Universal Abelian H-spaces

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ABSTRACT

The question of the existence of universal homotopy commutative and homotopy associative H-spaces (called Abelian H-spaces) is studied. Such a space $T(X)$ would prolong a map from X into an Abelian H-space to a unique H-map from T into X. Examples of such pairs (X, T) are given and conditions are discussed which limit the possible spaces X for which such a T can exist. Contrary to published assertions, the Anick spaces are shown not to be universal Abelian H-spaces for the corresponding Moore spaces; however conditions are discussed which could lead to a universal property with respect to a more limited range of targets, and a restricted universal property is proven.

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By an Abelian H-space we mean a connected CW complex with an H-space structure that is homotopy associative and homotopy commutative. A space T provided with a map $i : X \to T$ will be called the Abelianization of X if T is an Abelian H-space which satisfies the following universal property: for any Abelian H-space Z and any map $f : X \to Z$ there is an H-map $\widehat{f} : T \to Z$, unique up to homotopy, such that $\widehat{f} i \sim f$. Given a space X, an Abelianization may or may not exist, but if it does it is unique. Thus an Abelianization plays a role for Abelian H-spaces analogous to the James construction for group like spaces.

The problem of constructing an Abelianization for suitable spaces has received much recent attention [1,12,14,16,17,22, 26,27]. This concept was first discussed in [12], which took up the question of whether the Anick space $T_{2n-1}(p^r)$ is the Abelianization of the Moore space $P^{2n}(p^r) = S^{2n-1} \cup_{p^r} e^{2n}$. The Anick spaces (see [1,2]) are certain H-spaces that lie in a fibration sequence:

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \to T_{2n-1}(p^r) \to \Omega S^{2n+1}$$

where π_n is the Cohen–Moore–Neisendorfer map of degree p^r. One of our conclusions is that $T_{2n-1}(p^r)$ is not an Abelianization for $P^{2n}(p^r)$. This conclusion is not consistent with the results of [26], where several arguments appear to have gaps.

In [12], a secondary EHP sequence was developed involving the spaces $T_{2n-1}(p^r)$ and spaces $T_{2n}(p^r)$ ($T_{2n}(p^r)$ is the fiber of the degree p^r map on S^{2n+1}). Some sort of universal property was needed to form the compositions necessary for the EHP machine. $T_{2n}(p^r)$ was shown to be the Abelianization of the Moore space $P^{2n+1}(p^r)$, but the properties of $T_{2n-1}(p^r)$ were unresolved.

The goal of this work is to study the conditions under which an Abelianization exists as well as to seek a more restricted universal property which is appropriate for the Anick spaces.

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In particular, we prove the following:

Proposition 1.2. *Suppose T is an Abelianization of X and $k = \mathbb{F}_p$ or Q. Then*

$$H_*(T; k) \cong \mathcal{S}\big(H_*(X; k)\big)$$

where $\mathcal{S}(V)$ is the symmetric algebra on the vector space V.

We will say that a graded Abelian group G has torsion exponent p^r if each element of G of finite order has order $\leqslant p^r$.

Proposition 2.2. *Suppose T is the Abelianization of X. Then the torsion exponent of $H^*(T)$ is the same as the torsion exponent of $H^*(X)$.*

The next result severely impacts the question of which spaces X allow an Abelianization.

Theorem 2.3. *Suppose that $H_*(X)$ has bounded torsion and that pth powers are trivial in $H^*(X; Z/p)$. Then if an Abelianization T exists for X, $H_{2n-1}(X)$ is free for all n.*

Every known example of an Abelianization T of a space X satisfies the condition that the H-map $\Omega SX \xrightarrow{h} T$ extending the inclusion $i: X \to T$ has a right homotopy inverse. We call such an Abelianization split. In Section 3 we study H-spaces T together with a map $h: \Omega SX \to T$ which is split. In this case we construct a fibration sequence:

$$\Omega SX \xrightarrow{h} T \to R \xrightarrow{\pi} SX \tag{3.1}$$

where R is a co-H-space. We also introduce the "universal Whitehead product" map (see [26]) $W = \nabla \omega$

$$\Omega SX * \Omega SX \xrightarrow{\omega} SX \vee SX \xrightarrow{\nabla} SX$$

where ω is the fiber of the inclusion: $SX \vee SX \to SX \times SX$ and ∇ is the folding map. We prove:

Theorem 3.7. *Suppose T is Abelian and split. Then if π factors through W, T is the Abelianization of X*

$$
\begin{array}{ccc}
 & \Omega SX * \Omega SX & \\
 & \nearrow \quad \Big\downarrow W & \\
R & \xrightarrow{\quad \pi \quad} & SX.
\end{array}
$$

In Section 4 we survey the known examples of Abelianization and analyze them from the point of view of Section 3.

In Section 5 we return to the problem of finding a suitable universal property for the Anick spaces. We will call an H-space Z a suitable target for $T_{2n-1}(p^r)$ iff each map $f: P^{2n}(p^r) \to Z$ has a unique extension to an H-map $\widehat{f}: T_{2n-1}(p^r) \to Z$. To this goal we introduce a torsion condition. Recall that $\pi_m(Y; Z/p^r) \simeq [S^{m-1} \cup_{p^r} e^m, Y]$ is the mth homotopy group with coefficients in Z/p^r [23]:

Definition 5.6. A space Y is (n, r)-subexponential if for each $s \geqslant 1$,

$$p^{r+s-1} \pi_{np^s}\big(Y; Z/p^{r+s}\big) = 0.$$

We then conclude that a generalized Eilenberg–MacLane space is a suitable target for $T_{2n-1}(p^r)$ iff it is $(2n, r)$ subexponential. In fact:

Theorem 5.7.

(a) *If Z is any space which is $(2n, r)$-subexponential, then every map $f: P^{2n}(p^r) \to \Omega Z$ extends to a map $\widehat{f}: T_{2n-1}(p^r) \to Z$.*
(b) *If Z is a $(2n, r)$-subexponential H-space and there is an H-map*

$$\widehat{f}: T_{2n-1}\big(p^r\big) \to Z$$

extending $f: P^{2n}(p^r) \to Z$, then $\widehat{f}$ is unique up to homotopy.

The question of whether the Anick spaces are Abelian H-spaces is as yet unresolved. In [26] an argument for the affirmative was presented, but it relied on some of the same unpublished details used in the claim that the Anick spaces are the Abelianization of the Moore space. In Section 6 we prove:

Corollary 6.4. *If $T_{2n-1}(p^r)$ is homotopy associative and Z is a $(2n, r)$-subexponential double loop space, then Z is a suitable target for $T_{2n-1}(p^r)$.*

Throughout this paper, all spaces will be connected, localized at a fixed prime p, and of finite type. I would like to thank Jim Lin for help with Section 2, Fred Cohen, and especially Joe Neisendorfer for many helpful discussions.

1. In this section we will assume that we are given a space X with an Abelianization T. Given a graded vector space V, we will write $\mathcal{S}(V)$ for the symmetric algebra on V. This is the free commutative algebra generated by V and is a tensor product of polynomial algebras on a basis for V modulo the relation $2x^2 = 0$ when x has odd degree. If V is a co-algebra, $\mathcal{S}(V)$ is a Hopf algebra.

For later purposes we will only assume that the map $i : X \to T$ is universal for a selected list of target spaces.

Proposition 1.1. *Suppose $k = \mathbb{F}_p$ or Q, and $i : X \to T$ is universal for the targets $\Omega^2 S^2 X$ and $K(k, m)$ for all m. Then*

$$H_*(T; k) \cong \mathcal{S}\big(H_*(X; k)\big).$$

Proof. Since T is Abelian, there is a map of Hopf algebras:

$$\theta : \mathcal{S}\big(H_*(X; k)\big) \to H_*(T; k).$$

We first show that θ is a monomorphism. Using the universal property we construct an H-map:

$$\gamma : T \to \Omega^2 S^2 X$$

extending the inclusion of X in $\Omega^2 S^2 X$. We claim that the composition:

$$\mathcal{S}\big(H_*(X; k)\big) \xrightarrow{\theta} H_*(T; k) \to H_*\big(\Omega^2 S^2 X; k\big)$$

is a monomorphism. In the case that $k = \mathbb{F}_p$, this follows from the work of Cohen [7, 3.2] where the author shows that $H_*(\Omega^2 S^2 X; \mathbb{F}_p)$ is a symmetric algebra on certain generators which include the generators of $H_*(X; k)$. Thus θ is a monomorphism in this case. In case $k = Q$, choose a basis $\{x_i\}$ for $H_*(X; Q)$ and let $\widehat{x}_i$ be a dual basis. Then the map:

$$\Pi \widehat{x}_i : X \to \Pi K\big(Q, |x_i|\big)$$

is a monomorphism in rational homology. Using the universal property for each factor, we construct an extension:

Since $H_*(\Pi K(Q, |x_i|))$ is the symmetric algebra on the x_i, it follows that θ is a monomorphism in this case as well.

