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Discrete Mathematics 283 (2004) 37–43
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Ramsey numbers in complete balanced multipartite graphs. Part I: Set numbers

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Received 17 September 2001; received in revised form 11 September 2002; accepted 9 February 2004

Abstract

The notion of a graph theoretic Ramsey number is generalised by assuming that both the original graph whose edges are arbitrarily bi-coloured and the sought after monochromatic subgraphs are complete, balanced, multipartite graphs, instead of complete graphs as in the classical definition. We previously confined our attention to diagonal multipartite Ramsey numbers. In this paper the definition of a multipartite Ramsey number is broadened still further, by incorporating off-diagonal numbers, fixing the number of vertices per partite set in the larger graph and then seeking the minimum number of such partite sets that would ensure the occurrence of certain specified monochromatic multipartite subgraphs.
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MSC: 05C55; 05D10

Keywords: Set multipartite Ramsey number; Multipartite graph

1. Introduction

The classical graph theoretic Ramsey number $r(m,n)$ may be defined as the smallest natural number p with the property that, if the edges of the complete graph K_p are arbitrarily coloured using the colours red and blue, then a red K_m or a blue K_n will be forced as subgraph. We generalise this definition by taking both the original graph whose edges are to be bi-coloured and those which are sought as monochromatic subgraphs to be complete, balanced, multipartite graphs. When adopting this generalisation, we follow the natural approach to fix the cardinality of each partite set in the larger graph and to seek the minimum number of partite sets of this cardinality that would ensure the occurrence of certain specified monochromatic multipartite subgraphs, as is done in the following definition (which is an off-diagonal generalisation of the definition used by Burger, et al. [1]). We call this minimum a set multipartite Ramsey number. In the definition we denote a complete, balanced, multipartite graph consisting of n partite sets and l vertices per partite set by $K_{n\times l}$.

Definition 1 (*Set multipartite Ramsey numbers*). Let j, l, n, s and t be natural numbers with $n,s \geqslant 2$. Then the set multipartite Ramsey number $M_j(K_{n\times l}, K_{s\times t})$ is the smallest natural number ξ such that an arbitrary colouring of the edges of $K_{\xi\times j}$, using the two colours red and blue, necessarily forces a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph.

Note that the subgraphs in Definition 1 need not be vertex-induced subgraphs, i.e. additional edges (of any colour) may be present between vertices of the same partite set within the forced monochromatic multipartite subgraphs. This definition is a generalisation of that of the classical Ramsey numbers in the sense that if $r(\sigma,\lambda)=\tau$, then $M_1(K_{\sigma\times 1}, K_{\lambda\times 1})=\tau$. The following symmetry property of off-diagonal set multipartite Ramsey numbers, whose analogy is well-known in the classical case, is a result of the ubiquity of the word "colour".

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doi:10.1016/j.disc.2004.02.004

Proposition 1 (Symmetry property). *If the multipartite Ramsey number $M_j(K_{n\times l}, K_{s\times t})$ exists, then $M_j(K_{n\times l}, K_{s\times t}) = M_j(K_{s\times t}, K_{n\times l})$.* □

Our goal in this paper is to determine new, small, off-diagonal set multipartite Ramsey numbers. After establishing the boundedness of these numbers, as well as some of their basic properties, in Section 2, we briefly review all known set multipartite Ramsey numbers and establish some new numbers in Section 3. In Sections 4 and 5 we turn to the problems of determining, respectively, lower and upper bounds for larger numbers.

2. Boundedness and basic properties

The question of the boundedness of set multipartite Ramsey numbers is settled first, and the proof is based upon the known existence of the classical Ramsey numbers. To achieve this, we need the notion of an expansive colouring.

Definition 2 (*Expansive colourings*). A colouring of the edges of $K_{k\times j}$ is called an expansive colouring if, for every pair of partite sets of $K_{k\times j}$, the edges between all vertices in these partite sets have the same colour. Therefore every expansive colouring of $K_{k\times j}$ corresponds to exactly one edge colouring of K_k (this may be seen by contracting each partite set of $K_{k\times j}$ to a single vertex), and we say that the expansive colouring of $K_{k\times j}$ is induced by the corresponding colouring of K_k (this definition is due to Day, et al. [5]).

