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Global actions of Lie symmetries for the nonlinear heat equation

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A B S T R A C T

By restricting to a natural class of functions, we show that the Lie point symmetries of the nonlinear heat equation exponentiate to a global action of the corresponding Lie group. Remarkably, in most of the cases, the action turns out to be linear.

1. Introduction

Lie symmetries provide a powerful and systematic tool for the analysis of partial differential equations. For instance, the method of reduction of variables via Lie point symmetries is an extremely useful technique for simplifying or solving PDE's [1,2,7,8,10,11,13,14,17,15,18,21,3]. Of course, the entire theory is based on the notion of local one-parameter actions of Lie groups. As a result, the various algorithms give rise to infinitesimal symmetries that only generate a Lie algebra. Typically, the corresponding local one-parameter actions do not exponentiate to a global action of the corresponding Lie group. As a result, the enormous body of literature devoted to the study of Lie groups is frequently not applicable to the study of symmetries of PDE's.

However in [5,6], M. Craddock made an important discovery. He found that, in certain cases, a global representation was made possible by restricting to an appropriate subset of the solution space. This allows the full weight of representation theory to be brought to bear. For instance when this machinery is applied to the wave equation, representation theory naturally picks out a distinguished orthonormal basis that is extremely well behaved with respect to energy and momentum (actually consisting of smooth rational solutions when the space dimension is odd) [9]. Similarly nice results are achieved in the case of the heat and Schrödinger equations with links to the harmonic oscillator [19,20].

Of course the cases mentioned above consist of linear PDE's and so it is not surprising that representation theory can be used on the problem. In this paper we examine the nonlinear heat equation

$$u_t = \big(k(u)u_x\big)_x$$

where $k(u)$ is not a constant. The classification of Lie point symmetries and equivalence transformations are well known [10,16,15,4,14]. The Lie point symmetries of this equation fall into four categories, each of which will be examined. It turns out that three of the cases generate solvable groups while one case has a semisimple component. A priori, there is no reason to suppose that an algebra of Lie point symmetries exponentiates to an action of the entire corresponding group. Nevertheless in each case, we show that the Lie algebra of symmetries extends to a *global* action of the corresponding Lie group by restricting to a natural subclass of functions. In each of the four cases, we explicitly write down this global action. Remarkably in three of these cases, including the one with the semisimple component, we show that the action of the

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group is actually given by a *linear* action. In the remaining case where the group acts nonlinearly, we show that the action is given by the composition of a linear action with a relatively simple (nonlinear) translation.

2. Equivalence transformations

The equivalence transformations for

$$u_t = \big(k(u)u_x\big)_x,$$

$k(u)$ not a constant, are given by

$$\bar{t} = at + e, \qquad \bar{x} = bx + f, \qquad \bar{u} = cu + g, \bar{k} = \frac{b^2}{a}k$$

where $abc \neq 0$ [10,16,15,14]. Up to these transformations, the classification of the Lie point symmetries breaks into four cases: the generic case, $k = e^u$, $k = u^\sigma$ with $\sigma \neq 0, -\frac{4}{3}$, and $k = u^{-\frac{4}{3}}$. The first three cases lead to solvable Lie algebras and will be studied in Section 3. The last case leads to an algebra with an $\mathfrak{sl}(2,\mathbb{R})$ component and will be examined in Section 4. All cases end up producing a global action of the corresponding Lie group on a natural class of functions. In all but the second case, the resulting global action is the restriction of a linear action.

3. Solvable cases

3.1. Generic case

In the generic case, the symmetry Lie algebra is three-dimensional and spanned by

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}.$$

As to be expected, this case is particularly simple.

To find a globalization of the corresponding local one-parameter actions, consider the solvable group $G_1 \cong \mathbb{R}^+ \rtimes \mathbb{R}^2$ given by

$$G_1 = \left\{ \begin{pmatrix} r & 0 & v \\ 0 & r^2 & w \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, r, v, w \in \mathbb{R},\ r > 0 \right\}$$

along with the subgroups

$$D_1 = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, r \in \mathbb{R},\ r > 0 \right\},$$
$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, t \in \mathbb{R} \right\}.$$

