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# CONVERGENCE RATES FOR RECORD TIMES AND THE ASSOCIATED COUNTING PROCESS

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Let $X_1, X_2, \ldots$ be independent random variables with a common continuous distribution function. Rates of convergence in limit theorems for record times and the associated counting process are established. The proofs are based on inversion, a representation due to Williams and random walk methods.

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i.i.d. random variables * continuous distribution function * record time * counting process * inversion * strong law * central limit theorem * remainder term estimate * law of the iterated logarithm * convergence rate

## 1. Introduction

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables, whose distribution function $F$ is continuous. Much attention has been devoted to the sequences of sample maxima, record times, interrecord times and record values, which are defined as follows:

The *sample maxima* are

$$Y_n = \max\{X_1, X_2, \ldots, X_n\}, \quad n \geq 1.$$

The *record times* are $L(1) = 1$ and, recursively,

$$L(n) = \min\{k: X_k > X_{L(n-1)}\}, \quad n \geq 2.$$

The *interrecord times* are $\Delta(1) = L(1) = 1$ and

$$\Delta(n) = L(n) - L(n-1), \quad n \geq 2.$$

The *record values* are

$$X_{L(n)}, \quad n \geq 1.$$

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The present paper is devoted to the study of the sequence of record times and the associated *counting process* $\{\mu(n), n \geq 1\}$ defined by

$$\mu(n) = \# \text{ records in } [1, n] = \max\{k\colon L(k) \leq n\}.$$

The term counting process is inspired from renewal theory.

We first collect some asymptotic properties of $\mu(n)$.

**Theorem 1.** (i) $\mu(n)/\log n \xrightarrow{\text{a.s.}} 1$ *as* $n \to \infty$.

(ii) *The sequence* $\{(\mu(n)/\log n)^r, n \geq 3\}$ *is uniformly integrable and for all* $r > 0$, $E(\mu(n)/\log n)^r \to 1$ *as* $n \to \infty$.

(iii) $(\mu(n) - \log n)/\sqrt{\log n} \xrightarrow{d} \mathrm{N}(0, 1)$ *as* $n \to \infty$.

(iv) *The sequence* $\{|(\mu(n) - \log n)/\sqrt{\log n}|^r, n \geq 3\}$ *is uniformly integrable,*

$$E\left|\frac{\mu(n) - \log n}{\sqrt{\log n}}\right|^r \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^r \, \mathrm{e}^{-x^2/2} \, \mathrm{d}x \quad \textit{for all } r > 0$$

*and*

$$E\left(\frac{\mu(n) - \log n}{\sqrt{\log n}}\right)^k \to 0 \quad \textit{for } k = 1, 3, 5, \ldots .$$

(v) $$\limsup_{n\to\infty} \left(\liminf_{n\to\infty}\right) \frac{\mu(n) - \log n}{\sqrt{2 \log n \log\log\log n}} = +1 \quad (-1) \quad \textit{a.s.} \qquad \square$$

The odd numbered conclusions are proved in the pioneering paper Rényi (1962). The starting point is that if we set, for $k \geq 1$,

$$I_k = \begin{cases} 1 & \text{if } X_k \text{ is a record,} \\ 0 & \text{otherwise,} \end{cases}$$

then $\{I_k, k \geq 1\}$ are independent random variables, such that $P(I_k = 1) = 1 - P(I_k = 0) = k^{-1}$ and

$$\mu(n) = \sum_{k=1}^{n} I_k, \quad n \geq 1 \tag{1.1}$$

(see also Resnick, 1987, Section 4.1).

The conclusions follow from classical limit theorems for sums of independent, uniformly bounded random variables (note that $|I_k - 1/k| \leq 1$). More precisely, (i) follows from the Kolmogorov three-series theorem and Kronecker's lemma, see e.g. Petrov (1975, Chapter IX), (iii) follows from Lyapounov's condition or by checking the appropriate moment generating function and (v) follows from Kolmogorov's law of the iterated logarithm, see e.g. Petrov (1975, Chapter X). Moreover, the convergence of the moment generating function of $\mu(n)$, normalized as in (i) and (iii), also implies uniform integrability of the normalized sequences of the counting process and, hence, the convergence of moments as described in the even numbered conclusions. These, latter, facts do not seem to have been mentioned explicitly before.

The corresponding result for the sequence of record times is as follows.

**Theorem 2.** (i) $(\log L(n))/n \to^{\text{a.s.}} 1$ *as* $n \to \infty$.

(ii) *The sequence* $\{((\log L(n))/n)^r, n \geq 1\}$ *is uniformly integrable and*

$$E\left(\frac{\log L(n)}{n}\right)^r \to 1 \quad \textit{as } n \to \infty \textit{ for all } r > 0.$$

(iii) $(\log L(n) - n)/\sqrt{n} \to^{\text{d}} \mathrm{N}(0,1)$ *as* $n \to \infty$.

(iv) *The sequence* $\{|(\log L(n) - n)/\sqrt{n}|^r, n \geq 1\}$ *is uniformly integrable,*

$$E\left|\frac{\log L(n) - n}{\sqrt{n}}\right|^r \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^r \, \mathrm{e}^{-x^2/2} \, \mathrm{d}x \quad \textit{for all } r > 0$$

*and*

$$E\left(\frac{\log L(n) - n}{\sqrt{n}}\right)^k \to 0 \quad \textit{for } k = 1, 3, 5, \ldots .$$

(v) $$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \frac{\log L(n) - n}{\sqrt{2n \log\log n}} = +1 \quad (-1) \quad \textit{a.s.} \qquad \square$$

Again, (i), (iii) and (v) are due to Rényi (1962); the conclusions follows from their counterparts in Theorem 1 and inversion, see formula (2.5) below. The results (ii) and (iv), which do not seem to have been mentioned explicitly before, follow from Williams' representation, which is presented in Section 3.

A limit theorem states that convergence holds. Given a limit theorem, a natural question is: "What is the rate of convergence?" The aim of this paper is to provide answers to this question with respect to Theorem 1(i), (iii), (v) and Theorem 2(i), (ii), (v). The classical reference for remainder term estimates in the ordinary central limit theorem is Esseen (1945). Some references for results corresponding to the other two limit laws for random walks are Baum, Katz and Read (1962), Baum and Katz (1965), Davis (1968a, b), Slivka (1969), Gut (1978, 1980) and Russo (1988).

