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Cycle decompositions: From graphs to continua

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Abstract

We generalise a fundamental graph-theoretical fact, stating that every element of the cycle space of a graph is a sum of edge-disjoint cycles, to arbitrary continua. To achieve this we replace graph cycles by topological circles, and replace the cycle space of a graph by a new homology group for continua which is a quotient of the first singular homology group H_1. This homology seems to be particularly apt for studying spaces with infinitely generated H_1, e.g. infinite graphs or fractals.

Keywords: Infinite graph; Homology; Continuum

1. Introduction

1.1. Overview

In a recent series of papers, Diestel et al. showed that many well-known theorems about cycles in finite graphs remain true for infinite graphs provided one replaces the classical graph-theoretical concepts by topological analogues. For example, instead of graph cycles one uses topological circles. This approach has been very fruitful, not only extending theorems from the finite to the infinite case (see e.g. [5,6,16]), but also having further applications [18] and opening new directions [4,11,10,12,19]. See [7] for a survey on this project.

This paper is motivated by an attempt to generalise some of these graph-theoretical facts to continuous objects. And indeed, our main result is a generalisation of one of the most basic tools

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in the aforementioned project of Diestel et al., Theorem 1.3 below, from graphs to arbitrary continua. In order to achieve this generalisation we introduce a new homology that generalises the cycle space of graphs to arbitrary metric spaces. We use this homology to conjecture a characterisation of the continua embeddable in the plane.

1.2. Background and motivation

The cycle space $\mathcal{C}(G)$ of a finite graph G coincides with its first, simplicial or singular, homology group. As an example of the usefulness of this concept in graph theory, let me mention the following classical theorem of MacLane, providing an algebraic characterisation of the graphs embeddable in the plane.

Theorem 1.1. *(See MacLane [22,6].) A finite graph G is planar if and only if its cycle space $\mathcal{C}(G)$ has a 2-basis.*

Here, a 2-basis is a set B generating $\mathcal{C}(G)$ such that no edge of G is used by more than two elements of B. See [6] for more.

If the graph is infinite though, then Theorem 1.1 does not hold any more if $\mathcal{C}(G)$ is still taken to be the first simplicial or singular homology group [5]. However, Diestel and Kühn [8,9] introduced a new homology for infinite graphs, called the *topological cycle space* $\mathcal{C}(G)$, which allows a verbatim generalisation of Theorem 1.1:

Theorem 1.2. *(See Bruhn and Stein [5].) A locally finite graph G is planar if and only if its topological cycle space $\mathcal{C}(G)$ has a 2-basis.*

The topological cycle space allows for such generalisations of all the fundamental facts about the cycle space of a finite graph. It is defined as a vector space, over $\mathbb{Z}_2$, consisting of sets of edges of the graph. Namely, it contains those edge-sets of G that form topological circles in the end-compactification $|G|$ of G, as well as the sums of these edge-sets, where we allow sums of infinitely many summands as long as they are well defined. An important innovation in the approach of Diestel and Kühn is that even if one is interested in the graph G only, it is helpful to consider the larger space $|G|$ that also contains the *ends* of G. The interested reader can find more details and results about the topological cycle space in [6, Chapter 8.5] or [7]; these details are however not necessary for understanding the current paper. The topological cycle space $\mathcal{C}(G)$ of G is larger than the first simplicial homology group of G, since the latter does not have any element comprising infinitely many edges.

It is far less obvious, but true [10], that $\mathcal{C}(G)$ is on the other hand smaller than the first singular homology group of $|G|$. Consider for example the graph G of Fig. 1, which is a one-way infinite 'ladder'. The end-compactification $|G|$ of G is in this case its one-point compactification (graphs are considered as 1-complexes throughout the paper). Thus there is a loop σ in $|G|$, depicted in Fig. 1, starting at the top-left vertex v, winding around each of the infinitely many 4-gonal faces of G, reaching the point at infinity, then returning to v, and finally winding around the whole graph once in the clockwise direction without using any of the perpendicular edges. It turns out [10] that σ does not belong to the trivial element of $H_1(|G|)$, but it does correspond to the trivial element of $\mathcal{C}(G)$: it traverses each edge the same number of times in each direction; thus, seen as an element of $\mathcal{C}(G)$, it is the empty set of edges. A similar example can be obtained in the Hawaiian earing by contracting a spanning tree of G to a point. This pathological behaviour

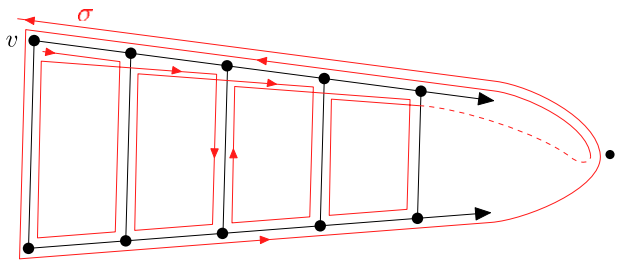

Fig. 1. A loop σ that is not null-homologous although we would like it to be.

of σ is due to the fact that although it winds around any hole the same number of times in each direction, it does so in such a complicated order that one cannot 'disentangle' it by adding only finitely many boundaries of 2-simplices. To put it in a different way, the homology class of σ is a product of infinitely many commutators.

This example shows that $\mathcal{C}(G)$ is indeed smaller than the first singular homology group of $|G|$ as claimed. However, this discrepancy between $\mathcal{C}(G)$ and $H_1(|G|)$ should by no means be considered as a shortcoming of $\mathcal{C}(G)$; for example, it is important for the truth of Theorem 1.2: the set of edge-sets of the 4-gonal faces of Fig. 1 form a 2-basis, but it cannot represent a loop like σ. It turns out, and is not hard to check, that $\mathcal{C}(G)$ is canonically isomorphic to the first Čech homology group of $|G|$; see [10] for details.

We would like to generalise graph-theoretical theorems like Theorem 1.1 to continuous spaces. The main aim of this paper is to achieve such a generalisation for the following fact, which has been a cornerstone in the aforementioned project of Diestel et al.

Theorem 1.3. *(See Diestel and Kühn [9].) Let G be a locally finite graph. Then every element of $\mathcal{C}(G)$ is a disjoint union of edge-sets of circles in $|G|$.*

Theorem 1.3 has found several applications in the study of $\mathcal{C}(G)$ [5,8,20] and elsewhere [16], and at least four proofs have been published; see [17] for an exposition.

Now in order to be able to generalise theorems like Theorem 1.3 or Theorem 1.1 to continuous spaces, we have to overcome two major difficulties: firstly, reformulate the assertions to rid them of any concepts, e.g. edges, that only make sense for graphs, and secondly, choose the right homology theory.

To see how the first difficulty can be overcome, suppose that the graph G in Theorem 1.3 is finite. We could then reformulate the assertion as follows

$$\text{Every element of } \mathcal{C}(G) \text{ has a representative of minimal length.} \tag{1.1}$$

Here, a *representative* is a formal sum of edge-sets of cycles. Indeed, this formulation is equivalent to that of Theorem 1.3 if G is finite: a representative of minimal length cannot have two summands C_1, C_2 containing the same edge e, for then we could delete e, and any other common edges, from both C_1, C_2 and combine the remaining paths into a new cycle or new set of cycles whose total length is smaller, since we saved some length by removing e.

Formulation (1.1) has the advantage that it makes sense for objects other than graphs if one replaces $\mathcal{C}(G)$ by some suitable homology group. The question now is, which homology should one use to extend this assertion beyond graphs. For example, singular homology will not do

because of the example of Fig. 1: the loop σ has finite length if we metrize that space using the Euclidean metric, but there are loops homologous to σ with arbitrarily small length, namely, those obtained by translating σ to the right by one or more squares. Singular homology can fail to satisfy (1.1) even if it is finitely generated, see Example 6.3.

1.3. A new homology

In view of the above discussion it is clear that in order to make assertion (1.1) true in general we need a homology group that excludes some 'redundant' elements of singular homology. In fact such an approach is often followed when dealing with 'wild' spaces, e.g. spaces with an uncountably generated fundamental group: in these cases many elements of the homotopy or homology groups do not capture some 'hole' of the space but rather represent a complicated way to wind around infinitely many holes, and one wants to omit these elements in order to obtain a smaller group that still reflects the structure of the space; see [12–14] for some examples. In many cases the better-known shape groups [23] also provide such simplifications of the corresponding homotopy or homology groups.

Earlier constructions of homology groups are not well-suited for our purposes as they either obviously fail to satisfy (1.1) or it is not clear how to assign lengths to their representatives. In this paper, we will introduce a homology group H_d that comes with a natural notion of length, has the topological cycle space as a special case (Section 11) and, more importantly, makes assertion (1.1) true for all compact metric spaces.

We define H_d as a quotient of the first singular homology group H_1. For example, we would like to identify the class of σ in the example of Fig. 1 with the trivial class. In order to decide which classes should be identified, we introduce a natural distance function on H_1, and identify any two elements if their distance is zero. This distance function is defined as follows. Intuitively, if two 1-cycles are not homologous, then there are some 'holes' in our space that witness this fact, and we assign a distance to the corresponding pair of classes of H_1 reflecting the 'size' of these holes. More precisely, the distance between two classes $c, d \in H_1$ is defined to be the minimal total area of a – possibly infinite – set of metric discs and cylinders that we could glue to our space X to make c and d homologous. These metric discs and cylinders must bear a metric such that this glueing does not affect the metric of X. See Section 3 for the formal definitions. In Section 6 we display some examples that justify this definition by showing that modifying it would make assertion (1.1) false.

An important feature of this distance function is that an infinite commutator product as the one of Fig. 1 can have distance zero to the trivial element. For example, patching all but finitely many of the 4-gonal faces in Fig. 1 by adding the missing trapeze would render σ null-homologous, and this can be accomplished by adding arbitrarily little area if we skip a lot of the 4-gonal faces.

The aforementioned distance function gives rise to a metric on H_d after the identifications have taken place, which turns H_d into a metrizable topological group. We will also consider the completion $\hat{H}_d$ of H_d, which will have the effect of strengthening our main result.

1.4. Main result

We can now state our main result.

Theorem 1.4. *For every compact metric space X and $C \in H_d(X)$, there is a σ-representative $(z_i)_{i\in\mathbb{N}}$ of C whose length is at most the infimum of the lengths of all representatives of C.*

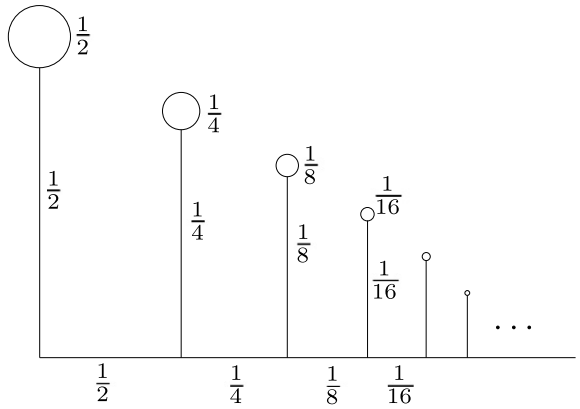

Fig. 2. A compact subspace of the real plane. The numbers denote the lengths of the corresponding segments.

Here, a σ-representative can intuitively be thought of as a sum of infinitely many 1-cycles z_i. Formally, a σ-representative of C is defined as a sequence $(z_i)_{i\in\mathbb{N}}$ whose initial subsequences give rise to a sequence $(\sum_{j\leqslant i} z_j)_{i\in\mathbb{N}}$ of 1-cycles the homology classes of which converge to C with respect to the metric of H_d; see Section 3 for details. The *length* of a σ-representative is the sum of the lengths of the simplices in z_i, the latter lengths being defined in the standard way (see Section 2).

For example, consider the subspace X of the real plane depicted in Fig. 2. Let σ be a closed 1-simplex $\sigma : [0, 1] \to X$ that traverses each of the infinitely many circles in this space precisely once and has finite length. Let $\beta \in H_1(X)$ denote the homology class of the 1-cycle 1σ. Note that for every representative of β there is a further representative of smaller length, obtained by avoiding to traverse some of the perpendicular segments. Thus no representative achieves a minimum length. Still, Theorem 1.4 yields a σ-representative $(z_i)_{i\in\mathbb{N}}$ of minimum length: let for example each z_i be a closed simplex winding around the ith circle once in a straight manner.

Theorem 1.4 implies Theorem 1.3. This can be shown by a similar argument as the one we used for the equivalence of the latter and (1.1) for finite G, except that if G is infinite we assign lengths to its edges to make their total length summable; see Section 11 for details. In fact, we obtain a strengthened version of Theorem 1.3. Furthermore, with Theorem 1.4 we generalise Theorem 1.3 to non-locally-finite graphs, achieving a goal set by the author in [19, Section 5]; see Section 11 for more.

For the proof of Theorem 1.4 we obtain an intermediate result which might be of independent interest. This result states that if $(H, +)$ is an abelian metrizable topological group, and a function $\ell : H \to \mathbb{R}^+$ is given satisfying certain natural properties that derive their intuition from the behaviour of lengths in geometry, then every element h of H can be 'decomposed' as a sum $h = \sum h_i$ so that $\ell(h) = \sum \ell(h_i)$ and no h_i can be decomposed further. See Section 8 for details.

1.5. Further problems and remarks

In this section we discuss some related conjectures for which strong evidence is available.

With Theorem 1.4 we extended a basic graph-theoretical tool to arbitrary compact metric spaces. It remains to try to exploit this in order to also extend results whose proofs are based on or related to this tool. A conjecture of this kind is offered in [19, Conjecture 6.1]. A further example is the following conjecture, which seeks an algebraic characterisation of the Peano continua embeddable in the plane, similar to that of Theorem 1.1.

Conjecture 1.5. *Let X be a compact, locally connected, metrizable space that is locally embeddable in S^2. Then X is embeddable in S^2 if and only if there is a simple set S of circles in X and a metric d inducing the topology of X so that the set $U := \{[\![\chi]\!] \in \hat{H}_d(X,d) \mid \chi \in S\}$ spans $\hat{H}_d(X,d)$.*

See [19, Conjecture 6.2] for more on this conjecture. For example, X here could be the Sierpinski triangle, in which case we could choose S to be the set of its triangular face boundaries, corroborating the conjecture.

A further question motivated by our main result is whether something similar holds for higher dimensions. It is straightforward to see how to generalise the definition of H_d: instead of topological discs and cylinders one has to use their higher dimensional analogues. Our proof cannot prove this, but many of our intermediate steps still work.

Problem 1.6. Generalise Theorem 1.4 to higher dimensions.

See Section 12 for more on this problem.

