Regularity of solutions of convolution equations

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Received 7 February 2007
Available online 2 June 2007
Submitted by Richard M. Aron

Abstract

This paper investigates the regularity of solutions of convolution equations in the frame of classes of ultradifferentiable functions and ultradistributions. We improve previous work by Bonet, Chou, Fernández, Galbis, Meise and others.

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Keywords: Ultradistributions; Convolution operators; Hypoellipticity; Regularity

1. Introduction

The purpose of this article is to consider the following problem: Let $\mu$ be an ultradistribution with compact support in $\mathbb{R}^N$, consider the convolution equation $\mu \ast v = f$ and assume that $f$ belongs to a certain class of ultradifferentiable functions which need not be related to the class defining the ultradistribution $\mu$. When does $v$ belong to the same class of ultradifferentiable functions of $f$? We work in the context of ultradifferentiable functions as defined by [5] and the precise definitions are given in Section 2. For a weight function $\omega$ in the sense of Braun, Meise and Taylor [5], we denote by $\mathcal{D}'_\omega$ and $\mathcal{D}'_{\{\omega\}}$ the spaces of $\omega$-ultradistributions of Beurling and Roumieu type, respectively. We will write $\mathcal{D}'_\omega$ when we refer to both classes of ultradistributions. The notation for spaces of ultradifferentiable functions $\mathcal{E}_\omega$, $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_\omega$ is analogous. We do not use the standard notation $\mathcal{E}_{\ast}$ and $\mathcal{D}'_{\ast}$ to denote both classes Beurling and Roumieu because throughout this work the results involve two different weights. For every ultradistribution with compact support $\mu \in \mathcal{E}'_\omega$, we consider the convolution operator $S_\mu : \mathcal{D}'_\omega \to \mathcal{D}'_\omega$, defined by $S_\mu(v) = \mu \ast v$. We say that $S_\mu$ is $\sigma$-hypoelliptic, $\sigma$ being a weight which could be different from $\omega$, whenever $(S_\mu)^{-1}(E_\sigma) \subset E_\sigma$. Case $\omega = \sigma$, according to [1], $\mu$ is said to be an $\omega$-hypoelliptic ultradistribution for the Beurling or Roumieu classes. If $\sigma(x) = \log(1 + |x|^2)$ we identify $\mathcal{E}(\sigma)$ with the space of all the smooth functions $\mathcal{E}$, and we write $S_\mu$ for the convolution operator when it is defined on the classical space $D'$ of distributions. We say that $S_\mu$ is hypoelliptic whenever

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\textsuperscript{1} The research of the authors is supported by FEDER and MCYT, Project No. MTM2004-02262, and the net MTM2006-26627-E.

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doi:10.1016/j.jmaa.2007.05.053
(\(S_{\omega}^0\))\(^{-1}\)(\(\mathcal{E}\)) \(\subset\) \(\mathcal{E}\). Hypoellipticity and \((\omega)\)-hypoellipticity convolution operators were characterized by Chou [7] for spaces of ultradistributions in the Komatsu sense. The hypoellipticity of a convolution operator in the classical space of distributions of Schwartz is covered by Hörmander [11, 16.6]. Bonet, Fernández and Meise characterized in [1] the \(\omega\)-hypoelliptic ultradistributions \(\mu\) in terms of the behaviour of its Fourier transform \(\hat{\mu}\) and its zero set \(V(\hat{\mu})\). They did it in the more general framework of the ultradistributions in the sense of Braun, Meise and Taylor [5]. Our aim is to extend the results of [1,7] to results about regularity with respect to a class of ultradifferentiable functions in a space of ultradistributions associated to a different weight. In particular, we deal with the hypoellipticity in spaces of ultradistributions in the sense of Braun, Meise and Taylor, extending properly the results in [7] as Bonet, Fernández and Meise made with the \(\omega\)-hypoellipticity.

2. Preliminaries

In this preliminary section we introduce weight functions, the space of ultradifferentiable functions, the space of ultradistributions and we fix the notation which will be used in the sequel. The following definitions go back to Braun, Meise and Taylor [5].

**Definition 1.** A weight function is a continuous increasing function \(\omega : [0, \infty[ \to [0, \infty[\) satisfying:

1. \((\alpha)\) there exists \(K \geq 1\) with \(\omega(2t) \leq K(\omega(t) + 1)\) for all \(t \geq 0\),
2. \((\beta)\) \(\int_0^\infty \frac{\omega(t)}{1+t^2} \, dt < \infty\),
3. \((\gamma)\) \(\log(1+t^2) = o(\omega(t))\) as \(t\) tends to \(\infty\),
4. \((\delta)\) \(\varphi : t \to \omega(e^t)\) is convex.

For a weight function \(\omega\) we define \(\tilde{\omega} : \mathbb{C}^N \to [0, \infty[\) by \(\tilde{\omega}(z) = \omega(|z|)\) and again call this function \(\omega\), by abuse of notation. Here \(|z| = \sum_{j=1}^N |z_j|\). The function \(\varphi^* : [0, \infty[ \to \mathbb{R}\), \(\varphi^*(s) := \sup\{s t - \varphi(t) : t \geq 0\}\) is called the Young conjugate of \(\varphi\). There is no loss of generality to assume that \(\omega\) vanishes on \([0, 1]\). Then \(\varphi^*\) has only non-negative values and \(\varphi^{**} = \varphi\). Each weight function satisfies \(\lim_{t \to \infty} \frac{\omega(t)}{t} = 0\) by Remark 1.3 of [14].

**Definition 2.** Let \(\omega\) be a weight function. For a compact set \(K \subset \mathbb{R}^N\) and \(\lambda > 0\) let

\[\mathcal{E}_\omega(K, \lambda) := \{ f \in C^\infty(K) : \|f\|_{K, \lambda} < \infty\}\]

where

\[\|f\|_{K, \lambda} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}^N_0} |f^{(\alpha)}(x)| \exp\left(-\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right)\].

For an open set \(U \subset \mathbb{R}^N\) define

\[\mathcal{E}_{(\omega)}(U) := \{ f \in C^\infty(K) : \|f\|_{K,m} < \infty \text{ for each } K \subset U \text{ and each } m \in \mathbb{N}\}\]

and

\[\mathcal{E}_{[\omega]}(U) := \{ f \in C^\infty(K) : \text{ for each } K \subset U \text{ there is } m \in \mathbb{N} \text{ with } \|f\|_{K, \frac{1}{m}} < \infty\}\].

