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GENERALIZED CHROMATIC POLYNOMIALS

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There are several polynomials used in the study of combinatorics including Birkhoff's chromatic polynomial for graphs, Stanley's order polynomial for partially ordered sets, and Tutte's dichromatic polynomial for graphs. We develop a common basis for these polynomials which depends on a sequence of standard reference objects in the category under consideration, the sequence being called a chromatic complex. The values of the polynomial then indicate the number of morphisms in the category from the object in question to the objects in the chromatic complex.

Introduction

In G.D. Birkhoff's 1912 paper [1] there is defined for the first time a chromatic polynomial associated to a plane partitioned into regions. Since the only aspect of the partition that is important to coloring problems is the adjacency graph of the partition, this chromatic polynomial is actually associated to a graph. The starting point for the present article on generalized chromatic polynomials is the well-known observation that Birkhoff's chromatic polynomial of a graph G is characterized as the function $B(G; n)$ whose value at the integer n is the number of graph morphisms from G to the complete graph on n vertices. By a graph morphism from G to H is meant a function from the vertices of G to the vertices of H along with a function from the edges of G to the edges of H such that the assignments respect incidence. If we interpret the n vertices of the complete graph as n colors, then it is clear that $B(G; n)$ is the number of ways to color the graph G with n colors.

Similar polynomials can be defined for many other mathematical objects. For instance, the order polynomial for partially ordered sets is such a function. This polynomial was defined by Stanley [8, p. 42]. The order polynomial of a partially ordered set P has the value at n which is the number of morphisms (functions which preserve weak order) from P to the linearly ordered set of n elements. Somewhat dual to that polynomial is the one whose value at n is the number of morphisms from the linearly ordered set of $n+1$ elements to P.

Tutte [9] defined the dichromatic polynomial for graphs. This is a polynomial of two variables which contains more information about the graph than Birkhoff's chromatic polynomial. Although it cannot be interpreted as counting morphisms

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in a category, there is a related polynomial of two variables defined below which captures as much information as Tutte's dichromatic polynomial and whose value at a pair of integers (m, n) is the number of graph morphisms from the graph in question to the (m, n)th graph in a doubly indexed sequence of graphs.

In this article we develop a common basis and some of the theory for these polynomials. In Section 1 we define the chromatic category which will hold a sequence of standard reference objects. We use it in Section 2 to define a chromatic complex in the category under consideration which in turn we use to define the chromatic function associated to an object in that category. We also give conditions which are sufficient to guarantee that these functions are polynomials. Section 3 includes primary examples of chromatic polynomials such as Birkhoff's chromatic polynomial and Stanley's order polynomial. We develop recurrence relations for chromatic polynomials in Section 4. General polychromatic theory is introduced in Section 5 with applications to Tutte's dichromatic polynomial in Section 6. The last section includes comments about chromatic polynomials and related material.

1. The simplicial category and the chromatic category

The goal of this section is to develop a category which has a sequence of objects, the nth object being the 'Platonic' object of n colors. This category may be defined as a subcategory of the well-known simplicial category.

The notation for the simplicial category here closely follows that of Mac Lane [7, pp. 171–176]. Let $\boldsymbol{\Delta}$, the *simplicial category*, have as objects all finite ordinal numbers $[n]=\{0, 1, \ldots, n-1\}$ and as morphisms $f:[m]\to[n]$ all functions preserving weak order, that is, $i\leqslant j$ implies $f(i)\leqslant f(j)$. Let $\mathscr{X}$, the *chromatic category*, be the subcategory of $\boldsymbol{\Delta}$ consisting of the monomorphisms of $\boldsymbol{\Delta}$. Then a morphism in $\mathscr{X}$ preserves strict order, that is, $i<j$ implies $f(i)<f(j)$.

Let $d_i:[n]\to[n+1]$ be the usual ith face maps ($i=0, i, \ldots, n$) defined by

$$d_i(j)=\begin{cases} j & \text{if } j<i, \\ j+1 & \phantom{\text{if }} j\geqslant i. \end{cases}$$

Then $\mathscr{X}$ can be abstractly described as the category generated by the face maps d_i subject to identities

$$d_i\circ d_j=d_{j+1}\circ d_i \quad (\text{for } i\leqslant j).$$

The following proposition is easily verified.