We will show that θ is an epimorphism by showing that the dual

$$\theta^* : H^*(T; k) \to \mathcal{S}^*\big(H_*(X; k)\big)$$

is a monomorphism. Choose an element $\xi \in H^n(T; k)$ in the kernel of θ^* of least possible dimension.

Let $\mu : T \times T \to T$ be the H-space structure map, so

$$\mu^* : H^*(T; k) \to H^*(T \times T; k) \cong H^*(T; k) \otimes H^*(T; k)$$

defines the coalgebra structure. Then

$$\big(\theta^* \otimes \theta^*\big)\big(\mu^*(\xi)\big) = \nabla^*\big(\theta^*(\xi)\big) = 0.$$

Suppose $\mu^*(\xi) = \xi \otimes 1 + \sum \xi_i' \otimes \xi_i'' + 1 \otimes \xi$ where ξ_i' and ξ_i'' have dimension less than n. We then get:

$$\big(\theta^* \otimes \theta^*\big)\big(\mu^*(\xi)\big) = \sum \theta^*\big(\xi_i'\big) \otimes \theta^*\big(\xi_i''\big).$$

However $\theta^* \otimes \theta^*$ factors:

$$H^*(T; k) \otimes H^*(T; k) \xrightarrow{\theta^* \otimes 1} \mathcal{S}^* \otimes H^*(T; k) \xrightarrow{1 \otimes \theta^*} \mathcal{S}^* \otimes \mathcal{S}^*.$$

Since θ^* is a monomorphism in dimensions less than n this composition is a monomorphism when restricted to $H^i(T; k) \otimes H^j(T; k)$ with $i, j < n$. Since

$$\sum \theta^*(\xi_i') \otimes \theta^*(\xi_i'') = 0,$$

it follows that ξ is primitive. Represent ξ by an H-map

$$\xi : T \to K(k, n).$$

By the universal property, we need only show that $\xi i : X \to K(k, n)$ is trivial to conclude that ξ is trivial and hence θ^* is a monomorphism. But i^* factors:

$$H^*(T; k) \xrightarrow{\theta^*} \mathcal{S}^* \to H^*(X; k)$$

since the inclusion $H_*(X; k) \to H_*(T; k)$ factors through $\mathcal{S}(H_*(X; k))$. So $i^*(\xi) = 0$ since $\theta^*(\xi) = 0$. $\quad \square$

Consequently a much weaker universal property—for example restricting acceptable targets Z to be double loop spaces—is sufficient to calculate the homology. As a corollary, we have

Proposition 1.2. *Suppose T is the Abelianization of X and $k = \mathbb{F}_p$ or Q. Then*

$$H_*(T; k) \cong \mathcal{S}(H_*(X; k)).$$

2. In this section we will develop some general properties of an Abelianization T of X and conclude that there are some severe limitations on the spaces X that are available.

Recall that if Z is an H-space, a class $\xi \in H^m(Z; G)$ is called primitive if the corresponding map:

$$\xi : Z \to K(G, m)$$

is an H-map. This is equivalent to the equation:

$$\mu^*(\xi) = \pi_1^*(\xi) + \pi_2^*(\xi)$$

where $\mu : Z \times Z \to Z$ is the H-space structure map, and π_i is the ith projection. Let $PH^*(Z; G)$ be the subgroup of primitive cohomology classes.

Lemma 2.1. *Suppose T is the Abelianization of X. Then*

$$i^* : PH^*(T; G) \to H^*(X; G)$$

is an isomorphism.

Proof. This is immediate from the universal property. $\quad \square$

We will say that a space has cohomology torsion exponent p^r if each cohomology class of finite order has order dividing p^r. Since all spaces we are considering are of finite type, the cohomology torsion exponent is the same as the homology torsion exponent.

Proposition 2.2. *Suppose X has cohomology exponent p^r and T is the Abelianization of X. Then T has cohomology exponent p^r.*

Proof. Since the inclusion $X \to \Omega^2 \Sigma^2 X$ extends over T, $S^2 X$ is a retract of $S^2 T$, so the exponent of X is less than the exponent of T. Suppose X has cohomology exponent p^r and $\xi \in H^m(T)$ has order p^{r+1}. Suppose that m is the minimal dimension in which such a class ξ can occur. We will show that $p^r \xi$ is primitive. This will imply that $p^r \xi = 0$ by 2.1, completing the proof. To accomplish this we examine the cohomology group $H^m(T \wedge T)$ using a Künneth Theorem available for all spaces of finite type [20, 5.7.26]

$$\widetilde{H}^m(T \wedge T) \cong \bigoplus_{i+j=m} \widetilde{H}^i(T) \otimes \widetilde{H}^j(T) \oplus \bigoplus_{i+j=m+1} \text{Tor}\big(\widetilde{H}^i(T), \widetilde{H}^j(T)\big).$$

Since $\widetilde{H}^1(T)$ is free, all nonzero terms on the right involve cohomology groups of dimension less than m; hence $H^m(T \wedge T)$ has homology exponent at most p^r. Thus $\mu^*(p^r \xi)$ has no middle terms and $p^r \xi$ is primitive, completing the proof. $\quad \square$

The next result is particularly useful when either X is a co-H-space or all Steenrod operations are trivial in X.

Theorem 2.3. *Suppose $H_*(X)$ has bounded torsion and pth powers are trivial in $H^*(X; Z/p)$. Then if X has an Abelianization, $H_{2n-1}(X)$ is free for all n.*

Proof. Suppose T is an Abelianization of X. We first show that all pth powers are trivial in $H^*(T; Z/p)$. The map

$$\mathcal{P}: H^*(T; Z/p) \to H^*(T; Z/p)$$

defined by $\mathcal{P}(\xi) = \xi^p$ is a map of Hopf algebras, as is its dual

$$\mathcal{P}_*: H_*(T; Z/p) \to H_*(T; Z/p).$$

$\mathcal{P}_*$ is determined by its action on generators, all of which are in the image of $H_*(X; Z/p)$. Since $\mathcal{P}$ is trivial when applied to X, $\mathcal{P}_*$ is as well. Thus $\mathcal{P}_*$ and $\mathcal{P}$ are trivial when applied to T.

We will suppose now that there is an element $\widetilde{y} \in H_{2n-1}(X)$ of finite order with $r > 1$ and r maximal. Using the exact sequence:

$$\cdots \to H_{k+1}(X; Z/p) \xrightarrow{\partial} H_k(X) \xrightarrow{p} H_k(X) \xrightarrow{\rho} H_k(X; Z/p) \to \cdots \tag{2.4}$$

we choose $x' \in H_{2n}(X; Z/p)$ with $\partial x' = p^{r-1}\widetilde{y}$ and let $y' = \rho(\widetilde{y})$. Then $\beta^{(r)}(x') = y'$. Choose dual classes $\widehat{x}'$ and $\widehat{y}'$ with $\langle x', \widehat{x}'\rangle = 1 = \langle y', \widehat{y}'\rangle$ and $\beta^{(r)}\widehat{y}' = \widehat{x}'$. Now extend $\widehat{x}'$ and $\widehat{y}'$ to primitive cohomology classes $\widehat{x}$ and $\widehat{y}$ in $H^*(T; Z/p)$. Let $x = i_*(x')$ and $y = i_*(y')$ so $\langle x, \widehat{x}\rangle = 1 = \langle y, \widehat{y}\rangle$. We now appeal to the theorem of Browder:

Theorem 2.5. *([4, 5.4]) Let T be an Abelian H-space and $x \in E_{2n}^{(r)}$, $\beta^{(r)}(x) = y \in E_{2n-1}^{(r)}$ where $E_*^{(*)}$ is the Backstein spectral sequence. Then if $r > 1$, $\beta^{(r+1)}(x^p) = \{x^{p-1}y\}$—the class of $x^{p-1}y$ in $E_{2n-1}^{(r+1)}$.*

