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G–R-Sequences and Incidence Coalgebras of Posets of Full Binomial Type*

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A family of polynomial sequences, named G–R-sequences, is introduced and its connections with both graded coalgebras and posets of full binomial type are studied. Moreover, the G–R-sequences $p_n(x)$, such that p_n is a divisor of p_{n+1}, are characterized in terms of roots of unity and linearly recursive sequences. © 1986 Academic Press, Inc.

1. INTRODUCTION

The central rôle played in a number of topics by the elementary binomial theorem is well known. In particular, its form

$$(x-y)^n = \sum_{k=0}^{n} \binom{n}{k} (x-z)^k (z-y)^{n-k} \tag{1}$$

counts in two ways the number of one-to-one maps $f: A \to X$ such that $\operatorname{Im} f \cap Y = \emptyset$, where A, X, Y, Z are sets, $Y \subseteq Z \subseteq X$ and n, x, y, z are the cardinalities of A, X, Y, Z, respectively.

In their systematic investigation on the foundations of combinatorial theory, Goldman and Rota in [11] aimed to obtain a similar combinatorial derivation of various identities classically known as q–identities (see also [3, 4, 10, 12]). In particular, they studied the case of the list of homogeneous polynomials

$$P_n(x, y) = \prod_{i=0}^{n-1} (x - q^i y), \qquad P_0 = 1, \tag{2}$$

and proved the identity

$$P_n(x, y) = \sum_{k=0}^{n} \binom{n}{k}_q P_k(x, z)\, P_{n-k}(z, y) \tag{3}$$

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by counting in two ways the one-to-one linear transformations $f\colon A \to X$ such that $\operatorname{Im} f \cap Y = 0$, where A, X, Y, Z are now linear spaces over the finite field $GF(q)$, $Y \subseteq Z \subseteq X$, $n = \dim(A)$, x, y, z are the cardinalities of X, Y, Z and $\binom{n}{k}_q$ denotes the usual Gaussian coefficient.

In order to produce a detailed analysis of the previous analogy, the authors of the present paper found it suitable in [7] to introduce the following definitions.

DEFINITION 1. Let K be a field of characteristic zero; the polynomial sequence

$$P_n(x, y) = \sum_{i=0}^{n} p_n^i x^i y^{n-i}, \qquad n \geqslant 0 \tag{4}$$

(where: $p_n^i \in K$, $\quad p_n^n = \pi_n \neq 0$, $\quad p_0^0 = \pi_0 = 1$) is said to be a *homogeneous Goldman–Rota-sequence*, if there exist suitable constants $h_k^n \in K$ such that

$$P_n(x, y) = \sum_{k=0}^{n} h_k^n \, P_k(x, z) \, P_{n-k}(z, y). \tag{5}$$

Moreover, the corresponding non-homogenous sequence $p_n(x) := P_n(x, 1)$ will be simply called a *G–R-sequence.*

It is easy to see that the G–R-sequence $p_n(x)$ may be directly characterized by

$$p_n(xy) = \sum_{k=0}^{n} h_k^n \, p_k(x) \, y^k \, p_{n-k}(y) \tag{6}$$

in substitution for (5).

DEFINITION 2. The polynomial sequences

$$S_n(x, y) = \sum_{i=0}^{n} s_n^i \, x^i \, y^{n-i}, \qquad n \geqslant 0, \tag{7}$$

will be called a *homogeneous Goldman–Rota–Sheffer-sequence* associated with the G–R-sequence $p_n(x)$ if

$$S_n(x, y) = \sum_{k=0}^{n} h_k^n \, P_k(x, z) \, S_{n-k}(z, y). \tag{8}$$

The G–R–S-sequences are the corresponding non-homogeneous sequences $s_n(x) := S_n(x, 1)$.

It is plain that a G–R-sequence is also a (self-associated) G–R–S-sequence.