Although we can generalise our homology group H_d or $\hat{H}_d$ to higher dimensions, we do not obtain a homology theory in the sense of Eilenberg and Steenrod [15,21], since $H_d(X)$ depends not only on the topology of X but also on its metric. For the purposes of the current paper this is rather an advantage of H_d: since Theorem 1.4 holds for any choice of a compatible metric, we can affect the outcome of the application of the theorem by varying the metric. Still, it would be interesting to obtain a similar homology theory that does satisfy the axioms of Eilenberg and Steenrod by eliminating the dependence on the metric. Similarly, one could for example try to prove the following:

Conjecture 1.7. *Every Peano continuum X has a metric compatible with its topology such that the corresponding $\hat{H}_d$ coincides with the first Čech homology group of X.*

Theorem 11.1 below implies that this is true when X is the end-compactification of a locally finite graph. The condition that X be a Peano continuum is imposed because in a space that is not locally connected Čech homology may contain elements not represented by singular homology.

2. General definitions and basic facts

In this section we recall the standard definitions and facts that we will use later. Most of this is very well known but it is included for the convenience of the reader. For other standard terms used in the paper but not found in this section we refer to the textbooks [1] for topology, [21] for algebraic topology and [6] for graph theory.

For every metric space M, it is possible to construct a complete metric space M', called the *completion* of M, which contains M as a dense subspace. The completion M' of M has the following universal property [25]:

If N is a complete metric space and $f : M \to N$ is a uniformly continuous function, then there exists a unique uniformly continuous function $f' : M' \to N$ which extends f. The space M' is determined up to isometry by this property (and the fact that it is complete). (2.1)

Next, we recall the definition of the length of a topological path $\sigma : [a, b] \to M$ in a metric space (M, d). For a finite sequence $S = s_1, s_2, \dots, s_k$ of points in $[a, b]$, let $\ell(S) := \sum_{1 \leqslant i < k} d(\sigma(s_i), \sigma(s_{i+1}))$, and define the *length* of σ to be $\ell(\sigma) := \sup_S \ell(S)$, where the supremum is taken over all finite sequences $S = s_1, s_2, \dots, s_k$ with $a = s_1 < s_2 < \cdots < s_k = b$. This definition coincides with that of the 1-dimensional Hausdorff measure of $\sigma([a, b])$ when σ is injective.

The n-dimensional Hausdorff measure of M is defined by

$$HM^n(M) := \lim_{\delta \to 0} \inf \left\{ \sum_i diam(U_i)^n \;\middle|\; \bigcup_i U_i = M,\ diam(U_i) < \delta \right\},$$

the infimum taken over all countable covers $(U_i)_{i \in \mathbb{N}}$ of M by sets U_i of diameter less than δ.

We will also need the following.

Lemma 2.1 *(Heine–Cantor Theorem). Let M be a compact metric space, and let $f : M \to N$ be a continuous function, where N is a metric space. Then f is uniformly continuous.*

3. Definitions and basic facts: $\hat{H}_d$, σ-representatives, and length; statement of main result

Let X be any topological space, fixed throughout the paper, and consider its first singular homology group $H_1 = H_1(X; \Gamma)$ over a group Γ. Our results are stated and proved for Γ being any of the groups $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$. We restrict ourselves to those groups because we want to make use of the absolute value $|a|$ of an element a of Γ.

As mentioned in Section 1.3 we want to put a distance function on H_1 and identify any two elements if their distance is zero. This distance between two classes b, c measures the total area of the 'holes' that we have to 'patch' to make b equivalent to c, in a sense that we will soon make precise. Intuitively, we are going to glue some spaces of a special form to X in order to make b equivalent to c, and measure the area of those spaces. Another way of saying that 'we glue some spaces to X' is to say that 'we embed X into a larger space', and I found it more convenient to adhere to the second alternative. This motivates the following definition.

Definition 3.1. An *area extension* (X', ι) of X is a metric space X' in which X is embedded by an isometry $\iota : X \to X'$ such that each component of $X' \backslash \iota(X)$ is either a *metric disc* or a *metric cylinder*, i.e. a metric space homeomorphic to either $\{x \in \mathbb{R}^2 \mid |x| < 1\}$ or $\{x \in \mathbb{R}^2 \mid 1 < |x| < 2\}$. The *excess area* of this area extension is the sum of the areas of the components of $X' \backslash \iota(X)$.

The *area* of such a component can be defined as its 2-dimensional Hausdorff measure. However, in the area extensions that we will actually use, each such component is either a domain of $\mathbb{R}^3$ or a finite union of such domains. Thus the reader may choose, for simplicity, to append to the definition of an area extension that each component of $X' \backslash \iota(X)$ is a domain of $\mathbb{R}^n$, and use his favourite definition of *area*; our proofs still apply without modification. I chose the above more generic definition because it leads to a stronger main result without complicating the proofs.

The effect of a metric disc in an area extension is to make a loop bounding it null-homologous. Similarly, the effect of a metric cylinder is to make two loops homologous to each other. Note that the latter could also be achieved by using two discs to make each of the two loops null-homologous. Thus one could wonder if we really need to allow for metric cylinders in Definition 3.1. Example 6.3 below shows however that we do need these metric cylinders in order to make our main result true.

We now define a pseudo-metric d_1 on the singular homology group $H_1(X)$ of X. Given two elements $[\phi], [\chi]$ of $H_1(X)$, where ϕ and χ are n-chains, let $d_1([\phi], [\chi])$ be the infimum of the excess areas of all area extensions X' of X such that ϕ and χ belong to the same element of $H_1(X')$.

It follows easily by the definitions that

$$d_1 \text{ satisfies the triangle inequality.} \tag{3.1}$$

However, d_1 is not yet a metric, since there may exist $c \neq f \in H_1$ with $d(c, f) = 0$: for example, the homology class c of the loop of Fig. 1 satisfies $d(c, \mathbb{0}) = 0$ although $c \neq \mathbb{0}$ as proved in [10]. Still, declaring $c, f \in H_1$ to be equivalent if $d(c, f) = 0$ and taking the quotient with respect to this equivalence relation we obtain the group $H_d = H_d(X)$; the group operation on H_d can be naturally defined for every $c, d \in H_d$ by choosing representatives $\alpha \in c$ and $\beta \in d$ and letting $c + d := [\![\alpha + \beta]\!]$ be the class in H_d containing the element $\alpha + \beta$ of H_1. To see that this sum is well defined, i.e. does not depend on the choice of α and β, note that the union of two extensions of X of excess area at most ϵ each is an extension of X of excess area at most 2ϵ.

We will use the notation $[\![\gamma]\!]$, where γ is either an element of $H_1(X)$ or a 1-cycle, to denote the equivalence class in $H_d(X)$ containing γ or $[\gamma]$ respectively, where $[\chi]$ always denotes the element of $H_1(X)$ containing the 1-cycle χ.

Now d_1 induces a distance function on H_d, which we will, with a slight abuse, still denote by d_1: for any $[\![\phi]\!], [\![\chi]\!] \in H_d$ let $d_1([\![\phi]\!], [\![\chi]\!]) := d_1([\phi], [\chi])$; it is an easy consequence of (3.1) that this is well defined, and that d_1 is now a metric on H_d.

Definition 3.2. We now define a new homology group $\hat{H}_d = \hat{H}_d(X)$ of X to be the completion of H_d with respect to the metric d_1. The operation of $\hat{H}_d$ is defined, for every $C, D \in \hat{H}_d$, by $C + D := \lim_i (c_i + d_i)$ where $(c_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in C and $(d_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in D.

If X is compact and well-behaved then it might be the case that $\hat{H}_d$ is complete, which means that it coincides with H_d. However, this is not always the case; see Example 6.4. If $C \in H_d$ then we will sometimes, with a slight abuse of notation, still use the symbol C to denote the element of $\hat{H}_d$ corresponding to C, that is, the equivalence class of the constant sequence (C).

$\hat{H}_d$ is by definition a metrizable abelian topological group. If defined over $\mathbb{R}$ then it can also be viewed as a Banach space.

The operation $C + D$ in Definition 3.2 is well defined since, by (3.1), $c_i + d_i$ is a Cauchy sequence too and it does not depend on the choice of c_i and d_i.

The following observation, which is easy to prove, can be used to obtain an alternative definition for the addition operation $C + D$, where one first adds 1-cycles and then considers their homology classes rather than the other way round. Here (ϕ_i) and (χ_i) are sequences of 1-cycles.

Lemma 3.3. *Let* $(\|\phi_i\|)$ *and* $(\|\chi_i\|)$ *be Cauchy sequences in* H_d. *Then* $\lim\|\phi_i + \chi_i\| = \lim(\|\phi_i\| + \|\chi_i\|)$.

Before we go on to prove our main result about $\hat{H}_d$ we should pause to think whether we just identified all elements of H_1 to the zero element to obtain a trivial $\hat{H}_d$, which would make our main result void. In fact, this can happen in certain pathological spaces, but we will show that, for example, $\hat{H}_d(S^1)$ is not trivial, and this can be applied to show the non-triviality of $\hat{H}_d$ for many other spaces. See Section 6 for more.

A *σ-representative* of $C \in \hat{H}_d$ is an infinite sequence $(z_i)_{i\in\mathbb{N}}$ of 1-cycles $z_i \in Z_1$ such that the sequence $(\|\sum_{j\leqslant i} z_j\|)_{i\in\mathbb{N}}$ is a Cauchy sequence in C. One can think of a σ-representative as a "1-cycle" comprising the infinitely many 1-simplices z_i. Later on (Section 9.1) we will rigorously define infinite sums of elements of $\hat{H}_d$, and it turns out that $C = \sum_i \|z_i\|$ for every representative (z_i) of C.

For example, in Fig. 4, we can build a σ-representative of the class of the loop described there by letting z_i be a 1-simplex going around the ith circle.

One of the central concepts in our main Theorem 3.4 is the *length* of an element of $\hat{H}_d$. To define this we first need to define the length of a simplex, a 1-chain, and an element of H_1. With a considerable abuse of notation, we will denote the length of any of those objects by $\ell()$.

Since a simplex χ is by definition a topological path, we can use the standard definition of its length $\ell(\chi)$ as in Section 2. We can then define the length of a 1-chain $z = \sum_i a_i\chi_i$ by $\ell(z) := \sum_i |a_i|\ell(\chi_i)$, and consequently the length $\ell(\beta)$ of an elements β of H_1 by $\ell(\beta) := \inf_{z\in\beta} \ell(z)$. Finally, for $C \in \hat{H}_d$, we define $\ell(C) := \inf_{\beta_i} \lim_i \ell(\beta_i)$ where the infimum ranges over all sequences $(\beta_i)_{i\in\mathbb{N}}$ with $\beta_i \in H_1$ such that $(\|\beta_i\|)_{i\in\mathbb{N}}$ is a Cauchy sequence in C and $\lim_i \ell(\beta_i) \in \mathbb{R}^+ \cup \{\infty\}$ exists.

We can now state our main result, Theorem 1.4, in a stronger and more precise form. Recall that a 1-*simplex* is a continuous function $\sigma : [a, b] \to X$. If $\sigma(a) = \sigma(b)$ then σ is called a *closed simplex*, and if moreover σ is injective on $[a, b)$ then it σ is called a *circlex* (note the similarity to a *circle*, i.e. a homeomorph of S^1).

Theorem 3.4. *For every compact metric space* X *and* $C \in \hat{H}_d(X)$, *there is a* σ*-representative* $(z_i)_{i\in\mathbb{N}}$ *of* C *with* $\sum_i \ell(z_i) = \ell(C)$.

In particular, for every other σ*-representative* $(w_i)_{i\in\mathbb{N}}$ *of* C *we have* $\sum_i \ell(z_i) \leqslant \sum_i \ell(w_i)$.

Moreover, if $\ell(C) < \infty$ *then* $(z_i)_{i\in\mathbb{N}}$ *can be chosen so that each* z_i *is a circlex.*

As a consequence we can now simplify the definition of $\ell(C)$ once we have proved Theorem 3.4: the following assertion yields an equivalent definition.

Corollary 3.5. *For every compact metric space* X *and* $C \in \hat{H}_d(X)$ *we have* $\ell(C) = \inf \sum_i \ell(z_i)$, *the infimum ranging over all* σ*-representatives* $(z_i)_{i\in\mathbb{N}}$ *of* C.

Proof. Theorem 3.4 immediately yields $\ell(C) \geqslant \inf \sum_i \ell(z_i)$. The reverse inequality follows from the definition of $\ell(C)$: given any σ-representative $(z_i)_{i\in\mathbb{N}}$ of C we can let $\beta_i := [\sum_{j\leqslant i} z_j]$, and as $(\|\beta_i\|)_{i\in\mathbb{N}}$ is a Cauchy sequence in C by the choice of (z_i), we have $\ell(C) \leqslant \lim_i \ell(\beta_i) \leqslant \lim_i \ell(\sum_{j\leqslant i} z_j) = \sum_i \ell(z_i)$. □

4. Isoperimetric properties of lengths

In this section we prove two basic facts relating length and area in metric spaces. The reader is encouraged to skip this section during the first reading of the paper and come back when it becomes relevant.

The following lemma yields a kind of isoperimetric property for arbitrary metric spaces: it shows that any "hole" can be "filled in" by an area proportional to the square of the perimeter of the hole.

Lemma 4.1. *There is a universal constant U such that for every metric space (X, d_X) and every closed curve $\sigma : I \to X$ there is an area extension (X', ι) of excess area at most $U\ell^2(\sigma)$ in which σ is null-homotopic.*

Moreover, X' can be chosen so that $X'\backslash\iota(X)$ is a metric disc with diameter less than $\ell(\sigma)$.

Proof. Pick a (geometric) circle D of length $\ell := \ell(\sigma)$ in $\mathbb{R}^3$, and a continuous mapping f from D to I such that corresponding subpaths have equal lengths; that is, for any subarc D' of D we have $\ell(D') = \ell(\sigma \restriction f(D'))$. Let S be a closed hemisphere in $\mathbb{R}^3$ having D as its equator, and give S its path metric (i.e. the distance d_S of two points in S is defined to be the minimum length of an arc in S between these two points). Now in order to obtain the desired area extension X', we glue a copy of S along the image of σ in X using $\sigma \circ f$ as an identifying map. We still have to specify a metric d' for X'. Note that by the choice of d_S, for every pair of points x, y in the domain of $\sigma \circ f$ we have

$$d_S(x, y) \geqslant d_X\big(\sigma \circ f(x), \sigma \circ f(y)\big). \tag{4.1}$$

This allows us to extend the metric d_X of X into a metric d' of X' as follows. Let $d'(z, w) = d_X(z, w)$ for every pair of points z, w of X, including points that got identified with points of S. If $z \in S$ and $w \in X$, then let $d'(z, w) = \inf_y\{d_S(z, y) + d_X(\sigma \circ f(y), w)\}$, the infimum taken over all points y in D. Finally, if $z, w \in S$ let

$$d'(z, w) = \min\Big\{d_S(z, w), \inf_{y, y'}\big\{d_S(z, y) + d_X\big(\sigma \circ f(y), \sigma \circ f\big(y'\big)\big) + d_S\big(y', w\big)\big\}\Big\},$$

the infimum taken over all pairs of points y, y' in D, even if $y = y'$. It is an easy exercise to check, using (4.1), that d' is indeed a metric, and that (X', id) is an area extension of X of excess area at most the area of S.

