Eigenvalues of the Neutron Transport Operator for a Homogeneous Finite Moderator

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I. INTRODUCTION

The diffusion of neutrons in a homogeneous moderator is described by a linear transport equation

\[
\frac{\partial}{\partial t} N(r, v, t) = -v \cdot \nabla N(r, v, t) - v \Sigma_t(v) N(r, v, t) + \int_{v'} \Sigma_s(v' \rightarrow v) v' N(r, v', t) \, dv',
\]

(1.1)

where \( N(r, v, t) \) is the neutron density at time \( t \) with respect to the position \( r \) and the neutron velocity \( v = v\hat{\Omega} \) (\( \hat{\Omega} \) being a unit vector representing the direction of neutron motion), while \( \Sigma_t(v) \) and \( \Sigma_s(v' \rightarrow v) \) are positive functions by which macroscopic cross sections are denoted and

\[
\Sigma_t(v) = \Sigma_a(v) + \Sigma_s(v), \quad \Sigma_s(v) = \int_{v'} \Sigma_s(v' \rightarrow v') \, dv'.
\]

(1.2)

Here \( \Sigma_a \) and \( \Sigma_s \) represent the absorption and scattering cross sections respectively. Equation (1.1) is considered for \( r \) in a domain \( D \) of the moderator medium and the domain \( V \) for \( v \) extends to the whole of a three-dimensional Euclidean space since \( \Sigma_t(v) \) and \( \Sigma_s(v' \rightarrow v) \) are defined for all values of \( v \) and \( v' \) when the moderation of neutrons is considered. At the surface \( \Gamma \) of \( D \), \( N \) is assumed to satisfy the boundary condition that no neutrons can enter the medium \( D \) from outside.

Clearly the investigation of Eq. (1.1) under the boundary condition mentioned above as well as a given initial condition leads to the spectral study of the transport operator defined by the right hand side of Eq. (1.1) under the same boundary condition. In particular discrete eigenvalues are of interest for the study of the asymptotic behavior of \( N \) when \( t \) is large.
The spectrum of the transport operator for a moderator has been discussed by many authors for various cases. A simple but important case is that of a space-independent problem for which the spectrum has already been investigated thoroughly. (For example, see [1], [3].) A more general operator has been studied by Jörgens [4] under the assumptions that $D$ is a finite domain and that $0 < c_1 \leq v \leq c_2 < \infty$ for $V$, and has been shown to have a discrete spectrum only. The transport operator for a moderator, however, violates Jörgens' restrictions because neutrons of arbitrary velocities must be taken into account. Concerning the case of a finite moderator, the work by Albertoni and Montagnini [5] should be mentioned which treats a free gas moderator both with and without assumption of the isotropic scattering of neutrons, while the spectrum for a moderator slab has been discussed in [6].

In the present paper we consider a homogeneous moderator of a finite convex body with the assumption that the scattering of neutrons is isotropic. We will focus our attention on the discrete spectrum of the corresponding operator which lies in the half-plane $\text{Re} \lambda > -\lambda^*$, ($\lambda^*$ being defined in Eqs. (2.5)). The scattering models here employed are those proposed by Kuščer and Corngold [2] on the basis of Van Hove’s theory [7] with the incoherent approximation. Thus we will deal with, roughly speaking, all species of materials, a gas, a liquid and a solid.

II. FORMULATION OF THE PROBLEM

As stated above, we assume that the scattering is isotropic, so that the last term in the right hand side of Eq. (1.1) must be replaced by

$$\frac{1}{4\pi} \int_{\nu} \Sigma_s(v' \to v) v'N(r, v', t) dv',$$

where

$$\Sigma_s(v' \to v) = \int_{\nu} \Sigma_s(v' \to v) d\Omega'.$$

(2.1b)

Note that the definition of $\Sigma_s(v' \to v)$ in Eq. (2.1b) differs from the usual one.

The properties of $\Sigma_s(v)$ and $\Sigma_s(v' \to v)$ may affect the spectrum of the related operator. Conditions imposed on these functions in this paper are as follows.

(i) The principle of the detailed balance [8] holds so that the kernel

$$S(v, v') = vv'^2 \sqrt{\frac{M(v')}{M(v)}} \Sigma_s(v' \to v),$$

(2.2)
EIGENVALUES OF NEUTRON TRANSPORT OPERATOR

(2.1) The kernel $S(v, v')$ satisfies the following inequalities.

\[ \int_0^\infty S(v, v') dv' < C_1, \quad \int_0^\infty S^2(v, v') dv' < C_2, \]  

(2.3)

where $C_1$ and $C_2$ are positive constants independent of $v$. In the small velocity range, $v, v' \ll 1$, $S(v, v')$ takes the form

\[ S(v, v') \simeq A_v v', \quad v > v', \]  

(2.4a)

for a gas,

\[ S(v, v') \simeq A_t \ln \left( \frac{(v + v')^4 + \xi^2(v^2 - v'^2)^2}{(v - v')^4 + \xi^2(v^2 - v'^2)^2} \right), \]  

(2.4b)

for a liquid,

\[ S(v, v') \simeq A_s v v'(v^2 + v'^2), \]  

(2.4c)

for a solid, respectively. Here $A_v, A_t, A_s$ and $\xi$ are positive quantities specific to the moderator materials and $\xi > 1$.

(iii) The total cross section $\Sigma_t(v)$ is finite in the limit $v \to \infty$ while $v \Sigma_t(v)$ is an increasing function of $v$, the minimum $\lambda^*$ of which, therefore, is attained at $v = 0$. For $v \ll 1$,

\[ v \Sigma_t(v) \simeq \lambda^* + bv^2, \quad \text{for a gas and a solid,} \]  

(2.5a)

\[ v \Sigma_t(v) \simeq \lambda^* + bv, \quad \text{for a liquid,} \]  

(2.5b)

where $b$ is a positive constant.

