R}f = (\mathbf{X}, Q_f, \delta_f, I, \tau_f, Y, \beta_f),$$

where $\tau_f = r_f \eta I$, so that r_f is the reachability morphism of $\mathbf{R}f$, and $\beta_f\colon Q_f \to Y$ is obtained by using the fact that $r_f = \operatorname{coeq}(\alpha, \gamma)$ and $\alpha f = \gamma f$.

Suppose that $(a, c)\colon (\mathbf{X}, I, f, Y) \to (\mathbf{X}, I', f', Y')$ is a $\mathbf{Beh}(\mathbf{X})$-morphism and both f and f' satisfy postulates (1)–(4).

Consider the following diagram:

$$\begin{tikzcd}
\mathbf{X}^{@}E_f \arrow[rr, shift left, "\alpha^{\#}"] \arrow[rr, shift right, "\gamma^{\#}"'] & & \mathbf{X}^{@}I \arrow[rr, "f"] \arrow[dr, "r_f"] \arrow[ddd, "\mathbf{X}^{@}a"] & & Y \arrow[ddd, "c"] \\
& E_f \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\gamma"'] \arrow[d, dashed, "\psi"] & & Q_f \arrow[ur, "\beta_f"] \arrow[d, "b"] & \\
& E_{f'} \arrow[dr, shift left, "\alpha'"] \arrow[dr, shift right, "\gamma'"'] & & Q_{f'} \arrow[dr, "\beta_{f'}"] & \\
\mathbf{X}^{@}E_{f'} \arrow[rr, shift left, "(\alpha')^{\#}"] \arrow[rr, shift right, "(\gamma')^{\#}"'] & & \mathbf{X}^{@}I' \arrow[rr, "f'"'] \arrow[ur, "r_{f'}"] & & Y'
\end{tikzcd}$$

We have $f\alpha^{\#} = f\gamma^{\#}$ and so $cf\alpha^{\#} = cf\gamma^{\#}$. Thus $f'\mathbf{X}^{@}a\alpha^{\#} = f'\mathbf{X}^{@}a\gamma^{\#}$. Since α', γ' is the abstract Nerode equivalence of f', there exists a unique morphism ψ: $E_f \to E_{f'}$ such that $\alpha'\psi = \mathbf{X}^{@}a\alpha$ and $\gamma'\psi = \mathbf{X}^{@}a\gamma$. Now $r_{f'}\mathbf{X}^{@}a\alpha = r_{f'}\alpha'\psi = r_{f'}\gamma'\psi$ (since $r_{f'} = \text{coeq}(\alpha', \gamma')$) $= r_{f'}\mathbf{X}^{@}a\gamma$. Since $r_f = \text{coeq}(\alpha, \gamma)$, there exists a unique b: $Q_f \to Q_{f'}$ such that $br_f = r_{f'}\mathbf{X}^{@}a$.

We can now formulate our next result, which shows that the assignment $f \to \mathbf{R}f$ is functorial. Let $\mathbf{Beh}'(\mathbf{X})$ denote the full subcategory of $\mathbf{Beh}(\mathbf{X})$ with objects those behaviors satisfy postulates (1)–(4).

THEOREM 2.3. *There is a functor* $\mathbf{R}$: $\mathbf{Beh}'(\mathbf{X}) \to \mathbf{Mach}(\mathbf{X})$ *which assigns to each behavior* f *its minimal realization* $\mathbf{R}f$ *and to each* $\mathbf{Beh}'(\mathbf{X})$*-morphism* (a, c): $f \to f'$ *the* $\mathbf{Mach}(\mathbf{X})$*-morphism* (a, b, c): $\mathbf{R}f \to \mathbf{R}f'$, *where* b *is defined as above.*

Proof. We have to show that (a, b, c) is a $\mathbf{Mach}(\mathbf{X})$-morphism. By using the definition of b it is easily verified that $b\tau_f = \tau_{f'}a$.

We can also prove that $f'\mathbf{X}^{@}a\alpha = f'\mathbf{X}^{@}a\gamma$.

Since $r_f = \text{coeq}(\alpha, \gamma)$, there exists a unique h: $Q_f \to Y'$ such that $hr_f = f'\mathbf{X}^{@}a$. We have $\beta_{f'}br_f = \beta_{f'}r_{f'}\mathbf{X}^{@}a = f'\mathbf{X}^{@}a$ and $c\beta_f r_f = cf = f'\mathbf{X}^{@}a$. Thus $\beta_{f'}b = h = c\beta_f$.