The Euclidean area of S is $2\pi R^2$ for $R := \ell/2\pi$. Since we consider the path metric d_S on S, distances are greater by a factor of up to π compared to the Euclidean metric, thus $area(S) \leqslant 2\pi R^2\pi^2 = \pi\ell^2/2$ and we can take $U = \pi/2$. Moreover, the diameter of S is by construction $\ell(\sigma)/2$. This completes the proof.

A point that might require some clarification is that we are not assuming that the closed curve σ is injective in its interior. If it is not, then the closure of $X'\backslash X$ is not necessarily a closed disc, but still $X'\backslash X$ itself is an (open) metric disc as the reader can check, and so X' is indeed an area extension. □

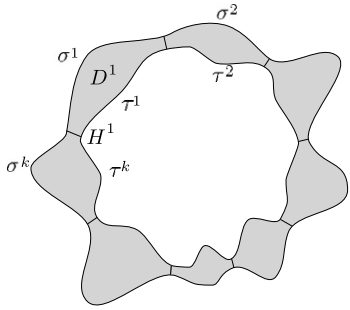

Fig. 3. Two curves that are close to each other can be made homotopic by a cylinder of relatively small area.

The following observation about the real numbers is an easy exercise.

$$\text{For every } \ell, \epsilon \in \mathbb{R}^+ \text{ there is an } r \in \mathbb{R} \text{ such that if } a_1, a_2, \dots, a_k \text{ are positive real numbers with } a_i < r \text{ for every } i \text{ and } \sum a_i = \ell, \text{ then } \sum a_i^2 < \epsilon. \tag{4.2}$$

Our previous lemma shows that a 'hole' of small perimeter can be patched using relatively little area. Our next result performs a similar task: it shows that if two holes are bounded by curves that are 'close' to each other, then the corresponding homology classes can be made equivalent using relatively little area. The following definition makes this concept of 'closeness' precise; see also Fig. 3.

Definition 4.2. Let $\sigma, \tau : I \to X$ be two closed curves in a metric space X. We will say that σ and τ are δ-*close*, if $|\ell(\sigma) - \ell(\tau)| < \delta$ and moreover there are subdivisions $\sigma^1, \sigma^2, \dots, \sigma^k$ and $\tau^1, \dots, \tau^k$ of σ and τ respectively that fulfill the following requirements for every $i \in \{1, \dots, k\}$:

(i) $\ell(\sigma^i) < \delta$ and $\ell(\tau^i) < \delta$,
(ii) $|\sum_{j \leqslant i} \ell(\sigma^j) - \sum_{j \leqslant i} \ell(\tau^j)| < \delta$, and
(iii) if p, q are the vertices of σ^i and p', q' are the vertices of τ^i then $d(p, p') < \delta/k$ and $d(q, q') < \delta/k$.

Define the *width* of a homotopy $h : [0, 1] \times I \to X$ to be

$$\sup\big\{d\big(h((0, y)), h((x, y))\big) \,\big|\, x \in [0, 1],\ y \in I\big\}.$$

We say that a curve $\sigma : I \to X$ has *constant speed* c if for every subinterval $[a, b]$ of I we have $\frac{\ell(\sigma([a,b]))}{b-a} = c$.

We can now state our next lemma.

Lemma 4.3. *For every $\epsilon, l \in \mathbb{R}_+$ with $\epsilon < l$ there is an $f(l, \epsilon) \in \mathbb{R}_+$ such that for every metric space X, and every two closed curves $\sigma, \tau : I \to X$ in X of length less than $l + \epsilon$ that are $f(l, \epsilon)$-close, there is an extension X' of X of excess area less than ϵ in which there is a homotopy h between σ and τ. Moreover, if σ and τ have constant speed then X' and h can be chosen so that the width of h is less than $5 f(l, \epsilon)$.*

Proof. Suppose the closed curves σ, τ of length less than $l+\epsilon$ are δ-close for some real number δ much smaller than l, and let $\sigma^1, \dots, \sigma^k$ and $\tau^1, \dots, \tau^k$ be subdivisions of σ and τ respectively as in Definition 4.2. In order to construct the desired extension X', start by adding to X, for every $1 \leqslant i \leqslant k$, an isometric copy H^i of the real interval $[0, m^i]$, where m^i is the distance in X between the first vertex p_i of σ^i and the first vertex p'_i of τ^i; then identify one endpoint of H^i with p_i and the other with p'_i, see Fig. 3. After having done so, note that concatenating, for every i, the "paths" σ^i, H^{i+1}, τ^i (inversed) and H^i we can obtain a closed curve c^i of length

$$\ell\big(c^i\big) < \delta(2 + 2/k). \tag{4.3}$$

As in the proof of Lemma 4.1, we can for every i glue a disc D^i of small diameter along the image of c^i to obtain an extension of X of excess area at most $U\ell^2(c^i)$ in which c^i is contractible. Uniting all these extensions – identifying points corresponding to the same point of X – we obtain the extension X' of excess area $V \leqslant U\sum \ell^2(c^i)$. It is easy to see that σ is indeed homotopic to τ in X'. Note that

$$\sum \ell\big(c^i\big) < \ell(\sigma) + \ell(\tau) + 2\sum \ell\big(H^i\big) \leqslant 2(l + \epsilon + \delta) < 3l,$$

and as the length of each c^i is bounded from above by (4.3), it follows from (4.2) that choosing δ small enough we can achieve $V < \epsilon$ as desired. Thus we can let $f(l, \epsilon) := \delta$ for such a δ.

To prove the second sentence of the assertion, suppose now that σ and τ have constant speed, and define X' as above. We are now going to construct the desired homotopy h. To begin with, let $h(0, x) = \sigma(x)$ and $h(1, x) = \tau(x)$. Moreover, for every $1 \leqslant i \leqslant k$ let h map the straight line segment L^i in $[0, 1] \times [0, 1]$ joining the preimages of the first vertices of σ^i, τ^i homeomorphically to H^i. Note that the segments L^i do not intersect each other except perhaps at their endpoints. Then, extend h continuously to the rest of $[0, 1] \times [0, 1]$, mapping the area bounded by L^i and L^{i+1} to the disc D^i.

We claim that the width of h is less than $5f(l, \epsilon)$. To see this, consider a point $p = (x, y) \in [0, 1] \times [0, 1]$, let D be the disc containing $h(p)$ (or one of the discs containing $h(p)$ if we were unlucky and $h(p)$ lies in some H^i) and let D' be a disc whose boundary contains $h(p')$ where $p' := (0, y)$. Requirement (ii) of Definition 4.2 provides a lower bound for the angles that the segments L^i form with the segment $\{0\} \times [0, 1]$; more precisely, requirement (ii) and the fact that σ and τ have constant speed implies that there is a point $t = (0, t_1) \in [0, 1] \times [0, 1]$ such that $h(t) \in D$ and the length μ of the restriction of $h(0, x) = \sigma(x)$ to the interval between t and p' is at most $2f(l, \epsilon)$. Indeed, let L^m be the segment separating p from p' in $[0, 1] \times [0, 1]$, and let $t = (0, t_1)$ and $t' = (0, t'_1)$ be the endpoints of L^m in $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$ respectively. By the choice of L^m we have $|t_1 - t'_1| \geqslant |t_1 - y|$. But recall that t, t' are the preimages of first vertices of σ^m, τ^m respectively, and since σ and τ have constant speed, and by (ii) of Definition 4.2 there holds $|\sum_{j \leqslant m} \ell(\sigma^j) - \sum_{j \leqslant m} \ell(\tau^j)| < f(l, \epsilon)$, it follows that the length μ of the restriction of $h(0, x) = \sigma(x)$ to the interval between t and p' is at most $2f(l, \epsilon)$ as claimed.

By the definition of length, this implies $d(h(t), h(p')) \leqslant 2f(l, \epsilon)$. Moreover, as both $h(t), h(p)$ lie in D, and D was chosen so that its diameter is at most $3f(l, \epsilon)$, we have $d(h(p), h(p')) \leqslant 3f(l, \epsilon) + 2f(l, \epsilon) = 5f(l, \epsilon)$. Since p was chosen arbitrarily, the last inequality proves that the width of h is at most $5f(l, \epsilon)$. □

5. Basic facts about lengths

In this section we prove some basic facts about lengths of homology classes, as defined in Section 3, which we will need later.

Our first task is to prove that $\ell(C)$ is attained by some sequence (β_i) for every $C \in \hat{H}_d$:

Observation 5.1. *For every $C \in \hat{H}_d$ there is a sequence $(\beta_i)_{i\in\mathbb{N}}$ with $\beta_i \in H_1$ such that $(\|\beta_i\|)_{i\in\mathbb{N}}$ is a Cauchy sequence in C and $\ell(C) = \lim_i \ell(\beta_i)$.*

Note that this observation follows immediately from our main result Theorem 3.4, by taking $\beta_i = [\sum_{j\leqslant i} z_j]$, but Theorem 3.4 is much stronger. As we will use Observation 5.1 in the proof of Theorem 3.4 we have to prove the former separately:

Proof of Observation 5.1. If $\ell(C) = \infty$ then the assertion is easily seen to be true, so suppose $\ell(C) < \infty$. Pick a non-constant sequence $(h_i)_{i\in\mathbb{N}}$, $h_i \in \mathbb{R}$, that converges to $\ell(C)$ from above. For $j = 1, 2, \ldots$, let $(\beta_i^j)_{i\in\mathbb{N}}$ be a sequence of elements of H_1 such that $(\|\beta_i^j\|)_{i\in\mathbb{N}}$ is a Cauchy sequence in C and $\lim_i \ell(\beta_i^j) < h_j$; such a sequence exists by the definition of $\ell(C)$. Pick an index $k \in \mathbb{N}$ such that $\ell(\beta_k^j) < h_j$ and $d_1(C, \|\beta_k^j\|) < 2^{-j}$, and let $\beta_j := \beta_k^j$; such a k exists by the choice of (β_i^j).

By construction, the sequence $(\beta_i)_{i\in\mathbb{N}}$ we just constructed is a Cauchy sequence in C and satisfies $\lim_i \ell(\beta_i) = \ell(C)$. □

Since for every $\beta \in H_1$ we can, by the definition of $\ell(\beta)$, find 1-cycles in β with lengths arbitrarily close to $\ell(\beta)$, we obtain with the above observation

Corollary 5.2. *For every $C \in \hat{H}_d$ there is a sequence of 1-cycles $(\phi_i)_{i\in\mathbb{N}}$ such that $(\|\phi_i\|)_{i\in\mathbb{N}}$ is a Cauchy sequence in C and $\ell(C) = \lim_i \ell(\phi_i)$.*

With Lemma 4.1 Corollary 5.2 easily yields

Observation 5.3. *If $C \neq \mathbb{0} \in \hat{H}_d$ then $\ell(C) > 0$.*

Next, we check that the lengths of elements of H_1 satisfy a triangle inequality:

Lemma 5.4. *Let X be a metric space and let ϕ, χ be two 1-chains in X. Then $\ell([\phi + \chi]) \leqslant \ell([\phi]) + \ell([\chi])$ (and thus $\ell([\phi - \chi]) \geqslant \ell([\phi]) - \ell([\chi])$).*

Proof. It is a trivial fact that if ϕ', χ' are 1-chains in X, then $\ell(\phi' + \chi') \leqslant \ell(\phi') + \ell(\chi')$. The assertion now easily follows from the definition of $\ell([\phi])$, since if $b_1, b_2 \in B_1'$ then $\ell(\phi + \chi + b_1 + b_2) \leqslant \ell(\phi + b_1) + \ell(\chi + b_2)$. □

From this we easily obtain a triangle inequality for elements of $\hat{H}_d$ too:

Corollary 5.5. *Let X be a metric space and let $C, D \in \hat{H}_d(X)$. Then $\ell(C + D) \leqslant \ell(C) + \ell(D)$ (and thus $\ell(C - D) \geqslant \ell(C) - \ell(D)$).*

6. Examples

In this section we show examples that explain some of our choices in the preceding definitions and statements.

We defined H_d as a quotient of H_1 by identifying pairs of elements with distance zero. This identification entails the danger of identifying all of H_1 with the trivial element, which would make H_d and $\hat{H}_d$ trivial and our main result void. And indeed, in certain pathological spaces X, e.g. when each element of $H_1(X)$ can be represented as an infinite product of commutators, this could happen; an example of such a spece can be found in [3]. However, the following basic example when $X = S^1$ shows that H_d and $\hat{H}_d$ are not trivial when it should not be:

Theorem 6.1. $\hat{H}_d(S^1) \cong H_1(S^1)$.

Proof. Let σ be a circlex in S^1. Then $H_1(S^1)$ is generated by the corresponding homology class $[\sigma]$. Thus all we need to show is that $[\sigma]$ is not identified with the trivial element $\mathbb{0}$ of $H_1(S^1)$; in other words, that there is a lower bound M such that every area extension of S^1 in which σ is null-homologous has excess area at least M.

So let (S', ι) be an area extension of S^1 in which σ is null-homologous. Thus there is a 2-chain $\mathcal{B}$ in S' whose boundary is σ. From now on we assume for simplicity that the group of coefficients on which $H_1(S^1)$ is based is $\mathbb{Z}$; the interested reader will be able to adapt our arguments to other groups of coefficients.

We may assume without loss of generality that $\mathcal{B}$ consists of a single 2-simplex ρ whose boundary is a subdivision of σ into three subsimplices, for otherwise we can combine pairs of 2-simplices of $\mathcal{B}$ together to get a shorter 2-chain. All we need to show now is that the area $A(P)$ of the image P of ρ is bounded from below by some constant M independent of S'. In fact, we will show that we can choose $M = 1$. Recall that we defined the area of a metric space to be its 2-dimensional Hausdorff measure, although the reader could probably arrive to the same conclusions using any alternative concept of area he is keen on. Note that this bound $M = 1$ is best possible, since it equals the 2-dimensional Hausdorff measure of the unit disc.