Proposition 1. *The chromatic category $\mathscr{X}$ has pullbakcs, and all pullbacks are* *compositions of pullback squares of the two forms*

$$\begin{tikzcd} {[n-1]} \arrow[r, "d_i"] \arrow[d, "d_j"'] & {[n]} \arrow[d, "d_{j+1}"] \\ {[n]} \arrow[r, "d_i"'] & {[n+1]} \end{tikzcd} \quad (\text{for } i \leqslant j)$$

and

$$\begin{tikzcd} {[n]} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & {[n]} \arrow[d, "d_i"] \\ {[n]} \arrow[r, "d_i"'] & {[n+1]} \end{tikzcd} \qquad \square$$

Note that a square of the second form merely states that each face map d_i is a monomorphism in $\mathscr{X}$.

The morphisms in $\mathscr{X}$ are easily counted. There are $\binom{n}{m}$ morphisms from $[m]$ to $[n]$ in $\mathscr{X}$.

2. Chromatic complexes, chromatic functions and chromatic polynomials

Let $\mathbb{C}$ be a category. A *chromatic complex* in $\mathbb{C}$ is a functor $\Delta:\mathscr{X}\to\mathbb{C}$ that preserves pullbacks. Let the object $\Delta([n])$ be denoted Δ_n and let the image of the face map d_i also be denoted d_i and called a *face map* in $\mathbb{C}$. Call the image of any morphism of $\mathscr{X}$ a *chromatic morphism* in $\mathbb{C}$. The following proposition follows directly from the previous one.

Proposition 2. *A chromatic complex in a category $\mathbb{C}$ is determined by a sequence of objects $\Delta_0, \Delta_1, \Delta_2, \ldots,$ and monomorphisms $d_i:\Delta_n \to \Delta_{n+1}$ $(i=0,\ldots,n)$ in $\mathbb{C}$ such that the square*

$$\begin{tikzcd} \Delta_{n-1} \arrow[r, "d_i"] \arrow[d, "d_j"'] & \Delta_n \arrow[d, "d_{j+1}"] \\ \Delta_n \arrow[r, "d_i"'] & \Delta_{n+1} \end{tikzcd}$$

is a pullback for $i \leqslant j$. $\square$

For the rest of this section let $\Delta:\mathscr{X}\to\mathbb{C}$ be a chromatic complex in a category $\mathbb{C}$, and let A be an object in $\mathbb{C}$. Call any morphism $f:A\to\Delta_n$ an *n-coloring* of A. Define the *chromatic function* of A as the function whose value at n, $\chi(A;n)$, is the number of n-colorings of A, that is, $\chi(A;n)=\operatorname{Card}\operatorname{Hom}_{\mathbb{C}}(A,\Delta_n)$. Say an n-coloring $f:A\to\Delta_n$ is *proper* if whenever f factors as $A\to\Delta_k\to\Delta_n$ where $\Delta_k\to\Delta_n$ is a chromatic morphism, then $\Delta_k\to\Delta_n$ is an identity map. In other words, a proper coloring does not factor through any face maps. Thus, in some sense, the proper colorings use all available colors.

Proposition 3. *Each coloring $f: A \to \Delta_n$ is uniquely factorable as a composition of a proper coloring $A \to \Delta_k$ and a chromatic morphism $\Delta_k \to \Delta_n$.*

Proof. Let k be least such that f factors as $A \to \Delta_k \to \Delta_n$ where the second morphism is chromatic. Then $A \to \Delta_k$ is a proper coloring. Uniqueness follows from the fact that pullbacks of chromatic morphisms are chromatic. □

Corollary 4. *Let $\alpha(A; n)$ denote the number of proper n-colorings of an object A. Then*

$$\chi(A; n) = \sum_{k=0}^{\infty} \alpha(A; k)\binom{n}{k}. \qquad \square$$

At this point we have a representation of chromatic functions by series involving binomial coefficients. These series may be formally converted into power series. Because of the finite nature of the applications, these power series are truncated to polynomials.