## 2. Remainder term estimates in the central limit theorems

**Theorem 3.** *For* $k \geq 2$ *we have*

$$\sup_n \left| P(\mu(k) \leq n) - \Phi\left(\frac{n - \log k}{\sqrt{\log k}}\right) \right| \leq \frac{1.9}{\sqrt{\log k}}.$$

**Remark 1.** Because of degeneracy the case $k = 1$ is excluded.

**Theorem 4.** *We have*

$$\sup_k \left| P(L(n) \geq k) - \Phi\left(\frac{n - \log k}{\sqrt{n}}\right) \right| \leq \frac{4.3}{\sqrt{n}}.$$

For the proofs we need the following estimates for the standard normal distribution function.

**Lemma 1.** (a) $\Phi(-x) \leqslant x^{-2}$ *for* $x > 0$.

(b) $$|\Phi(x) - \Phi(y)| \leqslant \frac{1}{\sqrt{2\pi}}|x-y| \quad \text{for } x, y \in \mathbb{R}.$$

(c) $$\left|\Phi\left(\frac{z}{x}\right) - \Phi\left(\frac{z}{y}\right)\right| \leqslant 1.25\left|\frac{x}{y} - 1\right| \quad \text{for } x, y > 0,\ z \in \mathbb{R}.$$

(d) $$\left|\Phi\left(\frac{x-y}{\sqrt{x}}\right) - \Phi\left(\frac{x-y}{\sqrt{y}}\right)\right| \leqslant \frac{1}{e\sqrt{2\pi}} \frac{1+\theta_0 y_0^{-1/4}}{(1-\theta_0 y_0^{-1/4})^{3/2}} \frac{1}{\sqrt{y}}$$
$$\leqslant \frac{2}{e\sqrt{2\pi}} \cdot \frac{1}{(1-\theta_0 y_0^{-1/4})^{3/2}} \cdot \frac{1}{\sqrt{y}}$$
*for all* $x, y > 0$ *such that* $|x-y| \leqslant \theta_0 y^{3/4}$, $y > y_0 > \theta_0^4 > 0$.

**Proof.** (a) and (b) are standard, (c) is Englund (1982, Lemma 2.9).

The proof of (d) follows the lines of Englund (1980, p. 1112). Let $x = y + \theta \cdot y^{3/4}$, with $|\theta| \leqslant \theta_0$. Then

$$\left|\frac{x-y}{\sqrt{x}} - \frac{x-y}{\sqrt{y}}\right| = |\theta| y^{1/4} |(1+\theta y^{-1/4})^{-1/2} - 1|$$
$$\leqslant \tfrac{1}{2}\theta^2(1-\theta_0 y_0^{-1/4})^{-3/2} \leqslant \tfrac{1}{2}\theta_0^2(1-\theta_0 y_0^{-1/4})^{-3/2}.$$

Taylor expansion and computations like in Englund (1980), formulas (2.26)-(2.28) yield $(0 < \delta < 1)$,

$$\left|\Phi\left(\frac{x-y}{\sqrt{x}}\right) - \Phi\left(\frac{x-y}{\sqrt{y}}\right)\right|$$
$$\leqslant \frac{1}{\sqrt{2\pi}} \cdot \tfrac{1}{2}\theta^2(1-\theta_0 y_0^{-1/4})^{-3/2} \exp\left\{-\frac{1}{2}\left(\frac{x-y}{\sqrt{y}}\right)^2\left(1+\delta\left(\sqrt{\frac{x}{y}}-1\right)\right)^2\right\}$$
$$\leqslant \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \cdot \frac{\theta^2}{(1-\theta_0 y_0^{-1/4})^{3/2}} \cdot \frac{1}{e \cdot \frac{1}{2} \cdot \theta^2\sqrt{y}/(1+\theta_0 y_0^{-1/4})},$$

and the conclusion follows. □

Now, recall the representation (1.1) of $\mu(k)$, and set

$$m_k = E\mu(k), \qquad s_k^2 = \operatorname{Var}\mu(k), \qquad \beta_k^3 = \sum_{j=1}^{k} E|I_j - 1/j|^3,$$

let $\gamma_k = \sum_{j=1}^{k} j^{-1} - \log k$ and note that $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k \downarrow \gamma = 0.577\ldots =$ Euler's constant. The following relations are easily verified:

$$m_k = \sum_{j=1}^{k} \frac{1}{j} = \log k + \gamma_k = \log k + \gamma + o(1) \quad \text{as } k \to \infty, \tag{2.1}$$

$$s_k^2 = \sum_{j=1}^{k} \frac{1}{j}\left(1 - \frac{1}{j}\right) = \log k + \gamma - \tfrac{1}{6}\pi^2 + o(1) \quad \text{as } k \to \infty, \tag{2.2}$$

$$\log k \geq s_k^2 \geq \log k + \gamma - \tfrac{1}{6}\pi^2 \geq \log k - 1.07, \tag{2.3}$$

$$\frac{\beta_k^3}{s_k^3} \leq \left(\sum_{j=1}^{k} \frac{1}{j}\left(1 - \frac{1}{j}\right)\right)^{-1/2} = s_k^{-1}. \tag{2.4}$$

**Proof of Theorem 3.** We first observe that it suffices to consider $k$-values such that $1.9/\sqrt{\log k} < 1$, i.e. $k \geq [e^{3.61}] + 1 = 37$, since otherwise the bound is trivial.