To prove the claimed bound for $A(P)$, we subdivide S^1, and σ, into four equal arcs X_1, Y_1, X_2, Y_2 of length $\pi/2$ each, traversed by σ in that order. Note that

$$d(x, X_1) \geqslant \sqrt{2} \text{ holds for every } x \in X_2, \text{ and similarly for } Y_1, Y_2. \tag{6.1}$$

We will use this observation to prove that $A(P)$ is at least half the area of a square with side length $\sqrt{2}$.

For this, define the mapping $f : P \to [0, \sqrt{2}]^2$ by $x \mapsto (\lceil d(x, X_1) \rceil, \lceil d(x, Y_1) \rceil)$ where $\lceil d \rceil := \max\{d, \sqrt{2}\}$. We may assume that the domain of ρ is also the square $[0, \sqrt{2}]^2$ rather than the standard 2-simplex. Now consider the function $f \circ \rho : [0, \sqrt{2}]^2 \to [0, \sqrt{2}]^2$, which is continuous since both f and ρ are. Note that the restriction of $f \circ \rho$ to the boundary of the square $[0, \sqrt{2}]^2$ is, by (6.1), a homeomorphism from that boundary onto itself. From this we will infer that

$$f \circ \rho \text{ is onto.} \tag{6.2}$$

There are perhaps many ways to prove this basic fact, and the reader might have a favourite one depending on their background. Here we sketch a proof using homology: suppose, to the contrary,

that some point $z \in [0, \sqrt{2}]^2$ is not in the image I of $f \circ \rho$, and let $Q := [0, \sqrt{2}]^2 \setminus \{z\}$. Note that Q is homotopy equivalent to S^1, and so $H_1(Q)$ is isomorphic to $H_1(S^1)$ [21, Corollary 2.11]. But $f \circ \rho$ is a 2-simplex of Q proving that its boundary is null-homologous, and so $H_1(Q)$ is trivial by the remark preceding (6.2). This contradiction establishes (6.2). Note that f must thus also be onto.

Now suppose that $A(P) < 1$, which means that for every δ there is a countable cover $(U_i)_{i\in\mathbb{N}}$ of P with $diam(U_i) < \delta$ and $\sum diam(U_i)^2 < 1$. Letting $V_i := f(U_i)$ we obtain a cover $(V_i)_{i\in\mathbb{N}}$ of $[0, \sqrt{2}]^2$ since f is onto. Moreover, by the definition of f and the triangle inequality we have $diam(V_i) \leqslant \sqrt{2}\, diam(U_i)$. Thus $\sum diam(V_i)^2 \leqslant 2 \sum diam(U_i)^2 < 2$ by the above assumption. This means that the area of $[0, \sqrt{2}]^2$ is less than 2, a contradiction.

This completes the proof that any area extension of S^1 in which σ is null-homologous has an excess area of at least 1, implying that $\hat{H}_d(S^1) \cong H_1(S^1)$. □

Using the same arguments one can generalise this to the following.

Corollary 6.2. *Let G be a locally finite 1-complex. Then $\hat{H}_d(G) \cong H_1(G)$.*

The following important example shows that Theorem 3.4 would become false if we banned metric cylinders from the definition of an area extension. Moreover, it shows that Theorem 3.4 fails if we replace $\hat{H}_d$ by the first singular homology group $H_1(X)$ even if $H_1(X)$ is finitely generated. This example could also contribute to a better understanding of Section 10, the geometric part of the proof of our main result.

Example 6.3. We will define our space X as a subspace of $\mathbb{R}^3$ with the Euclidean metric. It is similar to a well-known construction of [2] called the *harmonic archipelago*. The shape of X is reminiscent of the shape of an old-fashioned folding camera: for every even $i \in \mathbb{N}$ let D_i be the circle $\{(x, y, z) \in \mathbb{R}^3 \mid y = 2^{-i},\ x^2 + z^2 = 1\}$ and for every odd $i \in \mathbb{N}$ let D_i be the circle $\{(x, y, z) \in \mathbb{R}^3 \mid y = 2^{-i},\ x^2 + z^2 = 1/2 + 2^{-i}\}$. Moreover, for every $i \in \mathbb{N}$ let X_i be the closed cylinder in $\mathbb{R}^3$ with boundary $D_i \cup D_{i+1}$ that has minimum area among all such cylinders. Let X be the closure of $\bigcup X_i$ in $\mathbb{R}^3$, that is, X is the union of $\bigcup X_i$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid y = 0,\ 1/2 \leqslant x^2 + z^2 \leqslant 1\}$.

For every $i \in \mathbb{N}$ let σ_i be a circlex that travels once around D_i. Note that σ_i is homotopic to σ_j for every i, j. However, no σ_i is homologous to a circlex τ that travels once around the circle $D := \{(x, y, z) \in \mathbb{R}^3 \mid y = 0,\ x^2 + z^2 = 1/2\} \subset X$, because no 2-simplex can meet infinitely many X_i. Moreover, the two homology classes corresponding to τ and the σ_i cannot be made equivalent by glueing discs of arbitrarily small area to X without distorting its metric. Thus, if we modified the definition of area extension to only allow discs as components of $X' \setminus \iota(X)$, then Theorem 3.4 would fail for $C := [\![1\sigma_1]\!]$, as C has representatives with length arbitrarily close to $\pi = \ell(\tau)$, namely, the σ_i, but no representative of length π or less.

This example also shows that we cannot replace $\hat{H}_d(X)$ by $H_1(X)$ (and 'σ-representative' by 'representative') in the assertion of Theorem 3.4 even if $H_1(X)$ is finitely generated. Indeed, $H_1(X)$ is generated by 2 elements here, namely $[\sigma_1]$ and $[\tau]$, and $[\sigma_1]$ has no representative of minimum length.

If X is compact then in many cases we do not gain anything when we take the completion $\hat{H}_d(X)$ of $H_d(X)$. For example, if X is the space of Fig. 4 then $H_d(X)$ is already complete as

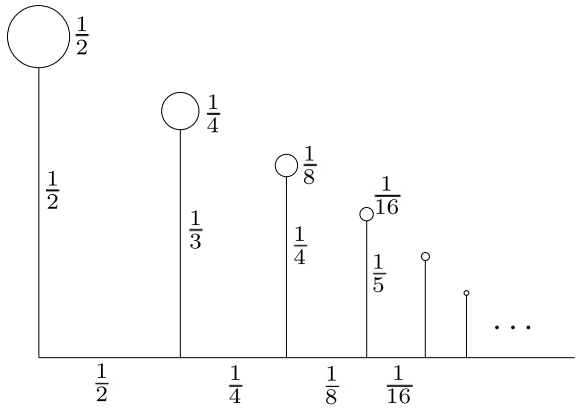

Fig. 4. A compact subspace of the real plane. The numbers denote the lengths of the corresponding segments.

the interested reader can check. There are however compact examples X where $H_d(X)$ is not complete:

Example 6.4. Let X be a metric space obtained as follows. Start with a topologist's sine curve S, pick a countably infinite 'cofinal' sequence $(u_i)_{i\in\mathbb{N}}$ of points of S, and attach a circle of length 2^{-i} at each point u_i. To see that $H_d(X)$ is not complete, let σ_i be a circlex corresponding to the circle attached at u_i, and note that $(\llbracket\sigma_i\rrbracket)_{i\in\mathbb{N}}$ is a Cauchy sequence that has positive distance from each element c of $H_1(X)$. Indeed, any such c must miss some circle, and Theorem 6.1 yields a lower bound for that distance.

For $C' \in H_d$ the element C of $\hat{H}_d$ corresponding to C' satisfies $\ell(C) \leqslant \inf\{\ell(\beta) \mid \beta \in C'\}$ by the definitions. The aim of our next example is to show that this inequality can be proper. This means that (the first sentence of) the assertion of Theorem 3.4, applied to a $C \in H_d$, is in fact stronger than that of Theorem 1.4.

Example 6.5. Consider the compact space $X \subseteq \mathbb{R}^2$ depicted in Fig. 4. It is easy to construct a closed 1-simplex $\sigma : [0,1] \to X$ that traverses each of the infinitely many circles in this space precisely once. Let $\beta \in H_1(X)$ denote the homology class of the 1-cycle 1σ, and note that for every 1-cycle $\chi \in \beta$ there holds $\ell(\chi) = \infty$ because of the perpendicular segments. It is not hard to see that for $C' := \llbracket\beta\rrbracket \in H_d(X)$ we have $\inf\{\ell(\beta) \mid \beta \in C'\} = \infty$. Now let τ_i be a circlex that travels once around the circle of length 2^{-i} in X, and let ψ_i denote the 1-chain $\sum_{j\leqslant i} \tau_j$. By Lemma 4.1 we can, for every i, 'patch' all circles of X of length less than 2^{-i} to obtain an area extension X_i of X of some excess area $v(i) < \infty$ in which the 1-cycles 1σ and ψ_i are homologous. Note that $\lim_i v(i) = 0$, thus $(\llbracket\psi_i\rrbracket)_{i\in\mathbb{N}}$ is a Cauchy sequence equivalent to the constant sequence $(C')_{i\in\mathbb{N}}$, which means that $(1\tau_i)_{i\in\mathbb{N}}$ is a σ-representative of the class $C \in \hat{H}_d(X)$ containing these sequences. Thus $\ell(C) \leqslant \sum_i \ell(\tau_i) = 1$.

Finally, it is worth mentioning that we cannot relax the assertion of Theorem 3.4 to require that X is just complete rather than compact. For example, the cylinder $\{(x,y,z) \in \mathbb{R}^3 \mid z \geqslant 1, x^2 + y^2 = 1 + 1/z\}$ with the Euclidean metric is complete, but it is easy to see that no non-trivial element of $\hat{H}_d$ has a σ-representative of minimum length.

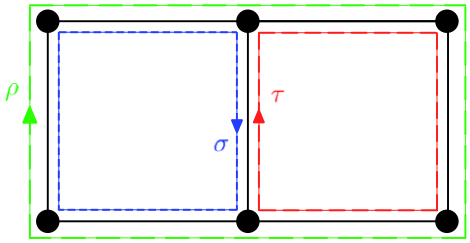

Fig. 5. A simple example showing that we need our primitive decompositions to be economical.

7. Sketch of the main proof

The proof of our main result, Theorem 3.4, consists of two major steps: the first step is algebraic, and shows that every $C \in \hat{H}_d$ can be 'decomposed' as a sum $\sum D_i$ of simpler elements of $\hat{H}_d$, called *primitive* elements, that are easier to work with. The second step is more geometric, and proves the assertion for these primitive elements.

Our intuition behind a primitive element is that it is a homology class corresponding to a single circle, and indeed we will prove, in Section 10, that every primitive element D has a representative consisting of a circlex z, and in fact one of the desired length $\ell(z) = \ell(D)$. We obtain z by a geometric construction: starting from a sequence of closed 1-simplices σ_i representing D whose lengths converge to $\ell(D)$, we exploit the compactness of our space to find a subsequence that converges pointwise to the desired 1-simplex z, and show that $[\![z]\!] = D$ by constructing arbitrarily small metric cylinders joining z to some σ_i. See also Example 6.3, where we could choose τ to be the desired circlex z.

Now having a decomposition $C = \sum D_i$ as above, we can try to combine all the circlexes z_i we got as representatives of each D_i to form a σ-representative of C. But will such a σ-representative have the desired total length $\sum \ell(z_i) = \ell(C)$? In general not, if our decomposition is arbitrary. For example, in the graph of Fig. 5 consider the class $C = [\![\sigma + \tau]\!]$. We could write $C = D_1 + D_2$ where $D_1 = [\![\sigma]\!]$ and $D_2 = [\![\tau]\!]$ are both primitive. Now σ, τ are circlexes that do attain the length of D_1, D_2 respectively, but we cannot combine them into a representative of C of minimum length, because $\ell(\sigma) + \ell(\tau) > \ell(C)$; indeed, C has the representative ρ whose length is smaller than $\ell(\sigma) + \ell(\tau)$ because it avoids the middle edge. This example shows that if we want to follow the above plan of first decomposing C as a sum of primitive elements and then combine shortest representatives of those elements into a σ-representative of C of the desired total length $\ell(C)$, then our decomposition has to be 'economical'. If our space is a graph then it is easy to say what 'economical' should mean: no edge should be used in more than one summands. In a general space this is less obvious, but there is an elegant way around it described in Section 9.2. We will prove, in Section 9, that every $C \in \hat{H}_d$ can be decomposed as a sum $\sum D_i$ where, not only the D_i are primitive, but also the decomposition is economical in this sense. This proof is algebraic, and we obtain a more general abstraction described in the next section.

8. Intermezzo: generalising to abelian metrizable topological groups

In this section we state an intermediate result, mentioned also in Section 1.4, that might be useful in other contexts too. It says that if a topological group H and an assignment $\ell : H \to \mathbb{R}^+$ (which can be thought of as an assignment of lengths) satisfy certain axioms, then every element

of H can be written as a sum of *primitive* elements, which we define below, and this sum is in a sense 'economical' (recall the discussion in the previous section).

The reader will lose nothing by assuming that $H = \hat{H}_d(X)$ throughout this section.

Given two elements C, D of H we will write $C \preccurlyeq D$ if $\ell(C) = \ell(D) + \ell(C - D)$. Note that

$$\text{if} \quad D \preccurlyeq C \quad \text{then } C - D \preccurlyeq C, \tag{8.1}$$

since $\ell(C - (C - D)) = \ell(D)$.

We will say that an element C of H is *primitive* if $C \neq \mathbb{0}$ and there is no $D \in H \setminus \{C, \mathbb{0}\}$ such that $D \preccurlyeq C$ holds.

The reader may choose to skip the rest of this section, since this is a corollary of our main result rather than something that we will need later.

Theorem 8.1. *Let $(H, +)$ be an abelian metrizable topological group, and suppose a function $\ell : H \to \mathbb{R}^+$ is given satisfying the following properties*:

(i) $\ell(C) = 0$ *if and only if* $C = \mathbb{0}$;
(ii) $\ell(C + D) \leqslant \ell(C) + \ell(D)$ *for every* $C, D \in H$;
(iii) *if* $D = \lim C_i$ *then* $\ell(D) \leqslant \liminf \ell(C_i)$;
(iv) *for some metric d of H there is a bound $U \in \mathbb{R}$ such that $d(C, \mathbb{0}) \leqslant U\ell^2(C)$ for every $C \in H$ (i.e. an isoperimetric inequality holds).*

Then every element C of H can be represented as a (possibly infinite) sum $C = \sum D_i$ of primitive *elements D_i so that $\ell(C) = \sum \ell(D_i)$.*

Infinite sums as in the conclusion of the theorem are formalised, in Section 9.1, using the concept of nets.