Corollary 5. *The chromatic function $\chi(A; n)$ of an object A is a polynomial in the variable n of degree d with finite coefficients if and only if A can be properly colored with at most d colors and there are only finitely many d-colorings of A.* □

3. Primary examples of chromatic polynomials

Example 1. Let $\mathbb{C}$ be the category of graphs. For the purposes of this article the term 'graph' refers to an undirected multigraph that may have loops. Let Δ_n be the complete graph on the n vertices $\{0, 1, \ldots, n-1\}$ and define the face map $d_i: \Delta_n \to \Delta_{n+1}$ so that on vertices it satisfies the usual definition that $d_i(j)$ equals j if $j < i$, but equals $j+1$ if $j \geq i$. Then Δ is a chromatic complex. The chromatic function of a graph G is then Birkhoff's chromatic function $B(G; n)$. It is a polynomial if and only if G is finite, in which case its degree is the number of vertices of G.

Example 2. Let $\mathbb{C}$ be the category of partially ordered sets. Let Δ_n be the linearly ordered set on N elements, and let the face maps be defined by the usual requirements. Then the chromatic function for a poset P is the order polynomial for the poset, $\chi(P; n)$. It is an actual polynomial if and only if P is finite in which case its degree is the number p of elements of P. Incidentally, $\alpha(P; p)$, the number of proper p-colorings of P, is the number of extensions of the partial order to a linear order.

Example 3. Let $\mathbb{C}$ be the category of finite distributive lattices, that is, finite lattices with a smallest element 0, a largest element 1, and finite meets and joins which satisfy the distributive axioms. Then $\mathbb{C}$ is dual to the category of finite partially ordered sets. The duality may be effected as follows. To the lattice L associate the partially ordered set of prime elements. (An element is *prime* if it is not 0 and is not the join of two distinct elements.) Conversely, associate to a partially ordered set P the distributive lattice of order ideals of P. (An *order ideal* of P is a subset of P closed downward.)

Let Δ_n be the linear lattice with n elements besides 0 and 1. The face maps again are the obvious ones. Then the chromatic polynomial of a distributive lattice L counts maps from L to Δ_n. Dualizing to the category of partially ordered sets, we find that for each partially ordered set P there is a polynomial whose value at n is the number of maps from the linearly ordered set on $n+1$ elements to P. The degree of this polynomial is one less than the number of elements in the longest linear chain in P.

4. Formulas deriving from coproducts and pushouts

In this section let $\mathbb{C}$ be a category that has finite colimits. Although the primary purpose here is to develop formulas for chromatic polynomials based on chromatic complexes, since there is no mention of the face maps, we only assume that we have a collection of 'objects of colors' in $\mathbb{C}$. A map from an object A to an object of colors T will be called a *T-coloring* of A. The function $\chi(A;T) = \operatorname{Card}\operatorname{Hom}_{\mathbb{C}}(A, T)$ will be called the *chromatic function* of A and denoted simply $\chi(A)$. We first consider the relation between coproducts and chromatic functions.

Proposition 6. *Suppose that an object A is a coproduct of two objects A_1 and A_2. Then $\chi(A) = \chi(A_1)\chi(A_2)$.*

Proof. A coloring of A corresponds to a pair of colorings, one of A_1 and one of A_2. $\square$

In order to describe the reduction formulas based on pushouts we need an appropriate concept of cover. (Perhaps 'cocover' would be a more accurate term.) We say that a finite family of morphisms $\{A \to A_i : i = 1, \ldots, n\}$ is a *color-cover* of an object A if the following two conditions hold.

(1) Each morphism $f: A \to A_i$ in the family is a *color-epimorphism*, that is, if two colorings g and h yield the same coloring when composed with f, $g \circ f = h \circ f$, then they are equal, $g = h$.

(2) Every coloring of A is the composition of a coloring of some A_i and a morphism $A \to A_i$ in the family.