Some remarks must be made on these conditions. The inequalities (2.3) are obtained by Eqs. (2.4) and by a physical consideration that, for $v, v' \gg 1$, the behavior of $S(v, v')$ should coincide with that for a free gas model [3]. The formulas (2.4) and (2.5) have been given by Kusčer and Corngold [2] using the theory of Van Hove. The positive quantities appearing in these formulas are given explicitly in their paper. Note that Eqs. (2.4c) and (2.5a) for a solid ignore the effects of elastic scattering. For a space-independent problem this is not restrictive, and even in the space-dependent problem the effects may be expected to be small, (for details, see [8].) The condition (iii) was originally derived for $\Sigma_t(v)$, but, here, it will be imposed on $\Sigma_t(v)$ since in general $\Sigma_t(v)$ of the moderator behaves like $\Sigma_t(v)$. If an extra absorber which violates Eqs. (2.5) is considered, then the conclusions obtained in this paper are not valid. However the analysis of such a case will be carried out easily by a similar procedure.

Now we will formulate our problem. Since the scattering is assumed to be isotropic, it is preferable to use variables $v$ and $\Omega$ in stead of $v$. Let $I_0$ be the
interval \((0, \infty)\) and \(U\) the set of points on the surface of the unit sphere with the center at the origin of \(V\). Then \(v \in I_v\) and \(\Omega \in U\) while \(D\) for \(r\) is a finite convex body. After some transformations the eigenvalue problem of our interest can be written as

\[
\lambda \psi = B\psi,
\]

\[
(B\psi)(r, v, \Omega) = -v\Omega \cdot \nabla \psi(r, v, \Omega) - v\Sigma_\epsilon(v) \psi(r, v, \Omega)
\]

\[+ \frac{v}{4\pi} \int_{I_v \times U} \frac{S(v, v')}{\sqrt{vv'}} \psi(r, v', \Omega) \, dv' \, d\Omega', \quad (2.6a)\]

\[
\psi(r, v, \Omega) = 0; \quad r \in I' \quad \text{and} \quad \Omega \text{ entering } D, \quad (2.6b)
\]

in which \(\psi\) is related to \(N\) by

\[
vN \, dr \, dv - \sqrt{v^2 M(v)} \, \psi \, dr \, dv \, d\Omega. \quad (2.7)
\]

The investigation of Eqs. (2.6) requires, first of all, the determination of a space of functions on which the operator \(B\) acts. Here we will work with Hilbert space \(L_2\) of complex, square summable functions with usual definitions of norms \(||\cdot||\) and inner products (\(,\)). Thus we will solve Eqs. (2.6) in \(L_2(D \times I_v \times U)\).

Define the function \(\varphi(r, v)\) by

\[
\varphi(r, v) = \int_U \psi(r, v, \Omega) \, d\Omega, \quad (2.8)
\]

which is an element of \(L_2(D \times I_v)\) if \(\psi \in L_2(D \times I_v \times U)\). Further let \(\Re \lambda \geq -\lambda^*\), then we can obtain an integral equation for \(\varphi(r, v)\), [9]:

\[
\varphi(r, v) = \frac{1}{4\pi} \int_D \int_{I_v} \frac{\exp \left[ -\frac{\lambda + v\Sigma_\epsilon(v)}{v} \frac{\| r - r' \|^2}{\sqrt{vv'}} \right] S(v, v') \varphi(r', v') \, dr' \, dv'}{\sqrt{vv'}} \, dr \, dv, \quad (2.9)
\]

or simply

\[
\varphi = G_\lambda \varphi. \quad (2.10)
\]

The relation between Eqs. (2.6) and (2.9) is stated in the following theorem.

**Theorem.** A necessary and sufficient condition for the existence of a solution of Eqs. (2.6) in \(L_2(D \times I_v \times U)\) with some \(\lambda, \Re \lambda \geq -\lambda^*\), is that Eq. (2.9) has a solution in \(L_2(D \times I_v)\) for that value of \(\lambda\).
EIGENVALUES OF NEUTRON TRANSPORT OPERATOR

We will omit the proof of this theorem since it is quite similar to that for the monoenergetic transport operator for an infinite slab [10]. Owing to this theorem, therefore, the eigenvalues in \( \text{Re} \lambda \geq - \lambda^* \) of the operator \( B \) can be investigated through Eq. (2.9).

III. THE EIGENVALUE SPECTRUM OF THE OPERATOR \( B \)

Now we will solve Eq. (2.9). For this purpose we study the operator \( G_\lambda \). Define the operators

\[
(E_\lambda \varphi)(r, v) = \frac{1}{4\pi} \int_D \exp \left[ - \frac{\lambda + \frac{\nu \lambda'}{v}}{v} |r - r'|^2 \right] \varphi(r', v) \, dr',
\]

and

\[
(S\varphi)(r, v) = \int_{I_v} \frac{S(v, v')}{\sqrt{vv'}} \varphi(r, v') \, dv'.
\]

It is obvious that \( S \) is a self-adjoint operator on \( L_2(D \times I_v) \) and \( G_\lambda = E_\lambda S \).

**Lemma 3.1.** Let \( \text{Re} \lambda \geq - \lambda^* \). Then \( G_\lambda \) is a bounded operator and

\[
\| G_\lambda \| \leq \ell C_3;
\]

where \( C_3 \) is a positive constant independent of \( \lambda \) and \( \ell \) denotes the maximum chord length of \( D \), which plays a role as a measure of the size of \( D \).

**Proof.** By Schwartz’ inequality we have for \( \text{Re} \lambda \geq - \lambda^* \)

\[
| E_\lambda \varphi |^2 \leq \frac{1}{(4\pi)^2} \int_D \int_D \frac{1}{|r - r'|^2} \, dr' \int_D \frac{1}{|r - r'|^2} |\varphi(r', v')|^2 \, dr'
\]

\[
\leq \frac{\ell}{4\pi} \int_D \int_D \frac{1}{|r - r'|^2} |\varphi(r', v')|^2 \, dr',
\]

whence we obtain

\[
\| E_\lambda \varphi \| \leq \ell \| \varphi \|.
\]

Further by Schwartz’ inequality

\[
| S\varphi |^2 \leq \int_{I_v} S(v, v') \frac{1}{v'} \, dv' \int_{I_v} S(v, v') \frac{1}{v} |\varphi(v, v')| \, dv'.
\]