Theorem 1 (Boundedness). *The multipartite Ramsey number $M_j(K_{n\times l}, K_{s\times t})$ exists and, in fact,*

$$\max\{r(n,s), \min\{\lceil l/j\rceil n, \lceil t/j\rceil s\}\} \leqslant M_j(K_{n\times l}, K_{s\times t}) \leqslant \binom{nl+st-2}{nl-1}$$

for all $j, l, t \geqslant 1$ and $n, s \geqslant 2$.

Proof. Let $w = r(n,s) \geqslant 2$. By the definition of w, there exists an edge bi-colouring of K_{w-1} that contains neither a red K_n nor a blue K_s as subgraph. The expansive colouring of $K_{(w-1)\times j}$ induced by this bi-colouring therefore contains neither a red K_n nor a blue K_s as subgraph. But $K_n \equiv K_{n\times 1} \subseteq K_{n\times l}$ and $K_s \equiv K_{s\times 1} \subseteq K_{s\times t}$ for any $l, t \geqslant 1$, and so the expansive colouring of $K_{(w-1)\times j}$ also contains neither a red $K_{n\times l}$ nor a blue $K_{s\times t}$ as subgraph. We conclude that $M_j(K_{n\times l}, K_{s\times t}) > w - 1$.

In order to establish the second lower bound, we prove that $K_{n\times l}, K_{s\times t} \nsubseteq K_{k\times j}$ if $k < \min\{\lceil l/j\rceil n, \lceil t/j\rceil s\}$. Consider first the case where $l \leqslant j$ and suppose that $k < \lceil l/j\rceil n$. Then $k < n$, but $K_n \subseteq K_{n\times l}$ for any $l \geqslant 1$, while $K_n \nsubseteq K_{k\times j}$. Therefore $K_{n\times l} \nsubseteq K_{k\times j}$. Consider now the case where $l = \alpha j + \beta$ for some integers $\alpha \geqslant 1$ and $0 \leqslant \beta < j$. We show, by attempting to construct n partite sets of a potential subgraph $K_{n\times l}$, that $K_{n\times l} \nsubseteq K_{k\times j}$, unless, possibly, if $k \geqslant \lceil l/j\rceil n$. Note that two vertices from any partite set of $K_{k\times j}$ may not occur in different partite sets of an attempted construction of $K_{n\times l}$ within $K_{k\times j}$, since there are no edges between such vertices. Hence we need at least $\lceil l/j\rceil$ partite sets from $K_{k\times j}$ for the construction of a single partite set of $K_{n\times l}$, and there will be $j - \beta$ superfluous vertices for each such construction. But there must be n partite sets to form $K_{n\times l}$, implying that $k \geqslant \lceil l/j\rceil n$. It can be shown in a similar way that $K_{s\times t} \nsubseteq K_{k\times j}$, unless, possibly, if $k \geqslant \lceil t/j\rceil s$.

Finally, consider the upper bound. From the existence theorem of Erdös and Szekeres [6] it follows that $r(nl, st) \leqslant \binom{nl+st-2}{nl-1} = u$, say. Hence, when arbitrarily bi-colouring the edges of K_u, a red K_{nl} or a blue K_{st} is forced as subgraph. But since $K_{n\times l} \subseteq K_{nl}$, $K_{s\times t} \subseteq K_{st}$ and $K_u \equiv K_{u\times 1} \subseteq K_{u\times j}$ for any $j \geqslant 1$, it follows that $K_{u\times j}$ necessarily contains a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph. □

Although the above existence theorem is constructive in the sense that it provides explicit bounds for $M_j(K_{n\times l}, K_{s\times t})$, these bounds are typically weak. It is possible to establish growth properties for set multipartite Ramsey numbers.