For the moment we study the case where there are two morphisms in the color-cover, $A \to A_1$ and $A \to A_2$. Let A_{12} be the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A_{12}. \end{tikzcd}$$

It is easily shown that $A_1 \to A_{12}$ and $A_2 \to A_{12}$ are also color-epimorphisms. Then by applying the functor $\mathrm{Hom}_{\mathbf{C}}(-, T)$ where T is an object of colors, we have the diagram

$$\begin{tikzcd} \mathrm{Hom}(A_{12}, T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}(A_1, T) \arrow[d] \\ \mathrm{Hom}(A_2, T) \arrow[r] & \mathrm{Hom}(A, T) \end{tikzcd}$$

which in the category of sets is both a pullback (since the previous diagram was a pushout) and a pushout (since condition 2 holds for the color-cover) of monomorphisms (since the previous diagram consisted of four color-epimorphisms). So we can treat $\mathrm{Hom}(A_1, T)$ and $\mathrm{Hom}(A_2, T)$ as subsets of $\mathrm{Hom}(A, T)$ with intersection $\mathrm{Hom}(A_{12}, T)$ and union $\mathrm{Hom}(A, T)$. Consequently, we have reduction formula

$$\chi(A) = \chi(A_1) + \chi(A_2) - \chi(A_{12}).$$

The above equation is just a special case of the principle of inclusion and exclusion. It generalizes to a color-cover of n morphisms as follows.

Theorem 7. *Let $\{A \to A_i : i = 1, \ldots, n\}$ be a color-cover of an object A. For each increasing sequence of integers, $1 \leq i_0 < i_1 < \cdots < i_r \leq n$, let the object $A_{i_0 i_1 \cdots i_r}$ be the colimit of the diagram consisting of the subfamily $\{A \to A_{i_j} : j = 1, \ldots, r\}$ of morphisms. Then we have the reduction formula*

$$\chi(A) = \sum (-1)^r \chi(A_{i_0 i_1 \cdots i_r}),$$

where the sum is taken over all increasing sequences bounded between 1 *and n.* □

When the objects of colors are the objects in a chromatic complex, we find that a similar reduction formula holds for counting proper colorings:

$$\alpha(A; k) = \sum (-1)^r \alpha(A_{i_0 i_1 \cdots i_r}; k).$$

It is interesting to note that the principle of inclusion and exclusion was expounded to mathematicians by Whitney [14] and used by him to derive Birkhoff's determinant formula for the classic chromatic polynomial, see [14, Section 6].

The following examples are continued from Section 3.

Example 1. Let G be a graph and let v and w be two vertices of G that are not connected by an edge. Let G_1 be G with an edge attached between v and w. Let G_2 be G with the vertices v and w identified. Then $\{G \to G_1, G \to G_2\}$ is a color cover of G. The graph G_{12} has the vertices v and w identified and has a loop attached to that identified vertex. The chromatic polynomial of G_{12} is zero. Thus, we have a standard reduction formula for chromatic polynomials of graphs: $\chi(G; k) = \chi(G_1; k) + \chi(G_2; k)$.

Example 2. Let P be a partially ordered set, and let x and y be two incomparable elements of P. Let P_1 be $P/(x \leqslant y)$, and let P_2 be $P/(x \geqslant y)$. Then we have a color-cover. The partially ordered set P_{12} is $P/(x = y)$. Hence, $\chi(P; k) = \chi(P_1; k) + \chi(P_2; k) - \chi(P_{12}; k)$.

Example 3. In terms of lattices this example closely follows Example 2. Let L be a distributive lattice and x and y two elements of L. Let L_1 be $L/(x \leqslant y)$, L_2 be $L/(x \geqslant y)$, and L_{12} be $L/(x = y)$. Then $\chi(L; k) = \chi(L_1; k) + \chi(L_2; k) - \chi(L_{12}; k)$. We can translate this identity into one for partially ordered sets and their associated polynomials. Let P be a partially ordered set and I and J two order ideals of P. Let P_1 be P with $I - J$ removed, let P_2 be P with $J - I$ removed, and let P_{12} be P with both $I - J$ and $J - I$ removed. Then the polynomial for P is that of P_1 plus that of P_2 minus that of P_{12}.

5. Polychromatic polynomials

In this section we consider a variation of the method above for constructing chromatic polynomials which allows for more than one variable. In the theory that follows, we consider only two variables (leading to dichromatic polynomials), but there are analogous definitions and propositions for more than two variables.