The elements of \(\mathcal{E}_{(\omega)}(U)\) (respectively \(\mathcal{E}_{[\omega]}(U)\)) are called \(\omega\)-ultradifferentiable functions of Beurling (respectively Roumieu) type. We denote \(\mathcal{E}_{(\omega)} := \mathcal{E}_{(\omega)}(\mathbb{R}^N)\) and \(\mathcal{E}_{[\omega]} = \mathcal{E}_{[\omega]}(\mathbb{R}^N)\). We write \(\mathcal{E}_\omega\) where \(\omega\) can be either \((\omega)\) or \([\omega]\).

For a compact set \(K\) in \(\mathbb{R}^N\) we put

\[\mathcal{D}_\omega(K) := \{ f \in \mathcal{E}_\omega : \text{ supp}(f) \subset K\}\]
endowed with the induced topology. For a fundamental sequence \((K_n)_{n \in \mathbb{N}}\) of compact subsets of \(\mathbb{R}^N\) we let
\[
\mathcal{D}_\omega := \text{ind}_{n \to \infty} \mathcal{D}_\omega(K_n).
\]
The elements of \(\mathcal{D}'(\omega)\) (respectively \(\mathcal{D}'(\omega_0)\)) are called \(\omega\)-ultradistributions of Beurling (respectively Roumieu) type.

**Example 3.** The following functions \(\omega : [0, \infty[ \to [0, \infty[\) are examples of weight functions:

1. \(\omega(t) = t^\alpha, \ 0 < \alpha < 1.\)
2. \(\omega(t) = (\log(1+t))^\beta, \ \beta > 1.\)
3. Let \((M_p)_{p \in \mathbb{N}_0}\) be a sequence of positive numbers which has the following properties:
   - \(M_j^2 \leq M_{j-1}M_{j+1}\) for all \(j \in \mathbb{N};\)
   - there exists \(A, H > 1\) with \(M_n \leq AH^n\) \(\min_{0 \leq j \leq n} M_{M_n-j}\) for all \(n \in \mathbb{N};\)
   - \(A > 0\) with \(\sum_{j=1}^{\infty} M_j^{-1}/M_j = A\) \(M_j/M_{j+1}\);

and define \(\omega_M : \mathbb{R} \to [0, \infty[\) by
\[
\omega_M(t) = \begin{cases} \sup_{j \in \mathbb{N}_0} \log \left( \frac{|t|/M_j}{M_j} \right) & \text{for } |t| > 0, \\ 0 & \text{for } t = 0. \end{cases}
\]

Then \(\omega_M\) is a continuous even function and by Meise and Taylor [13, 3.11] there exists a concave weight function \(\kappa\) with \(\omega_M(t) \leq \kappa(t) \leq C\omega_M(t) + C\) for some \(C > 0\) and all \(t > 0\). We have
\[
\mathcal{E}_{(M_j)} := \left\{ f \in C^\infty : \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} \left| D^{\alpha}(x) \right| < \infty \text{ for each } h > 0 \text{ and each } K \subset \mathbb{R}^N \text{ compact} \right\} = \mathcal{E}_{(\kappa)}.
\]

We have an analogous identity for the Roumieu spaces.

**Remark 4.**

(a) For \(\omega(t) = t^\alpha\), the classes \(\mathcal{E}_{(\omega)}\) (respectively \(\mathcal{E}_{(\omega_0)}\)) coincide with the Gevrey classes \(\Gamma^{(d)}\) (respectively \(\Gamma^{(d)}\)) for \(d := \frac{1}{\alpha}\).

(b) The following (continuous) inclusions hold \(\mathcal{E}_{(\omega)} \subset \mathcal{E}_{(\omega_0)}, \mathcal{D}_{(\omega)} \subset \mathcal{D}_{(\omega_0)}, \mathcal{E}_{(\omega)}' \subset \mathcal{E}_{(\omega_0)}', \mathcal{D}_{(\omega)}' \subset \mathcal{D}_{(\omega_0)}'.\) If \(\omega = o(\sigma)\), then \(\mathcal{E}_{(\sigma)} \subset \mathcal{E}_{(\omega)}\) [5, 4.7], \(\mathcal{D}_{(\sigma)} \subset \mathcal{D}_{(\omega)}, \mathcal{E}_{(\omega)}' \subset \mathcal{E}_{(\sigma)}', \mathcal{D}_{(\omega)}' \subset \mathcal{D}_{(\sigma)}'.\)

(c) Moreover, if \(\sigma\) and \(\omega\) are two arbitrary weights, one has \(\mathcal{A} \hookrightarrow \mathcal{E}_{(\sigma)} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{D}' \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{E}_{(\omega)}\) where we denote by \(\mathcal{A}\) the set of real analytic functions.

**Definition 5.** For \(\mu \in \mathcal{D}'(\omega)\) the \(\omega\)-singular support, denoted by \(\text{sing supp}_\omega(\mu)\), is the set of points in \(\mathbb{R}^N\) having no open neighbourhood \(U\) to which the restriction \(\mu|_U\) is in \(\mathcal{E}_{\omega}(U)\).

**Definition 6.** Let \(\mu \in \mathcal{E}'_{(\omega)}, \mu \neq 0.\) We define the convolution operators
\[
T^{\omega}_\mu : \mathcal{E}_{\omega} \to \mathcal{E}_{\omega}, \ \text{by } T^{\omega}_\mu(\varphi)(x) := (\mu * \varphi)(x) = \{\mu_y, \varphi(x-y)\},
\]
and
\[
S^{\omega}_\mu : \mathcal{D}'_{(\omega)} \to \mathcal{D}'_{(\omega)}, \ \text{by } S^{\omega}_\mu(\varphi) := (\mu, T^{\omega}_\mu(\varphi))
\]
where \(\tilde{\mu}(f) = \mu(f)\) and \(\tilde{f}(x) = f(-x).\)

By [5, Remark 6.2], \(S^{\omega}_\mu(f) = T^{\omega}_\mu(f)\) for each function \(f \in \mathcal{E}_{\omega}\).

**Definition 7.** Let \(\mu \in \mathcal{E}'_{(\omega)}\). Since \(\mathcal{E}_{\omega} \hookrightarrow \mathcal{D}' \hookrightarrow \mathcal{D}'_{(\omega)}\) for \(\omega, \sigma\) weight functions we say that \(S^{\omega}_\mu\) is \(\sigma\)-hypoelliptic whenever \((S^{\omega}_\mu)^{-1}(\mathcal{E}_{(\sigma)}) \subset \mathcal{E}_{(\sigma)}\).

According to [1], \(\mu\) is said to be \((\omega)\)-hypoelliptic (respectively \({\omega}\)-hypoelliptic) if \(S^{(\omega)}_\mu\) (respectively \(S^{(\omega)}_{\mu}\)) is \((\omega)\)-hypoelliptic (respectively \({\omega}\)-hypoelliptic).
The function \( \sigma(t) = \log(1 + t^2) \) is not a weight function. In this case we identify \( \mathcal{E}_\sigma \) with \( \mathcal{E} \), and \( S_\mu^\omega \) is called hypoelliptic if \( (S_\mu^\omega)^{-1}(\mathcal{E}) \subseteq \mathcal{E} \).

