Quaternion involutions and anti-involutions

Todd A. Ell\textsuperscript{a,1}, Stephen J. Sangwine\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} 5620 Oak View Court, Savage, MN 55378-4695, USA
\textsuperscript{b} Department of Electronic Systems Engineering, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, United Kingdom

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Abstract

An involution or anti-involution is a self-inverse linear mapping. In this paper we study quaternion involutions and anti-involutions. We review formal axioms for such involutions and anti-involutions. We present two mappings, one a quaternion involution and one an anti-involution, and a geometric interpretation of each as reflections. We present results on the composition of these mappings and show that the quaternion conjugate may be expressed using three mutually perpendicular anti-involutions. Finally, we show that projection of a vector or quaternion can be expressed concisely using three mutually perpendicular anti-involutions.

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1. Introduction

Involutions are usually defined simply as self-inverse mappings. A trivial example is quaternion (or indeed complex) conjugation, which is obviously self-inverse. In this paper we consider involutions and anti-involutions of the quaternions. The paper begins by reviewing briefly the classical basis of quaternions, and then reviews axioms for involutions which go beyond the simple definition of a self-inverse mapping. Section 4 then presents quaternion involutions and anti-involutions, and Section 5 presents their geometric properties. Section 6 presents informally the result of composing involutions and anti-involutions and Section 7 discusses the quaternion conjugate and shows that it may be expressed using three mutually perpendicular quaternion anti-involutions (that is, the conjugate can be expressed in terms of quaternion multiplications and additions only). Finally, Section 8 shows that the projection of a vector or quaternion may be expressed using anti-involutions.

The motivation behind this paper is primarily in the use of quaternions as an applied geometric algebra, particularly for handling geometrical manipulation in 3-dimensional space. Therefore we consider only the specific algebra of real quaternions (that is, quaternions over the field of reals). It is likely that the ideas collected and presented here can be generalized to quaternions over the field of complex numbers, an issue that we leave for a later paper.

* Corresponding author. Tel.: +44 1206 872401; fax: +44 1206 872900.
E-mail addresses: T.Ell@IEEE.org (T.A. Ell), S.Sangwine@IEEE.org (S.J. Sangwine).
1 Dr. T.A. Ell is a Visiting Fellow at the University of Essex.
2. Basics of quaternions

We review here some basic ideas and present the notation used in the rest of the paper. A real quaternion may be represented in Cartesian form as \( q = w + ix + jy + kz \) where \( i, j, k \) are mutually perpendicular unit vectors obeying the multiplication rules below discovered by Hamilton in 1843 [1,2], and \( w, x, y, z \), are real:

\[
i^2 = j^2 = k^2 = ijk = -1.
\]

The conjugate of a quaternion is \( \overline{q} = w - ix - jy - kz \).

The quaternion algebra \( \mathbb{H} \) is a normed division algebra. The modulus of a quaternion is the square root of its norm, \( |q| = \sqrt{w^2 + x^2 + y^2 + z^2} \), and every non-zero quaternion has a multiplicative inverse given by its conjugate divided by its norm: \( q^{-1} = \overline{q}/|q|^2 = (w - ix - jy - kz)/(w^2 + x^2 + y^2 + z^2) \). For a more detailed exposition of the basics of quaternions, we refer the reader to Ward’s book [3] or to O’Meara’s more abstract treatment [4, Chapter 5, Section 57].

An alternative and much more powerful representation for a quaternion is as a combination of a scalar and a vector part, analogous to a complex number, and this representation will be employed in the rest of the paper. In this form \( q = a + \mu b \), where \( \mu \) is a unit pure quaternion (which we interpret as a 3-space vector), and \( a \) and \( b \) are real. \( b \) is the modulus and \( \mu \) is the direction in 3-space of the vector part of the quaternion. In terms of the Cartesian representation:

\[
a = w, \quad b = \sqrt{x^2 + y^2 + z^2}, \quad \mu = \frac{ix + jy + kz}{b}.
\]

The square of any unit vector such as \( \mu \) is \(-1\), as is easily demonstrated and was first shown by Hamilton [5, pp. 203, 209], [6]. It follows that there are an infinite number of solutions to the equation \( x^2 = -1 \), as is well-known geometrically (each root corresponds to a point on the 3-sphere). This also follows from the Brauer–Cartan–Hua theorem [7, Ch. 6, p. 198] which states that in a division algebra \( D \), if a polynomial \( f(x) \) with coefficients in the centre of \( D \) (here reals) has more roots in \( D \) than its degree, then it has infinitely many roots in \( D \). Here \( f(x) = x^2 + 1 \), where \( x \) is a real quaternion, has at least three roots \((i, j \text{ and } k)\) which is more than the degree of the polynomial (2), so there must be an infinite number of roots.