Now write

\[
\int_{I_v} S(v, v') \frac{dv'}{v'} = \left[ \int_0^{v_0} + \int_{v_0}^{\infty} \right] S(v, v') \frac{1}{v'} \, dv',
\]

where

\[
(3.1a)
\]

\[
(3.1b)
\]

\[
(3.2)
\]

\[
(3.3)
\]

\[
(3.4)
\]

\[
(3.5)
\]

\[
(3.6)
\]
where \( v_0 \) is a small positive constant. The second integral of the right side of the above is finite for all values of \( v \) since Eq. (2.3) holds. Let \( v < v_0 \ll 1 \), then the first integral is also finite which follows from Eqs. (2.4). In the case \( v' < v_0 \ll v, v_0 \ll 1 \), the kernel \( S(v, v') \) will take the form

\[
S(v, v') \approx A v' e^{-\alpha v^2}, \quad \alpha > 0,
\]

since it behaves for \( v' \ll 1 \) like Eqs. (2.4) while, for \( v' \gg 1 \), it is of order \( \exp(-\alpha v^2) \), \( \alpha > 0 \), as suggested from the free gas model. Thus the first integral of the right-hand side of Eq. (3.6) is bounded by a positive constant independent of \( v \). Consequently we can have

\[
\int_{I_v} S(v, v') \frac{1}{v'} dv' \leq C_3
\]  

and from Eq. (3.5)

\[
\|S\varphi\| \leq C_3 \|\varphi\|.
\]

Combining Eqs. (3.4) and (3.9), we get Eq. (3.2) and the proof is completed.

Theorem 3.1. If the medium is sufficiently small, then there is no eigenvalues in \( \text{Re} \lambda \geq -\lambda^* \). That is, there exists an upper limit \( \ell^* \) to \( \ell \) such that, if \( \ell < \ell^* \), the operator \( B \) has no eigenvalue spectrum in \( \text{Re} \lambda \geq -\lambda^* \).

It should be noted, however, that for some cases of moderators \( G, L \) may hold for all values of \( \ell \) and \( \lambda \), \( \text{Re} \lambda \geq -\lambda^* \). Hence \( \ell^* = \infty \) can occur, the possibility of which will be discussed later.

In the case \( \ell \geq \ell^* \), the set of eigenvalues of \( B \) in \( \text{Re} \lambda \geq -\lambda^* \) is not empty. We will now investigate the locations as well as the number of these eigenvalues. First we will prove

Theorem 3.2. The eigenvalues of the operator \( B \) in \( \text{Re} \lambda \geq -\lambda^* \) are real.

Proof. We assume that the operator \( S \) is positive definite, namely,

\[
(\varphi, S\varphi) > 0, \quad \varphi \in L_2(D \times I_v).
\]

In general this assumption is valid as shown in [2], [3]. Hence there is a self-adjoint, positive definite operator \( K \) such that \( K^2 = S \). Putting \( u = K\varphi \) into Eq. (2.10), we obtain \( u = KE_s Ku \) and

\[
0 < (u, u) - (u, KE_s Ku),
\]
for each solution of Eq. (2.9). The inner product \((u, KE_x Ku)\) can be rewritten by the Fourier transformation technique [10],
\[
(u, KE_x Ku) = \int \int \frac{1}{\omega} \tan^{-1} \frac{\omega}{\lambda + \omega \Sigma_x(v)} |(Ku)(\omega, v)|^2 dv \, d\omega, \quad \omega = |\omega|,
\]
(3.12)

Here \(\hat{u}(\omega, v)\) is the Fourier transform of \(u(r, v)\) which is extended outside of \(D\) as zero. The integration for \(\omega\) is over the whole of a three-dimensional Euclidean space. By a similar argument in [10], we then find that, if \(\text{Im} \lambda \neq 0\), the imaginary part of \((u, KE_x Ku)\) does not vanish for any \(u \neq 0\), \(u \in L_2(D \times I_v)\), which contradicts to Eq. (3.11), and thereby the theorem follows.

Now consider an eigenvalue equation
\[
\rho \varphi = G_\lambda \varphi,
\]
(3.13)
where \(\lambda\) is a parameter. The eigenfunctions of this equation with a positive eigenvalue \(\rho = 1\) are solutions of Eq. (2.9).

**Lemma 3.2.** The operator \(G_\lambda\) is completely continuous for \(\text{Re} \lambda \geq -\lambda^*\) if the moderator is a gas or a solid, and for \(\text{Re} \lambda > -\lambda^*\) in the case of a liquid moderator.

**Proof.** The adjoint operator \(G_\lambda^*\) to \(G_\lambda\) is given by \(SE_\lambda\) where \(\bar{\lambda}\) is the complex conjugate of \(\lambda\). Consider a set of functions \(u = SE_\lambda \varphi, \varphi \in L_2(D \times I_v)\) and let \(\Delta r\) and \(\Delta v\) be displacement vectors for \(r\) and \(v\). Further define \(u = 0\) if \(r \notin D, v \notin I_v\). Then

\[
\left| \frac{\exp \left[ -\frac{\lambda + \nu' \Sigma_x(v')}{\nu'} |r + \Delta r - r'| \right]}{|r + \Delta r - r'|^2} \right| \sum_{v'} |S(v, v')| d\varphi \, d\nu' \\
\left( \frac{\exp \left[ -\frac{\lambda + \nu' \Sigma_x(v')}{\nu'} |r - r'| \right]}{|r - r'|^2} \right) \sum_{v'} |S(v, v')| d\varphi \, d\nu' \\
+ \frac{1}{4\pi} \int_{R^3} \int_{I_v} \left| \frac{\exp \left[ -\frac{\lambda + \nu' \Sigma_x(v')}{\nu'} |r + \Delta r - r'| \right]}{|r + \Delta r - r'|^2} \right| \sum_{v'} |S(v, v')| d\varphi \, d\nu' \\
= x_1 + x_2 + x_3,
\]
(3.14)
in which $R$ is a sphere $|r - r'| \leq \delta$ and $D = D - R$. Let $|\Delta r| < \delta/2$, then the difference in $z_1$ can be bounded by a positive constant $0 < \epsilon < 1$ for $|\Delta r| < \delta' < \delta/2 \ll 1$. Hence, by Schwartz' inequality,

$$\|z_1\| \leq \frac{|D|}{4\pi} \epsilon C_3 \|\varphi\|,$$

(3.15)

where $|D|$ is the measure of $D$. The second and third terms, $z_2$ and $z_3$, can be evaluated by Eq. (3.2) with $\delta' = 2\delta/3$ and $\delta' = \delta$;

$$\|z_2\| \leq \frac{3}{8} \delta C_3 \|\varphi\|, \quad \|z_3\| \leq \delta C_3 \|\varphi\|.$$