Now,

$$\begin{aligned} &\left| P(\mu(k) \leq n) - \Phi\left(\frac{n - \log k}{\sqrt{\log k}}\right) \right| \\ &\leq \left| P(\mu(k) \leq n) - \Phi\left(\frac{n - m_k}{s_k}\right) \right| + \left| \Phi\left(\frac{n - m_k}{s_k}\right) - \Phi\left(\frac{n - m_k}{\sqrt{\log k}}\right) \right| \\ &\quad + \left| \Phi\left(\frac{n - m_k}{\sqrt{\log k}}\right) - \Phi\left(\frac{n - \log k}{\sqrt{\log k}}\right) \right| \\ &= P_1 + P_2 + P_3. \end{aligned}$$

By the Berry–Esseen theorem (see Esseen, 1945; and, for the constant, Van Beek, 1972, p. 196), (2.4) and (2.3) we obtain

$$P_1 \leq 0.7975 \frac{\beta_k^3}{s_k^3} \leq 0.7975 s_k^{-1} \leq \frac{0.7975}{\sqrt{\log k - 1.07}} \leq \frac{0.92}{\sqrt{\log k}}.$$

Next, by Lemma 1(c) and (2.3) we have

$$\begin{aligned} P_2 &\leq 1.25 \left| \frac{s_k}{\sqrt{\log k}} - 1 \right| = 1.25 \frac{|s_k^2 - \log k|}{s_k + \sqrt{\log k}} \\ &\leq 1.25 \frac{1.07}{\sqrt{\log k - 1.07} + \sqrt{\log k}} \\ &\leq \frac{1.34}{\sqrt{\log k - 1.07} + \sqrt{\log k}} \leq \frac{0.73}{\sqrt{\log k}}. \end{aligned}$$

Finally, (recall that $\gamma_k \downarrow$) Lemma 1(b) implies that

$$P_3 \leq \frac{1}{\sqrt{2\pi}} \frac{\gamma_k}{\sqrt{\log k}} \leq \frac{1}{\sqrt{2\pi}} \frac{\gamma_{37}}{\sqrt{\log k}} \leq \frac{0.25}{\sqrt{\log k}}. \qquad \square$$

Before proving Theorem 4 we briefly describe the method of inversion, which has been much exploited in renewal theory; in fact, our proof below is modelled after Englund (1980). The important relation connecting the record time process and the counting process is

$$\{L(n) \geq k\} = \{\mu(k) \leq n\}. \tag{2.5}$$

The idea is that given a limit theorem for one of the processes the corresponding limit theorem for the other process follows from (2.5) by letting $k$ and $n$ tend to infinity jointly in a suitable manner.

**Proof of Theorem 4.** It is sufficient to consider $n$-values such that $4.3/\sqrt{n} < 1$, that is $n \geq 19$.

We connect $k$ and $n$ through the relation $\log k - n = \theta n^{3/4}$ and consider the three cases $\theta \geq \theta_0$, $\theta \leq -\theta_0$ and $|\theta| < \theta_0$, where, as it turns out later, a suitable choice of $\theta_0$ will be $\theta_0 = 0.74$.

Thus, we first assume that $\log k - n = \theta n^{3/4}$, $\theta \geq \theta_0$. Then, by Tjebyshev's inequality and (2.3),

$$\begin{aligned} P(L(n) \geq k) &= P(\mu(k) \leq n) = P(\mu(k) - m_k \leq n - m_k) \\ &\leq P(|\mu(k) - m_k| \geq m_k - n) \leq P(|\mu(k) - m_k| \geq \log k - n) \\ &\leq \frac{\operatorname{Var} \mu(k)}{(\log k - n)^2} = \frac{s_k^2}{\theta^2 n^{3/2}} \leq \frac{\log k}{\theta^2 n^{3/2}} = \frac{1}{\theta^2 \sqrt{n}} + \frac{1}{\theta n^{3/4}} \\ &\leq \left(\frac{1}{\theta_0^2} + \frac{1}{\theta_0 \sqrt[4]{19}}\right) \frac{1}{\sqrt{n}}. \end{aligned} \tag{2.6}$$

Moreover, by Lemma 1(a),

$$\Phi\left(\frac{n - \log k}{\sqrt{n}}\right) = \Phi(-\theta n^{1/4}) \leq \frac{1}{\theta_0^2 \sqrt{n}}. \tag{2.7}$$

The case $\log k - n = \theta n^{3/4}$, $\theta \leq -\theta_0$, is treated (almost) the same way. We have

$$\begin{aligned} 1 - P(L(n) \geq k) &= P(L(n) < k) = P(\mu(k) > n) = P(\mu(k) \geq n + 1) \\ &\leq \frac{\operatorname{Var} \mu(k)}{(n + 1 - m_k)^2} \leq \frac{\log k}{(n + 1 - \cdot\, k - \gamma_k)^2} \\ &\leq \frac{\log k}{(n - \log k)^2} \leq \left(\frac{1}{\theta_0^2} + \frac{1}{\theta_0 \sqrt[4]{19}}\right) \frac{1}{\sqrt{n}} \end{aligned} \tag{2.8}$$

and (since $\theta$ now is negative) that

$$1 - \Phi\left(\frac{n - \log k}{\sqrt{n}}\right) = \Phi(\theta n^{1/4}) \leq \frac{1}{\theta_0^2 \sqrt{n}}. \tag{2.9}$$

Finally, suppose that $|\log k - n| \leq \theta_0 n^{3/4}$. It follows from (2.5), Theorem 3 and Lemma 1(d) (with $y_0 = \sqrt[4]{19}$) that

$$\begin{aligned}
&\left| P(L(n) \geq k) - \Phi\left(\frac{n - \log k}{\sqrt{n}}\right) \right| \\
&= \left| P(\mu(k) \leq n) - \Phi\left(\frac{n - \log k}{\sqrt{n}}\right) \right| \\
&\leq \left| P(\mu(k) \leq n) - \Phi\left(\frac{n - \log k}{\sqrt{\log k}}\right) \right| + \left| \Phi\left(\frac{n - \log k}{\sqrt{\log k}}\right) - \Phi\left(\frac{n - \log k}{\sqrt{n}}\right) \right| \\
&\leq \frac{1.9}{\sqrt{\log k}} + \frac{2}{\mathrm{e}\sqrt{2\pi}} \frac{1}{(1 - \theta_0/\sqrt[4]{19})^{3/2}} \cdot \frac{1}{\sqrt{n}} \\
&\leq \left( 1.9 \cdot \sqrt{\frac{n}{\log k}} + \frac{2}{\mathrm{e}\sqrt{2\pi}} \frac{1}{(1 - \theta_0/\sqrt[4]{19})^{3/2}} \right) \frac{1}{\sqrt{n}} \\
&\leq \left( 1.9 \cdot \frac{1}{\sqrt{1 - \theta_0}} + \frac{2}{\mathrm{e}\sqrt{2\pi}} \cdot \frac{1}{(1 - \theta_0/\sqrt[4]{19})^{3/2}} \right) \frac{1}{\sqrt{n}}.
\end{aligned} \tag{2.10}$$