A *dichromatic complex* in a category $\mathbb{C}$ is a functor Δ from the product category $\mathscr{X} \times \mathscr{X}$ to $\mathbb{C}$ that preserves pullbacks. Denote the object $\Delta([m],[n])$ by $\Delta_{m,n}$ and let the image of the morphism (d_i, id) be denoted d_i and the image of the morphism (id, d_j) be denoted e_j. Call d_i and e_j *face maps* in $\mathbb{C}$, and call the image of any morphism of $\mathscr{X} \times \mathscr{X}$ a *chromatic morphism* in $\mathbb{C}$.

Proposition 8. *A dichromatic complex in a category $\mathbb{C}$ is determined by a doubly indexed sequence of objects $\Delta_{m,n}$ (m and n nonnegative integers) and monomorphism $d_i : \Delta_{m,n} \to \Delta_{m+1,n}$ and $e_j : \Delta_{m,n} \to \Delta_{m,n+1}$ ($i = 0, \ldots, m$ and $j = 0, \ldots, n$) such that the following two types of squares*

$$\begin{tikzcd} \Delta_{m-1,n} \arrow[r, "d_i"] \arrow[d, "d_j"'] \arrow[dr, phantom, "\lrcorner"] & \Delta_{m,n} \arrow[d, "d_{j+1}"] \\ \Delta_{m,n} \arrow[r, "d_i"'] & \Delta_{m+1,n} \end{tikzcd} \qquad \begin{tikzcd} \Delta_{m,n-1} \arrow[r, "e_i"] \arrow[d, "e_j"'] \arrow[dr, phantom, "\lrcorner"] & \Delta_{m,n} \arrow[d, "e_{j+1}"] \\ \Delta_{m,n} \arrow[r, "e_i"'] & \Delta_{m,n+1} \end{tikzcd}$$

are pullbacks for $i \leqslant j$, *and*

$$\begin{tikzcd} \Delta_{m,n} \arrow[r, "e_j"] \arrow[d, "d_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta_{m,n+1} \arrow[d, "d_i"] \\ \Delta_{m+1,n} \arrow[r, "e_j"'] & \Delta_{m+1,n+1} \end{tikzcd}$$

is a pullback for all i *and* j.

Proof. Every morphism in $\mathscr{X}\times\mathscr{X}$ is a composition of face maps (d_i, id) and (id, d_j). Hence pullbacks in $\mathscr{X}\times\mathscr{X}$ are composed of pullback squares formed from these face maps. Thus, a factor $\Delta : \mathscr{H}\times\mathscr{H} \to \mathbb{C}$ preserves all pullbacks if and only if it preserves the pullback squares formed from face maps. $\square$

Given an object A in the category $\mathbb{C}$, call any morphism $f : A \to \Delta_{m,n}$ an (m, n)-*coloring* of A. Define the *dichromatic function* of A by $\chi(A; m, n) = \operatorname{Card} \operatorname{Hom}_{\mathbb{C}}(A, \Delta_{m,n})$. Thus, the dichromatic function counts (m, n)-colorings. An (m, n)-coloring is *proper* if it does not factor through any of the face maps d_i or e_j.

Proposition 9. *Each coloring* $f : A \to \Delta_{m,n}$ *is uniquely factorable as a composition of a proper coloring* $A \to \Delta_{k,1}$ *and a chromatic morphism* $\Delta_{k,1} \to \Delta_{m,n}$. *Let* $\alpha(A; m, n)$ *denote the number of proper* (m, n)-*colorings of* A. *Then*

$$\chi(A; m, n) = \sum_{k,l=0}^{\infty} \alpha(A; k, 1)\binom{m}{k}\binom{n}{l}. \quad \square$$

Note that the formulas deriving from coproducts and pushouts found in Section 4 may be applied to dichromatic polynomials as well as to chromatic polynomials.