To prove that b is a $\mathbf{Dyn}(\mathbf{X})$-morphism we consider two cases, for each of which Lemma 2.1 is used.

(a) Suppose $\mathbf{X}r_f$ is an epimorphism; we can apply Lemma 2.1 with $\mathbf{G} = \mathbf{X}$, $f_1 = r_f$, $f_2 = r_{f'}$, $g_1 = \mathbf{X}^{@}a$, and $g_2 = b$ to obtain $b\delta_f = \delta_{f'}\mathbf{X}b$.

(b) Suppose $\mathbf{X}^{@}r_f$ is an epimorphism; we want to apply Lemma 3.1 with $\mathbf{G} = \mathbf{X}^{@}$, $f_1 = r_f$, $f_2 = r_{f'}$, $g_1 = \mathbf{X}^{@}a$, and $g_2 = b$.

By using Lemma 1.1 we see that the hypothesis of Lemma 2.1 is satisfied and we obtain $b\delta_f{}^{@} = \delta_f{}^{@}\mathbf{X}^{@}b$.

We now have to prove that (z) in the following diagram commutes:

$$\begin{tikzcd}
\mathbf{X}^{@}Q_f \arrow[dddd, "\mathbf{X}^{@}b"'] \arrow[drr, "\delta_f^{@}"] & & \\
& \mathbf{X}Q_f \arrow[ul, "\eta_1 Q_f"'] \arrow[r, "\delta_f"] \arrow[d, "\mathbf{X}b"] \arrow[dr, phantom, "(z)"] & Q_f \arrow[d, "b"] \\
& \mathbf{X}Q_{f'} \arrow[r, "\delta_{f'}"] \arrow[ddl, "\eta_1 Q_{f'}"'] & Q_{f'} \\
& & \\
\mathbf{X}^{@}Q_{f'} \arrow[uurr, "\delta_{f'}^{@}"'] & &
\end{tikzcd}$$

From Lemma 1.1 we have $\delta_f^{@}\eta_1 Q_f = \delta_f$ and $\delta_{f'}^{@}\eta_1 Q_{f'} = \delta_{f'}$. It is now a straightforward matter to verify that $b\delta_f = \delta_{f'}\mathbf{X}b$ and that $\mathbf{R}$ is indeed a functor.

Let $\mathbf{M}$ be a subcategory of $\mathbf{Mach}(\mathbf{X})$ such that every machine in $\mathbf{M}$ is coequalizer-reachable, $\mathbf{B} = \mathbf{EM}$ is a subcategory of $\mathbf{Beh}'(\mathbf{X})$ and $\mathbf{RB}$ is a subcategory of $\mathbf{M}$. Furthermore, we assume that all $\mathbf{Mach}(\mathbf{X})$-morphisms of the form $(1_I, k, 1_Y)$: $M \to M'$, with M and M' $\mathbf{M}$-objects, are in $\mathbf{M}$. The restriction of $\mathbf{E}$: $\mathbf{Mach}(\mathbf{X}) \to \mathbf{Beh}(\mathbf{X})$ [$\mathbf{R}$: $\mathbf{Beh}'(\mathbf{X}) \to \mathbf{Mach}(\mathbf{X})$] to $\mathbf{M} \to \mathbf{B}$ [$\mathbf{B} \to \mathbf{M}$] is also denoted by $\mathbf{E}$[$\mathbf{R}$].

THEOREM 2.4. $\mathbf{E}$: $\mathbf{M} \to \mathbf{B}$ *is a left inverse left adjoint to* $\mathbf{R}$: $\mathbf{B} \to \mathbf{M}$.

Proof. From the definitions of $\mathbf{E}$ and $\mathbf{R}$ it follows directly that $\mathbf{ER} = 1_{\mathbf{B}}$.