The upper bound thus obtained is

$$\max\left\{ \frac{2}{\theta_0^2} + \frac{1}{\theta_0\sqrt[4]{19}}, \frac{1.9}{\sqrt{1 - \theta_0}} + \frac{2}{\mathrm{e}\sqrt{2\pi}} \cdot \frac{1}{(1 - \theta_0/\sqrt[4]{19})^{3/2}} \right\} \frac{1}{\sqrt{n}}. \tag{2.11}$$

Numerical calculations for different values of $\theta_0$ show that for $\theta_0 = 0.74$ both estimates involved are slightly smaller than 4.3, which finishes the proof. □

Lars Holst has drawn my attention to the fact that $\mu(k)$ is more closely approximated by the $\mathrm{Po}(m_k)$-distribution than by the appropriate normal distribution. More precisely, let $V_k \in \mathrm{Po}(m_k)$. By the Stein-Chen method one then has

$$\sup_{A \subset Z_+} |P(\mu(k) \in A) - P(V_k \in A)| \leq (1 - \mathrm{e}^{-m_k}) \frac{\sum_{j=1}^k j^{-2}}{\sum_{j=1}^k j^{-1}} \leq \frac{\pi^2}{6 \log k} \tag{2.12}$$

and this rate is of the correct order of magnitude; see Barbour and Hall (1984).

This is interesting for the following reason. Namely, (2.12) shows that the discrepancy between $\mu(k)$ and the $\mathrm{Po}(m_k)$-distribution, in terms of variational distance, is of the order of magnitude $(\log k)^{-1}$. Now, by the ordinary Berry-Esseen theorem, the discrepancy between the $\mathrm{Po}(m_k)$-distribution and the normal distribution is of order of magnitude $(\log k)^{-1/2}$. The Poisson approximation of the counting process thus is better than the normal approximation; another way to express the situation is that the main contribution in the remainder term estimate in Theorem 3 stems from the normal approximation of the Poisson distribution.

However, in view of (1.1) this is not surprising. Namely, $\mu(k)$ can be interpreted (for $k$ large) as a collection of independent, rare, events, which is precisely what characterizes the Poisson process. Let us also recall that the point process of record times is asymptotically Poisson (see Resnick, 1987, p. 170).

## 3. Williams' representation

As a preliminary to the next section we now describe a representation due to Williams (1973), see also Resnick (1987, p. 194), after which we show how the even numbered conclusions in Theorem 2 follow and how the representation can be used to extend Theorem 2(v) to provide all limit points of the normalized sequence of record times.

Recall that $L(1)=1$. It is easy to see that

$$P(L(2)=l_2)=P(X_2,\dots,X_{l_2-1}<X_1<X_{l_2})=\frac{1}{l_2}\cdot\frac{1}{l_2-1} \tag{3.1}$$

and, similarly, for arbitrary $n$, that

$$P(L(k)=l_k,\, k=2,\dots,n)=\frac{1}{l_n(l_n-1)(l_{n-1}-1)\cdots(l_2-1)}, \tag{3.2}$$

from which it follows that $\{L(n),\, n\geq 1\}$ is a time-homogeneous Markov chain with transition probabilities

$$p_{ij}=P(L(n)=j\,|\,L(n-1)=i)=\frac{i}{j(j-1)},\quad j>i \tag{3.3}$$

(and $p_{ij}=0$ for $j\leq i$).

Williams' representation is the following explicit construction of a sequence of random variables with the same properties: Let $\{Z_k,\, k\geq 1\}$ be i.i.d. exp(1)-distributed random variables, set $A_1=1$ and

$$A_n=[A_{n-1}\exp\{Z_n\}]+1,\quad n\geq 2. \tag{3.4}$$

Then $\{A_n,\, n\geq 1\}$ is a time-homogeneous Markov chain and

$$P(A_n=j\,|\,A_{n-1}=i)=P(j-1\leq i\exp\{Z_n\}<j)=\frac{i}{j(j-1)},\quad j>i,$$

which shows that $\{A_n,\, n\geq 1\}$ is a representation of the sequence of record times. We thus can (and will) write $L(1)=1$ and

$$L(n)=[L(n-1)\exp\{Z_n\}]+1,\quad n\geq 2. \tag{3.5}$$

It now follows from (3.5) that $L(k-1)\exp\{Z_k\}<L(k)\leq L(k-1)\exp\{Z_k\}+1$ and thus, since $L(n)\geq n$, that

$$\exp\{Z_k\}<\frac{L(k)}{L(k-1)}\leq \exp\{Z_k\}+\frac{1}{L(k-1)}\leq \exp\{Z_k\}+\frac{1}{k-1}. \tag{3.6}$$

By taking logarithms and noting that $\log L(n)=\sum_{k=2}^{n}\log(L(k)/L(k-1))$ we obtain (after some computations)

$$\sum_{k=2}^{n} Z_k<\log L(n)\leq \sum_{k=2}^{n} Z_k+\log(n-1)+\gamma_{n-1}. \tag{3.7}$$

However, $\Gamma_n = \sum_{k=2}^{n} Z_k$ has a $\Gamma(n-1, 1)$-distribution, whose asymptotics are well known and this, together with (3.7) can now be used to derive asymptotics for $L(n)$.

As a first example, consider Theorem 2(ii) and (iv) (as promised at the end of Section 1). Since $\{\Gamma_n, n \geq 2\}$, normalized as there, is uniformly integrable (for general sums of i.i.d. random variables see e.g. Gut, 1988, Section I.4) it follows from (3.7) that the same is true for $\{\log L(n), n \geq 1\}$. These facts also imply that the moments converge as desired.

A second example is provided by the following result. Let $C(\{x_n\})$ denote the cluster set of the sequence $\{x_n\}$.