The conjugate of a quaternion in complex form is \( \overline{q} = a - \mu b \). Geometrically, this is obviously a reversal of the direction of the vector part in 3-space. The product of a quaternion with its conjugate gives the modulus, or square of the product. This follows directly from the fact that \( \mu^2 = -1 \); \((a + \mu b)(a - \mu b) = a^2 - \mu^2 b^2 = a^2 + b^2 = w^2 + x^2 + y^2 + z^2 \), but it is also proved by [4, Chapter 5].

The polar form of a quaternion is analogous to the polar form of a complex number, \( q = a + \mu b = \sqrt{a^2 + b^2} e^{\mu \theta} \), where \( \tan \theta = b/a \).

3. Involutions and anti-involutions

The formal definition of an involution is not easy to find but [8] gives a reasonably authoritative statement from which we reproduce the following axioms.\(^2\) In this section we denote an arbitrary involution (or anti-involution) by the transformation \( x \mapsto f(x) \). In subsequent sections we introduce a compact notation specific to the two cases.

**Axiom 1.** \( f(f(x)) = x \). An involution is its own inverse.

**Axiom 2.** An involution is linear: \( f(x_1 + x_2) = f(x_1) + f(x_2) \) and \( \lambda f(x) = f(\lambda x) \), where \( \lambda \) is a real constant.

**Axiom 3.** \( f(x_1 x_2) = f(x_2) f(x_1) \). The involution of a product is the product of the involutions of the product terms taken separately, but the order must be reversed.

An anti-involution is a self-inverse transformation similar to an involution, except that it does not obey **Axiom 3** as stated. Instead, it satisfies the following\(^3\):

\(^2\) Most mathematical reference works define an involution simply as a mapping which is its own inverse, which is a much broader definition than is used in this article. There are many trivial such mappings (e.g., \( x \mapsto -x \)). A non-trivial example is the transformation \( x \mapsto -y \tau x, |y| = 1 \) [9, Theorem 5.1, p. 141] which is a 4-space reflection in a hyperplane. Clearly a reflection is self-inverse, but although this transformation satisfies the first two axioms it does not satisfy **Axiom 3** and is therefore not an involution according to the definition used in this paper.

\(^3\) The distinction between involutions and anti-involutions can exist only in a non-commutative algebra, of course.
Axiom 4. \( f(x_1x_2) = f(x_1)f(x_2) \), i.e., the anti-involutions of the product terms taken separately must be in the same order as in the anti-involution of the product.

An anti-involution as defined here is also known as an involutory automorphism.\(^4\)

4. Quaternion involutions and anti-involutions

Proposition 5 (Benn and Tucker [10, Appendix A, p. 338]). The transformation \( q \mapsto -q\bar{q}v \), denoted from here on by \( [q]_v \), where \( q \) is an arbitrary quaternion, is an involution for any unit vector \( v \).

Proof. Axiom 1 is easily shown to be satisfied using the generalized conjugate rule \( \overline{pqr} = r\overline{q}\overline{p} \) [11, Section 20, p. 238]: \( -v\overline{(-v\bar{q}v)}v = v\bar{v}q\bar{v}v = v(-v)(-v)v \), which gives \( q \) since \( v^2 = -1 \). Since the conjugate distributes over a sum and reals commute with quaternions, Axiom 2 is easily seen to be satisfied. Axiom 3 can be shown to be satisfied as follows:

\[
[q_1q_2]_v = -v\bar{q_1}q_2v = -v\bar{q_2}q_1v
\]

and inserting unity in the middle (in the form of \(-v^2\)):

\[
= -v\bar{q_2}(-v)v\bar{q_1}v = (-v\bar{q_2}v)(-v\bar{q_1}v) = [q_2]_v[q_1]_v. \quad \Box
\]

Proposition 6. The transformation \( q \mapsto -qv \), denoted from here on by \([q]_v \), where \( q \) is an arbitrary quaternion, is an anti-involution for any unit vector \( v \).