Given any machine $M = (\mathbf{X}, Q, \delta, I, \tau, Y, \beta)$ in $\mathbf{M}$ we first have to define a suitable morphism Π_M: $M \to \mathbf{RE}M$. Let $r = \operatorname{coeq}(\alpha, \gamma)$ be the reachability morphism of M. Then α, γ are abstractly $f = \beta r$-equivalent. Let x, y: $E_f \to \mathbf{X}^{@}I$ be the abstract Nerode equivalence of $f = \mathbf{E}M$ and let $r_f = \operatorname{coeq}(x, y)$. Consider the following diagram:

$$\begin{tikzcd}
E \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] \arrow[d, dashed, "\psi"'] & \mathbf{X}^{@}I \arrow[r, "r"] \arrow[d, "1_{\mathbf{X}^{@}I}"] & Q \arrow[r, "\beta"] \arrow[d, "\Pi"] & Y \arrow[d, "1_Y"] \\
E_f \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & \mathbf{X}^{@}I \arrow[r, "r_f"] & Q_f \arrow[r, "\beta_f"] & Y
\end{tikzcd}$$

By an argument similar to the one in the discussion before Theorem 2.3, we first obtain a unique morphism ψ: $E \to E_f$ such that $x\psi = y\psi$, and then a unique morphism Π: $Q \to Q_f$ such that $\Pi r = r_f$. Adopting a procedure similar to that used for proving Theorem 2.3, we can show that $\Pi_M = (1_I, \Pi, 1_Y)$ is a $\mathbf{Mach}(\mathbf{X})$-morphism, while from our assumption about $\mathbf{M}$-morphisms it follows that Π_M is an $\mathbf{M}$-morphism.

We still have to verify the universal property of Π_M. Let $(\mathbf{X}, I', f', Y') = \mathbf{E}M'$, where $M' = (\mathbf{X}, Q', \delta', I', \tau', Y', \beta')$, be any behavior in $\mathbf{B}$ and suppose that $g = (a, b, c)$: $M \to \mathbf{RE}M'$ is any $\mathbf{M}$-morphism. We show that $\mathbf{E}g = (a, c)$: $\mathbf{E}M \to \mathbf{ERE}M' = \mathbf{E}M'$ is the unique $\mathbf{B}$-morphism for which $\mathbf{RE}g\Pi_M = g$.

Consider the following diagram:

$$\begin{tikzcd}
E \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] & \mathbf{X}^{@}I \arrow[r, "r"] \arrow[d, "1_{\mathbf{X}^{@}I}"] & Q \arrow[d, "\Pi"] \\
& \mathbf{X}^{@}I \arrow[r, "r_f"] \arrow[d, "\mathbf{X}^{@}a"] & Q_f \arrow[d, "\beta"] \\
& \mathbf{X}^{@}I' \arrow[r, "r_{f'}"] & Q_{f'}
\end{tikzcd}$$

β is obtained from the construction of $\mathbf{RE}g = \mathbf{R}(a, c)$, while Π is as above.

We have $r_{f'}\mathbf{X}^{@}a\alpha = \beta r_f\alpha = \beta\Pi r\alpha = \beta\Pi r\gamma = \beta r_f\gamma = r_{f'}\mathbf{X}^{@}a\gamma$. Since $r = \text{coeq}(\alpha, \gamma)$, there is a unique morphism $h: Q \to Q_{f'}$ for which $hr = r_{f'}\mathbf{X}^{@}a$. From Lemma 1.3 we have $br = r_{f'}\mathbf{X}^{@}a$, while from the above diagram we obtain $\beta\Pi r = r_{f'}\mathbf{X}^{@}a$. Thus $b = h = \beta\Pi$.

We now have $\mathbf{RE}g\Pi_M = (a, \beta, c)(1_I, \Pi, 1_Y) = (a, \beta\Pi = b, c) = g$.

The uniqueness part of the proof is straightforward. ▌

Let $\mathbf{S}$ be the full subcategory of $\mathbf{M}$ with objects minimal machines, i.e., let M be an $\mathbf{S}$-object if and only if for any realization M' in $\mathbf{M}$ of $\mathbf{E}M$ there exists a unique $\mathbf{M}$-morphism $M' \to M$. An $\mathbf{S}$-object M is therefore a terminal object in the full subcategory of $\mathbf{M}$ with objects realizations of $\mathbf{E}M$. Thus in this subcategory M is unique up to isomorphism.