Define the trivial character $\chi_1 : D_1 \to \mathbb{C}^\times$ by

$$\chi_1\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 1$$

and consider the representation of G_1

$$\operatorname{Ind}_{D_1}^{G_1} \chi_1 = \big\{\varphi \in C^\infty(G_1) \,\big|\, \varphi(gd) = \chi_1(d)^{-1}\varphi(g) \text{ for } g \in G_1,\ d \in D_1\big\}$$

with G_1-action given by

$$(g_1 \cdot f)(g_2) = f\big(g_1^{-1}g_2\big)$$

for $g_i \in G_1$. Using what would be called the noncompact picture if we were working in the semisimple case [12], let

$$\mathcal{I}_1 = \left\{ f \in C^\infty(\mathbb{R}^2) \;\middle|\; f(x,t) = \varphi\left(\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right) \text{ for some } \varphi \in \operatorname{Ind}_{D_1}^{G_1} \chi_1 \right\}.$$

By requiring that the map $\varphi \to f$ be an intertwining operator, $\mathcal{I}_1$ inherits an action of G_1 so that $\mathcal{I}_1 \cong \operatorname{Ind}_{D_1}^{G_1} \chi_1$. Writing

$$\begin{pmatrix} r & 0 & x \\ 0 & r^2 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that φ can be reconstructed from f by

$$\begin{aligned} \varphi\left(\begin{pmatrix} r & 0 & x \\ 0 & r^2 & t \\ 0 & 0 & 1 \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= f(x,t). \end{aligned}$$

It follows that $\mathcal{I}_1 = C^\infty(\mathbb{R}^2)$ in this case. Of course it is possible to move out of the smooth category by studying L^2-functions, though we do not pursue such ideas here.

Theorem 1. *The (linear) action of G_1 on $\mathcal{I}_1$ is given by*

$$\left(\begin{pmatrix} r & 0 & v \\ 0 & r^2 & w \\ 0 & 0 & 1 \end{pmatrix} \cdot f\right)(x,t) = f\left(\frac{x-v}{r}, \frac{t-w}{r^2}\right).$$

Proof. Observe that

$$\begin{pmatrix} r & 0 & v \\ 0 & r^2 & w \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{v-x}{r} \\ 0 & 1 & \frac{t-w}{r^2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \left(\begin{pmatrix} r & 0 & v \\ 0 & r^2 & w \\ 0 & 0 & 1 \end{pmatrix} \cdot f\right)(x,t) &= \varphi\left(\begin{pmatrix} r & 0 & v \\ 0 & r^2 & w \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \varphi\left(\begin{pmatrix} 1 & 0 & \frac{x-v}{r} \\ 0 & 1 & \frac{t-w}{r^2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= f\left(\frac{x-v}{r}, \frac{t-w}{r^2}\right) \end{aligned}$$

where φ corresponds to f under the isomorphism $\mathcal{I}_1 \cong \operatorname{Ind}_{D_1}^{G_1} \chi_1$. □

Corollary 1. *The linear action of G_1 on $\mathcal{I}_1$ gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear heat equation $u_t = (k(u)u_x)_x$ in the generic case.*

Proof. Let

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be a basis for the Lie algebra of G_1. Using Theorem 1, it follows that

$$\begin{aligned} &(e^{sR} \cdot f)(x,t) = f(e^{-s}x, e^{-2s}t), \\ &(e^{sV} \cdot f)(x,t) = f(x-s,t), \\ &(e^{sW} \cdot f)(x,t) = f(x,t-s). \end{aligned}$$

Applying $\frac{\partial}{\partial s}|_{s=0}$ shows that R, V, W act on $\mathcal{I}_1$ by the differential operators

$$R = -x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t},$$
$$V = -\frac{\partial}{\partial x},$$
$$W = -\frac{\partial}{\partial t}.$$

Under the prolongation formalism, an easy application of the chain rule shows that the vector field Lie point symmetry

$$h_1(x,t)\frac{\partial}{\partial x} + h_2(x,t)\frac{\partial}{\partial t}$$

on $\mathbb{R}^2 \times \mathbb{R}$ corresponds to the differential operator

$$-h_1(x,t)\frac{\partial}{\partial x} - h_2(x,t)\frac{\partial}{\partial t}$$

on $C^1(\mathbb{R}^2)$. Therefore the one-parameter groups corresponding to $\{R, V, W\}$ give rise to the symmetry vector fields

$$X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_1 = \frac{\partial}{\partial t}.$$

Since the Lie point symmetries of $u_t = (k(u)u_x)_x$ are spanned by $\{X_1, X_2, X_3\}$ in the generic case, we are done. □

3.2. $k = e^u$

In this case, the symmetry Lie algebra is four-dimensional and spanned by

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},$$
$$X_4 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}.$$