Note that Browder's theorem does not assert that $\{x^{p-1}y\} \neq 0$. Our first task will be to determine this. Suppose that $x^{p-1}y = \beta^{(s)}(z)$ for some z and $s \leqslant r$. Observe, by induction on k, that

$$\langle x^k y, \widehat{x}^k \widehat{y}\rangle = k!$$

since $\widehat{x}$ and $\widehat{y}$ are primitive. Thus

$$(p-1)! = \langle x^{p-1}y, \widehat{x}^{p-1}\widehat{y}\rangle = \langle \beta^{(s)}(z), \widehat{x}^{p-1}\widehat{y}\rangle = \langle z, \beta^{(s)}(\widehat{x}^{p-1}\widehat{y})\rangle.$$

But $\beta^{(s)}\widehat{y} = 0$ and $\beta^{(s)}\widehat{x} = 0$ for $s < r$, so we must have $s = r$. Since $\beta^{(r)}\widehat{y} = \widehat{x}$, we get

$$(p-1)! = \langle z, \widehat{x}^p\rangle.$$

Since pth powers are trivial in $H^*(T; Z/p)$, we conclude that no such z can exist and $\beta^{(r+1)}(x^p) \neq 0$ in $E_{2np-1}^{(r+1)}$. It follows that there is an element of order p^{r+1} in $H_*(T)$. By 2.2 there must be an element of $H_*(X)$ of order p^{r+1} contradicting the choice of r being maximal. This completes the proof in case $r > 1$.

Now suppose that each element of $H_{2n-1}(X)$ of finite order has order p. Write:

$$H_*(X) = F \oplus T_e \oplus T_o$$

where F is free with a basis $\{\widetilde{z}_1, \ldots, \widetilde{z}_m\}$, T_e is a sum of cyclic groups of even degree generated by $\{\widetilde{e}_1, \ldots, \widetilde{e}_\ell\}$ and T_o is a sum of cyclic groups of odd degree generated by $\{\widetilde{y}_1 \ldots \widetilde{y}_k\}$. Using 2.4, construct corresponding classes in $H_*(X; Z/p)$, z'_i, e'_2, y'_i, f'_i and x'_i with $\beta^{(s_i)}(f'_i) = e'_i$ and $\beta^{(1)}(x'_i) = y'_i$. Using a monomial basis, construct dual classes $\widehat{z}'_i, \widehat{e}'_i, \widehat{y}'_i, \widehat{f}'_i, \widehat{x}'_i$ in $H^*(X; Z/p)$ and extend these to primitive cohomology classes $\widehat{z}_i, \widehat{e}_i, \widehat{y}_i, \widehat{f}_i, \widehat{x}_i$ in $H^*(T; Z/p)$. Write z_i, e_i, y_i, f_i and x_i for the images of $z'_i, e'_i, y'_i, f'_i, x'_i$ in $H_*(T; Z/p)$.

Suppose now that $2n = |x_1| \leqslant |x_i|$ for all i, and we relabel x_1 as x. Since $\beta^{(s)}(x^p) = 0$, ∂x^p is divisible by p. Since $|\partial x^p|$ is odd, this implies that $\partial x^p = 0$ so $x^p = \rho(\bar{x})$ for some class $\bar{x} \in H_{2np}(T)$. We claim that $\bar{x}$ has infinite order. If not $x^p = \beta^{(s)}(\theta)$ for some $\theta \in E_{2np+1}^{(s)}$. Let $J \subset H_*(T; Z/p)$ be the subalgebra generated by e_i, f_i, z_i. Then for $\xi \in J$, $\beta^{(s)}(\xi) \in J$ if it is defined. Now

$$E^{(2)} = Z/p[x_1^p, \ldots, x_k^p] \otimes \Lambda(y_1 x_1^{p-1}, \ldots, y_k x_k^{p-1}) \otimes E^{(2)}(J).$$

For $s \geqslant 2$, $E^{(s)}$ consists only of classes of elements listed here. For dimensional reasons, any possible θ must be of the form

$$\theta = \sum \sigma_i x_i^p + \sigma'_i y_i x_i^{p-1} + \sigma$$

where σ_i, σ'_i and σ are elements of J. Furthermore, $|x_i| = 2n$ for each i that can occur in this expression. Now $|\sigma_i| = 1$ and $|\sigma'_i| = 2$. Suppose then that $\beta^{(i)}(\theta) = 0$ for all $i < s$; then we have

$$\beta^{(s)}(\theta) = \sum \sigma'_i \beta^{(s)}(y_i x_i^{p-1}) + \beta^{(s)}(\sigma)$$

which is in the ideal generated by J. Since x^p is not in this ideal, we conclude that $\beta^{(s)}(\theta) \neq x^p$ for any s or θ and $\bar{x}$ has infinite order.

Now let $H_*(T) = \overline{F} \oplus \overline{T}$ where $\overline{F}$ is free and $\overline{T}$ is a torsion group. We claim that $\overline{F} = \mathcal{S}_Z(F)$, the symmetric ring over Z of F. Consider the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{S}_Z(F) & \xrightarrow{\ \alpha\ } & H_*(T) \\
\downarrow & & \downarrow \\
\mathcal{S}_Z(F) \otimes Q \simeq \mathcal{S}_Q(F) & \xrightarrow[\ \simeq\ ]{} & H_*(T; Q).
\end{array}
$$

Since the left and bottom homomorphisms are monomorphisms, α is too and $\mathcal{S}_Z(F) \subset \overline{F}$. However the rank of $\mathcal{S}_Z(F)$ is the same as the rank of $\mathcal{S}_Q(F) \simeq H_*(T; Q) = \overline{F} \otimes Q$, so $\mathcal{S}_Z(F)$ and $\overline{F}$ have the same rank. Consequently $\overline{F} = \mathcal{S}_Z(F)$ and $\bar{x} \in \overline{F}$ must be of the form:

$$\bar{x} = f(\bar{z}_1, \ldots, \bar{z}_m)$$

for some polynomial f. Apply ρ to this equation to get

$$x_1^p = f(z_1, \ldots, z_m)$$

where $z_i = p\bar{z}_i$. This is a contradiction since $f(a_1 \ldots z_m) \neq 0$, but $x_1^p = \mathcal{P}(x_1) = 0$. $\quad\square$

Note: We have few examples of spaces which allow an Abelianization, and 2.2 and 2.3 suggest that the possibilities are limited. We suggest here one method of finding Abelianization. This comes from the observation that if X and $X \cup e^n$ both have an Abelianization, there is a fibration

$$T(X) \to T\big(X \cup_\theta e^n\big) \to T\big(S^n\big).$$

This is particularly useful if n is odd and $p > 3$. In this case $T(S^n) = S^n$ and $T(X \cup e^*)$ can be constructed from clutching data:

$$S^{n-1} \times T(X) \to T(X)$$

obtained from the map $\theta : S^{2n-1} \to X$ and the H-space structure on $T(X)$.

3. In this section we assume that T is an H-space with a fixed structure map $\overline{\mu} : T \times T \to T$, and we are given a map $i : X \to T$. We do not assume that T is the Abelianization of X. We do, however, assume that both X and T are $(n-1)$-connected for $n \geqslant 1$ and that $i_* : \pi_n(X, *) \to \pi_n(T, e)$ is a nonzero isomorphism. This is a necessary condition if T were the Abelianization of X by 2.1.

Recall the Hopf quasi fibration [8]:

$$T \to T * T \to ST$$

where the inclusion $T \to T * T$ is null homotopic. We construct an induced fibration sequence:

$$
\begin{array}{ccccccc}
\Omega ST & \xrightarrow{\ \mu\ } & T & \longrightarrow & T * T & \longrightarrow & ST \\
\big\uparrow{\scriptstyle \Omega Si} & & \big\| & & \big\uparrow{\scriptstyle j} & & \big\uparrow{\scriptstyle Si} \\
\Omega SX & \xrightarrow{\ h\ } & T & \longrightarrow & R & \xrightarrow{\ \pi\ } & SX,
\end{array}
\qquad (3.1)
$$

where the connecting map μ is a left homotopy inverse to the inclusion $T \to \Omega ST$. Note also that if T is homotopy associative, μ is an H-map (see [25], [15, Proposition A.3]) and consequently h an H-map as well.

Definition 3.2. We will say that T is split if the map $h : \Omega SX \to T$ has a right homotopy inverse.

Recall that an $(n-1)$ connected space X is called atomic if $\pi_n(X)$ is a nontrivial cyclic group and any self map inducing an isomorphism in π_n is a homotopy equivalence.