Given any $\mathbf{M}$-morphism $g: \mathbf{R}f \to \mathbf{R}f'$, there exists a unique $\mathbf{B}$-morphism $k: f \to f'$ such that $\mathbf{R}k = g$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathbf{R}f \arrow[r, "\Pi_{\mathbf{R}f} = 1_{\mathbf{R}f}"] \arrow[dr, "g"'] & \mathbf{RER}f = \mathbf{R}f \arrow[d, "\mathbf{R}k"] & \mathbf{ER}f = f \arrow[d, "k"] \\
& \mathbf{R}f' & \mathbf{ER}f' = f'
\end{tikzcd}$$

Thus $\mathbf{RB}$ is a full subcategory of $\mathbf{S}$.

We show that $\mathbf{RB}$ is an equivalent subcategory of $\mathbf{S}$. Clearly the inclusion functor $\mathbf{RB} \to \mathbf{S}$ is faithful. We have to show that any machine S in $\mathbf{S}$ is isomorphic to some machine in $\mathbf{RB}$. Since both S and $\mathbf{RE}S$ are terminal objects in the full subcategory of $\mathbf{M}$ with objects realizations of $\mathbf{E}S$, we obtain the desired isomorphism.

If $\mathbf{G}: \mathbf{P} \to \mathbf{Q}$ is any functor with a left adjoint left inverse, then $\mathbf{GP}$ is a reflective subcategory of $\mathbf{Q}$ [12, p. 93]. From Theorem 3.4 we therefore infer that $\mathbf{RB}$ is a reflective subcategory of $\mathbf{M}$. We now state the following result, which is proved in [8]. Suppose that $\mathbf{A}$ is an equivalent subcategory of a subcategory $\mathbf{C}$ of a category $\mathbf{D}$, and that $\mathbf{A}$ is a reflective subcategory of $\mathbf{D}$. Then $\mathbf{C}$ is a reflective subcategory of $\mathbf{D}$.

By using the above results we now have the following theorem.

THEOREM 2.6. *$\mathbf{S}$ is a reflective subcategory of $\mathbf{M}$.*

The construction of the minimal realization functor $\mathbf{R}$ is based on the well-known Nerode equivalence relation. We proceed by showing that any other minimal realization functor $\mathbf{L}: \mathbf{B} \to \mathbf{M}$ has $\mathbf{E}: \mathbf{M} \to \mathbf{B}$ as left adjoint.

THEOREM 2.7. *Let $\mathbf{L}: \mathbf{B} \to \mathbf{M}$ be a functor such that $\mathbf{L}f$ is an $\mathbf{S}$-object for each $\mathbf{B}$-object f, and $\mathbf{EL} = 1_{\mathbf{B}}$. Then $\mathbf{L}$ is a right adjoint of $\mathbf{E}$.*

Proof. Given any $\mathbf{B}$-object f, $\mathbf{M}$-object M, and $\mathbf{B}$-morphism $g: \mathbf{E}M \to f$, we have to find a unique $\mathbf{M}$-morphism $g^\star: M \to \mathbf{L}f$ such that $\mathbf{E}g^\star = g$.

$$\begin{tikzcd} f & \mathbf{EL}f = f = \mathbf{ER}f \arrow[l, "1_f"'] \\ & \mathbf{E}M \arrow[ul, "g"] \arrow[u, "\mathbf{E}\bar{g}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{L}f & \mathbf{R}f \arrow[l, "\theta^{-1}"'] \\ M \arrow[u, "g^*"] \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

By the universal property of $\mathbf{R}f$ there is a unique $\mathbf{M}$-morphism $\bar{g}: M \to \mathbf{R}f$ such that $\mathbf{E}\bar{g} = g$. But there is a unique morphism $\theta: \mathbf{L}f \to \mathbf{R}f$ and θ is an isomorphism. Since $\Pi_{\mathbf{L}f}: \mathbf{L}f \to \mathbf{REL}f = \mathbf{R}f$ we have $\theta = \Pi_{\mathbf{L}f}$.

Furthermore, from the definitions of $\mathbf{E}$ and $\Pi_{\mathbf{L}f}$ we have $\mathbf{R}\Pi_{\mathbf{L}f} = 1_f$, so that $\mathbf{E}(\theta^{-1}) = 1_f$. Thus if we let $g^\star = \theta^{-1}\bar{g}$, we obtain $\mathbf{E}g^\star = \mathbf{E}(\theta^{-1}\bar{g}) = \mathbf{E}\theta^{-1}\mathbf{E}\bar{g} = g$.