**Definition 8.** Let \( \mu \in \mathcal{E}_\omega' \). We define the Fourier–Laplace transform \( \hat{\mu} \) as

\[
\hat{\mu} : z \mapsto (\mu, v_z), \quad z \in \mathbb{C}^N,
\]

where \( v_z(x) = \exp(-i\langle x, z \rangle) \), \( x \in \mathbb{R}^N \). We denote the zero set of the Fourier transform \( \hat{\mu} \) as

\[
V(\hat{\mu}) := \{ z \in \mathbb{C}^N : \hat{\mu}(z) = 0 \}.
\]

**Definition 9.** Let \( \omega \) be a weight function.

(a) An ultradistribution \( \mu \in \mathcal{E}_\omega' \) is called slowly decreasing for \( \omega \) if there exists \( A \geq 1 \) such that for each \( x \in \mathbb{R}^N \)

there exits \( t \in \mathbb{R}^N \) with \( |x - t| \leq A\omega(|x|) \) and \( |\hat{\mu}(t)| \geq \exp(-A\omega(|t|)) \).

(b) An ultradistribution \( \mu \in \mathcal{E}_\omega' \) is called slowly decreasing for \( \omega \) if for each \( n \in \mathbb{N} \) there exists \( x_n > 0 \) such that

for each \( x \in \mathbb{R}^N \) with \( |x| \geq x_n \) there exists \( t \in \mathbb{R}^N \) with \( |x - t| \leq n^{-1}\omega(|x|) \) and \( |\hat{\mu}(t)| \geq \exp(-n^4\omega(|t|)) \).

Accordingly to our notation, we write that \( \mu \) is slowly decreasing for \( \omega \) when \( \omega \) can be either \( \omega \) or \( |\omega| \).

We recall from [2, 2.9, 3.4] and [6, 2.4, 2.7] that \( \mu \in \mathcal{E}_\omega' \) is slowly decreasing for \( \omega \) if, and only if, the convolution operator \( S_\mu^\omega \) is surjective.

We finish this section with the definition of ultradifferential operator, which goes back to Braun [4].

**Definition 10.** Let \( G \in \mathcal{H}(\mathbb{C}^N) \) be an entire function such that \( \log G(z) = O(\omega(|z|)) \) (respectively \( \log G(z) = o(\omega(|z|)) \)) as \( |z| \) tends to infinity. Then

\[
TG(\varphi) := \sum_{\alpha \in \mathbb{N}_0^N} (-i)^{|\alpha|} \frac{D^\alpha G(0)}{\alpha!} \varphi^{(\alpha)}(0)
\]

defines an ultradistribution \( TG \in \mathcal{E}_\omega' \) (respectively \( TG(\varphi) \in \mathcal{E}_\omega' \)). The operator \( G(D) : \mathcal{E}_\omega \to \mathcal{E}_\omega \), defined by \( G(D) f := TG \ast f \) is called an ultradifferential operator of class \( \omega \).

3. Nearly hypoelliptic convolution operators

Let \( \sigma \) and \( \omega \) be weight functions with \( \sigma \leq \omega \) and let \( \mu \in \mathcal{E}_\omega' \). We consider the diagram

\[
\begin{array}{ccc}
S_\mu^\omega : & \mathcal{D}_\omega' & \to \mathcal{D}_\omega' \\
\cup & \cup & \\
\mathcal{E} & \mathcal{E} & \\
\cup & \cup & \\
\mathcal{E}_\sigma & \mathcal{E}_\sigma & \\
\cup & \cup & \\
T_\mu^\omega : & \mathcal{E}_\omega & \to \mathcal{E}_\omega.
\end{array}
\]

In this section we will characterize when \( (S_\mu^\omega)^{-1}(\mathcal{E}_\sigma) \) is included in \( \mathcal{E}_\sigma \). We also study the limit case when \( (S_\mu^\omega)^{-1}(\mathcal{E}) \) is included in the space \( \mathcal{E} \). The proofs of the results of this section are influenced by [1].

The next result is a generalization of [11, 16.6.1], where the statement is obtained for the classical space of distributions. The proof follows from a careful inspection of the given one in [1, 2.3], in whose statement the stronger condition \( \operatorname{Ker} S_\mu^\omega \subseteq \mathcal{E}_\omega \) is required, which however is not necessary in the arguments.

**Proposition 11.** If \( \mu \in \mathcal{E}_\omega' \) satisfies \( \operatorname{Ker} S_\mu^\omega \subseteq \mathcal{E} \), then

\[
\lim_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\Im z|}{\omega(z)} = \infty.
\]
In order to prove the analogous result in the Roumieu case we present the following lemma.

**Lemma 12.** Let \( \mu \in \mathcal{E}'_{(\omega)} \). The following assertions are equivalent:

(i) \( \liminf_{z \in \text{V}(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \ z|}{\omega(z)} > 0 \).

(ii) For all weight function \( \sigma \) such that \( \sigma = o(\omega) \),

\[
\lim_{z \in \text{V}(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \ z|}{\sigma(z)} = \infty.
\]

**Proof.** Clearly (i) implies (ii). To prove that (ii) implies (i) we argue by contradiction and we assume that (ii) holds and

\[
\liminf_{z \in \text{V}(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \ z|}{\omega(z)} = 0.
\]

We get a sequence \((z_n)_n \subset \mathbb{C}\) such that \((|z_n|)_n\) is an increasing sequence which tends to infinity, \( \hat{\mu}(z_n) = 0 \) and \(|\text{Im} \ z_n|/\omega(z_n) < 1/n\). Let \( g(x) := (1/n) \omega(x) \) if \( x \in [|z_n|, |z_{n+1}|]\), for each \( n \in \mathbb{N} \). By [5, 1.7], there exists a weight \( \sigma \) such that \( g = o(\sigma) \) and \( \sigma = o(\omega) \). There exists \( n_0 \) such that \( \sigma(|z_n|) \geq g(|z_n|) \) for each \( n \geq n_0 \). Now, for each \( n \in \mathbb{N} \) with \( n \geq n_0 \), we have

\[
\frac{|\text{Im} \ z_n|}{\sigma(z_n)} \leq \frac{|\text{Im} \ z_n|}{g(|z_n|)} = n \frac{|\text{Im} \ z_n|}{\omega(z_n)} \leq 1,
\]

and this is a contradiction with (ii). \( \square \)