(3.16)

Combining Eqs. (3.15) and (3.16), then we get

$$\|u(r + \Delta r, v) - u(r, v)\| \leq \|z_1\| + \|z_2\| + \|z_3\|$$

$$\leq \left(\frac{|D|}{4\pi} \epsilon + \frac{5}{2} \delta\right) C_3 \|\varphi\|.$$

(3.17)

On the other hand, we find, by Schwartz' inequality,

$$\|u(r, v + \Delta v) - u(r, v)\|^2 \leq \frac{1}{(4\pi)^2} \int_{D \times I_0} \frac{|\varphi(r', v')|^2}{|r - r'|^2} d\tau' d\varphi' \int_{D \times I_0} \exp\left(-\frac{\lambda + v'\Sigma_t(v')}{v'} |r - r'|^2\right)$$

$$\times \left|\frac{S(v + \Delta v, v') - S(v, v')}{\sqrt{v + \Delta v} \sqrt{v'}}\right|^2 d\tau' d\varphi',$$

(3.18)

which gives

$$\|u(r, v + \Delta v) - u(r, v)\|^2$$

$$\leq \ell \int_{I_0 \times I_0} \operatorname{Re} \frac{1}{\lambda + v'\Sigma_t(v')} \left|\frac{S(v + \Delta v, v') - S(v, v')}{\sqrt{v + \Delta v} \sqrt{v'}}\right|^2 d\tau' d\varphi' \|\varphi\|^2$$

$$= \ell J^2 \|\varphi\|^2.$$

(3.19)

Now assume that the inequality

$$\int_0^\infty \int_0^\infty \left|\frac{S(v, v')}{\sqrt{\operatorname{Re} \lambda + v'\Sigma_t(v')} \sqrt{v'}}\right|^2 < +\infty,$$

(3.20)

holds, then $J < \epsilon < 1$ for $|\Delta v| \ll 1$. Therefore, there exists a constant $0 < \epsilon < 1$ such that

$$\|u(r + \Delta r, v + \Delta v) - u(r, v)\|$$

$$\leq \|u(r + \Delta r, v + \Delta v) - u(r, v + \Delta v)\| + \|u(r, v + \Delta v) - u(r, v)\|$$

$$< \epsilon \|\varphi\|,$$

(3.21)
for some $|\Delta r|, |A\nu| \leq 1$, if Eq. (3.20) holds. Equation (3.21) shows that
the set of functions $u = G_\lambda^* \varphi, \|\varphi\| \leq C_4 < +\infty$, is equicontinuous in
$L_q(D \times I_\nu)$. In view of Lemma 3.1, further, these functions form an
uniformly bounded set in $L_q(D \times I_\nu)$. Consequently $G_\lambda^*$ is completely conti-
uous. The adjoint operator of a completely continuous operator is also
completely continuous. Hence so is $G_\lambda$ if Eq. (3.20) holds.

The validity of Eq. (3.20) can be examined by the aid of the condition (ii)
and (iii) of Section II, together with the formula (3.7), and the lemma is seen
to be assured.

Consider the operator $H_\lambda = KE_\lambda K$ and let $\lambda$ be real and $\lambda > -\lambda^*$. Then by Eq. (3.12) and the above lemma, $H_\lambda$ is self-adjoint, positive definite
and completely continuous, so that the equation

$$\rho u = H_\lambda u, \quad u \in L_q(D \times I_\nu),$$

for an arbitrary fixed $\lambda > -\lambda^*$ has a denumerable sequence of positive
eigenvalues accumulating at $\rho = 0$;

$$\rho_1 > \rho_2 > \cdots > \rho_n > \cdots > 0.$$  \hspace{1cm} (3.23)

It is clear that the set (3.23) is identical with the set of the eigenvalues of
Eq. (3.13). Now we must solve the equation $\rho(\lambda) = 1$. It is easy to prove that
each $\rho_n(\lambda)$ is a continuous function of $\lambda$. Further we can have

**Lemma 3.3.** Let $\lambda \geq -\lambda^*$, then each $\rho_n(\lambda)$ decreases with the increase of $\lambda$
and tends to zero as $\lambda$ tends to infinity.

**Proof.** The operator $H_\lambda - \lambda^* > -\lambda^*$ is positive definite
which can be seen by the application of Eq. (3.12). Hence $\rho_n(\lambda') > \rho_n(\lambda)$
if $\lambda > \lambda' \geq -\lambda^*$. That $\rho_n(\lambda) \to 0$ as $\lambda \to \infty$ follows also from Eq. (3.12).

**Lemma 3.4.** Each $\rho_n(\lambda) (\lambda > -\lambda^*)$ is an increasing function of $\ell$.

**Proof.** We will make use of the max-minimum theorem [10];

$$\rho_n(\lambda) = \max_{S_n} \min_{u \in S_n} (u, H_{\lambda} u), \quad \|u\| = 1,$$

where $S_n$ is an $n$-dimensional subspace of $L_q(D \times I_\nu)$. Let $D' \supset D$ or $\ell' > \ell$.
The suffixes $\ell'$ and $\ell$ will be used to specify the domains $D'$ and $D$. Let $S_n'$ be a
linear manifold generated by the first $n$ eigenfunctions of $H_{\lambda'}$ which are
declared to vanish outside of $D$. Then $S_n' \subset L_q(D' \times I_\nu)$ and every $u \in S_n'$ satisfies $(u, H_{\lambda'} u) = (u, H_{\lambda'} u)$. Hence

$$\rho_n(\lambda) > \min_{u \in S_n'} (u, H_{\lambda'} u) = \min_{u \in S_n'} (u, H_{\lambda'} u) = \rho_{n'}(\lambda),$$

which proves the lemma.
For the cases of a gas and a solid we see from Lemma 3.2 that each $\rho_n(-\lambda^*)$ is finite and $\rho_n(-\lambda^*) \to 0$ as $n \to \infty$. Consequently the equation $\rho_n(\lambda) = 1$ has at most one root $\lambda_n$ in the interval $[-\lambda^*, \infty)$. The number of $\lambda_n$'s is finite if $\ell^* \leq \ell < \infty$, and increases as the domain $D$ becomes large which can be seen from Lemma 3.4.