Proposition 3.3. *Suppose T and SX are atomic.*[1] *Then the following are equivalent*:

(a) *T is split*;
(b) *the inclusion $Si : SX \to ST$ has a left homotopy inverse*;
(c) *given a map $f : X \to Z$ where Z is an H-space, there is an extension $\widehat{f} : T \to Z$; i.e., $\widehat{f} i \sim f$.*

   Note: We are not asserting that $\widehat{f}$ is an H-map in (c) or that it is unique.

Proof. (a) $\Rightarrow$ (b). Let $g : T \to \Omega SX$ be a right homotopy inverse for h. Suppose that X is $(n-1)$-connected and $\pi_n(X) \neq 0$. Then g and h induce inverse isomorphisms in $\pi_n = H_n$. Thus the adjoint of g

$$\widetilde{g} : ST \to SX$$

induces an isomorphism of H_{n+1}. Consequently the composition:

$$SX \xrightarrow{Si} ST \xrightarrow{\widetilde{g}} SX$$

induces an isomorphism in H_{n+1}. Since SX is atomic, $e = \widetilde{g}(Si)$ is an equivalence. Then $e^{-1}\widetilde{g}$ is a left homotopy inverse to Si.
   (b) $\Rightarrow$ (c). Let $\widetilde{g} : ST \to SX$ be a left homotopy inverse to Si. Let $\widetilde{f}$ be the composition:

$$T \xrightarrow{g} \Omega SX \xrightarrow{f_\infty} Z$$

where g is the adjoint of $\widetilde{g}$ and f_∞ is an extension of f given by a choice of association in the James construction. Then $\widetilde{f} i \sim f$.
   (c) $\Rightarrow$ (a). Using (c), construct a map $g : T \to \Omega SX$ such that gi is the inclusion of X in ΩSX. From the naturality of the inclusion into the loops on the suspension, we get a homotopy commutative diagram:

Consequently $hgi \sim i$. Since $i_* \cdot \pi_n(X, *) \to \pi_n(T, e)$ is an isomorphism it follows that

$$(hg)_* : \pi_n(T) \to \pi_n(T)$$

is an isomorphism as well. Since T is atomic, hg is an equivalence and $g(hg)^{-1}$ is a right homotopy inverse for h and hence defines a splitting. $\square$

Remark 3.4. In the case that T is the Abelianization of X, it is easy to see that if T is atomic, X is atomic; for any map $f : X \to X$ induces an H-map $\widehat{f} : T \to T$. If f induces an isomorphism in $\pi_n(X)$, then $\widehat{f}$ will induce an isomorphism in $\pi_n(T)$. Hence $\widehat{f}$ is an H-equivalence. Since $PH^*(T; Z/p) \cong H^*(X; Z/p)$, it follows that f induces isomorphisms in cohomology and hence it is an equivalence.

Proposition 3.5. *If T is split, $R \vee ST \simeq SX \ltimes T$ and hence R is a co-H-space.*

Proof. Each fibration:

$$F \to E \to SK$$

is given by a clutching construction (see [11, 1b]) and there is a natural homotopy equivalence:

$$E/F \simeq SK \ltimes F.$$

Applying this we see that

$$R/T \simeq SX \ltimes T.$$

Since T is split, the map $T \to R$ is null homotopic, so $R/T \simeq R \vee ST$. $\square$

Proposition 3.6. *Suppose T is split and Z is an H-space. Then two H-maps $f_0, f_1 : T \to Z$ are homotopic iff $f_0 i \sim f_1 i : X \to Z$.*

---

[1] Being atomic implies that $\pi_n(T) \cong \pi_{n+1}(SX)$ is cyclic.

Proof. Let $a_k : J_k(X) \times T \to J(X) \times T \to \Omega SX \times T \to T$ be the restriction of the action map from the lower fibration in 3.1 and $\theta = a_1 : X \times T \to T$. Comparing the action maps for the two fibrations in 3.1 gives $\theta(x, t) = \overline{\mu}(i(x), t)$. Now by [15, A3] we have:

$$a_k(x_1, \ldots, x_k, t) = \theta\big(x_1, a_{k-1}(x_2, \ldots, x_k, t)\big).$$

Define $h_k : J_k(X) \to T$ by $h_k(x_1, \ldots, x_k) = a_k(x_1, \ldots, x_k, *)$. Then we have

$$h_k(x_1 \ldots x_k) = \theta\big(x_1, h_{k-1}(x_2, \ldots, x_k)\big) = \overline{\mu}\big(i(x_1), h_{k-1}(x_2, \ldots, x_k)\big).$$

Thus we have homotopy commutative diagrams with $i = 0$ or 1:

$$
\begin{array}{ccccc}
X \times J_{k-1}(X) & \xrightarrow{\;i \times h_{k-1}\;} & T \times T & \xrightarrow{\;f_i \times f_i\;} & Z \times Z \\
{\scriptstyle a}\big\downarrow & & {\scriptstyle \overline{\mu}}\big\downarrow & & \big\downarrow \\
J_k(X) & \xrightarrow{\quad h_k \quad} & T & \xrightarrow{\quad f_i \quad} & Z.
\end{array}
$$

Suppose $f_0 i \sim f_1 i$ and $f_0 h_{k-1} \sim f_1 h_{k-1}$. Since the top rows are homotopic for $i = 0, 1$, we see that $f_0 h_k a \sim f_1 h_k a$. Since $S(a)$ has a right homotopy inverse and Z is an H-space, we conclude that $f_0 h_k \sim f_1 h_k$. However these homotopies may not be compatible. This is similar to a phantom map situation where the filtration is not by skeletons or compact subsets but by the James filtration. This can be handled by the same methods (see [9]). Define

$$V = \bigvee_{k=0}^{\infty} J_k(X)$$

and construct a cofibration sequence:

$$V \to J(X) \to C(X) \to SV \to S J(X).$$

The map $SV \to S J(X)$ has a right homotopy inverse, so the map $S J(X) \to SC(X)$ is null homotopic. Now the difference

$$S(f_0 h) - S(f_1 h) : S J(X) \to SZ$$

is null homotopic when restricted to SV since $f_0 h_k \sim f_1 h_k$ for all k. Thus it factors over $SC(X)$ and is consequently inessential. Thus $S(f_0 h) \sim S(f_1 h)$. Since Z is an H-space $f_0 h \sim f_1 h : \Omega SX \to T \to Z$. Since T is split, we have $f_0 \sim f_1$. □

We see that if T is split extensions exist and if we have an extension which is an H-map, it is unique. The hard part is to construct extensions which are H-maps. The next result addresses this issue.

Recall [26], the universal Whitehead product W, which is the composition:

$$\Omega SX * \Omega SX \xrightarrow{\;\omega\;} SX \vee SX \xrightarrow{\;\nabla\;} SX$$

where ω is the fiber of the inclusion of $SX \vee SX$ in $SX \times SX$ and ∇ is the folding map.

Theorem 3.7. *Suppose T is Abelian and split. Then if π factors through W, T is the Abelianization of X*

$$
\begin{array}{ccc}
& & \Omega SX * \Omega SX \\
& \nearrow & \big\downarrow {\scriptstyle W} \\
R & \xrightarrow{\quad \pi \quad} & SX.
\end{array}
$$

Proof. Suppose Z is Abelian and $f : X \to Z$. Extend f to an H-map $f_\infty : \Omega SX \to Z$. Now consider the diagram:

$$
\begin{array}{ccccc}
\Omega(SX \vee SX) & \xrightarrow{\;\Omega\nabla\;} & \Omega SX & \xrightarrow{\;f_\infty\;} & Z \\
{\scriptstyle \Omega j}\big\downarrow & & & & \big\uparrow \\
\Omega(SX \times SX) \cong \Omega SX \times \Omega SX & & \xrightarrow{\;f_\infty \times f_\infty\;} & & Z \times Z.
\end{array}
\tag{3.8}
$$

Both components $\Omega(SX \times SX) \to Z$ in the bottom row are H-maps. The composition of the bottom row and the right vertical arrow is the sum of these two H-maps. Since Z is Abelian, this composition is an H-map. Thus all maps in this diagram are H-maps. Thus to check that it is homotopy commutative, it suffices to check the restriction to $X \vee X$, where both sides are $f\nabla$. From this we conclude that $(f_\infty)\Omega W \sim *$, since $\Omega W = (\Omega\nabla)(\Omega\omega)$ and $(\Omega j)(\Omega\omega) \sim *$. The hypothesis that π factors though W implies that $f_\infty \Omega \pi \sim *$.