**Proposition 13.** Let \( \mu \in \mathcal{E}'_{(\omega)} \). If \( \text{Ker} S_{\mu}^{(\omega)} \subset \mathcal{E} \), then

\[
\liminf_{z \in \text{V}(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \ z|}{\omega(z)} > 0.
\]

**Proof.** Since \( \mu \in \mathcal{E}'_{(\omega)} \) there exists a weight \( \omega_0 = o(\omega) \) such that \( \mu \in \mathcal{E}'_{(\omega_0)} \) (cf. [5, 7.6]). For each weight \( r \geq \omega_0 \), \( r = o(\omega) \) we have that \( \text{Ker} S_{\mu}^{(r)} \subset \text{Ker} S_{\mu}^{(\omega)} \subset \mathcal{E} \) and from Proposition 11 it follows that

\[
\lim_{z \in \text{V}(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \ z|}{r(z)} = \infty.
\]

Then

\[
\lim_{z \in \text{V}(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \ z|}{\sigma(z)} = \infty,
\]

for all weight function \( \sigma \) with \( \sigma = o(\omega) \). Apply Lemma 12 to conclude. \( \square \)

**Proposition 14.** Let \( \sigma \) be a weight function or \( \sigma(t) = \log(1 + t^2) \) and \( \mu \in \mathcal{E}'_{(\omega)} \).

1. If \( (S_{\mu}^{(\omega)})^{-1}(\mathcal{E}(\sigma)) \subset \mathcal{E} \), then there exist \( A, R > 0 \) such that

\[
|\hat{\mu}(x)| \geq \exp(-A\sigma(x))
\]

for each \( x \in \mathbb{R}^N, |x| \geq R \).

2. If \( (S_{\mu}^{(\omega)})^{-1}(\mathcal{E}(\sigma)) \subset \mathcal{E} \), then for each \( m \in \mathbb{N} \) there exists \( R_m \) such that

\[
|\hat{\mu}(x)| \geq \exp(-\frac{\sigma(x)}{m})
\]

for each \( x \in \mathbb{R}^N, |x| \geq R_m \).
Proof. (1) Arguing by contradiction, we construct a sequence \((x_j)_j \subset \mathbb{R}^N\) such that \(|x_j| > 2|x_{j-1}| > 2^j\) for every \(j \geq 2\) and \(|\hat{\mu}(x_j)| < \exp(-j\sigma(x_j))\) for each \(j \in \mathbb{N}\). Define \(v := \sum_j \exp(i \cdot x_j)\). The condition \(|x_j| > 2|x_{j-1}| > 2^j\) allows us to apply Sampson and Zielezny Lemma (cf. [15, p. 141], [1, 2.4]) to obtain that \(v\) belongs to \(D'(\mathbb{R}^N)\) and hence to \(D'_\omega\), but is not a \(C^\infty\) function on \(\mathbb{R}^N\). The same arguments used in the proof of [1, 2.5], show that the series

\[
\mu \ast v = \sum_{j=1}^{\infty} \hat{\mu}(x_j) \exp(i \cdot x_j)
\]

is convergent in \(D'_\omega\). Hence, if we show that \(\sum_j \hat{\mu}(x_j) \exp(i \cdot x_j)\) lies in \(E(\sigma)\), we get a contradiction. Notice that \(E(\sigma) \hookrightarrow \mathcal{E} \hookrightarrow D' \hookrightarrow D'_\omega\) because \(D_{\omega} \hookrightarrow D\) with dense range.

Case 1: If \(\sigma\) is a weight function, we claim that the series is Cauchy, and then convergent, in \(\mathcal{E}(\sigma)\), and therefore in \(\mathcal{E}\). To see that, let \(\varphi_\sigma(t) := \exp(t^2)\), and let \(\varphi_\sigma^*\) denote the young conjugate of \(\varphi_\sigma\). Let \(\alpha \in \mathbb{N}_0^N, m \in \mathbb{N}\), and \(j > m\) such that \(\sigma(x_j) > 1\)

\[
|\mu(x_j) \exp(i \cdot x_j)|^{(\alpha)} \exp(-m\varphi_\sigma^{**}(|\alpha|/m)) \leq |\mu(x_j)||x_j|^{|\alpha|} \exp(-m\varphi_\sigma^{**}(|\alpha|/m))
\]

\[
\leq \exp(-j\sigma(x_j) + |\alpha| \log |x_j| - m\varphi_\sigma^{**}(|\alpha|/m))
\]

\[
\leq \exp(-j\sigma(x_j) + m\varphi_\sigma^{**}(\log |x_j|))
\]

\[
= \exp(-(j + m)\sigma(x_j))
\]

\[
\leq \exp(-j + m).
\]

This shows that the series is convergent in \(\mathcal{E}(\sigma)\).

Case 2: If \(\sigma(t) = \log(1 + t^2)\), the sequence \((x_j)_j\) satisfies

\[
\hat{\mu}(x_j) = \frac{1}{(1 + |x_j|^2)^j}
\]

for each \(j \in \mathbb{N}\). And then for \(\alpha \in \mathbb{N}_0^N\),

\[
|\mu(x_j) \exp(i \cdot x_j)|^{(\alpha)} \leq \frac{|x_j|^{|\alpha|}}{(1 + |x_j|^2)^j}.
\]

For \(j\) bigger than \(|\alpha|\),

\[
|\mu(x_j) \exp(i \cdot x_j)|^{(\alpha)} \leq \frac{1}{|x_j|^2 - |\alpha|} \leq \frac{1}{2^{2j - |\alpha|}}.
\]

Hence the series (1) is convergent in \(\mathcal{E}(\sigma)\).

(2) Proceeding by contradiction, we find \(m \in \mathbb{N}\) and a sequence \((x_j)_j \in \mathbb{N}\) in \(\mathbb{R}^N\) such that \(|x_j| > 2|x_{j-1}| > 2^j\) such that \(|\hat{\mu}(x_j)| \geq \exp(-\sigma(x_j)/m)\). We again define \(v := \sum_j \exp(i \cdot x_j)\), which belongs to \(D'_\omega\) \(\mathcal{E}\). Now one can proceed similarly as in (1) to obtain that the series is convergent in \(\mathcal{E}_\sigma(K, m + 1)\) for each compact subset \(K\) of \(\mathbb{R}^N\).

Remark 15. Proposition 14 implies that for \(\mu \in \mathcal{E}'(\omega)\), and \(\omega \leq \sigma\) weight functions, if \(S^{(\omega)}_\mu\) is \((\sigma)\)-hypoelliptic then \(T^{(\sigma)}_\mu : \mathcal{E}(\sigma) \rightarrow \mathcal{E}(\sigma)\) is surjective and hence \(S^{(\sigma)}_\mu : D'(\sigma) \rightarrow D'(\sigma)\) is surjective too ([1, 2.1], [2, 2.9]).