**Theorem 5.**

$$C\left(\left\{\frac{\log L(n) - n}{\sqrt{2n \log \log n}}, n \geq \mathrm{e}^{\mathrm{e}}\right\}\right) = [-1, 1] \quad a.s.$$

**Proof.** Set, for $n \geq 2$,

$$\Gamma_n^* = \frac{\sum_{k=2}^{n} Z_k - n}{\sqrt{2n \log \log n}},$$

with $\{Z_k, k \geq 1\}$ as before. Then

$$C(\{\Gamma_n^*, n \geq 3\}) = [-1, 1] \quad \text{a.s.}$$

(see De Acosta, 1983, for a nice proof). This, together with (3.7) and Lemma III.11.1 of Gut (1988) proves the conclusion. □

## 4. Convergence rates in the LLN and LIL for record times

One way to study the rate of convergence in, say, Theorem 2(i) is to study the convergence/divergence of $\sum a_n P(|\log L(n) - n| > \varepsilon n)$, where $\varepsilon > 0$ and $a_n$ is some numerical sequence, typically $n$ raised to some power. Another way is to investigate last exit times and the number of boundary crossings; for Theorem 2(i) this corresponds to studying $T(\varepsilon) = \sup\{n: |\log L(n) - n| > \varepsilon n\}$ and $N(\varepsilon) = \text{Card}\{n: |\log L(n) - n| > \varepsilon n\}$ (with $\sup \emptyset = 0$ and $\text{Card}\, \emptyset = 0$). Note that we always have $N(\varepsilon) \leq T(\varepsilon)$ and that the a.s. finiteness of $T(\varepsilon)$ for all $\varepsilon > 0$ is equivalent to Theorem 2(i).

In this section we present results related to the sequence of record times on the convergence/divergence of sums of the above kind and provide conditions for existence/nonexistence of moments for the last exit times and the number of boundary crossings. In the following section corresponding results are given for the counting process. Some results on moments of the number of boundary crossings as defined in Theorems 7–9 below have also been obtained by Nayak (1984).

**Theorem 6.** (i)(a) *For all* $\varepsilon>0$ *there exists* $t_0$, $0<t_0<1$, *such that for* $t<t_0$ *we have*

$$\sum_1=\sum_{n=1}^{\infty} e^{tn}P(|\log L(n)-n|>n\varepsilon)<\infty.$$

*In particular, there exist* $A>0$ *and* $\rho<1$ *such that*

$$P(|\log L(n)-n|>n\varepsilon)<A\rho^n, \quad n=1,2,\ldots.$$

*Furthermore*:

(b) $\sum_2=\sum_{n=1}^{\infty} e^{tn}P\left(\sup_{k\geqslant n}\left|\frac{\log L(k)-k}{k}\right|>\varepsilon\right)<\infty$ *for* $t<t_0$.

(ii) *Let* $T_L(\varepsilon)=\sup\{n: |\log L(n)-n|>n\varepsilon\}$. *For all* $\varepsilon>0$, $t<t_0$ *we have*

$E\exp\{T_L(\varepsilon)\}<\infty$.

(iii) *Let* $N_L(\varepsilon)=\text{Card}\{n: |\log L(n)-n|>n\varepsilon\}$. *For all* $\varepsilon>0$, $t<t_0$ *we have*

$E\exp\{N_L(\varepsilon)\}<\infty$.

**Proof.** (i) It follows from Williams' representation (and (3.7)) that

$$|\log L(n)-n|\leqslant|\Gamma_n-(n-1)|+\log n+1, \tag{4.1}$$

where $\Gamma_n\in\Gamma(n-1,1)$. Now, for $0<t<1$, we have

$$\begin{aligned} E\, e^{t|\Gamma_{n+1}-n|}&=\int_0^{\infty} e^{t|x-n|}\frac{1}{\Gamma(n)}x^{n-1}e^{-x}\,dx\\ &=\int_0^{n} e^{t(n-x)}\frac{1}{\Gamma(n)}x^{n-1}e^{-x}\,dx+\int_n^{\infty} e^{t(x-n)}\frac{1}{\Gamma(n)}x^{n-1}e^{-x}\,dx\\ &\leqslant e^{tn}\cdot(1+t)^{-n}+e^{-tn}\cdot(1-t)^{-n}. \end{aligned}$$

By Markov's inequality we thus obtain

$$P(|\Gamma_{n+1}-n|>n\varepsilon)\leqslant(e^{t(1-\varepsilon)}(1+t)^{-1})^n+(e^{-t(1+\varepsilon)}(1-t)^{-1})^n. \tag{4.2}$$

A simple investigation shows that $e^{t(1-\varepsilon)}(1+t)^{-1}<1$ for all $t>0$ when $\varepsilon\geqslant1$ and for $t<\varepsilon(1-\varepsilon)^{-2}$ when $\varepsilon<1$ and that $e^{-t(1+\varepsilon)}(1-t)^{-1}<1$ for $t<\varepsilon(1-\varepsilon)^{-2}$. This, together with the fact that $\log n+1=o(n)$ as $n\to\infty$, completes the proof of (i)(a).

Relation (i)(b) follows from (i)(a) and the fact that

$$\sum_2\leqslant\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} e^{tn}P(|\log L(k)-k|>k\varepsilon)\leqslant(1-e^{-t})^{-1}\sum_1. \tag{4.3}$$

(ii) First recall that, by Theorem 2(i), we have $P(T_L(\varepsilon)<\infty)=1$ for all $\varepsilon>0$. The conclusion follows from the observation that

$$\{T_L(\varepsilon)\geqslant n\}=\left\{\sup_{k\geqslant n}\left|\frac{\log L(k)-k}{k}\right|>\varepsilon\right\}, \tag{4.4}$$

partial summation and (i)(b) (cf. Gut, 1980, Section 8).

(iii) Immediate, since $N_L(\varepsilon) \leqslant T_L(\varepsilon)$. $\square$

**Remark 2.** It further follows from (i)(a) and arguments like those of Gut (1980) and Williams' representation that $\sum_n e^{tn} P(\max_{k \leqslant n} |\log L(k) - k| > n\varepsilon) < \infty$ for all $\varepsilon > 0$ and $t <$ some $t_0$, $t_0 > 0$. A similar remark applies to the results below.

A related result concerning the law of the iterated logarithm is next. The proofs of (i), (ii) and (iii) follow from the corresponding results for random walks and (3.7). The proofs of (i) and (iv) are inspired by Davis (1968a) and Slivka (1969).