The remainder of the proof follows [12, Part I, Lemma 3.5]. Let $g : T \to \Omega SX$ be a right homotopy inverse to h. Since T is homotopy associative, h is an H-map, so $h(1 - gh) \sim *$. This implies that there is a map $s : \Omega SX \to \Omega R$ such that $(\Omega \pi)s \sim 1 - gh$. Consequently

$$f_\infty - f_\infty gh \sim f_\infty(1 - gh) \sim f_\infty(\Omega \pi)s \sim *.$$

So $f_\infty \sim f_\infty gh$. Let $\widehat{f} = f_\infty g : T \to Z$. Let $i' : X \to \Omega SX$ be the inclusion. Let $\widehat{i} : X \to \Omega ST$ be the adjoint of Si. Then by 3.1, $hi' \sim \mu \widehat{i} \sim i$. Consequently $\widehat{f} i = f_\infty gi \sim f_\infty ghi' \sim f_\infty i' \sim f$ so $\widehat{f}$ is an extension of f. To see that $\widehat{f}$ is an H-map, consider the diagram:

$$
\begin{array}{ccccc}
\Omega SX \times \Omega SX & \xrightarrow{h \times h} & T \times T & \xrightarrow{\widehat{f} \times \widehat{f}} & Z \times Z \\
\downarrow & & \downarrow & & \downarrow \\
\Omega SX & \xrightarrow{\quad h \quad} & T & \xrightarrow{\quad \widehat{f} \quad} & Z
\end{array}
$$

where the verticle maps are the H-space structure maps. The left hand square and the rectangle commute up to homotopy. Since $h \times h$ has a right homotopy inverse, the right hand square does as well so $\widehat{f}$ is an H-map. To see that $\widehat{f}$ is unique, suppose $\widehat{f}' : T \to Z$ is another H-map extending f. Then $\widehat{f}'h : \Omega SX \to Z$ is an H-map excluding f, so $\widehat{f}'h \sim \widehat{f}h$. Since h has a right homotopy inverse, $\widehat{f}' \sim \widehat{f}$. □

Corollary 3.9. *Suppose T is Abelian and π factors through W. Then there is a homotopy commutative diagram*:

$$
\begin{array}{ccccc}
R & \xrightarrow{\alpha} & \Omega SX * \Omega SX & \xrightarrow{\beta} & R \\
& {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle W} & \swarrow {\scriptstyle \pi} & \\
& & SX & &
\end{array}
$$

with $\beta\alpha$ a homotopy equivalence.

Proof. By 3.7, π factors through W. To see that W factors through π, consider the diagram (3.8) with T in place of Z and h in place of f_∞. It follows that $h(\Omega W) \sim *$. So ΩW factors through $\Omega \pi$. Since $\Omega SX * \Omega SX$ is a co-H-space, W factors through π. Thus we have constructed α and β. It remains to show that $\beta\alpha$ is an equivalence. Looping the diagram, we get:

$$
\begin{array}{ccccc}
\Omega R & \xrightarrow{\Omega\alpha} & \Omega(\Omega SX * \Omega SX) & \xrightarrow{\Omega\beta} & \Omega R. \\
& {\scriptstyle \Omega\pi} \searrow & \downarrow {\scriptstyle \Omega W} & \swarrow {\scriptstyle \Omega\pi} & \\
& & \Omega SX & &
\end{array}
$$

Since $\Omega \pi$ induces a monomorphism in mod p homology $(\Omega\beta)(\Omega\alpha)$ does as well. Since ΩR is of finite type, this implies that $(\Omega\beta)(\Omega\alpha)$ induces an isomorphism in mod p homology. Likewise $(\Omega\beta)(\Omega\alpha)$ induces an isomorphism in rational homology, so it is an equivalence and $\beta\alpha$ is an equivalence. □

Remark 3.10. When T is the Abelianization of X, the algebraic structure can be described as follows: Let L be the free Lie algebra on $H_*(X; k)$ and $U(L)$ its universal enveloping algebra. Then $H_*(\Omega SX; k) \cong U(L)$ and $H_*(T; k) = U(L/[L, L])$. $[L, L] \subset L$ is the free Lie algebra on the desuspension of $\widetilde{H}_*(R; k)$ and $H_*(\Omega R; k) = U([L, L])$.

Remark 3.11. Since $H_*(\Omega SX; k) \to H_*(T; k)$ is the standard map from a tensor algebra to its Abelianization in the case that T is the Abelianization of X, $H_*(\Omega R; k)$ is the universal enveloping algebra $U([L, L])$ where L is the Lie algebra given by $H_*(\Omega SX; k)$. Thus the image of the map

$$H_*(R; k) \to H_*(\Omega SX; k)$$

is $[L, L]$ where L is $H_*(\Omega SX; k)$ considered as a Lie algebra L.

4. In this section we will briefly review the known examples of spaces X with an Abelianization T. These occur in various places in the literature. They are all split and we will examine them in terms of the fibration sequence 3.1.
   (a) $X = S^{2n}$, $T = \Omega S^{2n+1}$, $R = *$. T is Abelian iff $p > 2$.
   (b) $X = S^{2n+1} = T$, $R = S^{4n+3}$, $\pi = [i, i] : S^{4n+3} \to S^{2n+2}$. T is Abelian iff $p > 3$.
   (c) $X = P^{2n+1}(p^r)$, $T = T_{2n}(p^r) = S^{2n+1}\{p^r\}$—the fiber of the degree p^r map $p^r : S^{2n+1} \to S^{2n+1}$. The structure map is given by:

$$\pi : R = \bigvee_{k \geqslant 1} S^{2nk+2} X \xrightarrow{\vee ad^{k-1}(u)([v,v])} P^{2n+2}(p^r)$$

where $v : P^{2n+1}(p^r) \to P^{2n+1}(p^r)$ is the identity map and

$$u = \beta^r v : P^{2n}(p^r) \to P^{2n+1}(p^r).$$

This splitting first appeared in [6, 1.1]. T is Abelian if $p > 3$ [21]. (See also [14].)

(d) $X = S^{2m} \cup_\theta e^{2n+1}$, $T =$ the fiber of $S\theta : S^{2n+1} \to S^{2m+1}$. This is an immediate generalization of (c), however the proof of the splitting is quite different [14] and simpler. The structure map is given by:

$$\pi : R = \bigvee_{k \geqslant 0} S^{2mk+2n+2} X \xrightarrow{\vee ad^{(k)}(u)([v,v])} SX$$

where u and v are defined similarly to (c) and the corresponding Whitehead products with coefficients in X are in the sense of [13]. The proof that T is Abelian for $p > 3$ is due to Grbic [17].

(e) $X = S^{2n-1} \cup_{\omega_n} e^{2np-2}$—$2np - 2$ skeleton of $\Omega^2 S^{2n+1}$. $T = \Omega J_{p-1}(S^{2n})$, where $J_{p-1}(S^{2n})$ is the $(p-1)$st filtration of the James construction $J(S^{2n})$

$$J_{p-1}(S^{2n}) = S^{2n} \cup e^{4n} \cup \cdots \cup e^{2n(p-1)},$$
$$R = \bigvee_{k \geqslant 0} S^{(2np-2)k+4n-1} \vee \bigvee_{k \geqslant 0} S^{(2np-2)(k+1)+2n} \xrightarrow{\pi} SX$$

is given by $ad^{(k)}(v)([u,u]) \vee ad^{(k+1)}(v)(u)$. Here by $[u,u]$ we mean the composition:

$$S^{4n-1} \xrightarrow{[i,i]} S^{2n} \to SX$$

and by $[v,\phi]$, for $\phi : S^m \to SX$, we mean the unique map (up to homotopy in the homotopy commutative square):

$$
\begin{array}{ccc}
S^{2np-2+m} & \xrightarrow{[v,\phi]} & SX \\
\uparrow & & \uparrow{\scriptstyle \phi\vee 1} \\
S^m X & \xrightarrow{\omega} & S^m \vee SX.
\end{array}
$$

This result appears in a number of places [12,14,19].

(f) $X = S^{2n+1} \cup_{\omega_n} e^{2np-2} \cup_{pi} e^{2n-1}$—the $2np - 1$ skeleton of $\Omega^2 S^{2n+1}$. In this case T is the Selick space F_2 (see [24]) given by the pull back:

$$
\begin{array}{ccc}
F_2 & \longrightarrow & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow{\scriptstyle H} \\
S^{2np-1} & \longrightarrow & \Omega^2 S^{2np+1}.
\end{array}
$$

This is the most complicated example and has been thoroughly worked out by Grbic [16].