Proposition 2 (Growth properties). *Let $n, s, \alpha, \gamma \geqslant 2$ and j, k, l, t, β and δ be natural numbers. Then*

(1) *$M_j(K_{n\times l}, K_{s\times t}) \leqslant M_j(K_{\alpha\times\beta}, K_{\gamma\times\delta})$ if $n \leqslant \alpha$, $l \leqslant \beta$, $s \leqslant \gamma$ and $t \leqslant \delta$. Strict inequality holds if at least one of the strict inequalities $n < \alpha$ or $s < \gamma$ holds.*
(2) *$M_j(K_{n\times l}, K_{s\times t}) \leqslant M_k(K_{n\times l}, K_{s\times t})$ if $k \leqslant j$.*

Proof. (1) Let $M_j(K_{\alpha\times\beta},K_{\gamma\times\delta})=w$ and suppose $n\leqslant\alpha$, $l\leqslant\beta$, $s\leqslant\gamma$ and $t\leqslant\delta$. Then an arbitrary bi-colouring of the edges of $K_{w\times j}$ necessarily contains a red $K_{\alpha\times\beta}$ (and hence a red $K_{n\times l}$) or a blue $K_{\gamma\times\delta}$ (and hence a blue $K_{s\times t}$) as subgraph. Consequently $M_j(K_{n\times l},K_{s\times t})\leqslant w$.

Now suppose that $M_j(K_{n\times l},K_{s\times t})=M_j(K_{\alpha\times\beta},K_{\gamma\times\delta})=v$ (say), but that $n<\alpha$, $l\leqslant\beta$, $s\leqslant\gamma$ and $t\leqslant\delta$. Then there exists a bi-colouring G of the edges of $K_{(v-1)\times j}$ that contains neither a red $K_{n\times l}$ nor a blue $K_{s\times t}$ as subgraph. Form a new multipartite edge colouring H by connecting each of the vertices of $K_{1\times j}$ to all the vertices of G by means of red edges. Then H contains no red $K_{(n+1)\times l}$ and hence no red $K_{\alpha\times\beta}$ as subgraph. Furthermore, H also contains no blue $K_{s\times t}$ and hence no blue $K_{\gamma\times\delta}$ as subgraph. But H is a bi-colouring of the edges of $K_{v\times j}$, which contradicts the definition of $M_j(K_{n\times l},K_{s\times t})$, and we conclude that if $n<\alpha$, $l\leqslant\beta$, $s\leqslant\gamma$ and $t\leqslant\delta$, then $M_j(K_{n\times l},K_{s\times t})<M_j(K_{\alpha\times\beta},K_{\gamma\times\delta})$. The desired result then follows from the symmetry property of Proposition 1.

(2) If $M_k(K_{n\times l},K_{s\times t})=w$ and $k\leqslant j$, then an arbitrary bi-colouring of the edges of $K_{w\times k}$ necessarily contains a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph. But, since $K_{w\times k}\subseteq K_{w\times j}$ if $k\leqslant j$, there must also be a red $K_{n\times l}$ or a blue $K_{s\times t}$ in an arbitrary bi-colouring of the edges of $K_{w\times j}$, so that $M_j(K_{n\times l},K_{s\times t})\leqslant w$. □

There are similar results to those of Propositions 2(1) and 2(2) for the classical Ramsey numbers. Note that the strictness of inequality property mentioned in Proposition 2(1) does *not* necessarily hold when at least one of the strict inequalities $l<\beta$ or $t<\delta$ holds. Exactly when strict inequality occurs (as well as minimal bounds on the gaps in such strict inequalities) is characterised by the next result.

Theorem 2 (Gaps between Ramsey numbers). *For all natural numbers $n\geqslant 3$, $s\geqslant 2$ and $j,l,t\geqslant 1$, $M_j(K_{n\times l},K_{s\times t})\geqslant M_j(K_{(n-1)\times l},K_{s\times t})+s\lceil t/j\rceil-1$.*

Proof. Let $v=M_j(K_{(n-1)\times l},K_{s\times t})$ and $w=s\lceil t/j\rceil-1$. Then there exists a bi-colouring G of the edges of $K_{(v-1)\times j}$ which contains neither a red $K_{(n-1)\times l}$ nor a blue $K_{s\times t}$ as subgraph. Form a new multipartite edge-colouring H by colouring all edges of $K_{w\times j}$ blue and connecting each of the vertices of $K_{w\times j}=F$ to all vertices of G by means of red edges.