Since it is the group $\hat{H}_d$ we are interested in this paper, we will give a formal proof of Theorem 8.1 only for the special case when $H = \hat{H}_d$ (more precisely, when H is the subgroup of elements of $\hat{H}_d$ with finite length). In this case Theorem 8.1 is tantamount to Corollary 9.7 below. However, the reader interested in Theorem 8.1 in its full generality will easily be able to check that the same proof applies, since no other properties of $\hat{H}_d$ are used in the proof of Corollary 9.7 than the conditions of Theorem 8.1.

One can relax condition (iv) above a bit by replacing it with the following

(iv′) if $C \in H$ is fragmentable then $C = \mathbb{0}$.

The term *fragmentable* is defined in Section 9.3 below.

9. Splitting homology classes into primitive subclasses

The main result of this section, Corollary 9.7, is that every $C \in \hat{H}_d$ can be written as a sum of primitive elements $D_i \preccurlyeq C$. This is the first step of the proof of our main result as sketched in Section 7. Recall the definitions of *primitive* and $\preccurlyeq$ from Section 8.

9.1. Infinite sums in $\hat{H}_d$

For the proof of our main result we are going to use some standard machinery related to *nets* in order to be able to rigorously define sums of infinitely many elements of $\hat{H}_d$. Let us first recall the necessary definitions.

A *net* in a topological space Y is a function from some directed set A to Y. A *directed set* is a nonempty set A together with a reflexive and transitive binary relation, that is, a preorder, with the additional property that every pair of elements has an upper bound in A. One can think of a net as a generalisation of the concept of a sequence, and one is usually interested in the convergence of such a generalised sequence: we say that the net (x_α) *converges* to the point $y \in Y$, if for every neighborhood U of y there is a $\beta \in A$ such that $x_\alpha \in U$ for every $\alpha \geqslant \beta$. See [26] for more details. In our case, the topological space Y in which our nets will take their values will always be $\hat{H}_d$, bearing the topology induced by the metric d_1.

We will say that an infinite family $\{C_i\}_{i\in I}$ of elements of $\hat{H}_d$ is *unconditionally summable* if for every $\epsilon > 0$ there is a finite subset F of $\mathcal{I}$ so that for every two finite sets $A, B \supseteq F$ there holds $d(\sum_{i\in A} C_i, \sum_{i\in B} C_i) < \epsilon$; in other words, if the family $\{\sum_{i\in F} C_i\}_{F\in\mathcal{F}}$ is a Cauchy net, where $\mathcal{F}$ is the set of finite subsets of $\mathcal{I}$ preordered by the inclusion relation. Since $\hat{H}_d$ is complete, it is well known that if $\{C_i\}_{i\in I}$ is unconditionally summable then the net $\{\sum_{i\in F} C_i\}_{F\in\mathcal{F}}$ converges to an element $C \in \hat{H}_d$, see [24, Proposition 2.1.49]. In this case, we call C the *sum* of the family $\{C_i\}_{i\in I}$ and write $C = \sum_{i\in\mathcal{I}} C_i$. Note that if $\mathcal{I}$ is countable then for every enumeration $a_1, a_2, \dots$ of $\mathcal{I}$ there holds

$$\sum_{i\in\mathcal{I}} C_i = \lim_i \sum_{1\leqslant j\leqslant i} C_{a_j}. \tag{9.1}$$

Our next lemma generalises the triangle inequality for $\hat{H}_d$ (Corollary 5.5) to infinite sums using the notions we just defined.

Lemma 9.1. *Let $\{C_i\}_{i\in I}$ be an unconditionally summable family of elements of $\hat{H}_d$. Then $\ell(\sum_{i\in\mathcal{I}} C_i) \leqslant \sum_{i\in\mathcal{I}} \ell(C_i)$.*

Proof. If $\sum_{i\in\mathcal{I}} \ell(C_i) = \infty$ then there is nothing to show, so suppose that $\sum_{i\in\mathcal{I}} \ell(C_i) < \infty$.

We may assume without loss of generality that $\ell(C_i) > 0$ holds for every $i \in \mathcal{I}$, for if $\ell(C_i) = 0$ then $C_i = \mathbb{0}$ by Observation 5.3. Thus, we may also assume that $\mathcal{I}$ is countable, and let $a_1, a_2, \dots$ be an enumeration of $\mathcal{I}$.

Let $C := \sum_{i\in\mathcal{I}} C_i$ and let $D_i := \sum_{1\leqslant j\leqslant i} C_{a_j}$ for every i. We have $C = \lim_i D_i$ by (9.1). By the definition of $\ell(C)$ we then have

$$\ell(C) \leqslant \lim \ell(D_i). \tag{9.2}$$

Applying Corollary 5.5 (several times) to D_i we obtain $\ell(D_i) \leqslant \sum_{1\leqslant j\leqslant i} \ell(C_{a_j}) < \sum_{i\in\mathcal{I}} \ell(C_i)$. Combining this with (9.2) yields $\ell(C) \leqslant \sum_{i\in\mathcal{I}} \ell(C_i)$ as desired. □

9.2. Splitting homology classes into shorter ones

We introduce the notation $C = D \oplus E$ to denote the assertion that $C = D + E$ and $\ell(C) = \ell(D) + \ell(E)$. Note that this definition implements the intuition outlined in Section 7 that $D + E$ is an 'economical' way to split C. It follows by the definitions that

$$D \preccurlyeq C \quad \text{if and only if} \quad C = D \oplus (C - D). \tag{9.3}$$

More generally, the notation $C = \bigoplus_{i\in\mathcal{K}} D_i$ (or $D_1 \oplus \cdots \oplus D_k$), where $\mathcal{K}$ is a possibly infinite set of indices, denotes the assertion that $C = \sum_{i\in\mathcal{K}} D_i$ and $\ell(C) = \sum_{i\in\mathcal{K}} \ell(D_i)$.

Our next lemma shows that, in a sense, $\oplus$ behaves well with respect to composition:

Lemma 9.2. *Let $C, D, E, F, G \in \hat{H}_d$ be such that $C = D \oplus E$ and $E = F \oplus G$. Then the following assertions hold*:

(i) $C = D \oplus F \oplus G$;
(ii) $C = (D + F) \oplus G$; *and*
(iii) $D + F = D \oplus F$.

In particular, $F, G, (D + F) \preccurlyeq C$.

Proof. By the assumptions we have $\ell(C) = \ell(D) + \ell(E) = \ell(D) + \ell(F) + \ell(G)$, which yields (i).

Note that $C = D + F + G$. By Corollary 5.5 we have $\ell(C) = \ell((D + F) + G) \leqslant \ell(D + F) + \ell(G)$, and $\ell(D + F) \leqslant \ell(D) + \ell(F)$. Now since we have already proved that $\ell(C) = \ell(D) + \ell(F) + \ell(G)$, equality must hold in both above inequalities. The first of these equalities yields (ii) and the second yields (iii). □

This nice behaviour of $\oplus$ extends to infinite sums too:

Lemma 9.3. *If $\ell(C) < \infty$ and $C = \bigoplus_{i\in\mathcal{K}} D_i$ then for every subset $\mathcal{M} \subseteq \mathcal{K}$ there holds $\sum_{i\in\mathcal{M}} D_i = \bigoplus_{i\in\mathcal{M}} D_i$ and $C = \sum_{i\in\mathcal{M}} D_i \oplus \sum_{i\in\overline{\mathcal{M}}} D_i$, where $\overline{\mathcal{M}} := \mathcal{K}\backslash\mathcal{M}$.*

Proof. By Corollary 5.5 we have $\ell(C) \leqslant \ell(\sum_{i\in\mathcal{M}} D_i) + \ell(\sum_{i\in\overline{\mathcal{M}}} D_i)$ and by Lemma 9.1 we have $\ell(\sum_{i\in\overline{\mathcal{M}}} D_i) \leqslant \sum_{i\in\overline{\mathcal{M}}} \ell(D_i)$. Combining the last two inequalities we obtain

$$\begin{aligned}\ell\Big(\sum_{i\in\mathcal{M}} D_i\Big) &\geqslant \ell(C) - \ell\Big(\sum_{i\in\overline{\mathcal{M}}} D_i\Big) = \sum_{i\in\mathcal{K}} \ell(D_i) - \ell\Big(\sum_{i\in\overline{\mathcal{M}}} D_i\Big)\\ &\geqslant \sum_{i\in\mathcal{K}} \ell(D_i) - \sum_{i\in\overline{\mathcal{M}}} \ell(D_i) = \sum_{i\in\mathcal{M}} \ell(D_i),\end{aligned}$$

where we used our assumption that $\sum_{i\in\mathcal{K}} \ell(D_i) = \ell(C) < \infty$. Applying Lemma 9.1 again we also have

$$\ell\Big(\sum_{i\in\mathcal{M}} D_i\Big) \leqslant \sum_{i\in\mathcal{M}} \ell(D_i),$$

hence equality holds in the last two inequalities, which proves that

$$\sum_{i\in\mathcal{M}} D_i = \bigoplus_{i\in\mathcal{M}} D_i.$$

Similarly, we have $\sum_{i\in\overline{\mathcal{M}}} D_i = \bigoplus_{i\in\overline{\mathcal{M}}} D_i$. The assertion

$$C = \sum_{i\in\mathcal{M}} D_i \oplus \sum_{i\in\overline{\mathcal{M}}} D_i$$

now easily follows from the definitions. □

9.3. Exploiting the isoperimetric inequality

We will say that an element $C \in \hat{H}_d$ is *δ-fragmentable*, for some $\delta \in \mathbb{R}_+$, if there is a finite family $\{D_i\}_{i\in\mathcal{K}}$, $D_i \in \hat{H}_d$, such that $C = \bigoplus_{i\in\mathcal{K}} D_i$ and for every i there holds $\ell(D_i) < \delta$. We will call C *fragmentable* if it is δ-fragmentable for arbitrarily small δ. It turns out that the only fragmentable element of $\hat{H}_d$ is $\mathbb{0}$:

Lemma 9.4. *If $C \in \hat{H}_d$ is fragmentable then $C = \mathbb{0}$.*

Proof. Suppose C is fragmentable, and fix some $\epsilon > 0$ for which we want to show that $d_1(C, \mathbb{0}) < \epsilon$. Let $\{D_i\}_{i\in\mathcal{K}}$ be a family witnessing the fact that C is δ-fragmentable for some parameter δ that we will specify later.

For every $i \in \mathbb{N}$, we can, by the definition of $\ell(D_i)$, find elements of H_1 arbitrarily close (with respect to d_1) to D_i whose lengths are arbitrarily close to $\ell(D_i)$; more formally, it follows by the definitions that there is a class $\alpha_i \in H_1$ with $\ell(\alpha_i) < \ell(D_i) + \min(\delta, \epsilon/2|\mathcal{K}|)$ such that

$$d_1\big([\alpha_i], D_i\big) < \epsilon/2|\mathcal{K}|. \tag{9.4}$$

By the definition of $\ell(\alpha_i)$ there is a 1-chain $\chi_i \in \alpha_i$ such that $\ell(\chi_i) < \ell(\alpha_i) + \min(\delta, \epsilon/2|\mathcal{K}|)$. Combining this with our assumption that $\ell(D_i) < \delta$ and the choice of α_i we obtain

$$\ell(\chi_i) < \ell(\alpha_i) + \delta < \ell(D_i) + 2\delta < 3\delta. \tag{9.5}$$

By Lemma 4.1 there is an extension X_i of X of excess area at most $U\ell^2(\chi_i)$ in which χ_i is null-homologous. Combining these extensions X_i for every i we obtain an extension X' of X of excess area V at most $U\sum_{i\in\mathcal{K}} \ell^2(\chi_i)$ in which the 1-chain $\sum_{i\in\mathcal{K}} \chi_i$ is null-homologous. Note that by the choice of the χ_i and the α_i we have

$$\begin{aligned}\sum \ell(\chi_i) &< \sum \big(\ell(\alpha_i) + \epsilon/2|\mathcal{K}|\big) < \sum \big(\ell(D_i) + \epsilon/2|\mathcal{K}| + \epsilon/2|\mathcal{K}|\big)\\ &= \Big(\sum \ell(D_i)\Big) + \epsilon = \ell(C) + \epsilon,\end{aligned}$$

where we used our assumption that $\ell(C) = \sum \ell(D_i)$. This means that $\sum \ell(\chi_i)$ is bounded from above; thus by (4.2) and (9.5) choosing δ small enough we can make V arbitrarily small; in

particular, we could have chosen a δ for which $V < \epsilon/2$ holds, which would imply

$$d_1\Big(\Big[\!\Big[\sum \chi_i \Big]\!\Big], \mathbb{0}\Big) < \epsilon/2. \tag{9.6}$$

Since $C = \sum D_i$ we easily obtain by Corollary 5.5 and (9.4)

$$d_1\Big(C, \Big[\!\Big[\sum_{i\in\mathcal{K}} \chi_i \Big]\!\Big]\Big) = d_1\Big(\sum_{i\in\mathcal{K}} D_i, \Big[\!\Big[\sum_{i\in\mathcal{K}} \chi_i \Big]\!\Big]\Big) \leqslant \sum_{i\in\mathcal{K}} d_1\big(D_i, [\![\chi_i]\!]\big) \leqslant \sum_{i\in\mathcal{K}} \epsilon/2|\mathcal{K}| = \epsilon/2,$$

and combined with (9.6) this yields $d_1(C, \mathbb{0}) < \epsilon$, and proves that $C = \mathbb{0}$ in this case. □

9.4. A technical lemma

The following somewhat technical lemma will be used in the proof of the main result of this section; it allows us to prove, using Zorn's Lemma, the existence of maximal families with certain properties.