**Theorem 7.** (i) *For all* $r > 0$ *and* $\varepsilon > 0$ *we have*

$$\Sigma_3 = \sum_{n=3}^{\infty} n^{r-1} P(|\log L(n) - n| > \varepsilon\sqrt{n \log\log n}) = +\infty.$$

(ii) *For* $\varepsilon > \sqrt{2r}$, $r \geqslant 1$, *we have*

$$\Sigma_4 = \sum_{n=3}^{\infty} \frac{1}{n} (\log n)^{r-1} P(|\log L(n) - n| > \varepsilon\sqrt{n \log\log n}) < \infty.$$

(iii) *For* $\varepsilon > 2$ *we have*

$$\Sigma_5 = \sum_{n=3}^{\infty} \frac{1}{n} \log n P(|\log L(n) - n| > \varepsilon\sqrt{n \log\log n}) < \infty,$$

$$\Sigma_6 = \sum_{n=3}^{\infty} \frac{1}{n} P\left(\sup_{k \geqslant n} \left|\frac{\log L(k) - k}{\sqrt{k \log\log k}}\right| > \varepsilon\right) < \infty.$$

(iv) *If* $\varepsilon \leqslant \sqrt{2r}$ *then* $\Sigma_4 = +\infty$ *and if* $\varepsilon \leqslant 2$ *then* $\Sigma_5$ *and* $\Sigma_6$ *diverge.*

(v) *Let* $T_L(\varepsilon) = \sup\{n: |\log L(n) - n| > \varepsilon\sqrt{n \log\log n}\}$. *Then*

$$E(T_L(\varepsilon))^r = +\infty \quad \text{for all } \varepsilon > 0 \text{ and } r > 0,$$

$$E(\log T_L(\varepsilon))^r < \infty \quad \text{for } \varepsilon > \sqrt{2(r+1)},\ r > 0,$$

$$E(\log T_L(\varepsilon))^r = +\infty \quad \text{for } \varepsilon < \sqrt{2(r+1)},\ r > 0,$$

$$E \log\log T_L(\varepsilon) < \infty \quad \text{for } \sqrt{2} < \varepsilon < 2.$$

(vi) *Let* $N_L(\varepsilon) = \text{Card}\{n: |\log L(n) - n| > \varepsilon\sqrt{n \log\log n}\}$. *All conclusions in* (v) *remain valid with* $T_L(\varepsilon)$ *replaced by* $N_L(\varepsilon)$.

**Proof.** (i) By Theorem 4 we have

$$|P(\log L(n) - n \geqslant \varepsilon\sqrt{n \log\log n}) - \Phi(-\varepsilon\sqrt{\log\log n})| \leqslant 4.3/\sqrt{n}. \tag{4.5}$$

Furthermore, (see e.g. Feller, 1968, p. 175), we have, as $n\to\infty$,

$$\Phi(-\varepsilon\sqrt{\log\log n})=1-\Phi(\varepsilon\sqrt{\log\log n})$$
$$\sim\varepsilon^{-1}(2\pi\log\log n)^{-1/2}(\log n)^{-\varepsilon^2/2}. \tag{4.6}$$

Now,

$$\begin{aligned}\textstyle\sum_3 &\geq \sum_{n=3}^{\infty} n^{r-1}P(\log L(n)-n>\varepsilon\sqrt{n\log\log n})\\ &=\sum_{n=3}^{\infty} n^{r-1}(P(\log L(n)-n>\varepsilon\sqrt{n\log\log n})-\Phi(-\varepsilon\sqrt{\log\log n}))\\ &\quad+\sum_{n=3}^{\infty} n^{r-1}\Phi(-\varepsilon\sqrt{\log\log n}).\end{aligned} \tag{4.7}$$

Since, by (4.5), $\sum n^{r-1}|P(\log L(n)-n>\varepsilon\sqrt{n\log\log n})-\Phi(-\varepsilon\sqrt{\log\log n})|<\infty$, for $r<\frac{1}{2}$ and all $\varepsilon>0$ and the second sum diverges for all $r>0$ and $\varepsilon>0$ it follows that $\sum_3$ diverges for $r<\frac{1}{2}$ and all $\varepsilon>0$ and, hence, since $\sum_3$ increases as $r$ increases, for all $r>0$ and $\varepsilon>0$.

(ii) and (iii) Let $\varepsilon>\sqrt{2r}$. By (4.5) (and a similar expression for the other tail) and (4.6) we have

$$\begin{aligned}\textstyle\sum_4 &\leq \sum_{n=3}^{\infty}\frac{(\log n)^{r-1}}{n}|P(|\log L(n)-n|>\varepsilon\sqrt{n\log\log n})\\ &\qquad -2(1-\Phi(\varepsilon\sqrt{\log\log n}))|\\ &\quad+2\sum_{n=3}^{\infty}\frac{(\log n)^{r-1}}{n}(1-\Phi(\varepsilon\sqrt{\log\log n}))<\infty.\end{aligned}$$

For $r=2$ this yields $\sum_5<\infty$.

(iv) Suppose that $\varepsilon\leq\sqrt{2r}$. The proof of the divergence of $\sum_4$ is similar to the proof of (i). We omit the details.

Now suppose that $\varepsilon\leq 2$. The divergence of $\sum_5$ follows from that of $\sum_4$ (with $r=2$). The divergence of $\sum_6$ follows from (v) (proved below) with $r=1$, the obvious analogue of (4.4) and partial summation.

(v) The nonexistence of moments follows from (i). For the remainder of (v) in the random walk case, see Russo (1988), Theorems 1 and 2 (for $r=1$ see also Gut, 1980, Theorem 8.3). The conclusions thus hold with $\log L(n)-n$ replaced by $\Gamma_n-(n-1)$ and, hence, by Williams' representation (and (4.1)) as desired.

(vi) The divergence parts follow from the corresponding results for random walks, see Slivka (1969) and Russo (1988), respectively, and (4.1) in the usual manner. Note, however, that, since Lemma 1 of Slivka (1969) carries over to the present context, the nonexistence of moments for $N_L(\varepsilon)$ is also a consequence of (i). The convergence parts of (vi) follow from (v), since $N_L(\varepsilon)\leq T_L(\varepsilon)$.

The proof of the theorem thus is complete. □

**Remark 3.** Alternative proofs of (i)-(iii) may be obtained by using the corresponding results for random walks together with Williams' representation.