Now suppose it were possible that H contains a red $K_{n\times l}$ as subgraph. Since F can contain at most one partite set (or part thereof) of $K_{n\times l}$ (because F has no red edges), at least $n-1$ partite sets of $K_{n\times l}$ must be in G. But this contradicts the definition of G, and we conclude that H does not contain a red $K_{n\times l}$ as subgraph.

Since at least $s\lceil t/j\rceil$ partite sets of size j are necessary for a multipartite graph to contain $K_{s\times t}$ as subgraph, F (and hence also H) contains no blue $K_{s\times t}$ as subgraph. Consequently H is an edge bi-colouring of $K_{(v+w-1)\times j}$ containing neither a red $K_{n\times l}$ nor a blue $K_{s\times t}$ as subgraph. □

3. Known and new small set numbers

There are only a few set multipartite Ramsey numbers known to the authors. These are $M_1(K_{2\times 2},K_{3\times 3})=7$ due to Chartrand and Schuster [2], $M_1(K_{2\times 2},K_{4\times 1})=10$ due to Chvátal and Harary [4], $M_2(K_{2\times 2},K_{3\times 1})=4$ and $M_2(K_{2\times 2},K_{4\times 1})=7$ due to Harborth and Mengersen [13,14], $M_1(K_{2\times 2},K_{5\times 1})=14$ due to Greenwood and Gleason [10], $M_1(K_{2\times 2},K_{6\times 1})=18$ due to Exoo [8], $M_1(K_{2\times 3},K_{2\times 3})=18$ due to Harborth and Mengersen [12] and the complete class of $(K_{2\times 2},K_{2\times 2})$ set multipartite Ramsey numbers, as listed in Table 1.

Bounds for small, diagonal as yet undetermined set multipartite Ramsey numbers may be found in [15]. We establish two new classes of basic set numbers.

Proposition 3 (Basic set multipartite numbers).

(1) $M_j(K_{2\times 1},K_{s\times t})=\lceil t/j\rceil s$ *for all $j,t\geqslant 1$ and $s\geqslant 2$.*
(2) $M_j(K_{n\times 1},K_{s\times 1})=r(n,s)$ *for all $j\geqslant 1$ and $n,s\geqslant 2$.*

Proof. (1) Consider an arbitrary bi-colouring of the edges of the graph $K_{\lceil t/j\rceil s\times j}$. If this colouring contains a red edge, then the graph contains a red $K_{2\times 1}$ as subgraph. Else it contains a blue $K_{s\times t}$ as subgraph. Hence $M_j(K_{2\times 1},K_{s\times t})\leqslant\lceil t/j\rceil s$. Now colour the edges of $K_{(\lceil t/j\rceil s-1)\times j}$ blue. This colouring contains neither a red $K_{2\times 1}$ nor a blue $K_{s\times t}$ as subgraph, so that $M_j(K_{2\times 1},K_{s\times t})>\lceil t/j\rceil s-1$.

(2) $M_j(K_{n\times l},K_{s\times t})\geqslant r(n,s)$ for all $j,l,t\geqslant 1$ and $n,s\geqslant 2$ by Theorem 1, but $M_j(K_{n\times 1},K_{s\times 1})\leqslant M_1(K_{n\times 1},K_{s\times 1})=r(n,s)$ by Proposition 2(1). Consequently $M_j(K_{n\times 1},K_{s\times 1})=r(n,s)$ for all $j\geqslant 1$ and $n,s\geqslant 2$. □

Finally, we conclude this section by fully establishing the new class of $(K_{2\times 2},K_{3\times 1})$ set multipartite Ramsey numbers.

Table 1
The class of $(K_{2\times 2}, K_{2\times 2})$ multipartite Ramsey numbers

j	$M_j(K_{2\times 2}, K_{2\times 2})$
1	6^a
2	4^b
3	4^b
4	3^b
5	2^b
$\geqslant 6$	2^b

[a]Due to Chvátal and Harary [3].
[b]Due to Burger, et al. [1].