Lemma 9.5. *Let $\{D_\alpha\}_{\alpha<\gamma}$ be a family of elements D_α of $\hat{H}_d\backslash\{\mathbb{0}\}$, indexed by an ordinal γ, such that for every $\beta < \gamma$ there holds $\sum_{\alpha\leqslant\beta} D_\alpha = \bigoplus_{\alpha\leqslant\beta} D_\alpha$ and $\sum_{\alpha\leqslant\beta} D_\alpha \preccurlyeq C$ for some fixed $C \in \hat{H}_d$ with $\ell(C) < \infty$. Then $(D_\alpha)_{\alpha<\gamma}$ is unconditionally summable and there holds $\sum_{\alpha<\gamma} D_\alpha \preccurlyeq C$ and $\sum_{\alpha<\gamma} D_\alpha = \bigoplus_{\alpha<\gamma} D_\alpha$.*

Proof. Since $D_\alpha \neq \mathbb{0}$, Observation 5.3 implies that $\ell(D_\alpha) > 0$ for every $\alpha < \gamma$. As we are assuming that $\ell(\sum_{\alpha\leqslant\beta} D_\alpha) = \sum_{\alpha\leqslant\beta} \ell(D_\alpha)$ and that $\ell(\sum_{\alpha\leqslant\beta} D_\alpha) \leqslant \ell(C)$ for every $\beta \leqslant \gamma$, we have $\sum_{\alpha\leqslant\beta} \ell(D_\alpha) \leqslant \ell(C) < \infty$ for every $\beta < \gamma$, which implies that γ is countable and

$$\sum_{\alpha<\gamma} \ell(D_\alpha) \leqslant \ell(C) < \infty. \tag{9.7}$$

Let $a_1, a_2, \ldots$ be an enumeration of γ. To see that $(D_\alpha)_{\alpha<\gamma}$ is unconditionally summable, note that for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that for every $k > n$ there holds $\ell(\sum_{n<j<k} D_{a_j}) \leqslant \sum_{n<j<k} \ell(D_{a_j}) < \epsilon$, hence $d_1(\sum_{n<j<k} D_{a_j}, \mathbb{0}) < U\epsilon^2$ by Lemma 4.1 and the definition of d_1.

Thus $S := \sum_{\alpha<\gamma} D_\alpha$ is well defined (see Section 9.1), and by (9.1) we have

$$S = \lim_n D^n \quad \text{and} \quad C - S = \lim_n \big(C - D^n\big), \tag{9.8}$$

where $D^n := \sum_{1\leqslant j\leqslant n} D_{a_j}$.

We have to prove that $S \preccurlyeq C$, i.e. that $\ell(C) = \ell(S) + \ell(C - S)$. By the definition of $\ell()$ and (9.8) we obtain $\ell(S) \leqslant \liminf_n \ell(D^n)$ and $\ell(C - S) \leqslant \liminf_n \ell(C - D^n)$. However, (9.7) and Corollary 5.5 easily imply that both $(\ell(D^n))$ and $(\ell(C - D^n))$ converge, and so we can write

$$\ell(S) \leqslant \lim_n \ell\big(D^n\big) \quad \text{and} \quad \ell(C - S) \leqslant \lim_n \ell\big(C - D^n\big). \tag{9.9}$$

Combining this with the fact that $\ell(C) \leqslant \ell(S) + \ell(C - S)$, which we obtain from Corollary 5.5, we have

$$\ell(C) \leqslant \lim_n \ell\big(D^n\big) + \lim_n \ell\big(C - D^n\big) = \lim_n \big(\ell\big(D^n\big) + \ell\big(C - D^n\big)\big). \tag{9.10}$$

We claim that $D^n \preccurlyeq C$ holds for every n. Indeed, note that there is a $\beta < \gamma$ such that $a_j \leqslant \beta$ holds for every $j \leqslant n$. As we are assuming that $\sum_{\alpha\leqslant\beta} D_\alpha = \bigoplus_{\alpha\leqslant\beta} D_\alpha$, Lemma 9.3 (for the application of which we set $C = \sum_{\alpha\leqslant\beta} D_\alpha$, $\mathcal{K} = \beta$ and $\mathcal{M} = \{a_1, \dots, a_n\}$) implies that

$$D^n = \bigoplus_{1\leqslant j\leqslant n} D_{a_j} \tag{9.11}$$

and that $D^n \preccurlyeq \sum_{\alpha\leqslant\beta} D_\alpha$. As we are furthermore assuming that $\sum_{\alpha\leqslant\beta} D_\alpha \preccurlyeq C$ holds, the transitivity of $\preccurlyeq$ (see Lemma 9.2) implies that $D^n \preccurlyeq C$ as claimed.

This means that $\ell(D^n) + \ell(C - D^n) = \ell(C)$ for every $n \in \mathbb{N}$. Plugging this into (9.10) yields $\ell(C) \leqslant \lim_n \ell(C) = \ell(C)$. Thus equality must hold throughout in (9.9) and (9.10). This implies that $\ell(C) = \ell(S) + \ell(C - S)$ – i.e. $\sum_{\alpha<\gamma} D_\alpha \preccurlyeq C$ – and that $\ell(S) = \lim_n \ell(D^n)$. Using (9.11) the latter yields $\ell(S) = \ell(\sum_{\alpha<\gamma} D_\alpha) = \lim_n \sum_{1\leqslant j\leqslant n} \ell(D_{a_j}) = \sum_{\alpha<\gamma} \ell(D_\alpha)$ as desired. □

9.5. Existence of primitive decompositions

We can now complete the proof of the main result of this section, that every non-trivial element C of $\hat{H}_d(X)$ can be decomposed as a sum of primitive elements. We do this by first proving that we can find at least one primitive element in C, and then using Zorn's Lemma to find a maximal family of primitive elements in C. The former task is fulfilled by the following lemma.

Lemma 9.6. *For every $C \neq \mathbb{0} \in \hat{H}_d(X)$ with $\ell(C) < \infty$ there is $D \preccurlyeq C$ such that D is primitive.*

Proof. By Lemma 9.4, C is not λ-fragmentable for some $\lambda \in \mathbb{R}_+$, i.e. for every finite family $\{D_i\}_{i\in\mathcal{K}}$ such that $C = \bigoplus_{i\in\mathcal{K}} D_i$ there is a member D_j with $\ell(D_j) \geqslant \lambda$. Note that there is also no infinite family $\{D'_i\}_{i\in\mathcal{I}}$ such that $C = \bigoplus_{i\in\mathcal{I}} D'_i$ and for every i there holds $\ell(D_i) < \lambda$. For if such a family exists, then we can find a finite subfamily $\{D'_i\}_{i\in\mathcal{I}'}$ such that $\sum_{i\in\mathcal{I}'} \ell(D'_i) > \ell(C) - \lambda$, which implies $\sum_{i\in\mathcal{I}\setminus\mathcal{I}'} \ell(D'_i) < \lambda$, and hence $\ell(\sum_{i\in\mathcal{I}\setminus\mathcal{I}'} D'_i) < \lambda$ by Lemma 9.1. But then, extending $\{D'_i\}_{i\in\mathcal{I}'}$ by one member, namely $\sum_{i\in\mathcal{I}\setminus\mathcal{I}'} D'_i$, we obtain a finite family $\{D'_i\}_{i\in\mathcal{I}''}$ which satisfies $\ell(D'_i) < \lambda$ for every $i \in \mathcal{I}''$, and it is not hard to see that $C = \bigoplus_{i\in\mathcal{I}''} D'_i$ holds; this contradicts the fact that C is not λ-fragmentable.

Now let $\{D_\alpha\}_{\alpha<\gamma}$, $D_\alpha \in \hat{H}_d\setminus\mathbb{0}$ be an unconditionally summable family, indexed by an ordinal number γ, that is maximal with the following properties:

(i) $\sum_{\alpha<\beta} D_\alpha \preccurlyeq C$ for every $\beta \leqslant \gamma$;
(ii) $\sum_{\alpha<\beta} D_\alpha = \bigoplus_{\alpha<\beta} D_\alpha$ for every $\beta \leqslant \gamma$; and
(iii) $\sum_{\alpha<\gamma} \ell(D_\alpha) \leqslant \ell(C) - \lambda$.

To see that a maximal such family exists, apply Zorn's Lemma on the set of all such families ordered by the subfamily relation, using Lemma 9.5 in order to show that every chain has an upper bound. We are not yet assuming that this maximal family is non-trivial.

Let $D := \sum_{\alpha<\gamma} D_\alpha$, and note that $D \preccurlyeq C$ by (i). It is not hard to see that either D or $C - D$ (or both) is still not λ-fragmentable, for if both split into families with elements of lengths less that λ, then so does C; more formally, suppose there are finite families $\{D_i\}_{i\in\mathcal{M}}$ and $\{D_i\}_{i\in\mathcal{N}}$ such that $D = \bigoplus_{i\in\mathcal{M}} D_i$, $C - D = \bigoplus_{i\in\mathcal{N}} D_i$, and $\ell(D_i) < \lambda$ for every $i \in \mathcal{M}\cup\mathcal{N}$. We claim that $C = \bigoplus_{i\in\mathcal{M}\cup\mathcal{N}} D_i$. Easily, $C = \sum_{i\in\mathcal{M}\cup\mathcal{N}} D_i$. To see that $\ell(C) = \sum_{i\in\mathcal{M}\cup\mathcal{N}} \ell(D_i)$, recall that $\ell(C) = \ell(D) + \ell(C - D)$ by (i), that $\ell(D) = \sum_{i\in\mathcal{M}} \ell(D_i)$, and that $\ell(C - D) = \sum_{i\in\mathcal{N}} \ell(D_i)$. This proves that either D or $C - D$ is not λ-fragmentable.

But if $C - D$ is not λ-fragmentable, then it is primitive: for if there is an $F_0 \neq \mathbb{0}$ with $F_0 \preccurlyeq C - D$ and $F_0 \neq C - D$, then either F_0 or $F_1 := C - D - F_0$ has length at least λ since $C - D$ is not λ-fragmentable and, by (9.3), $C - D = F_0 \oplus F_1$. Assume without loss of generality that $\ell(F_1) \geqslant \lambda$; we can now enlarge the family $\{D_\alpha\}_{\alpha<\gamma}$ by one member, namely F_0, to obtain a new family $\{D_\alpha\}_{\alpha<\gamma^+}$ that contradicts the maximality of $\{D_i\}_{i\in I}$: to prove that $\{D_\alpha\}_{\alpha<\gamma^+}$ also satisfies requirement (i) it suffices to check that $\sum_{\alpha<\gamma^+} D_\alpha \preccurlyeq C$. We have $\sum_{\alpha<\gamma^+} D_\alpha = D + F_0$ by construction, and by assertion (ii) of Lemma 9.2 we obtain

$$D + F_0 \preccurlyeq C, \tag{9.12}$$

which proves that $\{D_\alpha\}_{\alpha<\gamma^+}$ satisfies (i). To prove that $\{D_\alpha\}_{\alpha<\gamma^+}$ also satisfies requirement (ii), it suffices again to consider the case $\beta = \gamma^+$; in other words, to prove that $D + F_0 = \bigoplus_{\alpha<\gamma^+} D_\alpha$. Thus we have to prove that $\ell(D + F_0) = \sum_{\alpha<\gamma^+} \ell(D_\alpha) = \ell(D) + \ell(F_0)$, where for the last equality we used the fact that (ii) holds for $\beta = \gamma$ and $D_\gamma = F_0$. But this follows from assertion (iii) of Lemma 9.2, and so $\{D_\alpha\}_{\alpha<\gamma^+}$ also satisfies (ii). Finally, to see that $\{D_\alpha\}_{\alpha<\gamma^+}$ satisfies (iii), note that $\sum_{\alpha\in\gamma^+} D_\alpha = D + F_0$, that $\ell(C) = \ell(D + F_0) + \ell(F_1)$ by (9.12), and that $\ell(F_1) \geqslant \lambda$. This completes the proof that if $C - D$ is not λ-fragmentable then $C - D$ is primitive, for otherwise the maximality of $\{D_\alpha\}_{\alpha<\gamma}$ is contradicted.

Thus, if $C - D$ is not λ-fragmentable then we are done, since $D \preccurlyeq C$ and so we also have $C - D \preccurlyeq C$ by (8.1). So suppose it is not, in which case it is D that is not λ-fragmentable. Recall that $\ell(D) \leqslant \ell(C) - \lambda$ by (iii).

To sum up, having assumed that C is not λ-fragmentable, we proved that either there is a primitive $B \preccurlyeq C$, in which case we are done, or there is a $D \preccurlyeq C$ that is also not λ-fragmentable (for the same λ) and satisfies $\ell(D) \leqslant \ell(C) - \lambda$. In the latter case, we can repeat the whole argument replacing C with $C_1 := D$; this will again yield either a primitive $B \preccurlyeq C_1$, or a $C_2 \preccurlyeq C_1$ that is also not λ-fragmentable and satisfies $\ell(C_2) \leqslant \ell(C_1) - \lambda \leqslant \ell(C) - 2\lambda$, and so on. But as $\ell(C)$ is finite and λ positive, this procedure must stop after finitely many steps, yielding a primitive $B \preccurlyeq C_j \preccurlyeq C_{j-1} \preccurlyeq \cdots \preccurlyeq C$. As $\preccurlyeq$ is transitive (Lemma 9.2) we obtain $B \preccurlyeq C$. This completes the proof. □

We can now state and prove the main result of this section.

Corollary 9.7. *For every $C \neq 0 \in \hat{H}_d(X)$ with $\ell(C) < \infty$ there is a family $\{D_i\}_{i\in I}$ of primitive elements of $\hat{H}_d(X)$ such that $C = \bigoplus_{i\in I} D_i$.*

Proof. Using Zorn's Lemma we find a maximal family $\{D_\alpha\}_{\alpha<\gamma}$ of primitive $D_\alpha \in \hat{H}_d$ such that

(i) $\sum_{\alpha<\beta} D_\alpha \preccurlyeq C$ for every $\beta \leqslant \gamma$; and
(ii) $\sum_{\alpha<\beta} D_\alpha = \bigoplus_{\alpha<\beta} D_\alpha$ for every $\beta \leqslant \gamma$.

Indeed, consider the set of all such families ordered by the subfamily relation, and apply Lemma 9.5 in order to show that every chain has an upper bound.

Let $D := \bigoplus_{i\in I} D_i$. We claim that $C - D = 0$. For suppose not. Then by Lemma 9.6 there is a primitive $F \preccurlyeq C - D$. Now extend the family $\{D_\alpha\}_{\alpha<\gamma}$ by one member $D_\gamma := F$. By (ii) of Lemma 9.2 the new family still satisfies (i). To prove that it also satisfies (ii) we only have to show that $\ell(\sum_{\alpha\leqslant\gamma} D_\alpha) = \sum_{\alpha\leqslant\gamma} \ell(D_\alpha) = \ell(D) + \ell(F)$ (where we used the fact that the original family satisfies (ii)), but this follows from (iii) of Lemma 9.2. Thus the extended family contradicts the maximality of $\{D_\alpha\}_{\alpha<\gamma}$, which proves that $C - D = 0$ and establishes our assertion. □

10. Proof for primitive elements

By Corollary 9.7 every non-trivial element C of $\hat{H}_d$ can be written as a sum of primitive elements D_i so that $\ell(C) = \sum \ell(D_i)$. All that remains to show is that our main theorem holds for those elements D_i:

Lemma 10.1. *If $D \in \hat{H}_d$ is primitive then there is a circlex z such that $D = [\![z]\!]$ and $\ell(z) = \ell(D)$.*

Proof. We are going to obtain the desired closed simplex z as a limit, in a sense, of a sequence of closed simplices σ_i^1 related to D. Our proof is organised in three steps. In the first step we construct this sequence (σ_i^1). In the second step we construct z and, at the same time, homotopies between z and the σ_i^1 in appropriate area extensions of X, implying that $[\![z]\!] = \lim[\![sig_i^1]\!]$. Finally, in a third step we show that $D = [\![z]\!]$ and that $\ell(z) = \ell(D)$. We then remark that the closed simplex z we constructed must indeed be a circlex.