**Theorem 3.3a.** Let the moderator be an either gas or a solid and let $\ell^* \leq \ell < \infty$. Then the operator $B$ has a finite, nonempty set of real eigenvalues in $\text{Re} \lambda \geq -\lambda^*$.

This theorem does not apply to the case of a liquid since Eq. (3.20) is violated at $\lambda = -\lambda^*$, so that further examinations are required. First we note that, by the condition (iii) of Section II, there exists a positive constant $\eta$ such that

$$0 < \eta \leq \frac{\lambda + v \Sigma_i(v)}{v}, \quad (3.26)$$

for $-\lambda^* \leq \lambda < \infty, 0 \leq v < \infty$. Now we define

$$(E_{\lambda} q)(r, v) = \frac{\eta}{4\pi} \frac{v}{\lambda + v \Sigma_i(v)} \int_D \frac{\exp[-\eta |r - r'|]}{|r - r'|^2} q(r', v) \, dr', \quad (3.27)$$

and consider the eigenvalue equation

$$\rho' u = KE_{\lambda}' Ku. \quad (3.28)$$

Since

$$\frac{1}{\omega} \tan^{-1} \frac{v\omega}{\lambda + v \Sigma_i(v)} = \frac{1}{\omega} \frac{v}{\lambda + v \Sigma_i(v)} \int_0^\omega \frac{d\omega}{1 + \left(\frac{v\omega}{\lambda + v \Sigma_i(v)}\right)^2}$$

$$> \frac{1}{\omega} \frac{v}{\lambda + v \Sigma_i(v)} \int_0^\omega \frac{d\omega}{1 + \left(\frac{\omega}{\eta}\right)^2} = \frac{\eta}{\omega} \frac{v}{\lambda + v \Sigma_i(v)} \tan^{-1} \frac{\omega}{\eta}, \quad (3.29)$$

$(u, KE_{\lambda} K u) - (u, KE_{\lambda}' K u) > 0$ follows from a similar formula to Eq. (3.12), that is, the operator $KE_{\lambda} K - KE_{\lambda}' K$ is positive definite for $\lambda > -\lambda^*$. Hence, in view of the minimax theorem of [11],

$$\rho_n(\lambda) > \rho_n'(\lambda), \quad \text{for each } n. \quad (3.30)$$

The $\rho'$-eigenvalues are also those of the equation

$$\rho' \varphi = E_{\lambda}' S \varphi, \quad (3.31)$$
which can be solved by the separation of variables. Write

$$\varphi(r, v) = \sqrt{\nu} \frac{f(r) g(v)}{\sqrt{\lambda + v \Sigma(v)}}$$,

then we obtain

$$\mu f(r) = \frac{\gamma}{4\pi} \int_D \frac{\exp\left[-\gamma |r - r'| \right]}{|r - r'|^2} f(r') \, dr', \quad (3.32)$$

and

$$v g(v) = \int_0^\infty \frac{S(v, v')}{\sqrt{\lambda + v \Sigma(v)}} \frac{g(v')}{\sqrt{\lambda + v' \Sigma(v')}} \, dv', \quad (3.33)$$

where $\rho' = \mu \nu$.

First it may be noted that Eq. (3.32) coincides formally with the integral form of the monoenergetic neutron transport equation for $D$, which has been discussed in [12]. According to this, it has a denumerable set of positive eigenvalues $\mu_i$, $(\mu_i > \mu_{i+1}, i = 1, 2, 3,...)$ and $\mu_i \to 0 (i \to \infty)$.

The second equation (3.33) is the same as that considered for a space-independent neutron transport equation, and the case of a liquid moderator was studied by Kuščer and Corngold [2] who have shown (1) that there is a denumerable set of positive eigenvalues $\nu_j$, $(\nu_j > \nu_{j+1}, j = 1, 2, 3,...)$ which has an accumulation point at $\nu = 0$ for $\lambda < -\lambda^*$, (2) that each $\nu_j$ is a continuous, decreasing function of $\lambda$ tending to zero in the limit $\lambda \to \infty$, and (3) that the number of $\nu_j$'s such that $\nu_j \geq k^* - \epsilon$ increases indefinitely as $\lambda \to -\lambda^* + 0$ where $\epsilon > 0$ and

$$k^* = \frac{1}{b} 2\pi a^* \sin^{-1} \frac{2\zeta}{\epsilon^2 + 1}, \quad (3.34)$$

The assertion (3) of the above, however, is not stated clearly in their paper, so that we will give the proof in the below. For convenience, let $\kappa_j(Q)$ be the $j$th positive eigenvalue of an integral operator $Q$ defined by a kernel $Q(v, v')$.

First we introduce the kernel

$$R_j(v, v') = \begin{cases} S_j(v, v'), & v, v' < \nu_v, \\ 0, & \text{otherwise}, \end{cases} \quad (3.35)$$

where $S_j(v, v')$ stands for the kernel appearing in Eq. (3.33) and $\nu_v$ is a positive constant determined later. The integral operator $R_j$ is easily seen to be self-adjoint, completely continuous and positive definite for $\lambda > -\lambda^*$.

**Lemma 3.5.** For each $j$, 

$$\nu_j(\lambda) > \kappa_j(R_j), \quad \text{for each} \quad \lambda > -\lambda^*. \quad (3.36)$$
PROOF. Let $S_j$ be a $j$-dimensional subspace of $L_2(0, \infty)$ and $S_j'$ the linear manifold spanned by the first $j$ eigenfunctions of $R_{\lambda}$. Then every $g(r) \in S_j'$ vanishes for $v_0 \leqslant v < \infty$ and $(g, S_0 g) = (g, R_{\lambda} g)$. Consequently, using the max-minimum theorem [10],

$$v_j(\lambda) = \max_{S_j} \min_{g \in S_j} (g, S_0 g) \geqslant \min_{g \in S_j'} (g, S_0 g) = \min_{g \in S_j} (g, R_{\lambda} g) = \kappa_j(R_{\lambda}),$$

(3.37)

which verifies the lemma.