Theorem 3 (The class of $(K_{2\times 2}, K_{3\times 1})$ set multipartite numbers).

(1) $M_1(K_{2\times 2}, K_{3\times 1}) = 7$.
(2) $M_2(K_{2\times 2}, K_{3\times 1}) = 4$.
(3) $M_j(K_{2\times 2}, K_{3\times 1}) = 3$ *for all* $j \geqslant 3$.

Proof. (1) Due to Chartrand and Schuster [2].

(2) Due to Harborth and Mengersen [13,14].

(3) Let $V = \{x_1, \ldots, x_9\}$ be the vertex set of $G \simeq K_{3\times 3}$, where $\{x_1, x_2, x_3\}$, $\{x_4, x_5, x_6\}$ and $\{x_7, x_8, x_9\}$ are independent sets. Suppose there exists a (red, blue)-colouring of the edges of G containing neither a red C_4 nor a blue K_3 and denote the subgraphs of G induced by the red and blue edges of this colouring by R and B respectively. Since there is no graph with an odd number of vertices, each of odd degree, we have the following two cases:

Case (a): $\Delta(B) \geqslant 4$. If in B some vertex has at least two neighbours in each of the other two partite sets, we have a contradiction immediately. Thus we may assume that $\Delta(B) = 4$ and that $N_B(x_1) = \{x_4, x_5, x_6, x_7\}$. Then x_4, x_5, x_6 are the vertices of a red K_3, for otherwise there is a blue K_3. Since there is no red C_4, we may assume that $x_2x_8 \in B$ or $x_2x_9 \in B$; we assume the former. If either x_2 or x_8 is adjacent to two of x_4, x_5, x_6 there is a red C_4; otherwise $N_B(x_2) \cup N_B(x_8) \neq \emptyset$ and there is a blue K_3, which is a contradiction.

Case (b): $\Delta(R) \geqslant 4$. Assume that $|N_R(x_1)| \geqslant 4$, and note that x_2 is adjacent in B to at least $|N_R(x_1)| - 1$ of the vertices in $N_R(x_1)$; otherwise there is a red C_4. If $|N_R(x_1)| \geqslant 5$, then x_2 has degree at least four in B, so we have case (a). Hence we may assume that $N_R(x_1) = \{v_1, v_2, v_3, v_4\}$ and $N_B(x_1) = \{v_5, v_6\}$. Since $\{x_2, x_3, v_5, v_6\}$ does not span a red C_4, it follows that $x_2v_5 \in B$. Then x_2 has degree at least four in B, and again we have case (a).

We conclude that no such extremal colouring exists, and hence $M_3(K_{2\times 2}, K_{3\times 1}) \leqslant 3$. But $M_j(K_{2\times 2}, K_{3\times 1}) \geqslant r(2,3) = 3$ for all $j \geqslant 1$ by Theorem 1, and the desired result therefore follows from Proposition 2(2). □

4. Lower bounds

In 1972, Chvátal and Harary [4] proved that for any graphs G and H without isolated vertices,

$$r(G,H) \geqslant (\chi(G) - 1)(c(H) - 1) + 1, \tag{4.1}$$

where $\chi(G)$ denotes the vertex chromatic number of G and $c(H)$ denotes the cardinality of the largest connected component of H. For complete, balanced, multipartite graphs we therefore have the following corollary.

Corollary 1. $M_1(K_{n\times l}, K_{s\times t}) \geqslant \max\{(n-1)(st-1), (s-1)(nl-1)\} + 1$ *for all* $n, s \geqslant 2$ *and* $l, t \geqslant 1$. □

Unfortunately the above result does not hold for $M_j(\cdot,\cdot)$, where $j > 1$. However, using the probabilistic method described by Erdös and Spencer [7], it is possible to prove the following general set multipartite Ramsey lower bound.