(g) We might try to generalize this to other spaces in the nested sequence:

$$\Omega J_{p-1}(S^{2n}) \subset F_2 \subset \Omega J_{p^2-1}(S^{2n}) \subset F_3 \subset \cdots.$$

However none of these spaces beyond F_2 is the Abelianization of some space X. This is because in each case there is an indecomposable homology class $x_2 \in H_{2np^2-2}(T; Z/p)$ such that $(\mathcal{P}^1)_*(x_2) \in H_{(2np-2)p}(T; Z/p)$ is decomposable. Thus x_2 must lie in the image of $i_* : H_*(X; Z/p) \to H_*(T; Z/p)$ modulo decomposables, while $(\mathcal{P}^1)_*(x_2)$ does not lie in the image of i_*.

(h) Let X be a p-local CW complex with ℓ cells, each of which has odd dimension such that $\ell < p - 2$. Then X has an Abelianization [27]. These spaces were first discussed in [5].

5. The origin of the study of Abelianization arose from an attempt to understand the mapping properties of the Anick spaces $T_{2n-1}(p^r)$ [12]. Recall that the space $T_{2n}(p^r)$ is the Abelianization of the Moore space $P^{2n+1}(p^r)$. It is clear from 2.2, 2.3, or 5.2 (below) that $T_{2n-1}(p^r)$ is not the Abelianization of any subspace. Nevertheless, the considerations of [12] suggest that it should have some universal mapping property.

Definition 5.1. Let $M(n,r)$ be the category of H-spaces Z such that there is a 1–1 correspondence between $[P^{2n}(p^r), Z]$ and the set of homotopy classes of H-maps from $T_{2n-1}(p^r)$ to Z.

We seek to understand this category. Examining the proofs of 2.2 and 2.3 suggests that we first inquire as to which Eilenberg–MacLane spaces belong to $M(n, r)$. We consider $K(G, k)$ where G is a finitely generated Abelian group. The homotopy classes of H-maps from $T_{2n-1}(p^r)$ to $K(G, k)$ are in 1–1 correspondence with $PH^k(T_{2n-1}(p^r); G)$. It suffices to consider the cases $G = Z_{(p)}$ and $G = Z/p^r$.

Theorem 5.2. *The primitive cohomology classes in $T_{2n-1}(p^r)$ are given by*:

(a) $PH^k(T_{2n-1}(p^r)) = \begin{cases} Z/p & \text{if } k = 2np^s, s \geqslant 0, \\ 0 & \text{otherwise}. \end{cases}$

(b) $PH^k(T_{2n-1}(p^r); Z/p^t) = \begin{cases} Z/p & \text{if } k = 2np^s \text{ and } t \geqslant r + s, \\ Z/p & \text{if } k = 2np^s - 1 \text{ and } t \geqslant r + s, \\ 0 & \text{otherwise}. \end{cases}$

Proof. (a) $H^k(T_{2n-1}(p^r)) \neq 0$ iff $k = 2ni$ and the projection map:

$$T_{2n-1}(p^r) \to \Omega S^{2n+1}$$

is onto in cohomology. Let $e_k \in H^{2nk}(\Omega S^{2n+1})$ be the dual of the kth power of some chosen (and fixed) generator in $H_{2n}(\Omega S^{2n+1})$. Then the diagonal map is given by:

$$\mu^*(e_k) = \sum_{i+j=k} e_i \otimes e_j.$$

Let v_k be the image of e_k in $H^{2nk}(T_{2n-1}(p^r))$. The order of v_k is $p^{\nu(k)+r}$ where $\nu(k)$ is the number of powers of p in k. Suppose now that α_k is an integer such that $\alpha_k v_k$ is primitive. If $k = p^s \ell$ with $(p, \ell) = 1$ and $\ell > 1$, $\mu^*(v_k)$ contains the term $v_{p^s} \otimes v_{p^s(\ell-1)}$ which has the same order as v_k. So $\alpha_k v_k$ cannot be a nonzero primitive in this case. However the middle terms of $\mu^*(v_{p^s})$ all have orders $p^{r+\nu(i)}$ where $0 < i < p^s$. These orders all divide p^{r+s-1} and include p^{r+s-1} when $i = p^{s-1}$. Thus $p^{r+s-1}v_{p^s}$ generates the primitives and has order p.

   (b) Since $H^{2nk}(T_{2n-1}; Z/p^t) \cong H^{2nk}(T_{2n-1}) \otimes Z/p^t$, $PH^{2nk}(T_{2n-1}; Z/p^t)$ is only nonzero when the primitive in $H^{2nk}(T_{2n-1})$ is not divisible by p^t. Since $H^{2np^s}(T_{2n-1})$ has order p^{r+s}, this implies that $t \geqslant r + s$.

   From the exact sequence:

$$0 \to H^{2nk-1}(T_{2n-1}; Z/p^t) \xrightarrow{\delta} H^{2nk}(T_{2n-1}) \xrightarrow{p^t} H^{2nk}(T_{2n-1})$$

we see that if $\theta \in H^{2nk-1}(T_{2n-1}; Z/p^t)$ is primitive, $\delta(\theta)$ is as well, so if θ is nonzero. $k = p^s$ and $\delta(\theta) = p^{r+s-1}v_{p^s}$ up to a unit. In case $t \geqslant r + s$, v_{p^s} is also in the image of δ; write $v_{p^s} = \delta(\gamma)$ so $\theta = p^{r+s-1}\gamma$ since δ is a monomorphism. In this case the image of θ under the homomorphism:

$$\mu^* - \pi_1^* - \pi_2^* : H^{2np^s-1}(T_{2n-1}; Z/p^t) \to H^{2np^s-1}(T_{2n-1} \times T_{2n-1}; Z/p^t)$$

must be zero since $\mu^* - \pi_1^* - \pi_2^*$ factors through $H^{2np^s-1}(T_{2n-1} \wedge T_{2n-1}; Z/p^t)$ and all the elements of this group have order dividing p^{r+s-1}. So in case $t \geqslant r + s$, θ is primitive. Suppose on the other hand that $d = r + s - t > 0$. We will show that there are no primitives in $H^{2np^s-1}(T_{2n-1}; Z/p^t)$. To see this note that Z/p^t is an injective Z/p^t module. Consequently

$$H^m(W; Z/p^t) \cong \mathrm{Hom}(H_m(W; Z/p^t), Z/p^t).$$

It follows that there is a nonzero primitive class θ in $H^{2np^s-1}(T_{2n-1}; Z/p^t)$ iff there is a class in $H_{2np^s-1}(T_{2n-1}; Z/p^t)$ which is not in the image of the homorphism:

$$\mu_* - \pi_{1*} - \pi_{2*} : H_{2np^s-1}(T_{2n-1} \times T_{2n-1}; Z/p^t) \to H_{2np^s-1}(T_{2n-1}; Z/p^t).$$

A calculation with the homology Serres spectral sequence for the fibration:

$$S^{2n-1} \to T_{2n-1} \to \Omega S^{2n+1}$$

with Z/p^t coefficients shows that there is a subalgebra

$$Z/p^t[v'; 2np^{t-r}] \otimes \bigwedge(u'; 2np^{t-r} - 1)$$

of $H_*(T_{2n-1}; Z/p^t)$. Since $H_{2np^s-1}(T; Z/p^t)$ has only one generator, it must be $(v')^{p^d-1}u'$ up to a unit. This is in the image of $\mu_* - \pi_{1*} - \pi_{2*}$ iff $d > 0$, so $PH^{2np^s-1}(T_{2n-1}; Z/p^t) = 0$ when $d > 0$. $\square$

Corollary 5.3. *Suppose E is a generalized Eilenberg–MacLane space. Then $E \in M(n, r)$ iff $\forall s > 0$ we have*:

(a) $p^{r+s-1}\pi_{2np^s}(E) = 0$;
(b) *the torsion subgroup of $\pi_{2np^s-1}(E)$ has exponent at most p^{r+s-1}.*

Proof. Since there are H-maps:

$$K(G_k, k) \xrightarrow{\alpha_k} E \xrightarrow{\beta_k} K(G_k, k)$$

with $\beta_k \alpha_k \sim 1$, $E \in \mathcal{M}(n, r)$ iff $K(G_k, k) \in \mathcal{M}(n, r)$ for each k. This is trivial if $k \leqslant 2n$. Thus for $k > 2n$ this is equivalent to the statement that there are no nontrivial H-maps from $T_{2n-1}(p^r)$ to $K(G_k, k)$. It suffices to check this in the cases $G_k = Z_{(p)}$ and $G_k = Z/p^t$. We apply 5.2. There is a nontrivial H-map from $T_{2n-1}(p^r)$ to $K(Z_{(p)}, k)$ iff $k = 2np^s$ with $s \geqslant 1$ so this condition is equivalent to the statement that $\pi_{2np^s}(E)$ is a torsion group for each $s \geqslant 1$. There are no nontrivial H-maps from $T_{2n-1}(p^r)$ to $K(Z/p^t, k)$ iff $t < r + s$ whenever $k = 2np^s$ or $2np^s - 1$ with $s \geqslant 1$. $\quad\square$

The problem with targets that are Eilenberg–MacLane spaces is consequently not one of existence but of uniqueness. The next result shows that these conditions affect uniqueness in a more general context.