If $v_0$ is sufficiently small, then formulas (2.4b) and (2.5b) apply to $S_0(v, v')$ and $v S_0(v)$. Thus for $v, v' < v_0 \ll 1$,

$$R_{\lambda}(v, v') \approx F_{\lambda}(v, v') = \frac{A_{\lambda}}{\sqrt{\lambda + \lambda^* + bv} \sqrt{\lambda + \lambda^* + bv'}} \times \ln \frac{(v + v')^4 + \xi^2(v^2 - v'^2)^2}{(v - v')^4 + \xi^2(v^2 - v'^2)^2},$$

(3.38a)

which can be expanded [2] as

$$F_{\lambda}(v, v') = \sum_{m=1}^{\infty} A_m F_{\lambda}^{(m)}(v, v'),$$

$$F_{\lambda}^{(m)}(v, v') = \frac{1}{\sqrt{\lambda + \lambda^* + bv} \sqrt{\lambda + \lambda^* + bv'}} \left( \frac{v'}{v} \right)^{2(m-1)}, \quad v \leqslant v',$$

(3.38b)

with $A_m > 0$.

LEMMA 3.6. The operator $F_{\lambda}^{(m)}$ is self-adjoint, completely continuous and positive definite for $\lambda > -\lambda^*$.

PROOF. The self-adjointness and complete continuity of $F_{\lambda}^{(m)}$ follow directly from the symmetry and the square-integrability of the kernel $F_{\lambda}^{(m)}(v, v')$. Further, this kernel is Green’s function of a second order differential operator, so the equation $\kappa g = F_{\lambda}^{(m)} g$ can be reduced to

$$\frac{d}{dv} \left( \frac{d}{dv} h(v) \right) + \frac{2a(2m-1)}{(bv + \lambda + \lambda^*)} \left( \frac{2m-1}{v} \right) h(v) = 0,$$

(3.39)

$$h(0) = \text{finite}, \quad h(v_0) + v_0 h'(v_0) = 0,$$

(3.40)

where $h(v) = \sqrt{\lambda + \lambda^* + bv} g(v)$ and $\alpha = 1/\kappa$, from which we obtain

$$2a(2m-1) \int_0^{v_0} \frac{|h(v)|^2}{bv + \lambda + \lambda^*} \, dv$$

$$= \int_0^{v_0} v |h'(v)|^2 \, dv + v_0 |h'(v_0)|^2 + (2m-1)^2 \int_0^{v_0} \frac{|h(v)|^2}{v} \, dv,$$

(3.41)
and thereby we see that \( \alpha > 0 \), i.e., all eigenvalues \( \tau_i(F_\lambda^{(m)}) \) are positive. Thus \( F_\lambda^{(m)} \) can be concluded to be positive definite.

The transformation of variables,

\[
t = 2 \sqrt{2m - 1} \ln \left( \sqrt{bv + \sqrt{bv + \lambda + \lambda^*}} \over \sqrt{\lambda + \lambda^*} \right)
\]

and

\[
y(t) = \sqrt{(2m - 1) bv} h(v) / \sqrt{bv + \lambda + \lambda^*},
\]

rewrites Eqs. (3.39) and (3.40) as

\[
y''(t) + \left( {2\alpha \over b} - (2m - 1) - q(t) \right) y(t) = 0;
\]

\[
y(0) = 0, \quad \left( 1 - {1 \over 4} {\lambda + \lambda^* \over bv_0 + \lambda + \lambda^*} \right) y(t_0) + \sqrt{(2m - 1) bv_0 \over bv_0 + \lambda + \lambda^*} y'(t_0) = 0,
\]

in which

\[
t_0 = 2 \sqrt{2m - 1} \ln \left( \sqrt{bv_0 + \sqrt{bv_0 + \lambda + \lambda^*}} \over \sqrt{\lambda + \lambda^*} \right),
\]

\[
q(t) = \left( (2m - 1)^2 - {1 \over 4} \right) \left( \sinh^2 {t \over 2 \sqrt{2m - 1}} + 1 \right) + {3 \over 16} \sinh^2 {t \over 2 \sqrt{2m - 1}} - \left( \sinh^2 {t \over 2 \sqrt{2m - 1}} + 1 \right),
\]

\[
(2m - 1) \sinh^2 {t \over 2 \sqrt{2m - 1}} \left( \sinh^2 {t \over 2 \sqrt{2m - 1}} + 1 \right)
\]

\[
(3.42)
\]

**Lemma 2.7.** Let \( \alpha_j^{(m)} \) be the \( j \)th eigenvalue (when enumerated in an increasing sequence) of Eq. (3.42). Then for each \( m \) and \( j \),

\[
0 < \alpha_j^{(m)} \leq {b \over 2} \left( (2m - 1) + \left( {\beta_{m,j} \over t_0} \right)^2 \right),
\]

\[
(3.44)
\]

Here \( \beta_{m,j} \) is the \( j \)th zero of the equation

\[
\left( 1 - {1 \over 4} {\lambda + \lambda^* \over bv_0 + \lambda + \lambda^*} \right) \sqrt{\pi} J_{2m-2}(z) + \sqrt{(2m - 1) bv_0 \over bv_0 + \lambda + \lambda^*} (\sqrt{\pi} J_{2m-2}(z))' = 0,
\]

\[
(3.45)
\]

where \( J_m(z) \) is the Bessel function of order \( m \).
Consider Eq. (3.42) with \((4m - 2)^2 - \frac{1}{t^2}\) in place of \(q(t)\). Then the solution is \(\frac{\sqrt{t}}{J_{4m-2}(\sqrt{2m - (2m - 1)/b t})}\) and the corresponding eigenvalue \(\alpha_{m}^{(m)}\) is given just by the last formula of Eq. (3.44). The inequality (3.44), then, follows from the fact \(q(t) \leq (4m - 2)^2 - \frac{1}{t^2}\) and the variational principle [11].