Proof. Axioms 1 and 2 are easily seen to be satisfied. Axiom 4 can be shown to be satisfied as follows, noting that \(-v^2 = 1\):

\[
[q_1q_2]_v = -qv_1q_2v = -vq_1(-vv)q_2v = (-vq_1v)(-vq_2v) = [q_1]_v[q_2]_v. \quad \Box
\]

Proposition 6 is a generalization of three transformations (which were referred to as involutions, although by the definitions in this paper they are in fact anti-involutions) given by Chernov [12]:

\[
\begin{align*}
\alpha(q) &= -iqi = [q]_i = w + ix - jy - kz \\
\beta(q) &= -jqj = [q]_j = w - ix + jy - kz \\
\gamma(q) &= -kqk = [q]_k = w - ix - jy + kz.
\end{align*}
\]

These transformations were used by Bülow [13,14], and it is from this source that we first became aware of them. Bülow also showed that the quaternion conjugate can be expressed in terms of these three (anti-)involutions, and thus in terms of multiplications and additions only, a result that we generalize in Theorem 11.

Since there are an infinite number of unit vectors, there are an infinite number of quaternion involutions (and anti-involutions).

Note that a transformation \( q \mapsto v_1\bar{q}v_2 \) (respectively \( q \mapsto v_1q\bar{v}v_2 \)) with \( v_1 \neq v_2 \) is its own inverse, but is not an involution (anti-involution) as considered in this paper, because it does not satisfy Axiom 3 (Axiom 4).

Pure quaternions can be identified, via an isomorphism, with 3-space real vectors, i.e., \( ix + jy + kz \mapsto [x, y, z]^T \). Using a wider definition of the term involution than we use here (note our statement in the preceding paragraph), there is also a useful isomorphism between the nine involutions:

\[
5. Geometry

We now present geometric interpretations of quaternion involution and anti-involution. We refer to a direction in 3-space as the axis of involution for reasons that will be clear from the geometry.

**Theorem 7.** For an arbitrary quaternion \( q = a + \mu b \), the involution \([q]_\nu\) given in Proposition 5 leaves the scalar part of \( q \) (that is, \( a \)) invariant, and reflects the vector part of \( q \) (that is, \( \mu b \)) in the plane normal to the axis of involution \( \nu \). Similarly, the effect of the anti-involution \([q]_\nu\) given in Proposition 6 is to leave the scalar part invariant and to reflect the vector part in the line defined by the axis of involution (equivalently to rotate the vector part by \( \pi \) about the axis of involution). \([q]_\nu\) is the conjugate of \([q]_\nu\).

**Proof.** For the involution:

\[
[q]_\nu = -\nu(a + \mu b)^\nu = -\nu(a - \mu b)^\nu = -\nu^2 a + \nu \mu \nu b
\]

and, since \( \nu \) is a unit vector, \( \nu^2 = -1 \) and:

\[
[q]_\nu = a + \nu \mu \nu b.
\]

We recognise \( \nu \mu \nu \) to be a reflection of \( \mu \) in the plane normal to \( \nu \) as shown by Coxeter [9, Theorem 3.1].

In the case of the anti-involution, there is no conjugate, and it is straightforward to see that the result is the conjugate of that given above: \([q]_\nu = a - \nu \mu \nu b \), with an oppositely directed vector part. \( \Box \)

Fig. 1 shows the geometry of the two cases in the plane common to \( \nu \) and \( \mu \).

**Corollary 8.** \([q]_\nu = \overline{q} \) if \( \nu \parallel \mu \), \([q]_\nu = q \) if \( \nu \perp \mu \).

**Corollary 9.** \([q]_\nu = \overline{q} \) if \( \nu \perp \mu \), \([q]_\nu = q \) if \( \nu \parallel \mu \).

6. Composition

We present some results informally on the composition of involutions and anti-involutions which are quaternion restatements of some well-known properties of rotations and reflections in 3-space. We first review some elementary results which are needed in what follows.