6. Polychromatic polynomials of graphs

There are two closely related polynomials of two variables associated to a graph G. In [13] H. Whitney studied the symbols m_{ij} associated to G; m_{ij} is the number of subgraphs of G (containing all the vertices of G) that have rank i and nullity j. If a graph has v vertices, e edges, and c components, then its *rank* is $v-c$ and its *nullity* is $e-v+c$. Define the *rank generating* function $R(G; x, y)$ of G by $R(G; x, y) = \sum m_{ij} x^{r-i} y^j$, where r is the rank of G.

Tutte introduced in [9] the dichromatic polynomial $T(G; x, y)$ of G which turns out to be related to the rank generating function by the formula $T(G; x, y) = R(G; x-1, y-1)$. Tutte's polynomial is characterized by properties T1 and T2.

T1. If a graph G has i isthmuses (bridges) and j loops, but no other edges, then $T(G; x, y) = x^i y^j$.

T2. If b is an edge of G which is neither an isthmus nor a loop, G' is formed from G by deleting b, and G'' is formed from G by contracting b along with its two vertices to a single vertex, then $T(G) = T(G') + T(G'')$.

See Tutte's article [10] for more information on the rank generating function and discussion of yet another dichromatic polynomial. See Chapter 15.4 of Welsh's book [11] for a generalization of these polynomials to matroids.

Neither of these two polynomials may be directly interpreted as dichromatic polynomials in the sense defined in the previous section. We can, however, find related polynomials which can be so interpreted. Consider the dichromatic complex in the category of graphs where $\Delta_{m,n}$ is the graph with n vertices, one edge between each pair of distinct vertices and m loops at each vertex. The face maps, as usual, are as expected. Let $U(G; m, n)$ be the corresponding dichromatic polynomial for a graph G.

Theorem 10. *The following properties hold for the dichromatic polynomial $U(G; m, n)$ defined just above for graphs:*

(1) *If G consists of a vertex only, then $U(G; m, n) = n$.*

(2) *If G is the coproduct (disjoint union) of G_1 and G_2, then $U(G) = U(G_1)U(G_2)$.*

(3) *If G' is obtained from G by deleting a loop, then $U(G; m, n) = mU(G'; m, n)$.*

(4) *If b is any edge of G, G' is obtained from G by deleting b and G'' is obtained from G by contracting b, then $U(G; m, n) = U(G'; m, n) + (m-1)U(G''; m, n)$.*

(5) *If G'' is obtained from G by contracting an isthmus, then $U(G; m, n) = (m+n-1)U(G''; m, n)$.*

(6) *If G has i isthmuses, j loops, no other edges, and c components, then $U(G; m, n) = n^c(m+n-1)^i m^j$.*

Proof. Statements (1) and (3) are obvious from the definitions. Statement (2) is an application of Proposition 6. In order to show statement (4) let G''' be obtained from G by identifying the vertices of b together so that b becomes a loop. Then the inclusion $G' \to G$ and the obvious morphism $G' \to G''$ form a color-cover of G', while the pushout of these two morphisms yields G'''. Hence, by Theorem 7, $U(G') = U(G) + U(G'') - U(G''')$. But statement (3) gives $U(G''') = mU(G)$. Thus, statement (4) follows. In the special case that the edge b is an isthmus, $U(G'; m, n) = nU(G''; m, n)$, whence statement (5). Statement (6) follows from statements (1), (2), (3), and (5). □

Theorem 11. *Let G be a graph with e edges, v vertices, and c components. Then Tutte's dichromatic polynomial $T(G; x, y)$ and the dichromatic polynomial $U(G; m, n)$ defined above Theorem 10 are related by the formulas*

$$T(G; x, y) = \frac{U(G; y, (x-1)(y-1))}{(y-1)^{v-c}((x-1)(y-1))^c},$$

and

$$U(G; m, n) = (m-1)^{v-c} n^c T(G; (m+n-1)/(m-1), m).$$

Proof. Using statements (4) and (6) of Theorem 10, the characterizing properties T1 and T2 are easily derived for the right side of the first formula; therefore, the formula is valid. The second formula is merely a translation of the first. □

Compare the following theorem to Tutte's discussion of partitions [10, Section 3].