It is possible to prove (cf. [7, Proposition 5]) that (8) is equivalent to

$$\sum_{j=0}^{k} h_j^n p_j^r s_{n-j}^{k-j} = \delta_r^k s_n^k, \qquad k \geqslant r \tag{9}$$

which, by putting $k = r$ and $s_n(x) = p_n(x)$, gives

$$p_n^k = h_k^n p_{n-k}^0. \tag{10}$$

In Section 2 we shall show that the coefficients h_k^n which occur from (5) to (10) can be assumed as structure constants of a comultiplication Δ_H over $K[x]$, so that a G–R-sequence introduces a structure of graded coalgebra C_H into $K[x]$ and, conversely, the G–R-sequences can be characterized in coalgebraic terms.

Regarding the foregoing instances of G–R-sequences, let us note that $(x-1)^n$ is the characteristic polynomial of the lattice of subsets of a finite n-set in the same way that $\prod_{i=0}^{n-1} (x - q^i)$ is the characteristic polynomial of the lattice of subspaces of an n-dimensional vector space over $GF(q)$. The latter sequence has also been studied by Andrews [4] in order to develop a theory for enumeration problems in finite vector spaces that is analogous to the theory of binomial enumeration of Mullin and Rota [14]. As a further example of a G–R-sequence one may consider the list $p_n(x) := x^{n-1}(x-1)$ which corresponds to $h_k^n = 1$; in this case, $p_n(x)$ is the characteristic polynomial of the chain of length n. In the next section we shall describe a more comprehensive class of G–R-sequences which may be regarded as the sequences of characteristic polynomial of suitable partial ordered sets.

However, not all the polynomial sequences with such a property are G–R-sequences; for instance, the sequence

$$p_n(x) = x(x-1) \cdots (x-n+1), \qquad n \geqslant 0, \tag{11}$$

associated with the lattice of partitions of an n-set. On the other hand all four examples considered are graded bases $p_n(x)$ of the vector space $K[x]$ such that $p_{n+1}(x)$ is divisible by $p_n(x)$. Proposition 5 below will describe the G–R-sequences with such a property.

2. G–R-Sequences, Coalgebras, and Their Combinatorial Interpretation

For the sake of simplicity, in the following we shall consider only G–R-sequences of monic polynomials ($\pi_n = 1$). Concerning the general case as well as the proofs of Proposition 2 and Proposition 3 see [7].

We first remark that there is a bijection between the G–R-sequences $p_n(x)$ and their allied constants h_k^n. In fact, putting $y=0$ and $z=1$ in (5), we get $x^n=\sum_{k=0}^n h_k^n p_k(x)$; this, together with $p_n(x)=\sum_{k=0}^n p_n^k x^k$, gives the change of basis from x^n to $p_n(x)$. This proves the following:

PROPOSITION 1. *Let $p_n(x)=\sum_{k=0}^n p_n^k x^k$ be a G–R-sequence associated with constants h_k^n; then we have*

$$^tH=P^{-1} \tag{12}$$

where $P=(p_n^k)$ and $H=(h_k^n)$, with $p_n^k=h_k^n=0$ for $k>n$.

Associated with a given G–R-sequence $p_n(x)$ let us consider the following two linear maps:

$$\begin{array}{rl} \Delta_H\colon K[x] \to K[x]\otimes K[x] & \text{(comultiplication or} \\ x^n \mapsto \sum_{k=0}^{n} h_k^n x^k \otimes x^{n-k} & \text{diagonalization)} \end{array} \tag{13}$$

and

$$\begin{array}{rl} \varepsilon\colon K[x] \to K & \text{(counit).} \\ x^n \mapsto \delta_0^n & \end{array} \tag{14}$$