The product of two unit pure quaternions, \( \nu_1 \) and \( \nu_2 \), is, in general, a full quaternion with argument \( \theta \) (in polar form, see Section 2) equal to the angle between the two unit pure quaternions. This is a special case of the full quaternion product and the result is clear if expressed in terms of the scalar (dot) and vector (cross) products of vector analysis: \( \nu_1 \nu_2 = -\nu_1 \cdot \nu_2 + \nu_1 \times \nu_2 \) [3, Section 2.2, p. 65]. Since these products are given for unit vectors by \( \cos \theta \) and \( \mu \sin \theta \), where \( \mu \) is perpendicular to the plane containing the two vectors \( \nu_1 \) and \( \nu_2 \), we may write the product of two unit
vectors as \(-\cos \theta + \mu \sin \theta = -\exp(-\mu \theta)\). If the two vectors are perpendicular, the scalar product is zero, and the cross product is perpendicular to both of the unit vectors and has unit modulus. In this case, changing the order of the two unit vectors reverses (negates) the cross product. Given a system of three mutually perpendicular unit vectors \(\mathbf{v}_1, \mathbf{v}_2\) and \(\mathbf{v}_3\), the product of any two gives the third (possibly negated, depending on the ordering of the product).

The composition of two involutions \([\mathbf{q}]_{\mathbf{v}_1}\) and \([\mathbf{q}]_{\mathbf{v}_2}\) is, in general, a rotation of the vector part of \(\mathbf{q}\) through twice the angle between the two involution axes (the scalar part remains invariant since neither of the two involutions modifies it). In the special case where \(\mathbf{v}_2 \perp \mathbf{v}_1\) the composition is an anti-involution \([\mathbf{q}]_{\mathbf{v}_3}\) where \(\mathbf{v}_3 = \mathbf{v}_2 \mathbf{v}_1\). These results can be seen as follows:

\[
\left([\mathbf{q}]_{\mathbf{v}_1}\right)_{\mathbf{v}_2} = -\mathbf{v}_2 (\mathbf{v}_1 \mathbf{q} \mathbf{v}_1) \mathbf{v}_2 = \mathbf{v}_2 \mathbf{v}_1 \mathbf{q} \mathbf{v}_1 \mathbf{v}_2 = \mathbf{p} \mathbf{q} \mathbf{p}
\]

where \(\mathbf{p} = \mathbf{v}_2 \mathbf{v}_1 = e^{\mu \theta}\). The transformation \(\mathbf{q} \mapsto \mathbf{p} \mathbf{q} \mathbf{p}\) is a rotation of the vector part of \(\mathbf{q}\) about the axis \(\mu\) through an angle \(2\theta\) as shown by Coxeter [9, Theorem 3.2]. This reduces to the anti-involution stated in the special case where \(\mathbf{v}_2 \perp \mathbf{v}_1\).

The composition of two anti-involutions \([\mathbf{q}]_{\mathbf{v}_1}\) and \([\mathbf{q}]_{\mathbf{v}_2}\) gives the same results.

The composition of an involution and an anti-involution, in general, results in a rotation as above and a conjugation, but in the special case where the two axes are perpendicular, the result is an involution \([\mathbf{q}]_{\mathbf{v}_3}\) about an axis \(\mathbf{v}_3\) perpendicular to the axes of the involution and anti-involution.

The composition of three mutually perpendicular involutions, \([\mathbf{q}]_{\mathbf{v}_i}\), \(i = 1, 2, 3\); \(\mathbf{v}_1 \perp \mathbf{v}_2 \perp \mathbf{v}_3\) conjugates \(\mathbf{q}\), and the composition of three mutually perpendicular anti-involutions, \([\mathbf{q}]_{\mathbf{v}_i}\), is an identity. Both results are easily demonstrated.

### 7. The quaternion conjugate

The quaternion conjugate may be seen as a special case of the involution \([\mathbf{q}]_{\mathbf{v}}\) in which the involution axis \(\mathbf{v}\) and the vector part of \(\mathbf{q}\) are parallel (Corollary 8), or as a special case of the anti-involution \([\mathbf{q}]_{\mathbf{v}}\) in which the axis \(\mathbf{v}\) and the vector part of \(\mathbf{q}\) are perpendicular (Corollary 9).

A much more general formulation is possible, however: The conjugate can be expressed in terms of multiplications and additions alone, using three arbitrary but mutually perpendicular anti-involutions, a result that we now prove.