Theorem 12. *For each partition P of the set of vertices of a graph G into nonempty subsets called 'parts' of P, let $m(P)$ be the number of parts of P, and let $e(P)$ be the number of edges of G having both ends in the same part of P. Then*

$$U(G; m, n) = \sum m^{e(P)} n_{m(P)}$$

where $n_{m(P)}$ is the product $n(n-1)(n-2)\cdots(n-m(P)+1)$ and the sum is taken over all partitions of the set of vertices of G.

Proof. We prove this theorem by leaving the category of graphs in favor of the category $\mathbb{C}$ of graphs equipped with a morphism to the graph G_2 having one vertex and two loops, α and β. An object of $\mathbb{C}$ may be viewed as a graph where some of the edges are α-edges while the rest are β-edges. A morphism in $\mathbb{C}$ must map α-edges to α-edges and β-edges to β-edges. We define a chromatic complex Δ in $\mathbb{C}$ by defining $\Delta_{m,n}$ as the graph with n vertices, m α-loops at each vertex, and an α-edge and β-edge between each pair of distinct vertices. The resulting chromatic polynomials $U(G; m, n)$ include the previous chromatic polynomials when the edges of an ordinary graph are declared to be all α-edges.

For each graph G and each pair of vertices in that graph there is a color-cover $\{G \to G', G \to G''\}$ where G' is derived from G by identifying the pair of vertices, and G'' is derived from G by attaching a β-edge between the pair of vertices. The pushout has a β-loop, hence, its polynomial is zero. Thus, we have the reduction formula $U(G) = U(G') + U(G'')$. We apply this formula to all pairs of vertices until $U(G)$ is expressed as a sum of polynomials of graphs, each graph having the property that every pair of distinct vertices is connected by a β-edge. Such graphs correspond to partitions P of the vertices of G. The polynomial of each of these graphs is easily seen to be $m^{e(P)} n_{m(P)}$. □

There is an obvious trichromatic complex in the category of graphs. Let $\Delta_{k,m,n}$ be the graph having n vertices, m loops at each vertex and k edges between each pair of distinct vertices. Let $W(G; k, m, n)$ be the trichromatic complex associated to a graph G. Unfortunately, $W(G)$ contains no more information than $U(G)$. Indeed, they are related by the identity $W(G; k, m, n) = k^e U(G; m/k, n)$ where e is the number of edges of G.

7. Commentary

The covers that are usually used in the study of chromatic polynomials of graphs consist of a set of subgraphs of the graph rather than quotients of the graph as used here. E.J. Farrell has used covers by subgraphs to define a general class of polynomials associated to graphs [4, 5] which includes characteristic and dichromatic polynomials. Although Farrell's techniques are only given for graphs, analogous techniques can be worked out for partially ordered sets and for lattices.

As graphs generalized to matroids (combinatorial geometries), chromatic polynomials of graphs generalize to those of matroids. Indeed, there is a well-developed chromatic theory for matroids; see Chapter 15 of Welsh's book [11] and his later article [12]. In applying the concepts defined here to matroids there is yet some problem in choosing the appropriate category of matroids. As part of the chromatic theory of matroids Brylawski introduced in [2] the concept of a Tutte–Grothendieck invariant by abstracting the recursion properties of Tutte's dichromatic polynomial for matroids. He then showed that the dichromatic polynomial is the universal Tutte–Grothendieck invariant. A similar concept of 'chromatic invariant' may be defined for the chromatic theory defined here. Such a chromatic invariant would satisfy the properties given in Proposition 6 and Theorem 7. We would then like to know when the associated chromatic polynomial is the universal chromatic invariant. Incidentally, the theory of chromatic invariants (relative to a class of covers) is essentially equivalent to K-theory when the category is abelian.

There seem to be few applications of chromatic theory outside of combinatorics. One potential application is to the theory of knots. J.H. Conway discovered a polynomial for knots (since called the Conway polynomial), closely related to the Alexander polynomial for knots, which satisfies recursion formulas similar to the ones for chromatic polynomials. Although there is no obvious way to interpret these recursion formulas in terms of chromatic theory, such an interpretation may be possible once an appropriate category of knots is constructed. Conway published this polynomial in [3]. L.H. Kauffman has written an excellent article [6] on Conway polynomials.

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