Theorem 5.4. *Suppose $\widehat{f}_1, \widehat{f}_2 : T_{2n-1}(p^r) \to Z$ are two H-maps extending $f : P^{2n}(p^r) \to Z$, where Z is an Abelian H-space. Suppose also that for each $s \geqslant 1$:*

(a) $p^{r+s-1}\pi_{2np^s}(Z) = 0$;
(b) *the torsion subgroup of $\pi_{2np^s-1}(Z)$ has exponent at most p^{r+s-1}*

then $f_1 \sim f_2$.

Proof. Let $g = \widehat{f}_1 - \widehat{f}_2$. Since Z is Abelian, g is an H-map. Let $Z^{[m]}$ be the mth Postnikov section of Z and let

$$\pi_m : Z \to Z^{[m]}$$

be the projection. We will show by induction on n that $\pi_m g \sim *$. This is clearly true when $m \leqslant 2n$. Consider the fibration sequence:

$$K\big(\pi_{m+1}(Z), m+1\big) \xrightarrow{\partial} Z^{[m+1]} \xrightarrow{\rho_m} Z^{[m]} \xrightarrow{k_m} K\big(\pi_{m+1}(Z), m+2\big).$$

Assuming that $\pi_m g \sim *$ we can factor $\pi_{m+1}g$ through $K(\pi_{m+1}(Z), m+1)$. Choose any map $\gamma : T_{2n-1}(p^r) \to K(\pi_{m+1}(Z), m+1)$ with $\partial\gamma \sim \pi_{m+1}g$. We claim that γ is an H-map. Let

$$\Delta(\gamma) : T_{2n-1}\big(p^r\big) \wedge T_{2n-1}\big(p^r\big) \to K\big(\pi_{m+1}(Z), m+1\big)$$

be the H-deviation. According to [28, 5.1 and 5.3], the H-space structure on Z induces an H-space structure on $Z^{[m]}$ for each m such that the maps $\pi_m : Z \to Z^{[m]}$ and $\rho_m : Z^{[m]} \to Z^{[m]}$ are H-maps. Consequently $\partial\gamma \sim \pi_{m+1}g$ is an H-map. Since ∂ is an H-map as well, $\partial\Delta(\gamma) \sim \Delta(\partial\gamma) \sim *$. Thus $\Delta(\gamma)$ factors through Ωk_m:

$$T_{2n-1}\big(p^r\big) \wedge T_{2n-1}\big(p^r\big) \xrightarrow{\epsilon} \Omega Z^{[m]} \xrightarrow{\Omega k_m} K\big(\pi_{m+1}(Z), m+1\big).$$

The map ϵ is adjoint to the composition:

$$ST_{2n-1}\big(p^r\big) \wedge T_{2n-1}\big(p^r\big) \to Z^{[m]} \xrightarrow{k_m} K\big(\pi_{m+2}(Z), m+2\big).$$

By [18, 4.5], $ST_{2n-1}(p^r) \wedge T_{2n-1}(p^r)$ is a one point union of Moore spaces up to homotopy. However for each Moore space P^ℓ and map $\varphi : P^\ell \to Z^{[m]}$ we have $k_m \varphi \sim *$; this follows since any nontrivial map $\varphi : P^\ell \to Z^{[m]}$ could only occur when $\ell \leqslant m + 1$. Consequently γ is an H-map. Applying 5.3 we see that $\gamma \sim *$ so $\pi_{m+1}g \sim *$. Since $H^k(T_{2n-1}(p^r))$ is finite for each k, there are no phantom maps (see [9] or [10]). Thus $g \sim *$. $\quad\square$

The conditions (a) and (b) in 5.3 and 5.4 will clearly play a role in the eventual understanding of the mapping properties of T_{2n-1}. We note the following.

Lemma 5.5. *Suppose Y is of finite type and $b > 0$. Then $p^a\pi_m(Y; Z/a+b) = 0$ iff $p^a\pi_m(Y) = 0$ and the torsion in $\pi_{m-1}(Y)$ has order at most p^a.*

Proof. Consider the ladder of long exact sequences:

$$\begin{array}{ccccccccc}
\pi_m(Y) & \xrightarrow{p^{a+b}} & \pi_m(Y) & \longrightarrow & \pi_m(Y;Z/p^{a+b}) & \longrightarrow & \pi_{m-1}(Y) & \xrightarrow{p^{a+b}} & \pi_{m-1}(Y) \\
\downarrow p^a & (A) & \downarrow p^a & (B) & \downarrow p^a & & \downarrow p^a & & \downarrow p^a \\
\pi_m(Y) & \xrightarrow{p^{a+b}} & \pi_m(Y) & \longrightarrow & \pi_m(Y;Z/p^{a+b}) & \longrightarrow & \pi_{m-1}(Y) & \xrightarrow{p^{a+b}} & \pi_{m-1}(Y).
\end{array}$$

Assume first that $p^a \pi_m(Y; Z/p^{a+b}) = 0$. Then from square (B) we see that $p^a \pi_m(Y) \subset p^{a+b}\pi_m(Y)$. Since $b > 0$, this implies that the elements of $p^a\pi_m(Y)$ are divisible by arbitrarily high powers of p. Since Y is of finite types, we conclude that $p^a\pi_m(Y) = 0$. Suppose there is an element $\xi \in \pi_{m-1}(Y)$ of order p^n where $n > a$. Then

$$p^{a+b}\big(p^{n-a-1}\xi\big) = p^{n+b-1}\xi = 0$$

so $p^{n-a-1}\xi$ is in the image of $\pi_m(Y; Z/p^{a+b})$. Since $p^a\pi_m(Y; Z/p^{a+b}) = 0$, it follows that $p^{n-1}\xi = p^a(p^{n-a-1}\xi) = 0$.

Conversely suppose that $p^a\pi_m(Y) = 0$ and all torsion in $\pi_{m-1}(Y)$ has order dividing p^a. Let $\xi: P^m(p^{a+b}) \to Y$ represent an element of $\pi_m(Y; Z/p^{a+b})$. Consider the diagram:

Let ξ' be the restriction of ξ to S^{m-1}. Since $p^{a+b}\xi' = 0$, ξ' has finite order. By hypothesis $p^a\xi'' = 0$ so an extension $\xi'': S^m \to Y$ exists. By hypothesis $p^a\xi'' = 0$ so the middle horizontal composition is null homotopic. This composition is $p^a\xi$. □

Definition 5.6. A space Y is (n,r)-subexponential if for each $s \geqslant 1$

$$p^{r+s-1}\pi_{np^s}\big(Y; Z/p^{r+s}\big) = 0.$$

With this definition in hand, we have:

Theorem 5.7.

(a) *If Z is any space which is $(2n,r)$-subexponential, then every map $f: P^{2n}(p^r) \to \Omega Z$ extends to a map $\widehat{f}: T_{2n-1}(p^r) \to Z$.*
(b) *If Z is a $(2n,r)$-subexponential H-space and there is an H-map*

$$\widehat{f}: T_{2n-1}\big(p^r\big) \to Z$$

extending $f: P^{2n}(p^r) \to Z$, then $\widehat{f}$ is unique up to homotopy.