PROPOSITION 2. *With reference to Definition* 1, *if $h_0^n \neq 0$ for every $n\in \mathbf{N}$, then $C_H:=(K[x],\Delta_H,\varepsilon)$ is a (graded) coalgebra, i.e., the following diagrams are commutative*:

$$\begin{array}{ccc} C_H & \xrightarrow{\Delta_H} & C_H\otimes C_H \\ {\scriptstyle \Delta_H}\downarrow & & \downarrow{\scriptstyle \Delta_H\otimes I} \\ C_H\otimes C_H & \xrightarrow{I\otimes\Delta_H} & C_H\otimes C_H\otimes C_H \end{array} \qquad (\textit{coassociativity}) \tag{15}$$

$$\begin{array}{ccccc} K\otimes C_H & \xleftarrow{\varepsilon\otimes I} & C_H\otimes C_H & \xrightarrow{I\otimes\varepsilon} & C_H\otimes K \\ & {\scriptstyle \psi}\nwarrow & \uparrow{\scriptstyle \Delta_H} & \nearrow{\scriptstyle \psi} & \\ & & C_H & & \end{array} \qquad (\textit{counitary property}) \tag{16}$$

($H=(h_k^n)$, ψ is the canonial isomorphism and I is the identical map). When $h_0^n=0$ for some n, then only the right-hand side of diagram (16) *commutes, in which case C_H will be called a right graded coalgebra.*

Note that the commutativity of (15) is equivalent to

$$h_k^n h_r^k = h_r^n h_{k-r}^{n-r}. \tag{17}$$

Let us denote by $C_H^* := (K[[x]], m, u)$ the (right) graded algebra obtained by dualizing the (right) coalgebra C_H:

$$\begin{aligned} m\colon C_H^* \otimes C_H^* &\to C_H^* \\ x^i \otimes x^j &\mapsto h_i^{i+j} x^{i+j}, \end{aligned} \tag{18}$$

$$\begin{aligned} u\colon K &\to C_H^* \\ 1 &\mapsto x^0. \end{aligned} \tag{19}$$

Here $K[[x]]$ has been identified with the linear dual of $K[x]$ and the "series" x^n with the dual form of the "polynomial" x^n; consequently the series x^n is not in general the nth power of x^1 in C_H^* but, according to (17), we have $(x^1)^n = h_1^2 h_2^3 \cdots h_{n-1}^n x^n = h_1^2 h_1^3 \cdots h_1^n x^n$.

Note also that C_H is cocommutative (C_H^* is commutative) if and only if $h_k^n = h_{n-k}^n$; this is the case, because of (17), if $h_1^n \neq 0$ for every n.

Concerning the G–R-sequences in Section 1, note that C_H and C_H^* are respectively known as (i) the polynomial coalgebra and the algebra of divided power series, if $h_k^n = \binom{n}{k}$, $p_n(x) = (x-1)^n$; (ii) the coalgebra of divided powers and the algebra of power series if $h_k^n = 1$, $p_n(x) = x^{n-1}(x-1)$; (iii) the q-Eulerian coalgebra and the algebra of formal q-Eulerian series if $h_k^n = \binom{n}{k}_q$, $p_n(x) = \prod_{i=0}^{n-1} (x - q^i)$.

It is quite natural to associate with any series $s = \sum_{j \geq 0} s_j x^j \in C_H^*$ the linear map

$$\begin{aligned} \tilde{s} := m(- \otimes s)\colon C_H^* &\to C_H^* \\ x^n &\mapsto m(x^n \otimes s) = \sum_{k \geq n} s_{k-n} h_n^k x^k. \end{aligned} \tag{20}$$

In the following the series $\mu = \sum_{j \geq 0} \mu_j x^j$ defined as the multiplicative inverse (in C_H^*) of the series $\zeta = \sum_{j \geq 0} x^j$ will be of special interest; in the combinatorial interpretation we shall give later, they correspond to the usual Moebius function and zeta-function (see [2]).

We are now able to invert Proposition 2; in fact, Proposition 3 below shows how to construct the G–R–S-sequences (and then, the unique G–R-sequence) associated with a given graded coalgebra C_H.