The arguments of the proof of the following proposition are inspired in the comments after Definition 12 in [3] (cf. [2, 3.2]).

Proposition 16. Let \(\omega\) and \(\sigma\) be two weights such that \(\omega = o(\sigma)\) and let \(\mu \in \mathcal{E}'(\omega)\). If \((S^{(\omega)}_\mu)^{-1}(\mathcal{E}(\sigma)) \subset \mathcal{E}\), then there exists a weight \(r \geq \omega, r = o(\sigma)\) such that there exists \(R > 0\) for which \(|\hat{\mu}(x)| \geq \exp(-r(x))\) for each \(x \in \mathbb{R}^N, |x| \geq R\).
Proof. By [5, 7.6] there exists a weight \( \sigma_0 = o(\sigma) \) such that \( \mu \in E'_(\sigma_0) \). We can assume \( \sigma_0 \geq \omega \). Applying Proposition 14 we obtain an increasing sequence \( (R_m)_m \) tending to infinity such that \( |\hat{\mu}(x)| \geq \exp(-\sigma(x)/m) \) for each \( x \in \mathbb{R}^N, |x| \geq R_m \). We define
\[
g : [0, \infty) \to [0, \infty), \quad g(x) := \begin{cases} \frac{0}{\sigma(x)/m} & \text{for } x \in [0, R_1), \\ \frac{\sigma(x)}{m} & \text{for } x \in [R_m, R_{m+1}). \end{cases}
\]
By [5, 1.7] there exists a weight function \( r \) satisfying \( g = o(r) \), \( r = o(\sigma) \), \( \sigma_0 \leq r \). If we choose \( \ell \) such that \( g(t) \leq r(t) \) for all \( t \geq R_\ell \). Now we fix \( x \in \mathbb{R}^N \) with \( |x| \geq R_\ell \) and choose \( m \geq \ell \) such that \( |x| \in [R_m, R_{m+1}) \). Then
\[
|\hat{\mu}(x)| \geq \exp\left(-\frac{\sigma(x)}{m}\right) = \exp(-g(x)) \geq \exp(-r(x)). \quad \square
\]

Theorem 17. Let \( \mu \in E'_(\sigma)(\mathbb{R}^N) \) and \( \sigma \leq \omega \) weight functions. Then the following conditions are equivalent:

1. \((s_{(\omega)}^{(\mu)})^{-1}(E_\omega) \subset E_\sigma\)
2. \((s_{(\omega)}^{(\mu)})^{-1}(E_\sigma) \subset E\)
3. (a) \( \lim_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} z|}{\sigma(z)} = \infty \), and
   (b) there exist \( A, R > 0 \) such that \( |\hat{\mu}(x)| \geq \exp(-A\sigma(x)) \) for \( x \in \mathbb{R}^N, |x| \geq R \),
4. there exists \( F \in D'_(\sigma) \) with compact \( (\omega) \)-singular support and an analytic function \( g \) such that \( \mu * F = \delta + g \),
5. there exists \( E \in E'_(\omega) \) and \( \varphi \in D'(\omega) \) such that \( \mu * E = \delta + \varphi \).

Proof. It is clear that (1) implies (2), and (2) implies (3) is a consequence of Propositions 11 and 14.

(3) \( \Rightarrow (4) \) The hypothesis \( \sigma \leq \omega \) together with (3) imply
\[
\lim_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} z|}{\sigma(z)} = \infty, \quad \lim_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} z|}{\omega(z)} = \infty
\]
holds, and
\[
|\hat{\mu}(x)| \geq \exp(-A\sigma(x)), \quad |\hat{\mu}(x)| \geq \exp(-A\omega(x))
\]
also holds for \( x \in \mathbb{R}^N, |x| \geq R \). Thus \( \mu \) is \( (\omega) \)-hypoeulliptic by [1, 2.1]. By [1, 2.9], this yields that there exists \( F \in D'_(\omega) \) with compact \( (\omega) \)-singular support satisfying \( \mu * F = \delta + g \), \( g \) analytic. The function \( F \), again by [1, 2.9], is defined as follows
\[
\langle F, \varphi \rangle := \left( \frac{1}{2\pi} \right)^N \int_{|x| \geq R} \hat{\varphi}(x) \frac{\hat{\mu}(x)}{|\hat{\mu}(x)|} \, dx, \quad \varphi \in D'(\omega).
\]
This together with (3)(b) implies that \( F \in D'_(\sigma) \).

(4) \( \Rightarrow (5) \) We select \( \psi \in D(\sigma) \) such that \( \psi \equiv 1 \) in neighbourhood of a compact set \( K \) containing the \( (\omega) \)-singular support of \( F \). Set \( E := \psi F \in E'_(\sigma) \). As in the proof of (5) \( \Rightarrow (1) \) in [1, 2.1] we get
\[
\mu * E = \delta + \varphi, \quad \varphi \in D'(\omega).
\]

(5) \( \Rightarrow (1) \) Let \( h \in D'_(\omega) \) such that \( \mu * h = f \in E(\sigma) \). Consider the convolution \( G := E * (\mu * h) \in E_\sigma \). Since \( E, \mu \in E'_(\omega) \) and \( h \in D'_(\omega) \), the properties of the convolution yield
\[
G = (\delta + \varphi) * h = h + \varphi * h.
\]
Thus \( h \in E_\sigma \) since \( \varphi * h \in E_\omega \subseteq E(\sigma) \). \( \square \)

Corollary 18. Let \( \sigma \leq \omega \) be two weight functions and \( \mu \in E'_(\omega) \). If \( s_{(\omega)}^{(\mu)} \) is \( (\sigma) \)-hypoeulliptic or \( s_{(\omega)}^{(\mu)} \) is hypoeulliptic then \( \mu \) is \( (\omega) \)-hypoeulliptic.
Example 2.13(a) in [1] shows that, whenever \( \mu \in \mathcal{E}'_{(\omega)} \) and \( \sigma = o(\omega) \), regularity with respect to \( \mathcal{E}_{(\sigma)} \) is a stronger condition than \( (\omega) \)-hypoellipticity. Even more, we can state the following result:

Let \( \sigma, r \) and \( \omega \) be weight functions with \( \sigma = o(r) \) and \( r \leq \omega \). Then there exist an ultradistribution \( \mu \in \mathcal{E}'_{(\omega)} \) such that \( S_{(\omega)}^\mu \) is \( (r) \)-hypoelliptic but not \( (\sigma) \)-hypoelliptic. To prove this, one only has to make a slight modification of the arguments of [1, 2.13(a)] taking \( G(D) \) as an ultradifferential operator of class \( (r) \) and \( f \in \mathcal{E}'_r \).