Step I. The sequence (σ_i^1).

By Corollary 5.2 there is a sequence of 1-cycles $(\chi_i)_{i\in\mathbb{N}}$ such that $([\![\chi_i]\!])_{i\in\mathbb{N}}$ is a Cauchy sequence in D and $\ell(D) = \lim_i \ell(\chi_i)$.

By concatenating some of the simplices in χ_i if necessary, we may assume without loss of generality that every simplex in χ_i is closed. For every i enumerate the (closed) simplices in χ_i as $\sigma_i^1, \dots, \sigma_i^{k_i}$ in such a way that

$$\ell\big(\sigma_i^j\big) \geqslant \ell\big(\sigma_i^m\big) \quad \text{if } j < m. \tag{10.1}$$

For convenience, if $m > k_i$ then we let σ_i^m denote a trivial 1-simplex in X (thus $\ell(\sigma_i^m) = 0$ for $m > k_i$).

Let $\mathcal{M} \subseteq \mathbb{N}$ be the set of superscripts m such that $(\sigma_i^m)_{i\in\mathbb{N}}$ has no infinite subsequence $(\sigma_{\alpha_i}^m)_{i\in\mathbb{N}}$ such that $\lim_i \ell(\sigma_{\alpha_i}^m) = 0$. Note that, by (10.1),

$$\text{if} \quad m \in \mathcal{M} \quad \text{then } \{1, \dots, m-1\} \subset \mathcal{M}. \tag{10.2}$$

We begin with a simple and instructive fact indicating the significance of $\mathcal{M}$:

Claim. *If $\mathcal{M} = \emptyset$ then $D = \mathbb{0}$.*

Indeed, if $\mathcal{M} = \emptyset$ then there is an infinite subsequence $(\sigma_{\alpha_i})_{i\in\mathbb{N}}$ of $(\sigma_i)_{i\in\mathbb{N}}$ such that $\lim_i \ell(\sigma^1_{\alpha_i}) = 0$. We will show that for every $\epsilon > 0$ there holds $d_1(D, \mathbb{0}) < \epsilon$. For this, pick $j = \alpha_k \in \mathbb{N}$ large enough that

(i) $d_1(D, [\![\chi_j]\!]) < \epsilon/2$;
(ii) $\ell(\chi_j) < \ell(D) + \epsilon$; and
(iii) $\ell(\sigma^1_j) < \ell(D)\lambda$,

where $\lambda = \lambda(\epsilon) \in \mathbb{R}^+$ is some parameter that we will choose later. By (10.1) we have $\ell(\sigma^m_j) < \ell(D)\lambda$ for every $m \in \mathbb{N}$. By Lemma 4.1 there is for every m an area extension X_m of X of excess area at most $U\ell^2(\sigma^m_j)$ in which σ^m_j is null-homologous. Combining all these area extensions we obtain a single area extension X_ϵ of X of excess area at most $v := \sum_{m\in\mathbb{N}} U\ell^2(\sigma^m_j)$ in which χ_j is null-homologous. This means that

$$d_1\big([\![\chi_j]\!], \mathbb{0}\big) \leqslant v. \tag{10.3}$$

Since $\sum_{m\in\mathbb{N}} \ell(\sigma^m_j) = \ell(\chi_j) < \ell(D) + \epsilon$, given $\ell(D)$ and ϵ we can, by (4.2) and (iii), choose λ small enough that $v < \epsilon/2$. As $d_1(D, \mathbb{0}) \leqslant d_1(D, [\![\chi_j]\!]) + d_1([\![\chi_j]\!], \mathbb{0}) < \epsilon/2 + v$ by (i) and (10.3), and ϵ was chosen arbitrarily, we have proved the claim.

As we are assuming that D is primitive, the claim proves that $\mathcal{M} \neq \emptyset$, and thus $1 \in M$ by (10.2).

We may assume without loss of generality that

$$\sigma^1_i \text{ has constant speed for every } i. \tag{10.4}$$

We are going to construct z as a 'limit' of the σ^1_i (it will turn out that $\mathcal{M} = \{1\}$). For this, let $(\chi_{a_i})_{i\in\mathbb{N}}$ be a subsequence of $(\chi_i)_{i\in\mathbb{N}}$ such that $\lim_i \ell(\sigma^1_{a_i}) =: r$ exists. Note that we have already proved that $r > 0$. Moreover, $r < \infty$ holds since C is primitive and thus, easily, $\ell(C) < \infty$.

It is not hard to see that there is a subsequence $(\sigma^1_{b_i})_{i\in\mathbb{N}}$ of $(\sigma^1_{a_i})_{i\in\mathbb{N}}$ such that the restrictions

$$\sigma^1_{b_i} \restriction \mathbb{Q} \text{ converge pointwise.} \tag{10.5}$$

Indeed, let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q}$. Find a subsequence $(\tau^0_i)_{i\in\mathbb{N}}$ of $(\sigma^1_{a_i})_{i\in\mathbb{N}}$ such that the points $\tau^0_i(q_1)$ converge. Then find a subsequence $(\tau^1_i)_{i\in\mathbb{N}}$ of $(\tau^0_i)_{i\in\mathbb{N}}$ such that the points $\tau^1_i(q_2)$ also converge, and so on. Now letting $\sigma^1_{b_i} = \tau^i_i$ satisfies (10.5). (We could have chosen any dense countable subset of $[0, 1]$ instead of $\mathbb{Q}$.)

Step II. Construction of z and h.

By (10.4) and (10.5) it follows easily that

$$\begin{aligned}&\text{for every } \delta \text{ there is an } n \in \mathbb{N} \text{ such that } \sigma^1_{b_i} \text{ and } \sigma^1_{b_j} \text{ are } \delta\text{-close for every}\\ &i, j \geqslant n.\end{aligned} \tag{10.6}$$

Using Lemma 4.3 and (10.6) we can now construct a subsequence $(\sigma^1_{c_i})_{i\in\mathbb{N}}$ of $(\sigma^1_{b_i})_{i\in\mathbb{N}}$ such that for every i there is an area extension X'_i of X of excess area at most 2^{-i} in which $\sigma^1_{c_i}$ and

$\sigma^1_{c_{i+1}}$ are homotopic: for every $i = 0, 1, \dots$, use (10.6) to obtain a c_i such that $\sigma^1_{b_i}$ and $\sigma^1_{b_j}$ are $f(r, 2^{-i})$-close for every $i, j \geqslant c_i$, where the function f is that of Lemma 4.3. Choosing c_i larger if needed, we may also ensure that $c_i > c_{i-1}$ (where we set $c_{-1} := 0$), and that $\ell(\sigma^1_{b_i}) < r + 2^{-i}$ for every $i \geqslant c_i$. Then, by Lemma 4.3, there is indeed an extension X'_i as desired. Let h_i be a homotopy from $\sigma^1_{c_{i+1}}$ to $\sigma^1_{c_i}$ in X'_i as supplied by Lemma 4.3.

Combining all h_i together we can obtain a continuous function $h' : (0, 1] \times [0, 1] \to X'$, where $X' := \bigcup X'_i$. We are later going to "complete" h' into a homotopy $h : [0, 1] \times [0, 1] \to X'$ such that $h(0, x)$ is our desired simplex z. To define h', suppose that for every i we had chosen the domain of h_i to be $[2^{-i}, 2^{-(i+1)}] \times [0, 1]$. Intuitively, the interval $[2^{-i}, 2^{-i+1}]$ here corresponds to 'time'; think of time as running in the negative direction if you prefer the homotopies to be from $\sigma^1_{c_i}$ to $\sigma^1_{c_{i+1}}$ rather than the other way round. Now let $h' := \bigcup h_i$.

Let $R := \{2^{-i} \mid i \in \mathbb{N}\}$. We claim that

$$h' \restriction \big(R \times [0, 1]\big) \text{ is uniformly continuous.} \tag{10.7}$$

For suppose not. Then, there is some $\epsilon \in R_+$ and an infinite sequence of pairs $P_i = \{p_i, q_i\}$ of points in $R \times [0, 1]$ such that $d(h'(p_i), h'(q_i)) > \epsilon$ for every i and the distance between p_i and q_i converges to 0. Note that for every $s \in R$ the subspace $\{s\} \times [0, 1]$ is compact, thus the function $h' \restriction (\{s\}) \times [0, 1]$ is uniformly continuous by Lemma 2.1. This means that $\{s\} \times [0, 1]$ cannot contain an infinite subsequence of $(P_i)_{i\in\mathbb{N}}$ for any $s \in R$. Even more, $\{s\} \times [0, 1]$ cannot meet an infinite subsequence of $(P_i)_{i\in\mathbb{N}}$, because the distance between p_i and q_i converges to 0. It follows that $\{0\} \times [0, 1]$ contains an accumulation point $(0, x)$ of $(P_i)_{i\in\mathbb{N}}$, i.e. a point $(0, x)$ every neighbourhood of which contains infinitely many pairs P_i.

Now let δ be some (small) positive real number. Pick an $x' \in (\mathbb{Q} \cap [0, 1])$ such that $|x' - x| < \delta/2$, and consider the open ball $O := B_\delta((0, x'))$ in $[0, 1] \times [0, 1]$.

Let $R_O := O \cap (R \times \{x'\})$. Choosing δ small enough we can make sure that

$$\text{for every } s \in R_O \text{ there holds } \ell(\rho_s) < r + \epsilon, \tag{10.8}$$

where $\rho_s : [0, 1] \to X$ is defined by $x \mapsto h'(s, x)$; indeed, ρ_s coincides by definition with some σ^1_i, and $\lim_i \ell(\sigma^1_i) = r$.

As $O \ni x$, there is an infinite subsequence of $(P_i)_{i\in\mathbb{N}}$ contained in O. Moreover, by (10.5) $h'(R_O)$ has a unique accumulation point in X. Thus we can find a pair $P_j = \{p_j, q_j\}$ in O such that if s (respectively, s') is the element of R for which $p_j \in \{s\} \times [0, 1]$ (resp., $q_j \in \{s'\} \times [0, 1]$) holds, then $d(h'(s, x'), h'(s', x')) < \epsilon/2$.

Since ρ_s coincides with some σ^1_i, it has constant speed. As $\|p_j, (s, x')\| < 2\delta$, this together with (10.8) implies $d(h'((s, x')), h'(p_j)) < 2\delta(r + \epsilon)$; similarly, we also have $d(h'((s', x')), h'(q_j)) < 2\delta(r + \epsilon)$. Thus, by the triangle inequality applied to the four points $h'(p_j)$, $h'((s, x'))$, $h'((s', x'))$ and $h'(q_j)$ we obtain

$$d\big(h'(p_j), h'(q_j)\big) \leqslant 2\delta(r + \epsilon) + \epsilon/2 + 2\delta(r + \epsilon).$$

Since ϵ and r are fixed and we can choose δ freely, we can force this distance to be smaller than ϵ contradicting the choice of the P_i. This proves (10.7).

The completion of $R \times [0,1]$ is $(R \cup \{0\}) \times [0,1]$; thus, by (2.1) and (10.7), $h' \restriction (R \times [0,1])$ can be extended into a uniformly continuous function $h'' : (R \cup \{0\}) \times [0,1] \to X$. Next, we prove that

$$h := h' \cup h'' \text{ is continuous.} \tag{10.9}$$

Clearly, h is continuous at any point in $(0,1] \times [0,1]$. So pick $x \in \{0\} \times [0,1]$ and $\epsilon \in \mathbb{R}_+$. By the continuity of h'', there is a basic open neighbourhood O_ϵ of x in $R \times [0,1]$ that is mapped by h'' within the ball $B_{\epsilon/2}(h(x))$. Let $m_\epsilon \in \mathbb{N}$ be large enough that h_i has width less that $\epsilon/2$ for every $i \geqslant m_\epsilon$; such an m_ϵ exists by the second sentence of Lemma 4.3 and the choice of the h_i. Assume without loss of generality that O_ϵ does not meet $2^{-i} \times [0,1]$ for $i < m_\epsilon$. Extend O_ϵ into a set $O' \subseteq [0,1] \times [0,1]$ as follows. For every $i \geqslant m_\epsilon$ and every point $p = (2^{-i}, y) \in O$, put into O' the line segment L_p connecting p to the point $(2^{-(i+1)}, y)$. Note that for every point $y \in L_p$ we have $d(h'(y), h'(p)) \leqslant \epsilon/2$ since h' coincides with h_i on L_p by the definition of h' and h_i has width less that $\epsilon/2$. As $O \cap (R \times [0,1])$ is mapped by h'' within the ball $B_{\epsilon/2}(h(x))$, this implies that $h(O') \subseteq B_\epsilon(h(x))$.

But O' contains by construction an open subset of $[0,1] \times [0,1]$ containing x. This proves (10.9), which means that h is a homotopy in X' between the closed 1-simplex $h(0,x)$ and $\sigma^1_{m_0} = h(1,x)$. We now define $z(x) := h(0,x)$, which is going to be the simplex we are looking for.

Note that for every j the restriction $h \restriction ([0, 2^{-j}] \times [0,1])$ is a homotopy between z and $\sigma^1_{m_j}$ in X', but this homotopy does not use the area extensions $X'_1, \dots, X'_{j-1}$. Thus, as the area extension X'_i has by construction excess area 2^{-i} for every i, we obtain $d_1([\![\sigma^1_{m_j}]\!], [\![z]\!]) \leqslant 2^{-(j-1)}$ for every j by the definition of d_1, since σ_{m_j} and z are homotopic in the area extension $\bigcup_{i \geqslant j} X'_i$ of X. This proves that

$$\big([\![\sigma^1_{m_i}]\!]\big)_{i \in \mathbb{N}} \text{ is a Cauchy sequence with limit } Z := [\![z]\!]. \tag{10.10}$$

Step III.