PROPOSITION 3. *Let* $C_H = (K[x], \Delta_H, \varepsilon)$ *be a coalgebra, graded with* *reference to the basis x^n on $K[x]$ (i.e., (13) to (16) are satisfied); the linear transformation*

$$\hat{s} := \psi \circ (I \otimes s) \circ \Delta_H : C_H \to C_H$$
$$x^n \mapsto s_n(x) := \sum_{i=0}^{n} h_i^n s_{n-i} x^i \tag{21}$$

associated with the series $s = \sum_{j \geqslant 0} s_j x^j \in C_H^*$ maps the canonical basis x^n into a G–R–S-sequence $s_n(x)$. In particular, if $s = \mu$, one gets the G–R-sequence $p_n(x)$, the associated coalgebra of which (in the sense of Proposition 2) is just C_H.

The intrinsic simplicity of the above statement appears manifest if one remarks that by dualizing map (21) one obtains (20).

We now come to consider a class of graded coalgebras arising in a combinatorial setting, whose allied G–R-sequences have the interpretation referred to in Section 1. Most of the results used here, as well as a more general treatment of the algebraic and coalgebraic aspects of combinatorial structures, can be found in [2, 9, 13].

First recall that a coalgebra $C(\mathscr{P})$, the so-called incidence coalgebra of $\mathscr{P}$, is associated with any locally finite partial ordered set $\mathscr{P}$, hereafter abbreviated to "l. f. poset." The support of $C(\mathscr{P})$ is the free K-space spanned by the set of all intervals $[a, b]$ in $\mathscr{P}$; comultiplication Δ and counit ε are given by

$$\Delta([a, b]) = \sum_{a \leqslant t \leqslant b} [a, t] \otimes [t, b] \tag{22}$$

$$\varepsilon([a, b]) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

By dualizing $C(\mathscr{P})$ one obtains the more familiar "incidence algebra" $C^*(\mathscr{P})$ of $\mathscr{P}$.

In the remaining part of this section we shall only consider l. f. posets of full binomial type. After Doubilet–Rota–Stanley [9], an l. f. poset $\mathscr{P}$ is said to be *of full binomial type* if (i) all the maximal chains in a given interval $[a, b]$ have the same length (Jordan–Dedekind chain condition); (ii) all the intervals of length n possess the same number, say B_n, of maximal chains; (iii) $\mathscr{P}$ contains a 0-element. The lattices considered in Section 1 (except the lattice of partitions of an n-set) are l. f. posets of full binomial type.

Without loss of generality, we may assume that $\mathscr{P}$ has infinite length. Two intervals $[a, b]$ and $[c, d]$ of $\mathscr{P}$ are said to be *equivalent*, $[a, b] \sim [c, d]$, if they have the same length. Later we shall make use of the following simple lemmas.

LEMMA 1. *Any two equivalent intervals* $[a, b] \sim [c, d]$ *of an l.f. poset of full binomial type* $\mathscr{P}$ *have the same level numbers (of the second kind).*

Proof. Let $[a, b]$ be an arbitrary interval of length n in $\mathscr{P}$, W_k be the number of elements of rank k in $[a, b]$, and $t \in [a, b]$ be one such element. It is easy to check that the maximal chains in $[a, b]$ containing t are $B_k B_{n-k}$; hence, $W_k = B_n/(B_k B_{n-k})$. ∎

We shall call the "level number indicator" of $[a, b]$ the polynomial

$$\iota(x) = \sum_{k=0}^{n} W_k x^{n-k} = \sum_{k=0}^{n} \frac{B_n}{B_k B_{n-k}} x^{n-k} \qquad (n = \text{length of } [a, b]). \tag{24}$$

LEMMA 2. *In the hypotheses of Lemma* 1, *there exists a bijection* $\phi\colon [a, b] \to [c, d]$ *such that if* $a \leqslant a_1 \leqslant b_1 \leqslant b$ *then* $[a_1, b_1] \sim [\phi(a_1), \phi(b_1)]$.