In [7, 4.2.2], Chou showed that in the context of ultradistributions of the form \( D'_r \), \( (M_r)_r \) being a sequence which satisfies the Komatsu conditions [12], an \( M_r \)-hypoelliptic ultradistribution \( \mu \) is hypoelliptic if and only if for each distribution \( F \in D' \) there exists \( G \in D' \) such that \( \mu * G = F \). We see below that this result can be extended to our framework.

**Proposition 19.** Let \( \sigma \leq \omega \) weight functions (or \( \sigma(x) = \log(1 + x^2) \)) and let \( \mu \in \mathcal{E}'_{(\omega)} \) be \( (\omega) \)-hypoelliptic. The following are equivalent:

(i) \( (S_{(\omega)}^\mu)^{-1}(\mathcal{E}_{(\sigma)}) \subseteq \mathcal{E}_{(\sigma)} \),

(ii) \( D'_r \subseteq \mathcal{E}'_{(\sigma)} \).

**Proof.** Suppose that (i) is true and fix \( F \in D' \). By Theorem 17 there exists \( G \in \mathcal{E}'_{(\sigma)} \) and \( \varphi \in D_{(\omega)} \) such that \( \mu * G = \delta + \varphi \). We observe that \( G * F \in D'_{(\sigma)} \) and \( \varphi * F \in D_{(\sigma)} \). Since \( S_{(\omega)}^\mu \) is \( (\omega) \)-hypoelliptic, \( \mu \) is \( (\omega) \)-slowly decreasing [1, 2.1] and then \( S_{(\omega)}^\mu \) is surjective [2, 2.5]. We take \( H \in \mathcal{E}_{(\sigma)} \) such that \( \mu * H = -\varphi * F \). Therefore \( \mu * ((G * F - H) = F \) and \( G * F - H \in D'_{(\sigma)} \).

Now we assume that (ii) holds. From [1, 2.9] there exists \( E \in D'_{(\omega)} \) and \( g \) real analytic such that \( \mu * E = \delta + g \). By the assumption, there exists \( F \in D'_{(\sigma)} \) such that \( \mu * F = \delta + g \). Therefore, \( E - F \in \text{Ker} \, S_{(\omega)}^\mu \subseteq \mathcal{E}_{(\omega)} \). This implies that the \( (\omega) \) singular support of \( E \) coincides with the \( (\omega) \) singular support of \( F \) (cf. [1, 2.11]). The conclusion follows from the implication \( (4) \Rightarrow (1) \) in Theorem 17. \( \square \)

**Proposition 20.** Let \( \omega \) and \( \sigma \) be two weights such that \( \sigma = o(\omega) \) and let \( \mu \in \mathcal{E}'_{(\sigma)} \). The following assertions are equivalent:

1. \( (S_{\omega}^\mu)^{-1}(\mathcal{E}_{(\sigma)}) \subseteq \mathcal{E}_{(\sigma)} \),
2. (a) \( \liminf_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \, \hat{\mu} (\omega(z))|}{|\omega(z)|} > 0 \), and
   (b) there exist \( A, R > 0 \) such that \( |\hat{\mu}(x)| \geq \exp(-A \sigma(x)) \) for \( x \in \mathbb{R}^N, |x| \geq R \).

**Proof.** We consider a weight function \( r \) with \( \sigma = o(r) \), \( r = o(\omega) \) such that \( \mu \in \mathcal{E}'_{(r)} \subseteq \mathcal{E}'_{(\omega)} \). Since \( D'_{(\sigma)} = \bigcup_{r \geq r, \tau = o(\omega)} D'_{(\tau)} \) [5, 7.6], condition (1) is equivalent to

\[
(S_{\omega}^\mu)^{-1}(\mathcal{E}_{(\sigma)}) \subseteq \mathcal{E}_{(\sigma)},
\]

for all \( r \geq r, \tau = o(\omega) \). From Theorem 17, this is equivalent to

1. \( \lim_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \, \hat{\mu} (\omega(z))|}{|\omega(z)|} = \infty, \forall r \geq r, \tau = o(\omega), \) and
2. there exist \( A, R > 0 \) such that \( |\hat{\mu}(x)| \geq \exp(-A \sigma(x)) \) for \( x \in \mathbb{R}^N, |x| \geq R \).

The argument of the proof of Lemma 12 permits to conclude that assertion (i) is equivalent to

\[
\liminf_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \, \hat{\mu} (\omega(z))|}{|\omega(z)|} > 0. \sqcup \sqcap
\]

**Proposition 21.** Let \( \omega \) and \( \sigma \) be two weights such that \( \sigma \leq \omega \) and let \( \mu \in \mathcal{E}'_{(\sigma)} \). The following assertions are equivalent:

1. \( (S_{(\omega)}^\mu)^{-1}(\mathcal{E}_{(\sigma)}) \subseteq \mathcal{E}_{(\sigma)} \),
2. (a) \( \lim_{z \in V(\hat{\mu}), |z| \to \infty} \frac{|\text{Im} \, \hat{\mu} (\omega(z))|}{|\omega(z)|} = \infty, \) and
(b) for each $m \in \mathbb{N}$ there exists $R_m > 0$ such that

$$|\hat{\mu}(x)| \geq \exp\left(-\frac{\sigma(x)}{m}\right)$$

for each $x \in \mathbb{R}^N$, $|x| \geq R_m$.

**Proof.** (1) implies (2) by Propositions 11 and 14. Now, we suppose (2). By condition (b), arguing as in Proposition 16, we obtain a weight function $\sigma_0 = o(\sigma)$ such that there exists $R > 0$

$$|\hat{\mu}(x)| \geq \exp(-\sigma_0(x)) \quad \text{for } x \in \mathbb{R}^N, \ |x| \geq R.$$  

From Theorem 17 we get

$$\left(S_{\mu}^{(\omega)}(\mathcal{E}_{(\sigma)})\right)^{-1}(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r)}, \quad \forall r \geq \sigma_0, \ r = o(\sigma).$$

Now we apply $\mathcal{E}_{(\sigma)} = \bigcap_{r \geq \sigma_0, \ r = o(\sigma)} \mathcal{E}_{(r)}$ [1, 3.5] to get the result. \(\square\)

As a particular case of the above theorem we have a characterization of the ultradistributions $\mu \in \mathcal{E}_1'$ such that $S_{\mu}^{(\omega)}$ is $(\omega)$-hypoelliptic. However, the arguments in Proposition 20 do not permit to obtain such a characterization for the ultradistributions $\mu \in \mathcal{E}_1''$ such that $S_{\mu}^{(\omega)}$ is $(\omega)$-hypoelliptic.