To find a globalization of these local one-parameter actions, consider the solvable group $G_2 \cong (\mathbb{R}^+)^2 \rtimes \mathbb{R}^2$ given by

$$G_2 = \left\{ \begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, r_i, v, w \in \mathbb{R},\ r_i > 0 \right\}$$

along with the subgroups

$$D_2 = \left\{ \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, r_i \in \mathbb{R},\ r_i > 0 \right\},$$
$$N_2 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, t \in \mathbb{R} \right\}.$$

Define the trivial character $\chi_2 : D_2 \to \mathbb{C}^\times$ by

$$\chi_2\left(\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 1$$

and consider the representation of G_2

$$\operatorname{Ind}_{D_2}^{G_2} \chi = \left\{ \varphi \in C^\infty(G_2) \,\middle|\, \varphi(gd) = \chi_2(d)^{-1}\varphi(g) \text{ for } g \in G_2,\ d \in D_2 \right\}$$

with G_2-action given by $(g_1 \cdot f)(g_2) = f(g_1^{-1} g_2)$ for $g_i \in G_2$. Using what would be called the noncompact picture if we were working in the semisimple case, let

$$\mathcal{I}_2 = \left\{ f \in C^\infty(\mathbb{R}^2) \;\middle|\; f(x,t) = \varphi\left(\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right) \text{ for some } \varphi \in \operatorname{Ind}_{D_2}^{G_2} \chi_2 \right\}.$$

By requiring that the map $\varphi \to f$ be an intertwining operator, $\mathcal{I}_2$ inherits an action of G_2 so that $\mathcal{I}_2 \cong \operatorname{Ind}_{D_2}^{G_2} \chi_2$. Writing

$$\begin{pmatrix} r_1 & 0 & x \\ 0 & r_2 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that φ can be reconstructed from f by

$$\begin{aligned} \varphi\left(\begin{pmatrix} r_1 & 0 & x \\ 0 & r_2 & t \\ 0 & 0 & 1 \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= f(x,t). \end{aligned}$$

It follows that $\mathcal{I}_2 = C^\infty(\mathbb{R}^2)$ in this case.

Theorem 2. *The (linear) action of G_2 on $\mathcal{I}_2$ is given by*

$$\left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \cdot f\right)(x,t) = f\left(\frac{x-v}{r_1}, \frac{t-w}{r_2}\right).$$

Proof. Observe that

$$\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{v-x}{r_1} \\ 0 & 1 & \frac{t-w}{r_2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1^{-1} & 0 & 0 \\ 0 & r_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \cdot f\right)(x,t) &= \varphi\left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \varphi\left(\begin{pmatrix} 1 & 0 & -\frac{v-x}{r_1} \\ 0 & 1 & \frac{t-w}{r_2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1^{-1} & 0 & 0 \\ 0 & r_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= f\left(\frac{x-v}{r_1}, \frac{t-w}{r_2}\right) \end{aligned}$$

where φ corresponds to f under the isomorphism $\mathcal{I}_2 \cong \operatorname{Ind}_{D_2}^{G_2} \chi_2$. $\square$

To complete our picture, let

$$\left(\tau\left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot f\right)(x,t) = f(x,t) + \ln\left(\frac{r_1^2}{r_2}\right).$$

This defines a (nonlinear) action of G_2 on $\mathcal{I}_2$ since

$$\left(\tau\left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 & 0 & v' \\ 0 & s_2 & w' \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot f\right)(x,t) = f(x,t) + \ln\left(\frac{r_1^2 s_1^2}{r_2 s_2}\right)$$

and

$$\left(\tau\left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot \left(\tau\left(\begin{pmatrix} s_1 & 0 & v' \\ 0 & s_2 & w' \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot f\right)\right)(x,t)$$
$$= \left(\tau\left(\begin{pmatrix} s_1 & 0 & v' \\ 0 & s_2 & w' \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot f\right)(x,t) + \ln\left(\frac{r_1^2}{r_2}\right)$$
$$= f(x,t) + \ln\left(\frac{s_1^2}{s_2}\right) + \ln\left(\frac{r_1^2}{r_2}\right).$$