Theorem 4 (Lower bounds). *For all* $n, s \geqslant 2$ *and* $j, l, t \geqslant 1$,

$$M_j(K_{n\times l}, K_{s\times t}) > \min\left\{ \sqrt[nl]{n!\,(l!)^n\, 2^{l^2\binom{n}{2}-1}},\ \sqrt[st]{s!\,(t!)^s\, 2^{t^2\binom{s}{2}-1}} \right\} \Big/ j.$$

Proof. Suppose

$$kj \leqslant \min\left\{ \sqrt[nl]{n!\,(l!)^n\,2^{l^2\binom{n}{2}-1}},\ \sqrt[st]{s!\,(t!)^s\,2^{t^2\binom{s}{2}-1}} \right\},$$

then it follows that

$$kj(kj-1)\cdots(kj-nl+1) < (kj)^{nl} \leqslant n!\,(l!)^n\,2^{l^2\binom{n}{2}-1}$$

and similarly

$$kj(kj-1)\cdots(kj-st+1) < s!\,(t!)^s\,2^{t^2\binom{s}{2}-1}.$$

Therefore

$$\binom{kj}{nl}\frac{(nl)!}{n!(l!)^n}2^{-l^2\binom{n}{2}} < \frac{1}{2} \quad\text{and}\quad \binom{kj}{st}\frac{(st)!}{s!(t!)^s}2^{-t^2\binom{s}{2}} < \frac{1}{2}. \tag{4.2}$$

Now consider a random bi-colouring of the edges of $K_{k\times j}$, where the probability of each edge being coloured red is $1/2$. The number of edges in any $K_{n\times l}$ (or $K_{s\times t}$) substructure of $K_{k\times j}$ is $l^2\binom{n}{2}$ (or $t^2\binom{s}{2}$), so that there are $2^{l^2\binom{n}{2}}$ (or $2^{t^2\binom{s}{2}}$) possible colourings of that substructure.

Let $P[K_{n\times l}^{(i,\ \mathrm{red})}]$ denote the probability of the event that the ith $K_{n\times l}$ substructure is entirely red, and use a similar notation for the event that an entirely blue substructure $K_{s\times t}$ occurs. Then

$$P[K_{n\times l}^{(i,\ \mathrm{red})}] = 2^{-l^2\binom{n}{2}} \text{ and } P[K_{s\times t}^{(h,\ \mathrm{blue})}] = 2^{-t^2\binom{s}{2}}$$

for all i and h, respectively, so that

$$\begin{aligned} P\left[\left(\bigvee_i K_{n\times l}^{(i,\ \mathrm{red})}\right) \vee \left(\bigvee_h K_{s\times t}^{(h,\ \mathrm{blue})}\right)\right] &\leqslant \sum_i P[K_{n\times l}^{(i,\ \mathrm{red})}] + \sum_h P[K_{s\times t}^{(h,\ \mathrm{blue})}] \\ &\leqslant \binom{kj}{nl}\frac{(nl)!}{n!(l!)^n}2^{-l^2\binom{n}{2}} + \left(kj\right)\frac{(st)!}{s!(t!)^s}2^{-t^2\binom{s}{2}} \\ &< \frac{1}{2}+\frac{1}{2} \\ &= 1 \end{aligned}$$

by (4.2), where $P[\vee_i K_{n\times l}^{(i,\mathrm{red})}]$ and $P[\vee_h K_{s\times t}^{(h,\mathrm{blue})}]$ denote the probabilities of, respectively, the events that at least some $K_{n\times l}$ substructure of $K_{k\times j}$ is entirely red, and that at least some $K_{s\times t}$ substructure of $K_{k\times j}$ is entirely blue. Therefore *some* edge bi-colouring of $K_{k\times j}$ is in the complement of the event $(\vee_i K_{n\times l}^{(i,\ \mathrm{red})})\vee(\vee_h K_{s\times t}^{(h,\ \mathrm{blue})})$, showing that $M_j(K_{n\times l}, K_{s\times t}) > k$. $\square$

Although the method used in Theorem 4 guarantees the existence of edge bi-colourings of small multipartite graphs that contain neither a red $K_{n\times l}$ nor a blue $K_{s\times t}$, the method provides no qualitative information as to how such bi-colourings may be constructed. This bound becomes weak when nl and st differ significantly in magnitude, since then an upper bound of $\frac{1}{2}$ on both $\binom{kj}{nl}((nl)!/n!(l!)^n)2^{-l^2\binom{n}{2}}$ and $\binom{kj}{st}((st)!/s!(t!)^s)2^{-t^2\binom{s}{2}}$ in (4.2) seems inefficient.