**Lemma 10.** The sum of three mutually perpendicular anti-involutions \([\mathbf{q}]_{\mathbf{v}_i}\), \(i = 1, 2, 3\) applied to a vector negates the vector (reverses its direction). That is, given a set of three mutually perpendicular unit vectors \(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\) and an arbitrary vector \(\mu\):

\[
[\mu]_{\mathbf{v}_1} + [\mu]_{\mathbf{v}_2} + [\mu]_{\mathbf{v}_3} = -\mu. \tag{4}
\]

**Proof.** Let \(\mu = \eta_1 + \eta_2 + \eta_3\) where \(\eta_i \parallel \mathbf{v}_i\), \(i = 1, 2, 3\). In other words, resolve \(\mu\) into three vectors\(^5\) parallel to the three mutually perpendicular unit vectors \(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\). Substitute this representation of \(\mu\) into the left-hand side of Eq. (4):

\[
[\eta_1 + \eta_2 + \eta_3]_{\mathbf{v}_1} + [\eta_1 + \eta_2 + \eta_3]_{\mathbf{v}_2} + [\eta_1 + \eta_2 + \eta_3]_{\mathbf{v}_3} = -\mu.
\]

**Axiom 2** allows us to apply the anti-involutions separately to the three components:

\[
[\eta_1]_{\mathbf{v}_1} + [\eta_2]_{\mathbf{v}_2} + [\eta_3]_{\mathbf{v}_3} + [\eta_1]_{\mathbf{v}_2} + [\eta_2]_{\mathbf{v}_1} + [\eta_3]_{\mathbf{v}_2} + [\eta_1]_{\mathbf{v}_3} + [\eta_2]_{\mathbf{v}_3} + [\eta_3]_{\mathbf{v}_1} = -\mu.
\]

We now make use of **Corollary 9**. In this case we are applying the anti-involutions to a vector, so the corollary states that an anti-involution with axis parallel to a vector is an identity, and an anti-involution with axis perpendicular to the vector reverses, or negates, the vector:

\[
\eta_1 - \eta_2 - \eta_3 - \eta_1 + \eta_2 - \eta_3 - \eta_1 - \eta_2 + \eta_3 = -\mu
\]

and cancelling out, we obtain: \(-\eta_1 - \eta_2 - \eta_3 = -\mu\), which is the assumption we made at the start of the proof. \(\square\)

---

\(^5\) The three vectors \(\eta_i\) are not, in general, of unit modulus.
The following theorem is a generalization of a similar result given in [13, Definition 2.2, p. 12] in which the three specific anti-involutions of Eq. (3) were used.

**Theorem 11.** Given a set of three mutually perpendicular unit vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \), the conjugate of \( q \) may be expressed as:

\[
\overline{q} = \frac{1}{2} \left( [q]_{\mathbf{v}_1} + [q]_{\mathbf{v}_2} + [q]_{\mathbf{v}_3} - q \right).
\]

**Proof.** Replace \( q \) in the right-hand side of Eq. (5) by its ‘complex’ form \( a + \mathbf{b} \):

\[
\overline{q} = \frac{1}{2} \left( (a + \mathbf{b})_{\mathbf{v}_1} + (a + \mathbf{b})_{\mathbf{v}_2} + (a + \mathbf{b})_{\mathbf{v}_3} - (a + \mathbf{b}) \right).
\]

We now apply the three anti-involutions separately to the components of \( q \) using Axiom 2, and noting from Theorem 7 that the scalar part \( a \) is invariant under the anti-involutions:

\[
\overline{q} = \frac{1}{2} \left( a + [\mathbf{u}]_{\mathbf{v}_1} b + a + [\mathbf{u}]_{\mathbf{v}_2} b + a + [\mathbf{u}]_{\mathbf{v}_3} b - a - \mathbf{b} \right).
\]

Gathering terms together and factoring out \( b \), we have:

\[
\overline{q} = a + \frac{1}{2} \left( [\mathbf{u}]_{\mathbf{v}_1} + [\mathbf{u}]_{\mathbf{v}_2} + [\mathbf{u}]_{\mathbf{v}_3} - \mathbf{u} \right) b
\]

and the right-hand side is equal to \( \overline{q} = a - \mathbf{u} b \) by Lemma 10. \( \Box \)

### 8. Projection using involutions

Finally, we now demonstrate the utility of quaternion involutions by presenting formulae for projection of a vector into or perpendicular to a given direction.