Proof. Part (a) follows directly from [1, 4.7]. Part (b) follows from 5.4 and 5.5. □

6. In this section we will discuss some of the possible options for a universality property for $T_{2n-1}(p^r)$. We begin with the observation that $ST_{2n-1}(p^r) \not\simeq P^{2n+1}(p^r)$. In fact, there is a space $G_{2n}(p^r)$ which is a retract of $ST_{2n-1}(p^r)$ and $P^{2n+1}(p^r) \subset G_{2n}(p^r) \subset ST_{2n-1}(p^r)$. $G_{2n}(p^r)$ contains cells of dimension $2np^i$ and $2np^i + 1$ for each $i \geqslant 0$ and no other cells in positive dimensions [18, 4.4c]. In particular, there is no map $\phi: T_{2n-1}(p^r) \to \Omega P^{2n+1}(p^r)$ which induces an isomorphism in π_{2n-1}. In terms of 5.7(a) this is correlated to the fact that [6]:

$$p^r\pi_{2np-1}\big(\Omega P^{2n+1}\big(p^r\big); Z/p^{r+1}\big) \neq 0.$$

We might ask for an H-map:

$$T_{2n-1}\big(p^r\big) \to \Omega^2 P^{2n+2}\big(p^r\big)$$

inducing an isomorphism in π_{2n-1}. There clearly is a map since $SG_{2n}(p^r)$ splits as a one point union of Moore spaces [18, 4.5].

Proposition 6.1. *There is a unique H-map* $\widehat{f} : T_{2n-1}(p^t) \to \Omega^2 P^{2n+2}(p^r)$ *extending the inclusion of* $P^{2n}(p^r)$ *iff the suspension map:*

$$T_{2n-1}(p^r) \xrightarrow{E} \Omega T_{2n}(p^r)$$

is an H-map.

Proof. If E is an H-map, we compose E with the loops on the map

$$g : T_{2n}(p^r) \to \Omega P^{2n+2}(p^r)$$

(example 4c) to obtain

$$T_{2n-1}(p^r) \to \Omega T_{2n}(p^r) \xrightarrow{\Omega g} \Omega^2 P^{2n+2}(p^r).$$

Conversely, since $T_{2n}(p^r)$ is a retract of $\Omega P^{2n+2}(p^r)$ we see that we can construct an H-map

$$T_{2n-1}(p^r) \to \Omega^2 P^{2n+2}(p^r) \xrightarrow{\Omega h} \Omega T_{2n}(p^r).$$

In fact, there is only one possible H-map $T_{2n-1}(p^r) \to \Omega T_{2n}(p^r)$ and only one possible H-map $T_{2n-1}(p^r) \to \Omega^2 P^{2n+2}(p^r)$ extending respectively the inclusions of $P^{2n}(p^r)$ by 5.7(b). □

Proposition 6.2. *Suppose* $T_{2n-1}(p^r)$ *is homotopy associative. Then the suspension map*

$$E : T_{2n-1}(p^r) \to \Omega T_{2n}(p^r)$$

is an H-map.

Note: In [26, 1.2], the author states that $T_{2n-1}(p^r)$ is homotopy associative for $p \geqslant 5$. Unfortunately, the argument given relies on some of the same assertions that appear in the universality argument and more details are needed to justify this claim.

Proof. The inclusion $f : G_{2n} \to ST_{2n-1}$ yields the homotopy commutative diagram of fibrations (as in 3.1)

(See also [18, 4.4].) Since T_{2n-1} is homotopy associative,

$$\partial : \Omega ST_{2n-1} \to T_{2n-1}$$

is an H-map; now $h : \Omega G_{2n} \to T_{2n-1}$ is the composition:

$$\Omega G_{2n} \xrightarrow{\Omega f} \Omega ST_{2n-1} \xrightarrow{\partial} T_{2n-1}$$

so h is also an H-map. Also from [18, 4.4] we have a homotopy commutative diagram of fibration sequences:

From this we see that $Eh \sim \Omega\varphi$ is an H-map. Now consider the diagram:

Since Eh is an H-map, the rectangle commutes up to homotopy. Since h is an H-map, the left square does as well. But then since $h \times h$ has a right homotopy universe, the right hand square does and E is an H-map. □

Proposition 6.3. *Suppose Z is a double loop space and Z is $(2n, r)$-exponential. Then each map $f : P^{2n}(p^r) \to Z$ has a unique extension to an H-map $\widehat{f} : T_{2n-1}(p^r) \to Z$ iff $E : T_{2n-1}(p^r) \to \Omega T_{2n}(p^r)$ is an H-map.*

Proof. Suppose E is an H-map and $f : P^{2n}(p^r) \to Z$. Since Z is a double loop space, we can extend f to an H-map:

$$\overline{f} : \Omega^2 P^{2n+2}(p^r) \to Z.$$

Since $T_{2n}(p^r)$ is a retract of $\Omega P^{2n+2}(p^r)$, we can form $\widehat{f}$ as the composition:

$$T_{2n-1}(p^r) \xrightarrow{E} \Omega T_{2n}(p^r) \xrightarrow{\Omega i} \Omega^2 P^{2n+2}(p^r) \xrightarrow{\overline{f}} Z.$$

By 5.4, $\widehat{f}$ is unique.

Conversely, if extensions always exist when the target is an (n, r)-subexponential double loop space, we can construct an H-map:

$$T_{2n-1}(p^r) \to \Omega^2 P^{2n+2}(p^r)$$

since $p^r \pi_{2np-1}(\Omega^3 P^{2n+2}(p^r); Z/p^{r+1}) = 0$. Now compose this map with the loops on the map $\Omega P^{2n+2}(p^r) \to T_{2n}(p^r)$. □

Corollary 6.4. *If $T_{2n-1}(p^r)$ is homotopy associative and Z is a double loop space which is $(2n, r)$-subexponential, then each map $f : P^m(p^r) \to Z$ extends uniquely to an H-map $\widehat{f} : T_{2n-1}(p^r) \to Z$.*

This is a desirable universal property. However, it is not sufficient for the program in [12]. For that purpose we would need to know that $T_m(p^r)$ is an acceptable target for any m.

We propose the following strengthening of 5.6.

Definition 6.5. A space Z is strongly (n, r)-subexponential if $p^{r+k}\pi_i(Z) = 0$ for all $i \leqslant np^{k+1}$ and $k \geqslant 0$.

Conjecture 6.6. $P^{2n}(p^r) \xrightarrow{i} T_{2n-1}(p^r)$ *is universal for targets which are strongly $(2n, r)$-subexponential Abelian H-spaces.*

Proposition 6.7. $T_{2n-1}(p^r)$ *is strongly $(2n, r)$-subexponential.*

Proof. Consider the fibration sequence:

$$W_n \to T_{2n-1}(p^r) \to \Omega T_{2n}(p^r)$$

from [1,12,18]. By the results of Cohen, Moore and Neisendorfer [6], $p\pi_i(W_n) = 0$. Neisendorfer has proven that

$$p^r \pi_i(T_{2n}(p^r)) = 0$$

(see [21]), so we conclude that

$$p^{r+1}\pi_i(T_{2n-1}(p^r)) = 0.$$

Thus it suffices to show that

$$p^r \pi_i(T_{2n-1}(p^r)) = 0 \quad \text{for } i \leqslant 2np.$$

For $i \leqslant 2np$, $\pi_i(W_n) = 0$ except when $i = 2np - 3$ and (in case $p = 3$) $i = 2np$. We will examine these groups. According to [1,18], T_{2n-1} is a retract of ΩG_{2n}. The $2np + 1$ skeleton of G_{2n} is:

$$P^{2n+1}(p^r) \cup CP^{2np}(p^{r+1}).$$

Now $\pi_{2np-3}(T_{2n-1}(p^r)) \subset \pi_{2np-2}(G_{2n}) \cong \pi_{2np-2}(P^{2n+1}(p^r))$. This group has order at most p^r. Likewise $\pi_{2np}(T_{2n-1}(p^r))$ is contained in $\pi_{2np+1}(G_{2n})$ which is a quotient of $\pi_{2np+1}(P^{2n+2}(p^r))$. This group also has exponent p^r. □

If Conjecture 6.6 holds, we could show that $p^r \cdot \pi_*(T_{2n-1}(p^r)) = 0$ and consequently each map $f : P^{m+1}(p^r) \to T_i$ has a unique extension to an H-map $\widehat{f} : T_m(p^r) \to T_i$. This is sufficient for the purposes of [12].

It is interesting to note that this condition is precisely the conclusion of a theorem of Barratt [3].

Theorem 6.8 (Barratt). *Suppose SX is n-connected and has characteristic p^r. Then SX is strongly (n, r)-subexponential.*

This invites the question of whether other n-connected suspensions of characteristic p^r support a map to an Abelian H-space which is universal for targets that are strongly (n, r)-subexponential H-spaces. Proposition 1.1 suggests that if so, these Abelian H-spaces would have homology which is a symmetric algebra.

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