## 5. Convergence rates in the LLN and LIL for the counting process

**Theorem 8.** (i) *For* $\varepsilon > 0$ *we have*

$$\Sigma_7 = \sum_{n=3}^{\infty} \frac{1}{n} \log n P(|\mu(n) - \log n| > \varepsilon \log n) < \infty,$$

$$\Sigma_8 = \sum_{n=3}^{\infty} \frac{1}{n} P\left(\sup_{k \geq n} \left|\frac{\mu(k) - \log k}{\log k}\right| > \varepsilon\right) < \infty.$$

(ii) *Let* $T_\mu(\varepsilon) = \sup\{n\colon |\mu(n) - \log n| > \varepsilon \log n\}$. *Then*

$$E \log T_\mu(\varepsilon) < \infty \quad \textit{for all } \varepsilon > 0.$$

(iii) *Let* $N_\mu(\varepsilon) = \mathrm{Card}\{n\colon |\mu(n) - \log n| > \varepsilon \log n\}$. *Then*

$$E \log N_\mu(\varepsilon) < \infty \quad \textit{for all } \varepsilon > 0.$$

**Proof.** The proof is based on exponential bounds. First, however, we recall some notation and estimates from Sections 1 and 2, namely

$$m_n = E\mu(n) = \sum_{k=1}^{n} k^{-1} \quad \text{and} \quad \log n \leq m_n \leq \log n + 1, \tag{5.1}$$

$$s_n^2 = \mathrm{Var}\,\mu(n) = \sum_{k=1}^{n} k^{-1}(1 - k^{-1}) \quad \text{and} \quad \log n + \gamma - \tfrac{1}{6}\pi^2 \leq s_n^2 \leq \log n. \tag{5.2}$$

Since $\{I_k - k^{-1}, k \geq 1\}$ are independent random variables with mean 0 which are uniformly bounded by 1, a minor modification of the proof of Gut (1980, Lemma 2.2) yields

$$P(|\mu(n) - m_n| > x) \leq 2 \exp\{-tx + \tfrac{1}{2}t^2 s_n^2(1 + \tfrac{1}{2}t)\}, \quad 0 < t \leq 1,$$

which, together with (5.2), provides the exponential bound

$$P(|\mu(n) - m_n| > x) \leq 2 \exp\{-tx + \tfrac{1}{2}(t^2 \log n)(1 + \tfrac{1}{2}t)\}, \quad 0 < t \leq 1. \tag{5.3}$$

(i) We first suppose that $0 < \varepsilon < 1$. An application of (5.3) with $x = \varepsilon \log n$ and $t = \varepsilon$ now shows that

$$P(|\mu(n) - m_n| > \varepsilon \log n) \leq 2 \exp\{-\tfrac{1}{2}\varepsilon^2(1 - \tfrac{1}{2}\varepsilon) \log n\} \leq 2n^{-\varepsilon^2/4}, \tag{5.4}$$

from which it follows that

$$\sum_{n=3}^{\infty} \frac{1}{n} \log n P(|\mu(n) - m_n| > \varepsilon \log n) \leq 2 \sum_{n=3}^{\infty} \frac{1}{n} \log n \cdot n^{-\varepsilon^2/4} < \infty. \tag{5.5}$$

Since the sum is decreasing as $\varepsilon$ increases we conclude that the sum in (5.5) converges for all $\varepsilon > 0$. In view of (5.1) it is now a trivial matter to replace $m_n$ by $\log n$, which proves that $\Sigma_7 < \infty$ as desired.

The convergence of $\sum_8$ follows as in the random walk case, cf. Baum and Katz (1965), Davis (1968a, b) and Gut (1978, 1980), together with (5.1). We omit the details. Note, however, that, since $\log\frac{1}{2}n \sim \log n$ as $n\to\infty$, the computations are somewhat easier (than in Davis, 1968a, b, and Gut, 1980).

(ii) Immediate from the fact that $\sum_8 < \infty$, the relation

$$\{T_\mu(\varepsilon)\geq n\}=\left\{\sup_{k\geq n}\left|\frac{\mu(k)-\log k}{\log k}\right|>\varepsilon\right\} \tag{5.6}$$

(cf. (4.4)) and partial summation.

(iii) Immediate from (ii) and the fact that $N_\mu(\varepsilon)\leq T_\mu(\varepsilon)$. □

**Remark 4.** $\sum_7$ (obviously) also converges without the factor $\log n$.

**Remark 5.** Since $\mu(n)\geq 0$ it follows that only the upper tail of $\mu(n)-\log n$ contributes for $\varepsilon>1$.

One could also investigate the sums

$$\sum_9=\sum_{n=3}^{\infty} n^{r-1}P(|\mu(n)-\log n|>\varepsilon\log n),$$

$$\sum_{10}=\sum_{n=3}^{\infty} n^{r-1}P\left(\sup_{k\geq n}\left|\frac{\mu(k)-\log k}{\log k}\right|>\varepsilon\right),$$

and the related moments of $N_\mu(\varepsilon)$ and $T_\mu(\varepsilon)$ for $r>0$, $\varepsilon>0$. However, we only have some negative results. Namely, the lower exponential bound, (5.1) and (5.2) yield

$$\begin{aligned}P(\mu(n)-\log n>\varepsilon\log n)&\geq P((\mu(n)-m_n)/s_n>\varepsilon(1+\eta)\sqrt{\log n})\\&\geq \exp\{-\tfrac{1}{2}\varepsilon^2(1+\eta)^2\log n\cdot(1+\gamma)\},\end{aligned} \tag{5.7}$$

for $\eta$ small, $n$ large and $\gamma>0$, from which it follows that

$$\sum_{10}\geq\sum_9=+\infty \quad \text{for } \varepsilon\leq\sqrt{2r}. \tag{5.8}$$

Moreover, by Slivka (1969, Lemma 1), which carries over to the present context, we have

$$E(T_\mu(\varepsilon))^r\geq E(N_\mu(\varepsilon))^r=+\infty \quad \text{for } \varepsilon\leq\sqrt{2r}. \tag{5.9}$$

A reasonable conjecture is that everything is finite for $\varepsilon>\sqrt{2r}$, but we have not been able to prove that this is the case.