The uniqueness of $g^\star$ follows from the uniqueness of $\bar{g}$ and the fact that θ is an isomorphism. ∎

We observe that $\mathbf{S}$ is closed under isomorphism. Since any two right adjoints of the behavior functor $\mathbf{E}: \mathbf{M} \to \mathbf{B}$ are naturally isomorphic, we have the following characterization of minimal realization functors, i.e., functors $\mathbf{L}: \mathbf{B} \to \mathbf{M}$ with $\mathbf{EL} = 1_{\mathbf{B}}$ which can be factored as $\mathbf{B} \to \mathbf{S} \xrightarrow{i} \mathbf{M}$.

COROLLARY 2.8. *A functor $\mathbf{L}: \mathbf{B} \to \mathbf{M}$ such that $\mathbf{EL} = 1_{\mathbf{B}}$ is a minimal realization functor if and only if it is a right adjoint of $\mathbf{E}: \mathbf{M} \to \mathbf{B}$.*

We now discuss results on universal realization obtained by Adámek in [1]. According to Adámek, an input-process $\mathbf{X}$ admits coequalizer minimal realization if every behavior has a minimal coequalizer realization. $\mathbf{X}$ admits universal realization if there exists a functor $\mathbf{R}: \mathbf{Beh(X)} \to \mathbf{RMach(X)}$ right adjoint to $\mathbf{ERMach(X)} \to \mathbf{Beh(X)}$ ($\mathbf{RMach(X)}$ is the full subcategory of $\mathbf{Mach(X)}$ with objects all reachable machines). Finally $\mathbf{X}$ admits external Nerode realization if every behavior satisfies the postulates for an external Nerode realization.

In [1] it is shown that every input process which admits external Nerode realization also admits universal realization. If we say that $\mathbf{X}$ admits abstract Nerode realization if every behavior satisfies the postulates for abstract Nerode realization, then we can obtain a similar result from Theorem 2.4.

COROLLARY 2.9. *If the input-process* **X** *admits abstract Nerode realization, then* **X** *admits universal realization.*

Proof. **X** admits abstract Nerode realization if and only if $\mathbf{Beh}'(\mathbf{X}) = \mathbf{Beh}(\mathbf{X})$. Furthermore, if we choose $\mathbf{M} = \mathbf{RMach}(\mathbf{X})$, then $\bar{\mathbf{B}} = \mathbf{EM} = \mathbf{Beh}(\mathbf{X})$, since every behavior has as reachable realization its free realization. The result now follows from Theorem 2.4. ∎

Under suitable (rather strong) conditions, Adámek has obtained a strong converse to his result, so that **X** admits universal realization if and only if **X** admits external Nerode realization. Various other realization procedures are also investigated in [1] but in general they are not functorial.

It is now a straightforward matter to obtain Adámek's result on external Nerode realization [1, Theorem 2.3]. Every behavior satisfying the postulates for an external Nerode realization also satisfies the postulates for an abstract Nerode realization [2, pp. 34, 35]. Thus if **X** admits external Nerode realization then it also admits abstract Nerode realization so that, by Corollary 2.9, **X** admits universal realization.

We conclude this section with an example. Let $\mathbf{K} = \mathbf{Set}$ and $\mathbf{X} = - \times X_0$ for a finite but fixed set X_0. Then $\mathbf{Mach}(\mathbf{X})$ is a category of sequential machines with a finite input alphabet. Let **M** be the subcategory of $\mathbf{Mach}(\mathbf{X})$ with objects (coequalizer-reachable) machines with the set $2 = \{0, 1\}$ as output set and a finite state set, and morphisms (a, b, c) with $c = 1_2$. Then **M** is a category of finite state acceptors while $\mathbf{EM} = \mathbf{B}$ is the category of regular sets over X_0 [8, p. 367]. Thus the adjoint situation between finite state acceptors and regular sets mentioned in [8] follows directly from Theorem 3.4. (In [8] the first component of **M**- and **B**-morphisms is also required to be surjective. This is not necessary in our theory, although **M** can be restricted in any such way.)

3. REFLEXIVE NERODE REALIZATION

In the previous section we showed that if a functor **X** admits abstract Nerode realization, then it admits universal realization. It was shown by Adámek that the result does not hold if we replace "abstract Nerode realization" by the more general "reflexive Nerode realization." In this section we investigate reflexive Nerode realization. It is shown that by a suitable restriction on the morphisms of the categories of machines and behaviors we do obtain universal realization for reflexive Nerode equivalences. This result enables us to obtain a hitherto unknown adjointness for tree automata.