The key observation about this new action is that it commutes with our original action of G_2 on $\mathcal{I}_2$ given in Theorem 2. In other words,

$$g_1 \cdot \left(\tau(g_2) \cdot f\right) = \tau(g_2) \cdot (g_1 \cdot f)$$

for $g_i \in G_2$ and $f \in \mathcal{I}_2$ since

$$\left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \cdot \tau\left(\begin{pmatrix} s_1 & 0 & v' \\ 0 & s_2 & w' \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot f\right)(x,t) = \left(\tau\left(\begin{pmatrix} s_1 & 0 & v' \\ 0 & s_2 & w' \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot f\right)\left(\frac{x-v}{r_1}, \frac{t-w}{r_2}\right)$$
$$= f\left(\frac{x-v}{r_1}, \frac{t-w}{r_2}\right) + \ln\left(\frac{s_1^2}{s_2}\right)$$

and

$$\left(\tau\left(\begin{pmatrix} s_1 & 0 & v' \\ 0 & s_2 & w' \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \cdot f\right)(x,t) = \left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \cdot f\right)(x,t) + \ln\left(\frac{s_1^2}{s_2}\right)$$
$$= f\left(\frac{x-v}{r_1}, \frac{t-w}{r_2}\right) + \ln\left(\frac{s_1^2}{s_2}\right).$$

As a result, we get an action of $G_2 \times G_2$ on $\mathcal{I}_2$ given by $(g_1, g_2) \cdot f = g_1 \cdot \tau(g_2) \cdot f$. Under the diagonal map $\Delta : G \to G \times G$, we therefore get a composite action of G_2 on $\mathcal{I}_2$ given by

$$\delta(g) \cdot f = g \cdot \tau(g) \cdot f.$$

Explicitly, we see that

$$\left(\delta\left(\begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot f\right)(x,t) = f\left(\frac{x-v}{r_1}, \frac{t-w}{r_2}\right) + \ln\left(\frac{r_1^2}{r_2}\right). \tag{3.1}$$

Corollary 2. *The nonlinear action of G_2 on $\mathcal{I}_2$ given by δ gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear heat equation $u_t = (e^u u_x)_x$.*

Proof. Let

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be a basis for the Lie algebra of G_2. Using Eq. (3.1), it follows that

$$\left(e^{sR_1} \cdot f\right)(x,t) = f\left(e^{-s}x, t\right) + 2s,$$
$$\left(e^{sR_2} \cdot f\right)(x,t) = f\left(x, e^{-s}t\right) - s,$$
$$\left(e^{sV} \cdot f\right)(x,t) = f(x-s,t),$$
$$\left(e^{sW} \cdot f\right)(x,t) = f(x,t-s).$$

Applying $\frac{\partial}{\partial s}|_{s=0}$ shows that

$$\frac{\partial}{\partial s}\big(e^{sR_1}\cdot f\big)(x,t)|_{s=0} = -x\frac{\partial}{\partial x}f(x,t) + 2,$$
$$\frac{\partial}{\partial s}\big(e^{sR_2}\cdot f\big)(x,t)|_{s=0} = -t\frac{\partial}{\partial t}f(x,t) - 1,$$
$$\frac{\partial}{\partial s}\big(e^{sV}\cdot f\big)(x,t)|_{s=0} = -\frac{\partial}{\partial x}f(x,t),$$
$$\frac{\partial}{\partial s}\big(e^{sW}\cdot f\big)(x,t)|_{s=0} = -\frac{\partial}{\partial t}f(x,t).$$

Under the prolongation formalism, an easy application of the chain rule shows that the vector field Lie point symmetry

$$h_1(x,t)\frac{\partial}{\partial x} + h_2(x,t)\frac{\partial}{\partial t} + h_3(x,t,u)\frac{\partial}{\partial u}$$

on $\mathbb{R}^2 \times \mathbb{R}$ gives rise to a local one-parameter group action on f whose partial with respect to s at $s=0$ is given by

$$-h_1(x,t)\frac{\partial}{\partial x}f(x,t) - h_2(x,t)\frac{\partial}{\partial t}f(x,t) + h_3\big(x,t,f(x,t)\big).$$

Therefore the one parameter groups corresponding to $\{R_1, R_2, V, W\}$ give rise to the symmetry vector fields

$$X_3 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u},$$
$$\frac{1}{2}X_3 - \frac{1}{2}X_4 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_1 = \frac{\partial}{\partial t}.$$

Since the Lie point symmetries of $u_t = (e^u u_x)_x$ are spanned by $\{X_1, X_2, X_3, X_4\}$, we are done. □

3.3. $k = u^\sigma$, $\sigma \neq 0, -\frac{4}{3}$

In this case, the symmetry Lie algebra is four-dimensional and spanned by

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},$$
$$X_4 = \frac{\sigma}{2}x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