Proof. Lemma 1 enables us to define a bijection ϕ_k from the kth level of $[a, b]$ to the kth level of $[c, d]$. Obviously, the map $\phi(t) := \phi_k(t)$ with $k =$ rank of t, satisfies the conditions in the lemma. ∎

From Lemma 2 we deduce that the subspace J of $C(\mathscr{P})$ spanned by the collection $\{[a, b] - [c, d] : [a, b] \sim [c, d]\}$ is a coideal (really, a maximal coideal) of $C(\mathscr{P})$, i.e., $\Delta(J) \subseteq J \otimes C(\mathscr{P}) + C(\mathscr{P}) \otimes J$ and $\varepsilon(J) = 0$. In fact, if $[a, b] \sim [c, d]$ we have

$$\begin{aligned}
&\Delta([a, b] - [c, d])\\
&= \sum_{a \leqslant t \leqslant b} [a, t] \otimes [t, b] - \sum_{c \leqslant s \leqslant d} [c, s] \otimes [s, d]\\
&= \sum_{a \leqslant t \leqslant b} [a, t] \otimes [t, b] - \sum_{a \leqslant t \leqslant b} [\phi(a), \phi(t)] \otimes [\phi(t), \phi(b)]\\
&= \sum_{a \leqslant t \leqslant b} \{([a, t] - [\phi(a), \phi(t)]) \otimes [t, b]\\
&\qquad + [\phi(a), \phi(t)] \otimes ([t, b] - [\phi(t), \phi(b)])\} \in J \otimes C(\mathscr{P}) + C(\mathscr{P}) \otimes J.
\end{aligned}$$

By identifying the polynomial x^n with the class of all intervals $[a, b] \subseteq \mathscr{P}$ of length n in the quotient coalgebra of $C(\mathscr{P})$ modulo J, one obtains a graded coalgebra $C_H = (K[x], \Delta_H, \varepsilon)$, called the *maximally reduced incidence coalgebra* of $\mathscr{P}$. In accordance with (22) and (23) we have

$$\Delta_H(x^n) = \sum_{k=0}^{n} \frac{B_n}{B_k B_{n-k}} x^k \otimes x^{n-k}, \qquad h_k^n = W_k = B_n/(B_k B_{n-k}) \tag{25}$$

$$\varepsilon(x^n) = \delta_0^n \tag{26}$$

We may now state the following:

PROPOSITION 4. *A cocommutative graded coalgebra* $C_H = (K[x], \Delta_H, \varepsilon)$ *given by* (25) *and* (26) *is associated with any l.f. poset of full binomial type* $\mathscr{P}$. *If* $p_n(x)$ *is the G–R-sequence associated with* C_H *and* $s_n(x)$ *is the G–R–S-sequence relative to the series* $\zeta = \sum_{n \geq 0} x^n$, *then any interval of length n in* $\mathscr{P}$ *has* $p_n(x)$ *as its characteristic polynomial and* $s_n(x)$ *as its level number indicator.*

Proof. The second part of the statement remains to be proved. In the transformation from the incidence algebra $C^*(\mathscr{P})$ to C_H^*, the zeta-function $\bar{\zeta}$ and the Moebius function $\bar{\mu}$ become the series ζ and $\mu = \zeta^{-1}$ considered above and we have $\bar{\zeta}(a, b) = \zeta(x^n) = 1$ and $\bar{\mu}(a, b) = \mu(x^n)$ if $[a, b] \subseteq \mathscr{P}$ has length n. Then, in particular,

$$B_n/(B_k B_{n-k})\, \mu(x^k) = W_k \bar{\mu}(a, \tau) = \sum_{\substack{a \leq t \leq b \\ k = \operatorname{rank}(t)}} \bar{\mu}(a, t)$$

where $\tau \in [a, b]$ and $\operatorname{rank}(\tau) = k$. On the other hand, from (21) of Proposition 3 we deduce

$$p_n(x) = \sum_{k=0}^{n} B_n/(B_k B_{n-k})\, \mu(x^k)\, x^{n-k}$$

and

$$s_n(x) = \sum_{k=0}^{n} B_n/(B_k B_{n-k})\, \zeta(x^k)\, x^{n-k} = \sum_{k=0}^{n} h_k^n x^{n-k}. \quad \blacksquare$$