We first provide some background. A pair of morphisms $t, u\colon A \to B$ is called *reflexive* if there exists a $v\colon B \to A$ such that $tv = 1_B = uv$. Given a behavior $f\colon \mathbf{X}^{@}I \to Y$, we call a pair $\alpha, \gamma\colon E_f \to \mathbf{X}^{@}I$ the *reflexive Nerode equivalence of* f if $f\alpha^{\#} = f\gamma^{\#}$ and whenever $f(\alpha')^{\#} = f(\gamma')^{\#}$ for a reflexive pair $\alpha', \gamma'\colon E' \to \mathbf{X}^{@}I$ there exists a unique $\psi\colon E' \to E_f$ such that the following diagram commutes:

$$\begin{tikzcd} E' \arrow[d, dashed, "\psi"'] \arrow[dr, shift left, "\alpha'"] \arrow[dr, shift right, "\gamma'"'] & \\ E_f \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\gamma"'] & \mathbf{X}^{@}I \end{tikzcd}$$

We now replace postulate (1) in Theorem 2.2 by the following:

(1′) *f has a reflexive Nerode equivalence* α, γ: $E_f \to \mathbf{X}^{@}I$.

The conclusion of the theorem now states that f has a minimal reflexive coequalizer-reachable realization (where a reflexive coequalizer is one coequalizing a reflexive pair) [2, p. 31].

Let (a, c): $(\mathbf{X}, I, f, Y) \to (\mathbf{X}, I', f', Y')$ be a **Beh**(**X**)-morphism which is split epi. Suppose that both f and f' satisfy postulates (1′)–(4). Consider the diagram preceding Theorem 2.3.

Since a is split epi, $\mathbf{X}^{@}a$ is split epi. Let k: $\mathbf{X}^{@}I \to \mathbf{X}^{@}I'$ be such that $\mathbf{X}^{@}ak = 1_{\mathbf{X}^{@}I'}$. Since α, γ is reflexive, there is a v: $\mathbf{X}^{@}I \to E_f$ such that $\alpha v = 1_{\mathbf{X}^{@}I} = \alpha v$. Now the pair $\mathbf{X}^{@}a\alpha$, $\mathbf{X}^{@}a\gamma$ is reflexive: $(\mathbf{X}^{@}a\alpha)(vk) = 1_{\mathbf{X}^{@}I'} = (\mathbf{X}^{@}a\gamma)(vk)$. We can now apply an argument similar to that used in the case of the abstract Nerode realization (before Theorem 2.3) to obtain a unique ψ: $E_f \to E_{f'}$ such that $\mathbf{X}^{@}a\alpha = \alpha'$, $\mathbf{X}^{@}a\gamma = \gamma'$. The rest of the construction and proof of Theorem 2.3 remain exactly the same.

Let **Beh***(**X**) be the subcategory of **Beh**(**X**) with objects all behaviors satisfying postulates (1′)–(4) and morphisms all behavior morphisms with split epi first component. Let **Mach***(**X**) be the subcategory of **Mach**(**X**) with the same objects as **Mach**(**X**) but with morphisms those machine morphisms with split epi first component.

THEOREM 3.1. *There is a functor* **R**: **Beh***(**X**) $\to$ **Mach***(**X**) *which assigns to each behavior f its minimal* (*reflexive-coequalizer reachable*) *realization* **R***f and to each* **Beh***(**X**)-*morphism* (a, c): $f \to f'$ *the* **Mach***(**X**)-*morphism* (a, b, c): $\mathbf{R}f \to \mathbf{R}f'$, *where b is defined as in Theorem* 2.3.

Let **M** be a subcategory of **Mach***(**X**) such that every machine in **M** is reflexive-coequalizer reachable, **B** = **EM** is a subcategory of **Beh***(**X**), and **RB** is a subcategory of **M**. Furthermore, we assume that all **Mach**(**X**) morphisms of the form $(1_I, x, 1_Y)$ are in **M**. The restrictions of **E**: **Mach**(**X**) $\to$ **Beh**(**X**) [**R**: **Beh***(**X**) $\to$ **Mach***(**X**)] to **M** $\to$ **B** [**B** $\to$ **M**] are also denoted by **E** [**R**].