To find a globalization of these local one-parameter actions, consider the solvable group $G_3 \cong (\mathbb{R}^+)^2 \rtimes \mathbb{R}^2$

$$G_3 = \left\{ \begin{pmatrix} r_1 & 0 & v \\ 0 & r_2 & w \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, r_i, v, w \in \mathbb{R},\ r_i > 0 \right\}$$

(here $G_3 = G_2$ though we keep separate notation for the sake of clarity) along with the subgroups

$$D_3 = \left\{ \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, r_i \in \mathbb{R},\ r_i > 0 \right\},$$
$$N_3 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, t \in \mathbb{R} \right\}.$$

Define the character $\chi_3 : D_3 \to \mathbb{C}^\times$ by

$$\chi\left(\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \left(\frac{r_1^2}{r_2}\right)^{\frac{1}{\sigma}}$$

and consider the representation of G_3

$$\operatorname{Ind}_{D_3}^{G_3}\chi_3=\left\{\varphi\in C^\infty(G)\;\middle|\;\varphi(gd)=\chi_3(d)^{-1}\varphi(g)\text{ for }g\in G_3,\ d\in D_3\right\}$$

with G_3-action given by $(g_1\cdot f)(g_2)=f(g_1^{-1}g_2)$ for $g_i\in G_3$. Using what would be called the noncompact picture if we were working in the semisimple case, let

$$\mathcal{I}_3=\left\{f\in C^\infty(\mathbb{R}^2)\;\middle|\;f(x,t)=\varphi\left(\begin{pmatrix}1&0&x\\0&1&t\\0&0&1\end{pmatrix}\right)\text{ for some }\varphi\in\operatorname{Ind}_{D_3}^{G_3}\chi_3\right\}.$$

By requiring that the map $\varphi\to f$ be an intertwining operator, $\mathcal{I}_3$ inherits an action of G_3 so that $\mathcal{I}_3\cong\operatorname{Ind}_{D_3}^{G_3}\chi_3$. Writing

$$\begin{pmatrix}r_1&0&x\\0&r_2&t\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&x\\0&1&t\\0&0&1\end{pmatrix}\begin{pmatrix}r_1&0&0\\0&r_2&0\\0&0&1\end{pmatrix},$$

we see that φ can be reconstructed from f by

$$\begin{aligned}\varphi\left(\begin{pmatrix}r_1&0&x\\0&r_2&t\\0&0&1\end{pmatrix}\right)&=\varphi\left(\begin{pmatrix}1&0&x\\0&1&t\\0&0&1\end{pmatrix}\begin{pmatrix}r_1&0&0\\0&r_2&0\\0&0&1\end{pmatrix}\right)\\&=r_1^{-\frac{2}{\sigma}}r_2^{\frac{1}{\sigma}}f(x,t).\end{aligned}$$

It follows that $\mathcal{I}_3=C^\infty(\mathbb{R}^2)$ in this case.

Theorem 3. *The (linear) action of G_3 on $\mathcal{I}_3$ is given by*

$$\left(\begin{pmatrix}r_1&0&v\\0&r_2&w\\0&0&1\end{pmatrix}\cdot f\right)(x,t)=r_1^{\frac{2}{\sigma}}r_2^{-\frac{1}{\sigma}}f\left(\frac{x-v}{r_1},\frac{t-w}{r_2}\right).$$

Proof. Observe that

$$\begin{pmatrix}r_1&0&v\\0&r_2&w\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&x\\0&1&t\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&-\frac{v-x}{r_1}\\0&1&\frac{t-w}{r_2}\\0&0&1\end{pmatrix}\begin{pmatrix}r_1^{-1}&0&0\\0&r_2^{-1}&0\\0&0&1\end{pmatrix}.$$

Thus

$$\begin{aligned}\left(\begin{pmatrix}r_1&0&v\\0&r_2&w\\0&0&1\end{pmatrix}\cdot f\right)(x,t)&=\varphi\left(\begin{pmatrix}r_1&0&v\\0&r_2&w\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&x\\0&1&t\\0&0&1\end{pmatrix}\right)\\&=\varphi\left(\begin{pmatrix}1&0&-\frac{v-x}{r_1}\\0&1&\frac{t-w}{r_2}\\0&0&1\end{pmatrix}\begin{pmatrix}r_1^{-1}&0&0\\0&r_2^{-1}&0\\0&0&1\end{pmatrix}\right)\\&=r_1^{\frac{2}{\sigma}}r_2^{-\frac{1}{\sigma}}f\left(\frac{x-v}{r_1},\frac{t-w}{r_2}\right)\end{aligned}$$

where φ corresponds to f under the isomorphism $\mathcal{I}_3\cong\operatorname{Ind}_{D_3}^{G_3}\chi_3$. □