**Lemma 3.8.** If \(\lambda + \lambda^*\) is sufficiently small, then

\[
\kappa_1(F_\lambda) \geq k^* - \epsilon, 
\]

where \(k^*\) is defined in Eq. (3.34) and \(0 < \epsilon \ll 1\), which may depend on \(j\).

**Proof.** We again make use of the max-minimum theorem. Let \(S_j'\) be an arbitrary \(j\)-dimensional subspace of \(L_2(0, \nu_0)\). Then,

\[
\kappa_j(F_\lambda) \geq \min_{g \in S_j'} (g, F_\lambda g) \geq \sum_{m=1}^{\infty} A_m \left( \min_{g \in S_j'} (g, F^{(m)}_\lambda g) \right). 
\]

(3.47)

The last step of the above is justified by the positive definitness of \(F^{(m)}_\lambda\). Further we can always find some integer \(j_m\) such that

\[
\min_{g \in S_j'} (g, F^{(m)}_\lambda g) \geq \kappa_{j+m}(F^{(m)}_\lambda) 
\]

(3.48)

since \(S_j'\) is finite dimensional and since \(F^{(m)}_\lambda\) is positive definite. Therefore, from Eqs. (3.44), (3.47), and (3.48) we have

\[
\kappa_j(F_\lambda) \geq \sum_{m=1}^{\infty} A_m \kappa_{j+m}(F^{(m)}_\lambda) 
\]

\[
\geq \frac{2}{b} \sum_{m=1}^{\infty} \frac{A_m}{2m - 1} \times \left[ 1 + \frac{(\beta_{m,j+j})^2}{4(2m - 1)^2 \left\{ \ln \frac{\sqrt{b}\nu_0 + \sqrt{b}\nu_0 + \sqrt{\lambda + \lambda^*}}{\sqrt{\lambda} + \sqrt{\lambda^*}} \right\}^2} \right]^{-1}, 
\]

(3.49)

Noting that \(\beta_{m,j}(2m - 1) \rightarrow \text{const.} (m \rightarrow \infty)\) for a fixed \(j\), then we obtain for \(\lambda + \lambda^* \ll 1\),

\[
\kappa_j(F_\lambda) \geq \frac{2}{b} \sum_{m=1}^{\infty} \frac{A_m}{2m - 1} - \epsilon, 
\]
EIGENVALUES OF NEUTRON TRANSPORT OPERATOR

where $0 < \epsilon \ll 1$. The sum of the above is equal to $k^* [2]$: 

$$\frac{2}{b^2} \sum_{m=1}^{\infty} \frac{A_m}{2m-1} = \frac{A_\ell}{b} \int_0^\infty \frac{1}{t} \ln \frac{(1 + t)^{4 + \xi^2(1 - t^2)^2}}{(1 - t)^{4 + \xi^2(1 - t^2)^2}} \, dt = k^*, \quad (3.51)$$

which complete the proof of the lemma.

Combining Eqs. (3.36) and (3.46), and noting that $\kappa_{\ell}(R_s) \sim \kappa_{\ell}(F_{\ell})$ for $v_0 \ll 1$, then we can conclude the assertion (3). For later use, further, we will prove the following lemma.

**Lemma 3.9.** The operator $S_{\lambda}$ for $\lambda = -\lambda^*$ has at most a finite number of discrete eigenvalues in $[k^* + \epsilon, \infty)$ where $\epsilon > 0$.

**Proof.** We shall show that the operator $S_{\lambda} - R_\lambda$ is completely continuous. In the region $v, v' > v_0$, the corresponding kernel $S_{\lambda}(v, v') - R_{\lambda}(v, v')$ is square-integrable even for $\lambda = -\lambda^*$ since $R_{\lambda}(v, v') = 0$ and $-\lambda^* + v\Sigma_t > 0$. The square-integrability of this kernel over $(0, v_0) \times (v_0, \infty)$ and $(v_0, \infty) \times (0, v_0)$ for $\lambda = -\lambda^*$ follows from $R_{\lambda}(v, v') = 0$ and the formula (3.7). In the region $v, v' < v_0$, it is obvious that $S_{\lambda}(v, v') - R_{\lambda}(v, v') = 0$. Thus the operator $S_{\lambda} - R_{\lambda}$ for $\lambda = -\lambda^*$ is of Hilbert-Schmit type, by which the lemma is assured.

Now we return to Eq. (3.31). The $\rho^*$-eigenvalue of this equation, clearly, is given by $\rho_n(\lambda) = \mu_n(\lambda)$, $\lambda = 1, 2, 3, \ldots$, so that, from the arguments given above, we see that the number of $\rho^*$-eigenvalues larger than $\mu_n k^*$ increases indefinitely as $\lambda \rightarrow -\lambda^* + 0$. By virtue of Eq. (3.30) this is also true for $\rho_n(\lambda)$. With this and Lemma 3.3, we then see that the solutions $\lambda_n$ of the equations $\rho_n(\lambda) = \mu_n k^*$, $(-\lambda^* < \lambda < \infty)$, are infinite in number and accumulate at $\lambda = -\lambda^*$. Hence, if $\mu_n k^* > 1$, the operator $B$ has an infinite number of real eigenvalues in $\text{Re} \lambda > -\lambda^*$ with an accumulation point at $-\lambda^*$.

The positive constant $k^*$ is specific to the moderator material while $\mu_1$ depends on $\ell$ or the size of $D$ as well as the quantity $\eta$. It is not difficult, however, to show that $\mu_1$ is an increasing function of $\ell$ and tends to unity in the limit $\ell \rightarrow \infty$ regardless of the value of $\eta$. On the other hand, if we put $\varphi(\rho, v) = \sqrt{v g(v)/\sqrt{\lambda + v \Sigma_t}}$ into Eq. (3.13) and if we let $\ell \rightarrow \infty$, then we obtain directly Eq. (3.33) with $\rho = v$. Consequently, referring to Lemma 3.4, we find that, if $k^* > 1$ and if the medium is large enough, $B$ has a denumerable set of real eigenvalues accumulating at $\lambda = -\lambda^*$, and that, if $k^* < 1$, the set of eigenvalues $\lambda_n$ is finite, which can be seen from Lemma 3.9.