With this new interpretation of **M** and **B**, Theorems 2.4–2.7 as well as Corollary 2.8 all remain valid. All the proofs remain unchanged.

We say that an input-process **X**: **K** $\to$ **K** admits reflexive Nerode realization if every behavior satisfies postulates (1′)–(4). **X** admits restricted universal realization if there is a functor **R**: **Beh***(**X**) $\to$ **Mach***(**X**) right adjoint for **E**: **Mach***(**X**) $\to$ **Beh***(**X**). In [1] Adámek proved that if either **K** is connected or **X***e* is epi for any coequalizer

e, then if $\mathbf{X}$ admits minimal realization it admits reflexive Nerode realization. Using this result we have the following theorem.

THEOREM 3.2. *Let either* $\mathbf{K}$ *be connected or* $\mathbf{X}e$ epi *for any coequalizer* e. *Then the following are equivalent*:

(a) $\mathbf{X}$ *admits minimal realization*;

(b) $\mathbf{X}$ *admits reflexive Nerode realization*;

(c) $\mathbf{X}$ *admits restricted universal realization.*

Proof. We only have to prove (a) $\Rightarrow$ (c). Let $\mathbf{M}$ be the full subcategory of $\mathbf{Mach}^\star(\mathbf{X})$ with objects all reachable machines. Then $\mathbf{EM} = \mathbf{Beh}^\star(\mathbf{X})$ since the free realization of any behavior is reachable. Since (a) $\Rightarrow$ (b), the minimal realization is functorial. The result now follows from Corollary 2.8 (with the interpretation as in this section). ▮

We note that if, like Arbib and Manes, we consider only the subcategory of $\mathbf{Mach}(\mathbf{X})$ with fixed input and output objects I and Y, and morphisms all machine morphisms of the form $(1_I, x, 1_Y)$, then the restriction on the first component of morphisms is trivially satisfied, and restricted universal realization is the same as universal realization.

An important consequence of Theorem 3.2 is that it can be used to obtain an adjointness for tree automata. For the sake of completeness we give some definitions.

A *label set* Ω is a set Ω together with a map ν which assigns to each $\omega \in \Omega$ a cardinal $\nu(\omega)$. When $\nu(\omega)$ is a natural number for all $\omega \in \Omega$, we call Ω *finitary*. The set Ω_n is the set of all $\omega \in \Omega$ with $\nu(\omega) = n$.

Given a label set we obtain a functor $\mathbf{X}_\Omega$: $\mathbf{Set} \to \mathbf{Set}$ with object mapping

$$\mathbf{X}_\Omega Q = \coprod_n \Big(\coprod_{\omega \in \Omega_n} Q^n \Big)$$

while the action on morphisms is given by

$$\begin{array}{ccc} Q & \qquad & \begin{tikzcd} \mathbf{X}_\Omega Q \arrow[d, dashed, "f\mathbf{X}_\Omega"] & Q^n \arrow[l, "in_{n,\omega}"'] \arrow[d, "f^n"] \\ \mathbf{X}_\Omega Q' & (Q')^n \arrow[l, "in_{n,\omega}"] \end{tikzcd} \\ {\scriptstyle f}\downarrow & & \\ Q' & & \end{array}$$

where $f^n(q_1, \ldots, q_n) = (f(q_1), \ldots, f(q_n))$.

A tree automaton is now simply an $\mathbf{X}_\Omega$-machine in set. The proof that $\mathbf{X}_\Omega$ is indeed an input process is given in [2]. Theorem 3.12 in [2] now states that, in our notation, for a finitary label set Ω, $\mathbf{X}_\Omega$ admits reflexive Nerode realization. By restricting our category of tree automata to those which have morphisms with surjective first component, we obtain the following from Theorem 3.2.

THEOREM 3.3. *For a finitary label set* Ω, $\mathbf{X}_\Omega$ *admits restricted universal realization.*

This result can now be used, for instance, to obtain an adjointness between derivation checkers and context-free languages [5, p. 254].

We notice that if we let $\Omega_1 = \mathbf{X}_0$ (i.e., if each $x \in X_0$ is the label of a unary operator), $\Omega_n = \varnothing$ for $n \neq 1$, an $\mathbf{X}_\Omega$-automaton is simply a sequential machine. In this case the above result is exactly the same as Goguen's early result for sequential machines [8], since he also required morphisms with surjective first components.

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