**Theorem 12.** An arbitrary vector \( \mathbf{u} \) may be resolved into two components parallel to, and perpendicular to, a direction in 3-space defined by a unit vector \( \mathbf{v} \):

\[
\mathbf{u}_{\parallel \mathbf{v}} = \frac{1}{2} \left( \mathbf{u} - [\mathbf{u}]_{\mathbf{v}} \right) = \frac{1}{2} (\mathbf{u} + [\mathbf{u}]_{\mathbf{v}}) \quad \mathbf{u}_{\perp \mathbf{v}} = \frac{1}{2} (\mathbf{u} + [\mathbf{u}]_{\mathbf{v}}) = \frac{1}{2} (\mathbf{u} - [\mathbf{u}]_{\mathbf{v}})
\]

where \( \mathbf{u}_{\parallel \mathbf{v}} \) is parallel to \( \mathbf{v} \) and \( \mathbf{u}_{\perp \mathbf{v}} \) is perpendicular to \( \mathbf{v} \), and \( \mathbf{u} = \mathbf{u}_{\parallel \mathbf{v}} + \mathbf{u}_{\perp \mathbf{v}} \).

**Proof.** From Theorem 7, \([\mathbf{u}]_{\mathbf{v}}\) is the reflection of \( \mathbf{u} \) in the plane normal to \( \mathbf{v} \) and \([\mathbf{u}]_{\mathbf{v}}\) is the reflection of \( \mathbf{u} \) in the line defined by \( \mathbf{v} \) as shown in Fig. 1. Taking the sum or difference of the original and reflected vectors cancels or sums the vector in the directions parallel and perpendicular to \( \mathbf{v} \) or vice versa. The factor of \( \frac{1}{2} \) divides the summed components to give the original vector resolved in the summed direction. \( \Box \)

Two of the four results in Theorem 12 (those using anti-involutions) were published in [17], but without explicit use of anti-involutions. The theorem may be generalised to quaternions as well as vectors, but since the scalar part of a quaternion is invariant under an involution or anti-involution, the result includes or excludes the scalar part according to whether the result of the involution or anti-involution is added to or subtracted from the original quaternion. Thus it is possible to ‘resolve’ a quaternion into a direction parallel or perpendicular to a unit vector, and to choose whether the scalar part is included in the parallel or perpendicular component.

The representation \( a + \mathbf{b} \) is independent of the coordinate system in that it expresses the quaternion in terms of the direction in 3-space of the vector part. However, the quaternion can be rewritten in terms of a set of orthogonal basis vectors, \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \), without recourse to a numerical representation. The three projections based on involutions and the conjugate provide the mechanism. That is, we may write a quaternion \( q = a + \mathbf{b} = a + \mathbf{b} \) as

\[
q = a + \mathbf{v}_1 \alpha + \mathbf{v}_2 \beta + \mathbf{v}_3 \gamma = a + \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3
\]

where \( \mathbf{b}_i \parallel \mathbf{v}_i \) and \( \alpha, \beta, \) and \( \gamma \) are real:

\[
a = \frac{1}{2} (q + \overline{q}); \quad \mathbf{b}_i = \frac{1}{2} (q - [\overline{q}]_{\mathbf{v}_i}), \quad i \in \{1, 2, 3\}.\]
In resolving the vector part into the three components \( b_1 \), we have exploited the fact that the scalar part of \( q \) is invariant under the involution, and is therefore cancelled out.

Finally, we note that this same idea allows us to extract all four components of a quaternion \( q = w + ix + jy + kz \) as follows:

\[
\begin{align*}
    w &= \frac{1}{2} (q + \bar{q}), \\
    x &= \frac{1}{2i} (q - [\bar{q}]_i), \\
    y &= \frac{1}{2j} (q - [\bar{q}]_j), \\
    z &= \frac{1}{2k} (q - [\bar{q}]_k),
\end{align*}
\]

where, in the case of the \( x, y \) and \( z \) components, the difference between \( q \) and the result of the involution is \( 2xi \), \( 2yj \) or \( 2zk \), and dividing by \( 2i, 2j \) or \( 2k \) respectively gives the result shown.

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References