Our next aim is to prove that

$$\ell(z) \leqslant r. \tag{10.11}$$

Recall that r was defined in Step I. Suppose, to the contrary, there is a finite sequence $S = s_1 < s_2 < \cdots < s_k$ of points in $[0,1]$ with $\sum_{1 \leqslant i < k} d(z(s_i), z(s_{i+1})) =: r' > r$. Clearly, we may assume that $s_j \in \mathbb{Q}$ for every j. Let $\epsilon := \frac{r'-r}{2k}$. By (10.5) and the construction of h we obtain that $\lim_i \sigma^1_{\beta_i}(s_j) = h(0, s_j) = z(s_j)$ for every j. Thus, choosing $i_0 \in \mathbb{N}$ large enough we can make sure that $d(\sigma^1_{\beta_i}(s_j), z(s_j)) < \epsilon$ for every j and every $i > i_0$. But then, the sequence S witnesses the fact that $\ell(\sigma^1_{\beta_i}) \geqslant r'$ for every $i > i_0$, which contradicts the choice of $(\sigma^1_i)_{i \in \mathbb{N}}$ and proves (10.11).

From (10.11) we will now easily yield

$$\ell(Z) = r. \tag{10.12}$$

Firstly, note that by (10.10) and the definition of $\ell(Z)$ we have $\ell(Z) \leqslant r$ by (10.11). Suppose that $\ell(Z) = r' < r$, and let $(\|\sigma'_i\|)_{i\in\mathbb{N}}$ be a Cauchy sequence in Z with $\lim \ell(\sigma'_i) = r'$. Replacing $\sigma^1_{c_i}$ in χ_{c_i} for every i by σ'_i we obtain a new sequence $(\chi'_i)_{i\in\mathbb{N}}$ from $(\chi_i)_{i\in\mathbb{N}}$, and it follows easily from (10.10) that $(\|\chi'_i\|)_{i\in\mathbb{N}} \in D$ since $(\|\chi_i\|)_{i\in\mathbb{N}} \in D$. But $\lim_i \ell(\chi'_{c_i}) = \lim_i \ell(\chi_{c_i}) - r + r' < \lim_i \ell(\chi_i)$, which contradicts the choice of $(\chi_i)_{i\in\mathbb{N}}$. Thus $\ell(Z) = r$ as claimed.

Similarly to the proof of (10.11) one can also easily prove that

$$z \text{ has constant speed.} \tag{10.13}$$

We now claim that $Z \preccurlyeq D$. Indeed, we have $\ell(D - Z) \geqslant \ell(D) - \ell(Z)$ by Corollary 5.5. Moreover, by Lemma 3.3 we have $D - Z = \lim(\|\chi_{a_i} - \sigma^1_{a_i}\|)$, and thus

$$\ell(D - Z) \leqslant \lim \ell\big(\chi_{a_i} - \sigma^1_{a_i}\big) = \lim \ell(\chi_{a_i}) - \lim \ell\big(\sigma^1_{a_i}\big) = \ell(D) - r = \ell(D) - \ell(Z),$$

where we used (10.12). Thus $Z \preccurlyeq D$ as claimed, and as D is primitive we obtain $Z = D$.

Finally, we claim that z is a circlex. Easily, the simplex z is closed since all the σ^1_i are. Suppose the image of z is not a circle. Then, there must be points $x \neq y \in [0, 1)$ such that $z(x) = z(y)$. Now consider the two simplices z_1 and z_2 obtained by subdividing z at these two points x, y, and define $Z_1 := \|z_1\|$ and $Z_2 := \|z_2\|$. Easily, $\ell(z) = \ell(z_1) + \ell(z_2)$. We will show that $Z_1 \preccurlyeq Z$. For this, note that $Z - Z_1 = \|z - z_1\|$ by Lemma 3.3, and so $Z - Z_1 = \|z_2\| = Z_2$. Thus

$$\ell(Z - Z_1) = \ell(Z_2) \leqslant \ell(z_2) = \ell(z) - \ell(z_1) \leqslant \ell(z) - \ell(Z_1) = \ell(Z) - \ell(Z_1),$$

and with Corollary 5.5 we obtain $\ell(Z - Z_1) = \ell(Z) - \ell(Z_1)$, i.e. $Z_1 \preccurlyeq Z$ as claimed. But as we have already shown that $Z = D$ and D was assumed to be primitive, we obtain $Z = Z_1$, and thus $\ell(z_1) = \ell(z)$ since $\ell(z) = \ell(Z)$. This means that $\ell(z_2) = 0$, which cannot be the case by (10.13). This contradiction proves that z is a circlex. □

Thus we have proved Lemma 10.1, which combined with Corollary 9.7 proves our main result Theorem 3.4:

Proof of Theorem 3.4. Suppose first that $\ell(C) < \infty$. Then we can apply Corollary 9.7 to obtain $C = \bigoplus_{i\in I} D_i$ where the D_i are primitive. Applying Lemma 10.1 to each D_i we obtain a circlex z_i with $D_i = \|z_i\|$ and $\ell(z_i) = \ell(D_i)$. Note that we have $\ell(C) = \sum \ell(D_i)$ by the definition of $\oplus$. Thus $\ell(C) = \sum \ell(z_i)$. It remains to check that $(z_i)_{i\in\mathbb{N}}$ is a σ-representative of C. Indeed, we have $C = \lim \sum_{j\leqslant i} D_i$ by (9.1), and substituting D_i by $\|z_i\|$ we obtain $C = \lim \sum_{j\leqslant i} \|z_i\|$, which means that $(z_i)_{i\in\mathbb{N}}$ is indeed a σ-representative of C by definition. This proves the assertion in this case.

The other case, when $\ell(C) = \infty$ is easier. All we need to show is the existence of a σ-representative of C. For this, let $(C_i)_{i\in\mathbb{N}}$ with $C_i \in H_d$ be a sequence in C, and for every C_i pick a 1-cycle c_i such that $\|z_i\| \in C_i$. Now putting $z_i := c_i - \sum_{j<i} c_j$, we obtain a σ-representative $(z_i)_{i\in\mathbb{N}}$ of C. □

11. Application to graphs

In this section we show that the topological cycle space $\mathcal{C}(G)$ described in the Introduction can be obtained as a special case of $\hat{H}_d$, and that our main result implies, in fact strengthens,

Theorem 1.3. The reader of this somewhat technical section is expected to be familiar with $\mathcal{C}(G)$ and the terminology and ideas of [6, Chapter 8.5].

Let us first prove

Theorem 11.1. *For every locally finite graph G there is a metric of $|G|$ such that $\hat{H}_d(|G|)$ is canonically isomorphic to $\mathcal{C}(G)$.*

The metric d_ℓ we are going to use in Theorem 11.1 is induced by an assignment $\ell : E(G) \to \mathbb{R}_{>0}$ of lengths to the edges of G. More precisely, any such assignment naturally induces a distance $d_\ell(x, y)$ between any two points x, y, and we let $|G|_\ell$ denote the completion of the corresponding metric space. For more details see [19], where the space $|G|_\ell$ is extensively studied. It turns out that choosing an appropriate assignment ℓ one obtains a metric space homeomorphic to $|G|$:

Theorem 11.2. *(See Georgakopoulos [19].) If G is locally finite and $\sum_{e\in E(G)} \ell(e) < \infty$ then $|G|_\ell \cong |G|$.*

Proof of Theorem 11.1 (sketch). Fix a normal spanning tree T of G. Choose $\ell : E(G) \to \mathbb{R}_{>0}$ such that $|G|_\ell \cong |G|$ and moreover the sums of the squares of the lengths of the fundamental cycles with respect to T is finite. For example, we could start with an assignment ℓ' with $\sum \ell'(e) < \infty$, which guarantees $|G|_\ell \cong |G|$ by Theorem 11.2, and then let $\ell(e) := \ell'(e)/m(e)$ where $m(e)$ is the number of fundamental cycles containing e.

We now define a map $f : \mathcal{C}(G) \to \hat{H}_d(|G|_\ell)$ which will turn out to be a canonical isomorphism. Given a $C \in \mathcal{C}(G)$, write C as the sum of a family $\mathcal{F}$ of fundamental cycles with respect to T; this is possible by [6, Theorem 8.5.8]. We will now construct a loop σ in $|G|_\ell$ whose class will become the image $f(C)$ of C. We begin with a loop τ in $|G|_\ell$ that traverses each edge of T once in each direction and traverses no other edges of G. To see that such a loop exists, replace each edge of T by a pair of parallel edges to obtain the auxiliary multigraph T', and apply [7, Theorem 2.5] to obtain a topological Euler tour τ' of T'. Now τ' clearly 'projects' to the desired loop τ. We then modify τ into σ by attaching to it the cycles in $\mathcal{F}$. To achieve this, assume that τ maps a non-trivial interval I_v to each vertex v of G. Now for every fundamental cycle $F \in \mathcal{F}$, let $v_F w_F$ be the chord of F, and assume without loss of generality that v_F is closer to the root of T than w_F. Modify τ so as to use the interval I_{v_F}, previously mapped to v_F, in order to travel once around F, starting and ending at v_F. Doing so for every $F \in \mathcal{F}$ we obtain the loop σ from τ. One still has to check that σ is indeed continuous, but this is not hard. We let $f(C) := [\![1\sigma]\!] \in \hat{H}_d(|G|)$.

The map f is well defined since T and τ are fixed, and every $C \in \mathcal{C}(G)$ has a unique representation as a sum of fundamental cycles with respect to T.

To see that f is injective, let $C \neq D \in \mathcal{C}(G)$. Then the representations of C and D as sums of fundamental cycles differ by at least one fundamental cycle, since there must be a chord e of T contained in one of C, D but not in the other. Now following the lines of Theorem 6.1 one can prove that $f(C) \neq f(D)$; indeed, $d_1(f(C), f(D))$ is bounded from below by a function of the length of e.

It remains to show that f is onto. Pick an element B of $\hat{H}_d(|G|)$ for which we would like to find a preimage. Let $(B_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in B. For every B_i choose a 1-cycle χ_i such that $[\![\chi_i]\!] = B_i$. Using the loop τ from our earlier construction, we can join all the simplices in χ_i into one loop ρ_i which, as τ is null-homotopic, is homologous to χ_i. Now let $C_i \in \mathcal{C}(G)$ be the

sum $\sum\{a_e F_e \mid e \in E(G)\backslash E(T)\}$ of fundamental cycles whose chords are traversed by ρ_i (here F_e denotes the fundamental cycle containing the chord e and a_e is the multiplicity of traversals of e by ρ_i).

We claim that $f(C_i)$ is the equivalence class of the constant sequence $(\|\rho_i\|)$. To begin with, recall that $f(C_i)$ is by definition the equivalence class of the constant sequence $(\|\sigma_i\|)$ for some loop σ_i that traverses the same chords of T as χ_i does. However, the two loops will in general not be homologous, since the order in which these chords are traversed may differ at infinitely many positions. But $\hat{H}_d$ has the ability of 'disentangling' infinite products of commutators, and indeed, we will show that $d_1(\|\rho_i\|, \|\sigma_i\|) = 0$. For this, recall that we chose the edge-lengths $\ell(e)$ so that the sum of the squares of the lengths of all the fundamental cycles is finite. Applying Lemma 4.1 to each fundamental cycle, we can construct an area extension of $|G|_\ell$ with finite excess area in which every fundamental cycle is null-homologous. This means that for every $\epsilon > 0$ there is an area extension X_ϵ of $|G|_\ell$ of excess area at most ϵ in which all but finitely many fundamental cycles are null-homologous. Note that in each such X_ϵ the loops ρ_i and σ_i are homologous, since they traverse the same chords, and all but finitely many of these chords do not matter in X_ϵ; thus the order in which they traverse the chords does not matter (recall that H_1 is abelian). This means that $d_1(\|\rho_i\|, \|\sigma_i\|) = 0$ as claimed.

We have thus found a sequence $C_i \in \mathcal{C}(G)$ such that $(f(C_i))$ converges to B, but we would like to have an element $C \in \mathcal{C}(G)$ with $f(C) = B$. To achieve this, we choose a subsequence (C_{a_i}) of (C_i) that converges, as a set, to an element C of $\mathcal{C}(G)$; such a subsequence exists by compactness. It is now straightforward to check that $f(C) = B$ as desired: we can bound $d_1(f(C), f(C_{a_i}))$ from above by any ϵ choosing i large enough. Indeed, choose i so that the sum of the squares of the lengths of the fundamental cycles with respect to chords in the symmetric difference $C - C_{a_i}$ is small compared to ϵ. Since the sequence $(f(C_i))$ converges to B this immediately yields $f(C) = B$. This completes the proof that f is onto, which makes it an isomorphism, and by construction a canonical one. □

Using this, one now easily obtains Theorem 1.3 as a corollary of our main result Theorem 3.4. Indeed, given $C \in \mathcal{C}(G)$ we apply Theorem 3.4 to $f(C)$, where f is the canonical isomorphism of Theorem 11.1, to obtain a σ-representative (z_i) of $f(C)$ with every z_i being a circlex. Now if two of these circlexes share an edge e, then we can remove e from both and combine the remaining arcs into a new closed simplex, thus obtaining a new σ-representative of smaller total length, contradicting Theorem 3.4. This proves that the z_i are edge-disjoint, and since f is canonical the $f^{-1}(z_i)$ correspond to the same circles of $|G|$ and sum up to C.

In fact, this way we get something slightly stronger than Theorem 1.3: for a given $C \in \mathcal{C}(G)$ there may be several ways to decompose it as a sum of edge-disjoint circles; see [17, p. 6] for an interesting example. Theorem 1.3 cannot distinguish between any of those ways, but our Theorem 3.4 can: it returns one of minimal length. As the total length of such a decomposition does not only depend on the edge-set (see [19, Example 4.5]), this fact can be used in order to control the decomposition we obtain by varying the edge-lengths.

Furthermore, with Theorem 1.4 we generalise, in a sense, Theorem 1.3 to non-locally-finite graphs. For such graphs there are many candidate topologies on which $\mathcal{C}(G)$ can be based, so there is no standard cycle space theory. Theorem 1.4 helps to overcome this difficulty by offering a general result that, for each choice of a topology, yields a corollary similar to Theorem 1.3. This approach is explained in [19, Section 5].

12. Higher dimensions

Our definition of $\hat{H}_d$ can be easily adapted to yield higher dimensional homology groups $\hat{H}_{d,n}$. One can then ask if an analogue of our main result Theorem 3.4 still holds in higher dimensions, but one should first choose a notion of n-dimensional content $vol()$, since there are several ways to generalise 'length' to higher dimensions. Having chosen such a notion, e.g. the n-dimensional Hausdorff measure, one could then try to prove the following.

Problem 12.1. For every compact metric space X and $C \in \hat{H}_{d,n}(X)$, there is a σ-representative $(z_i)_{i\in\mathbb{N}}$ of C with $\sum_i vol(z_i) = vol(C)$.

Most parts of our proof Theorem 3.4, in particular Theorem 8.1, could still be used in an attempt to prove Problem 12.1. To begin with, one would need to generalise the results of Section 4 for the chosen notion of content, which does not seem to be hard. The biggest difficulty though seems to be a generalisation of (10.7).

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