**Theorem 3.3b.** Let the moderator be a liquid. If $k^* > 1$, there exist a critical value $\ell^{**}$ for $\ell$ such that, for $\ell^{**} \leqslant \ell < \infty$, the operator $B$ has a
countable infinity of real, discrete eigenvalues in \( \text{Re} \lambda > - \lambda^* \) with an accumulation point at \( \lambda = - \lambda^* \), while, if \( \ell^{**} > \ell \geq \ell^* \), \( B \) has a finite number of discrete eigenvalues in \( (- \lambda^*, \infty) \). If \( \kappa^* < 1 \), the number of eigenvalues \( \lambda_n \) is finite as long as \( \ell^* < \ell < \infty \).

As for the multiplicity and the index of \( \lambda_n \), the following theorem holds, the proof of which is given in Appendix.

**Theorem 3.4.** Each eigenvalue of the operator \( B \) in \( \text{Re} \lambda > - \lambda^* \) is of finite multiplicity and has the index one.

Finally we will discuss the possibility of \( \ell^* = \infty \). As stated above, Eq. (3.33) can be obtained from Eq. (3.13) in the limit \( \ell \to \infty \), which is a neutron transport equation for an infinite moderator. Hence it follows from Lemma 3.4 that \( B \) has no eigenvalues in \( (- \lambda^*, \infty) \) if \( \lim_{\lambda \to \infty} \nu_1(\lambda) < 1 \), that is, if the corresponding space-independent transport operator has no eigenvalues in \( (- \lambda^*, \infty) \). Examples for this situation are found in [2].

**IV. REMARKS**

In the previous section, we have investigated the eigenvalues in \( \text{Re} \lambda > - \lambda^* \). The decomposition of the spectral plane for the operator \( B \) is completed by the following theorem. The proof can be easily carried out with the method used in [5], [10].

**Theorem 4.1.** The entire half-plane \( \text{Re} \lambda \leq - \lambda^* \) forms a continuum of the spectrum of the operator \( B \). The resolvent set of \( B \) consists of the right half-plane \( \text{Re} \lambda > - \lambda^* \) deleted by eigenvalues.

It should be noted that the result obtained in Section III for a gasous moderator is identical with those for a free gas moderator [5]. Further all theorems obtained in Section III apply also for an infinite moderator slab, which is shown in [6].

Finally we will consider the case of anisotropic scattering of neutrons. In this case we must study the operator

\[
(B\psi)(r, v) \equiv -v \cdot \nabla \psi(r, v) - v\Sigma_s(v) \psi(r, v) + \int \psi(S(v, v') \psi(r, v') dv',
\]

\[\psi(r, v) = 0 \quad \text{for} \quad r \in \Gamma \quad \text{and} \quad \Omega \text{ entering } D, \tag{4.1}\]

where

\[
N \, dr \, dv = \sqrt{M(v)} \, \psi \, dr \, dv, \quad S(v, v') = v' \sqrt{\frac{M(v')}{M(v)}} \Sigma_s(v' \to v). \tag{4.2}\]

\[
N \, dr \, dv = \sqrt{M(v)} \, \psi \, dr \, dv, \quad S(v, v') = v' \sqrt{\frac{M(v')}{M(v)}} \Sigma_s(v' \to v). \tag{4.3}\]
Concerning the case of a free gas moderator, the following theorem has been proved by Montagnini and Albertoni [5].

**Theorem 4.2.** Let $D$ be a finite convex domain. Then the entire half-plane $\text{Re} \lambda = - \lambda^*$ belongs to the spectrum of the operator $B$ defined by Eqs. (4.1) and (4.2). If $D$ is sufficiently small, the whole of the plane $\text{Re} \lambda > - \lambda^*$ is the resolvent set of $B$ and there are no eigenvalues in $\text{Re} \lambda > - \lambda^*$.

By a careful examination of the proof given by these authors, however, it can be found that Theorem 4.2 is valid so far as the inequalities
\[ \int_{\nu} S(\nu, \nu') \, d\nu' < \infty, \quad \int_{\nu} S^2(\nu, \nu') \, d\nu' < \infty, \tag{4.4} \]
hold. Thus the theorem can be generalized for other moderators since the conditions (4.4) seem to be nonrestrictive. The eigenvalues in $\text{Re} \lambda > - \lambda^*$ for the operator $B$ of Eqs. (4.1) and (4.2) are still left open to further investigations.

**Appendix**

Let $\psi_n(r, \nu, \Omega)$ be an eigenfunction of the operator $B$ corresponding to the eigenvalue $\lambda_n > - \lambda^*$. Then $\psi_n^*(r, \nu, \Omega) = \psi_n(r, \nu, -\Omega)$ gives an eigenfunction of the adjoint operator to $B$ for the eigenvalue $\lambda_n^*$. The index of $\lambda_n$ is unity if the inner product $(\psi_n^*, \psi_n)$ does not vanish [10]. After some calculations, we can find
\[
(\psi_n^*, \psi_n) = \frac{1}{(4\pi)^2} \int_{D \times D \times I_0} \frac{\exp \left( -\frac{\lambda_n + \nu \Sigma_1(\nu)}{\nu} \right) |r - r'|}{|r - r'|} \times S\varphi_n(r, \nu) \overline{S\varphi_n(r', \nu)} \, dr \, dr' \, d\nu', \tag{A.1}
\]
where $\varphi_n$ is defined by Eq. (2.8) with $\psi_n$. By the Fourier transformation technique as in Theorem 3.2, we get
\[
(\psi_n^*, \psi_n) = \int_{\omega} \int_{I_0} \frac{1}{|S\varphi_n(\omega, \nu)|^2 + \left( \frac{\nu}{\lambda_n + \nu \Sigma_1(\nu)} \right)^2} \, |S\varphi_n(\omega, \nu)|^2 \, d\omega \, d\nu. \tag{A.2}
\]
The integrand of the right-hand side is positive. Hence $(\psi_n^*, \psi_n) \neq 0$, which proves the latter part of Theorem 3.4. The finite multiplicity of $\lambda_n$ is a direct consequence of the complete continuity of the operator $G_\lambda$. 
REFERENCES