Finally, the corresponding, albeit less exhaustive, result related to the law of the iterated logarithm.

**Theorem 9.** (i) *Let* $h(x)>0$ *be such that* $h(x)=o((\log x)^{\delta})$ *for some* $\delta$, $0<\delta<1$, *as* $x\to\infty$. *For all* $\varepsilon>0$ *we have*

$$\sum\nolimits_{11}=\sum_{n=27}^{\infty}\frac{1}{nh(n)}P(|\mu(n)-\log n|>\varepsilon\sqrt{\log n\log\log\log n})=+\infty.$$

*In particular, we have, for all* $\varepsilon>0$ *and* $r\geqslant 0$, *that*

$$\sum\nolimits_{12}=\sum_{n=27}^{\infty}n^{r-1}P(|\mu(n)-\log n|>\varepsilon\sqrt{\log n\log\log\log n})=+\infty$$

*and, for all* $\delta$, $0<\delta<1$, *and* $\varepsilon>0$, *that*

$$\sum\nolimits_{13}=\sum_{n=27}^{\infty}\frac{1}{n(\log n)^{\delta}}P(|\mu(n)-\log n|>\varepsilon\sqrt{\log n\log\log\log n})=+\infty.$$

(ii) *For* $\varepsilon>\sqrt{2r}$, $r\geqslant 1$, *we have*

$$\sum\nolimits_{14}=\sum_{n=27}^{\infty}\frac{(\log\log n)^{r-1}}{n\log n}P(|\mu(n)-\log n|>\varepsilon\sqrt{\log n\log\log\log n})<\infty.$$

(iii) *If* $\varepsilon\leqslant\sqrt{2r}$ *then* $\sum_{14}=+\infty$.

(iv) *Let* $T_{\mu}(\varepsilon)=\sup\{n\colon|\mu(n)-\log n|>\varepsilon\sqrt{\log n\log\log\log n}\}$, *let* $h(x)$ *be defined as in* (i) *and set*

$$H(x)=\int^{x}\frac{1}{yh(y)}\,\mathrm{d}y.$$

*Then*

$$EH(T_{\mu}(\varepsilon))=+\infty\quad\textit{for all } \varepsilon>0.$$

*In particular,* $E(T_{\mu}(\varepsilon))^{r}=+\infty$ *for all* $r>0$, $\varepsilon>0$ *and* $E(\log T_{\mu}(\varepsilon))^{r}=+\infty$ *for all* $r>0$, $\varepsilon>0$.

*Moreover,*

$$E(\log\log T_{\mu}(\varepsilon))^{r}=+\infty\quad\textit{for } \varepsilon\leqslant\sqrt{2r},\ r\geqslant 1.$$

**Proof.** (i) Let $\log_3 n$ denote $\max\{\log\log\log n, 1\}$. A trivial modification of (5.7) shows that

$$P(\mu(n)-\log n>\varepsilon\sqrt{\log n\log_3 n})\geqslant\exp\{-\tfrac{1}{2}\varepsilon^{2}(1+\eta)^{2}\log_3 n\cdot(1+\gamma)\},\tag{5.10}$$

for $\eta$ small, $n$ large and $\gamma>0$, which proves (i).

(ii) We follow the proof of Theorem 7(ii) and (iii) with Theorem 3 playing the role of Theorem 4 there. Thus,

$$|P(\mu(n)-\log n>\varepsilon\sqrt{\log n\log_3 n})-\Phi(-\varepsilon\sqrt{\log_3 n})|\leqslant 1.9/\sqrt{\log n},\tag{5.11}$$

and similarly for the other tail. Moreover (cf. (4.6)),

$$\Phi(-\varepsilon\sqrt{\log_3 n})=1-\Phi(\varepsilon\sqrt{\log_3 n})\sim\varepsilon^{-1}(2\pi\log_3 n)^{-1/2}(\log\log n)^{-\varepsilon^{2}/2}\tag{5.12}$$

as $n\to\infty$. The conclusion follows.

Alternatively one can apply (5.3) with $x=\varepsilon\sqrt{\log n \log_3 n}$ and $t=\varepsilon\sqrt{\log_3 n/\log n}$ and (5.2). Namely, let $n$ be so large that $0<t<2\eta<1$. Then

$$P(|\mu(n)-m_n|>\varepsilon\sqrt{\log n \log_3 n})\leq 2\exp\{-\tfrac{1}{2}\varepsilon^2(1-\eta)\log_3 n\}. \tag{5.13}$$

By inserting this into the expression for $\sum_{14}$ and noting that $\eta$ may be chosen arbitrarily small it follows that $\sum_{14}<\infty$ as claimed.

(iii) The proof of the first part of Theorem 7(iv) is modified by replacing the use of (4.5), (4.6) and Theorem 4 there by (5.10), (5.11) and Theorem 3 here. As in the convergence part, one can, alternatively, use the lower exponential bound (5.10).

(iv) Use (i), (iii), the obvious modification of (5.6) and partial summation. □

**Remark 6.** Unfortunately we have no positive result concerning the integrability of some function of $T_\mu(\varepsilon)$. See, however, Russo (1988, Theorem 2) for the random walk case.

We noted in the beginning of this section that the a.s. finiteness of $T_L(\varepsilon)$, as defined in Theorem 6, for all $\varepsilon>0$ was equivalent to the strong law of large numbers for the record times. A similar statement can be made for the $T.(\varepsilon)$-variables defined in Theorems 7-9. The results on (non)integrability of $T.(\varepsilon)$ provide information on, in some vague sense, 'how strong' the corresponding laws are. For example, $T_L(\varepsilon)$ as defined in Theorem 6 possesses a moment generating function, which implies that $T_L(\varepsilon)$ is 'very finite'. The other extreme is $T_\mu(\varepsilon)$ as defined in Theorem 10. Although $P(T_\mu(\varepsilon)<\infty)=1$ for all $\varepsilon>\sqrt{2}$, we have $E\log T_\mu(\varepsilon)=+\infty$ for all $\varepsilon>0$, which means that $T_\mu(\varepsilon)$ is so large that even $\log T_\mu(\varepsilon)$, on the average, is extremely large; in fact, so large that the law of the iterated logarithm for the counting process 'barely' holds.

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