3. ON A PARTICULAR CLASS OF G–R-SEQUENCES

In this section we shall deal with G-R-sequences $p_n(x)$ of the form

$$p_n(x) = \prod_{i=0}^{n-1} (x - u_i), \qquad u_i \in K. \tag{27}$$

The following proposition enables us to construct all the sequences of this kind.

PROPOSITION 5. *Polynomial sequence* (27) *is a G–R-sequence if and only if either* (i) $u_i = 0$ *for every* $i \in \mathbf{N}$ *or* (ii) *every initial segment* $(u_0, u_1, \ldots, u_n)$ *of the sequence of scalars* $(u_i)_{i \in \mathbf{N}}$ *is also the initial segment of a linearly recursive sequence* $v = (v_i)_{i \in \mathbf{N}}$*—depending on n—with a characteristic polynomial of the form* $x^m - \rho$ $(m < n; \rho \in K)$ *and whose first m terms* $v_0 = u_0, \ldots, v_{m-1} = u_{m-1}$ *are the mth roots of unity.*

Regarding the notion of a linearly recursive sequence used here, see [5]; we should remember that not long ago it was proved (cf. [15]) that such sequences may be given a structure of Hopf algebra which is the dual of the polynomial one.

Note that the condition relating to $(u_i)_{i \in \mathbf{N}} \neq 0$ in Proposition 5 is equivalent to asserting that it may be generated by making use of the following prescriptions:

(a) $u_0 = 1$;

(b) if $u_0, ..., u_{m-1}$ $(m \geqslant 1)$ are the mth roots of unity then u_m is an arbitrary scalar and $u_n = (u_m)^s \cdot u_t$ (where $n = sm + t, t < m$) for $n < \operatorname{lcm}(p, m)$ or for every n depending on whether u_m is or is not a primitive pth root of unity, where p is not a divisor of m.

In order to prove Proposition 5, it is helpful to recall (cf. [6]) that if we put $p_n(x) = \sum_{k=0}^{n} p_n^k x^k = \prod_{i=0}^{n-1} (x - u_i)$, $p_0 = 1$, $P = (p_n^k)$, and $H = (h_k^n)$ is an arbitrary (infinite) matrix $(n, k \geqslant 0)$, then the following conditions are equivalent:

(i) ${}^t H = P^{-1}$;

(ii) the kth column h_k of H is a linearly recursive sequence whose characteristic polynomial is $p_{k+1}(x)$ and whose first terms are $h_k^n = \delta_k^n$, $0 \leqslant n \leqslant k$;

(iii) the matrices P and H are fully described by the recurrences

$$p_{n+1}^k = p_n^{k-1} - u_n p_n^k, \qquad p_0^k = \delta_0^k, \qquad p_n^{-1} = 0 \qquad (n, k \geqslant 0), \tag{28}$$

$$h_k^{n+1} = h_{k-1}^n + u_k h_k^n, \qquad h_k^0 = \delta_k^0, \qquad h_{-1}^n = 0 \qquad (n, k \geqslant 0). \tag{29}$$

Obviously, functions $p_n^k = p_n^k(u_0, ..., u_{n-1})$ [resp. $h_k^n = h_k^n(u_0, ..., u_k)$] are the elementary symmetric [resp. homogenous] functions of degree $n-k$ in the variables $u_0, u_1, ...$.