For a weight function $m$ and $\mu \in \mathcal{E}_1'' \subseteq \mathcal{E}_1''$, the condition $(S_{\mu}^{(m)})^{-1}(\mathcal{E}_{(m)}) \subset \mathcal{E}_{(m)}$ implies that $\mu$ is $\{m\}$-hypoelliptic. However, the converse is not true.

Indeed, keeping the notation of Proposition 2.13(b) in [1], the ultradistribution $\mu$ constructed in that proof satisfies $\mu \in \mathcal{E}_1'([\mathbb{R}^N])$, for a weight $m$ with $\omega = o(m)$, and $\mu$ is $(\omega)$-hypoelliptic, hence $\{m\}$-slowly decreasing, see [2, 3.2].

On the other hand, by the construction given there $\lim_{z \in V(\hat{\mu}), \ |z| \to \infty} \frac{|\text{Im} \ z|}{m(o(z))} = 1$. This implies that $\mu$ is $\{m\}$-hypoelliptic (see [1, 3.1]) but $\text{Ker} S_{\mu}^{(m)}$ is not included in $\mathcal{E}$ by Proposition 11.

We remark that the above example proves that there exists $\mu \in \mathcal{E}_1''(\mathbb{R}^N)$ which is $\{m\}$-hypoelliptic but not $\{m\}$-hypoelliptic.

**Proposition 22.** Let $\omega$ and $\sigma$ be two weights such that $\sigma \leq \omega$ and let $\mu \in \mathcal{E}_1'(\omega)$. The following assertions are equivalent:

1. $(S_{\mu}^{(\omega)})^{-1}(\mathcal{E}_{(\sigma)}) \subset \mathcal{E}_{(\omega)}$.
2. (a) $\liminf_{z \in V(\hat{\mu}), \ |z| \to \infty} \frac{|\text{Im} \ z|}{o(\omega(z))} > 0$, and
   (b) for each $m \in \mathbb{N}$ there exists $R_m > 0$ such that
      $$|\hat{\mu}(x)| \geq \exp\left(-\frac{\sigma(x)}{m}\right)$$
      for each $x \in \mathbb{R}^N$, $|x| \geq R_m$.

**Proof.** (1) implies (2) by Propositions 13 and 14. Under the assumptions of (2)(b), the proof of Proposition 16 implies the existence of a weight $r_0 = o(\sigma)$ such that there exists $R > 0$ for which

$$|\hat{\mu}(x)| \geq \exp(-r_0(x))$$

for each $x \in \mathbb{R}^N$, $|x| \geq R$. Now, from Proposition 20 we deduce that

$$(S_{\mu}^{(\omega)})^{-1}(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r)}, \quad \forall r = o(\sigma), \ r \geq r_0.$$  

Therefore we conclude (1) from [1, 3.5]. \(\square\)

**Remark 23.** Let $\sigma = o(\omega)$ be two weights functions and let $\mu \in \mathcal{E}_1'(\omega)$.

(a) $S_{\mu}^{(\omega)}$ is $(\sigma)$-hypoelliptic if and only if for each weight $r = o(\omega)$, $\sigma = o(r)$, $S_{\mu}^{(r)}$ is $(\sigma)$-hypoelliptic also.
(b) $S^o_\mu$ is $\{\sigma\}$-hypoelliptic if and only if there exists a weight $r = o(\sigma)$ such that $S^r_\mu$ is $(r)$-hypoelliptic. It is a consequence of the proofs of Propositions 16 and 21.

Using the above remark, it is not difficult to generalize Proposition 19 in the following way:

**Proposition 24.** Let $\sigma = o(\omega)$ weight functions (or $\sigma(x) = \log(1 + x^2)$) and let $\mu \in \mathcal{E}'(\omega)$ be $\omega$-hypoelliptic. The following are equivalent:

(i) $(S^o_\mu)^{-1}(\mathcal{E}_{\sigma}) \subseteq \mathcal{E}_{\sigma}$.

(ii) $\mathcal{D}'_{\sigma} \subseteq S^o_\mu(\mathcal{D}'_{\sigma})$.

4. Nearly elliptic convolution operators

Let $\omega$ and $\sigma$ be weight functions with $\omega = o(\sigma)$ and $\mu \in \mathcal{E}'(\omega) \subseteq \mathcal{E}'(\sigma)$. We consider the following diagram

\[
\begin{array}{ccc}
S^o_\mu : & \mathcal{D}'_{\sigma} & \longrightarrow & \mathcal{D}'_{\sigma} \\
& \cup & & \cup \\
& S^o_\mu : & \mathcal{D}'_{\omega} & \longrightarrow & \mathcal{D}'_{\omega} \\
& \cup & & \cup \\
& T^o_\mu : & \mathcal{E}_{\omega} & \longrightarrow & \mathcal{E}_{\omega} \\
& \cup & & \cup \\
& T^\sigma_\mu : & \mathcal{E}_{\sigma} & \longrightarrow & \mathcal{E}_{\sigma} \\
& \cup & & \cup \\
& \mathcal{A} & & \mathcal{A}.
\end{array}
\]

In this section we study regularity of the convolution operator $S^o_\mu$ with respect to a smaller class of ultradifferentiable functions than the corresponding one to the space of ultradistributions where the operator is defined. The limit case, an $\omega$-elliptic operator is that which has regularity with respect to the space of analytic functions, i.e., $(S^o_\mu)^{-1}(\mathcal{A}) \subseteq \mathcal{A}$.

We claim that the equivalence between conditions (1) and (3) in Theorem 17 does not apply if $\sigma \geq \omega$. In fact, every $(\omega)$-hypoelliptic ultradistribution $\mu \in \mathcal{E}(\omega)(\mathbb{R}^N)$ satisfies the condition (3) in this theorem. Thus, if this condition implied that $(S^o_\mu)^{-1}(\mathcal{E}(\sigma)) \subseteq \mathcal{E}(\sigma)$ for each $\sigma \geq \omega$, we would have that $(S^o_\mu)^{-1}(\mathcal{A}) \subseteq \bigcap_{\sigma \geq \omega} \mathcal{E}(\sigma)$. On the other hand, the space of real analytic function is the intersection of all classes of non-quasianalytic functions $\mathcal{E}(\sigma)$ (Bang–Mandelbrojt’s Theorem [7, 1.2.2]). Then, we would conclude that every $(\omega)$-hypoelliptic ultradistribution $\mu \in \mathcal{E}(\omega)$ is elliptic, and it is well known to be false.

**Proposition 25.** Let $\mu \in \mathcal{D}'(\omega)$. Let $\sigma$ be a weight function such that $\omega = o(\sigma)$. Then $\mu \in \mathcal{E}_{\sigma}$, if and only if $\mu * \varphi \in \mathcal{E}_{\sigma}$ for all $\varphi \in \mathcal{D}'(\omega)$.