It is anticipated that the above asymptotic bound is weak in general, just as the classical lower bound $r(k,k) \geqslant \sqrt{2^k}$ is weak, and that these bounds may be improved upon for small values of n, l, s and t by rather attempting to construct specific bi-colourings of multipartite graphs that contain neither a red $K_{n\times l}$ nor a blue $K_{s\times t}$ via computerised statistical searches. However, in the absence of such computerised searches, the above lower bound is the best known bound for large values of nl and/or st.

5. Upper bounds

Proof of the following upper bound for diagonal multipartite Ramsey numbers may be found in Burger, et al. [1] and in Stipp [15].

Theorem 5 (Diagonal bipartite upper bound).

$$M_j(K_{2\times l}, K_{2\times l}) \leqslant \left\lceil \frac{2l-1}{j} \right\rceil + \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{j} \right\rceil$$

for all $j, l \geqslant 1$. □

The above bound is valid only in the special case where the sought after monochromatic subgraphs are bipartite. Although we are unable to generalise the above result to the case of multipartite graphs in complete generality, it is possible to prove the following recursive upper bound.

Theorem 6 (Generalised recursive upper bound). *For any* $s \geqslant 2$ *and* $j \geqslant 1$,

$$M_j(G, K_{s\times 2}) \leqslant 2[M_1(G-v, K_{s\times 2})-1] + M_1(G, K_{(s-1)\times 2}), \tag{5.1}$$

where $G-v$ *is any connected graph.*

Proof. We first prove, by contradiction, that

$$M_1(G, K_{s\times 2}) \leqslant 2[M_1(G-v, K_{s\times 2})-1] + M_1(G, K_{(s-1)\times 2}).$$

Let $a = M_1(G-v, K_{s\times 2})$ and $b = M_1(G, K_{(s-1)\times 2})$, but suppose that $M_1(G, K_{s\times 2}) > 2(a-1)+b$. Then there exists an edge bi-colouring of $K_{2(a-1)+b}$ containing neither a red G nor a blue $K_{s\times 2}$ as subgraph. It is easy to see that the maximal red degree in this extremal colouring is $a-1$. Consider any vertex v and denote all vertices joined to v by means of red edges (blue edges, respectively) by R (B, respectively). Hence $|R| \leqslant a-1$ and $|B| \geqslant (a-1)+(b-1)$. Now let w be any vertex in R. There are at most $a-2$ vertices in B which are joined to w by means of red edges. Consequently there must be at least b vertices in B which are joined to w by means of blue edges; let $\langle H\rangle$ denote the subgraph induced by these vertices. Because $\langle H\rangle$ contains no red G as subgraph, it must contain a blue $K_{(s-1)\times 2}$ as subgraph. But this is a contradiction, since all vertices in H are joined to both v and w by means of blue edges, which would force a blue $K_{s\times 2}$ as subgraph of the above bi-colouring of $K_{2(a-1)+b}$. Now it follows by a similar argument to that in the proof of Proposition 2(2) that $M_j(G, K_{s\times 2}) \leqslant M_1(G, K_{s\times 2})$. □

6. Conclusion

In this paper the notion of a graph theoretic Ramsey number was generalised by replacing the requirement of a complete graph in the classical definition by that of a complete, balanced, multipartite graph following the general approach by Burger, et al. [1] in the diagonal special case. The notion of a set multipartite Ramsey number involved fixing the number of vertices per partite set in the larger graph and then seeking the minimum number of such partite sets that would ensure the occurrence of certain specified monochromatic multipartite subgraphs. The boundedness of these generalised Ramsey numbers was established and some new set numbers were found, as well as lower and upper bounds for other set numbers.

Acknowledgements

Research towards this paper was supported financially by the South African National Research Foundation under grant number GUN 2053755 and by Research Sub-Committee B at the University of Stellenbosch. Any opinions, findings and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the South African National Research Foundation.

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