Moreover, if $p_n(x)$ is also a G–R-sequence then from (i) and Propositions 1 and 3, it follows that the matrix H above is exactly that constructed with the structure constants of the graded coalgebra associated with $p_n(x)$. In such a case, by substituting (10) in (28) we obtain

$$u_n h_k^n p_{n-k}^0 = p_{n+1-k}^0 (h_{k-1}^n - h_k^{n+1})$$

and then, by (29),

$$u_n h_k^n p_{n-k}^0 = -p_{n+1-k}^0 u_k h_k^n;$$

hence, because $p_{n+1-k}^0 = -u_{n-k} p_{n-k}^0$ we have

$$h_k^n p_{n-k}^0 (u_n - u_k u_{n-k}) = 0. \tag{30}$$

Proof of Proposition 5. First note that as a consequence of (10) we get

$$u_m = 0 \Rightarrow u_{m+n} = 0.$$

Considering u_0, we have either $u_0 = 0$ (and then, $u_n = 0$ for every n) or $u_0 = 1$. In fact, putting $r = k = 0 \neq n$ in (17), we get either $h_0^n = 1$ or $h_0^n = 0$ (but $h_0^0 = h_n^n = 1$ because $P^{-1} = {}^tH$); on the other hand, by (ii) above the sequence $(h_0^n)_{n \in \mathbf{N}}$ is a geometrical progression with ratio u_0 and first term $h_0^0 = 1$; then it follows that either $h_0^n = 1 = u_0$ for every n or $h_0^{n+1} = 0 = u_0$ for every n.

Suppose now that $u_0, \ldots, u_{m-1}$ are the mth roots of unity (which is true at least for $m = 1$, if $u_0 \neq 0$) so that $p_m(x) = x^m - 1$. It is easy to check that the polynomial $P_{m+1}(x, y) = (x^m - y^m)(x - u_m y)$, whatever u_m may be, satisfies (5). If $u_m = 0$, there is nothing else to prove. Whereas, on the contrary, if $u_m \neq 0$, consider the mth column h_m of the matrix H; it is a linearly recursive sequence with characteristic polynomial $p_{m+1}(x) = x^{m+1} - u_m x^m - x + u_m$ and first terms $h_m^n = \delta_m^n$, $0 \leqslant n \leqslant m$. It follows that

$$h_m^n = h_m^{sm+t} = [1 + (u_m)^m + (u_m)^{2m} + \cdots + (u_m)^{(s-1)m}] (u_m)^t,$$
$$n = sm + t,\ t < m. \qquad (31)$$

Hence, $h_m^{sm+t} \neq 0$ and then, from (30),

$$u_n = u_{sm+t} = (u_m)^s \cdot u_t, \qquad n = sm + t, \qquad (32)$$

for all n, or only for $n < \operatorname{lcm}(m, p) = \tau$ if u_m should be a primitive pth root of unity, where p is not a divisor of m. In the latter case, the terms $u_0, \ldots, u_{\tau-1}$ are the τth roots of unity and the argument may be repeated. This completes the proof. ∎

It is plain to see that if the sequence $p_n(x) = \sum_{k=0}^{n} p_n^k x^k$, $p_n^n = 1$, is required to be a sequence of characteristic polynomials of intervals then $p_n^k \in \mathbf{Z}$ and the polynomial $p_2(x) = x^2 - 1$ must be excluded. Hence from Proposition 5 we deduce the following:

COROLLARY. *If a G–R-sequence $p_n(x)$ of form* (27) *has the combinatorial interpretation in terms of l.f. posets of full binomial type given in Section* 2, *then $p_n(x)$ must be of the form $p_n(x) = \prod_{i=0}^{n-1} (x - q^i)$.*

Thus, depending on whether $q = 0$, $q = 1$, or $q = p^n$ (p prime), we have the three lattices considered in Section 1. It must be noted, however, that several non-isomorphic posets of the required type can be associated with the same maximally reduced incidence coalgebra C_H and then with the same G–R-sequence. For instance, the coalgebra of divided powers is associated with both a chain and a planted tree.

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