**Proof.** Since $\mathcal{D}'(\omega) \subset \mathcal{E}' \subset \mathcal{E}'$, properties of the convolution establish that $\mathcal{E}_{\sigma} \ast \mathcal{D}'(\omega) \subset \mathcal{E}_{\sigma}$ [5, 6.4]. Now, we assume $\mu * \varphi \in \mathcal{E}_{\sigma}$ for all $\varphi \in \mathcal{D}'(\omega)$. Let $r$ be a weight function such that $\omega = o(r)$ and $r = o(\sigma)$. From [10, Lemma 2.3] we obtain an ultradifferential operator $G(D)$ of class $(r)$ and two functions $\chi \in \mathcal{D}(r)$ and $\Gamma \in \mathcal{D}[r]$ such that $G(D) \Gamma + \chi = \delta$. Then

\[ \mu = \mu \ast \delta = \mu \ast (G(D) \Gamma + \chi) = G(D)(\mu \ast \Gamma) + \mu \ast \chi. \]

Now, since $\chi, \Gamma \in \mathcal{D}(\omega)$, we obtain that $\mu \ast \chi \in \mathcal{E}_{\sigma}$ and $\mu \ast \Gamma \in \mathcal{E}_{\sigma}$. Finally, since $G(D)$ is an ultradifferential operator of class $\{\sigma\}$, and then also of class $\sigma$, we get the conclusion.  

The inclusion $\mathcal{A} \ast \mathcal{D} \subset \mathcal{A}$ is well known. The following result (cf. [9, 3.1.11]) characterize when an ultradistribution is an analytic function in terms of the convolution with ultradifferentiable functions. The proof is considerably easier than the given one in [9].

**Corollary 26.** Let $\mu \in \mathcal{D}'(\omega)$. If $\mu \ast \varphi \in \mathcal{A}$ for any $\varphi \in \mathcal{D}(\omega)$ then $\mu \in \mathcal{A}$.  

Proof. It is a consequence of Proposition 25 and the Bang–Mandelbrojt Theorem \( \bigcap_{\omega \geq \omega} E_{(\omega)} = A \) [7, I.2.2]. □

Proposition 27. Let \( \mu \in \mathcal{E}_{(\omega)}' \). Let \( \sigma \) be a weight function such that \( \omega = o(\sigma) \). If \( \text{Ker} T_{(\omega)}^\mu \subset E_{(\sigma)} \) then \( \text{Ker} S_{(\omega)}^\mu \subset E_{(\sigma)} \).

Proof. We claim that for each \( v \in \text{Ker} S_{(\omega)}^\mu \) and for each function \( \varphi \in D_{(\omega)} \) we have that \( v * \varphi \in \text{Ker} T_{(\omega)}^\mu \). Indeed, \( T_{(\omega)}^\mu (v * \varphi) = \mu * (v * \varphi) = (\mu * v) * \varphi = S_{(\omega)}^\mu (v) * \varphi = 0 \).
Therefore \( v * \varphi \in E_{(\sigma)} \) for every \( \varphi \in D_{(\omega)} \). The result is a consequence of Proposition 25. □

Proposition 28. Let \( \omega, \sigma \) be weight functions such that \( \omega = o(\sigma) \). Let \( \mu \in \mathcal{E}_{(\omega)}' \). If \( (T_{(\omega)}^\mu)^{-1}(E_{(\sigma)}) \subset E_{(\sigma)} \), then \( (S_{(\omega)}^\mu)^{-1}(E_{(\sigma)}) \subset E_{(\sigma)} \).

Proof. Let \( v \in D_{(\omega)}' \) such that \( \mu * v = f \in E_{(\sigma)} \) and let \( \varphi \in D_{(\omega)} \). Then, since \( E_{(\sigma)} * D \subset E_{(\sigma)} \), we get
\[
\mu * (v * \varphi) = (\mu * v) * \varphi = f * \varphi \in E_{(\sigma)},
\]
i.e. \( (T_{(\omega)}^\mu)(v * \varphi) \subset E_{(\sigma)} \). Our assumption implies that \( v * \varphi \in E_{(\sigma)} \) for each \( \varphi \in D_{(\omega)} \) and Proposition 25 yields the conclusion. □

Corollary 29. Let \( \omega \) and \( \sigma \) be weight functions wit \( \omega = o(\sigma) \). Let \( \mu \in \mathcal{E}_{(\omega)}' (\mathbb{R}^N) \). If \( (T_{(\omega)}^\mu)^{-1}(E_{(\sigma)}) \subset E_{(\sigma)} \), then \( \mu \) is \( \sigma \)-slowly decreasing.

Proof. It follows from Propositions 14 and 28. □

Theorem 30. Let \( \omega, \sigma \) be weight functions such that \( \omega = o(\sigma) \) and let \( \mu \in \mathcal{E}_{(\omega)}' \). Then \( (T_{(\omega)}^\mu)^{-1}(E_{(\sigma)}) \subset E_{(\sigma)} \) if, and only if, \( (S_{(\omega)}^\mu)^{-1}(E_{(\sigma)}) \subset E_{(\sigma)} \).

Proof. The no obvious implication is a consequence of the equality \( D_{(\sigma)}' = \bigcup_{\omega \leq \tau = o(\sigma)} D_{(\tau)}' \) [5, 7.6] and Proposition 28. □

Remark 31. Let \( \omega, \sigma \) be weight functions such that \( \omega = o(\sigma) \) and \( \mu \in \mathcal{E}_{(\omega)}' \).

(a) Then \( (T_{(\omega)}^\mu)^{-1}(E_{(\sigma)}) \subset E_{(\sigma)} \) if and only if \( \mu \) is \( \{\sigma\} \)-hypoelliptic.
(b) \( T_{(\omega)}^\mu \) is a \( (\sigma) \)-hypoelliptic operator if and only if \( S_{(\omega)}^\mu \) is \( (\sigma) \)-hypoelliptic.
(c) For the operators in (b) we do not have a characterization in terms of the behaviour of \( \hat{\mu} \), we only know necessary conditions derived from Propositions 13 and 14.

Also from Theorem 30 and the Bang–Mandelbrojt’s Theorem we can obtain the following result. Compare it with [8, Theorem 4].

Theorem 32. Let \( \omega \) be a weight and let \( \mu \in \mathcal{E}_{(\omega)}' \). \( T_{(\omega)}^\mu \) is elliptic if and only if \( S_{(\omega)}^\mu \) is elliptic for each weight \( r \geq \omega \).

Acknowledgments

The authors are indebted to J. Bonet for many comments and suggestions about this work. We also are grateful to C. Fernández and A. Galbis for their kind support. Finally, we also thank the referee for alerting us of a mistake in the comments before Proposition 22.

References