Corollary 3. *The linear action of G_3 on $\mathcal{I}_3$ gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear heat equation $u_t=(u^\sigma u_x)_x$.*

Proof. Let

$$R_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad R_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad V=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad\text{and}\quad W=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

be a basis for the Lie algebra of G_3. Using Theorem 3, it follows that

$$\big(e^{sR_1}\cdot f\big)(x,t)=e^{\frac{2s}{\sigma}}f\big(e^{-s}x,t\big),$$
$$\big(e^{sR_2}\cdot f\big)(x,t)=e^{-\frac{s}{\sigma}}f\big(x,e^{-s}t\big),$$
$$\big(e^{sV}\cdot f\big)(x,t)=f(x-s,t),$$
$$\big(e^{sW}\cdot f\big)(x,t)=f(x,t-s).$$

Applying $\frac{\partial}{\partial s}|_{s=0}$ shows that R_1, R_2, V, W act on $\mathcal{I}$ by the differential operators

$$R_1=-x\frac{\partial}{\partial x}+\frac{2}{\sigma},$$
$$R_2=-t\frac{\partial}{\partial t}-\frac{1}{\sigma},$$
$$V=-\frac{\partial}{\partial x},$$
$$W=-\frac{\partial}{\partial t}.$$

Under the prolongation formalism, an easy application of the chain rule shows that the Lie symmetry vector field

$$h_1(x,t)\frac{\partial}{\partial x}+h_2(x,t)\frac{\partial}{\partial t}+h_3(x,t)u\frac{\partial}{\partial u}$$

on $\mathbb{R}^2\times\mathbb{R}$ corresponds to the differential operator

$$-h_1(x,t)\frac{\partial}{\partial x}-h_2(x,t)\frac{\partial}{\partial t}+h_3(x,t)$$

on $C^1(\mathbb{R}^2)$. Therefore the one-parameter groups corresponding to $\{R_1, R_2, V, W\}$ give rise to the symmetry vector fields

$$\frac{2}{\sigma}X_4=x\frac{\partial}{\partial x}+\frac{2}{\sigma}u\frac{\partial}{\partial u},$$
$$\frac{1}{2}X_3-\frac{1}{\sigma}X_4=t\frac{\partial}{\partial t}-\frac{1}{\sigma}u\frac{\partial}{\partial u},$$
$$X_2=\frac{\partial}{\partial x},$$
$$X_1=\frac{\partial}{\partial t}.$$

Since the Lie point symmetries of $u_t=(u^\sigma u_x)_x$ are spanned by $\{X_1, X_2, X_3, X_4\}$, we are done. □

4. Nonsolvable case, $k=u^{-\frac{4}{3}}$

Let $A\cong\mathbb{R}^+\rtimes\mathbb{R}$ be the group of orientation preserving affine motions of $\mathbb{R}$ realized as

$$A=\left\{\begin{pmatrix} r & v\\ 0 & 1\end{pmatrix}\ \middle|\ r,v\in\mathbb{R},\ r>0\right\}$$

and consider the group

$$G=SL(2,\mathbb{R})\times A.$$

Define the subgroup $Q^+\subseteq G$ by

$$Q^+=\left\{\left(\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix},\begin{pmatrix} r & 0\\ 0 & 1\end{pmatrix}\right)\right\}$$

and the subgroup $N^-\subseteq G$ by

$$N^-=\left\{\left(\begin{pmatrix} 1 & 0\\ x & 1\end{pmatrix},\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}\right)\right\}.$$

Define the character $\chi : Q^+\to\mathbb{C}^\times$ by

$$\chi\left(\left(\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix},\begin{pmatrix} r & 0\\ 0 & 1\end{pmatrix}\right)\right)=a^3r^{\frac{3}{4}}$$

and consider the representation of G

$$\operatorname{Ind}_{Q^+}^G \chi = \{\varphi \in C^\infty(G) \mid \varphi(gq) = \chi(q)^{-1}\varphi(g) \text{ for } g \in G,\ q \in Q^+\}$$

with G-action given by $(g_1 \cdot f)(g_2) = f(g_1^{-1}g_2)$ for $g_i \in G$. Using what would be called the noncompact picture if we were working in the semisimple case, let

$$\mathcal{I} = \left\{ f \in C^\infty(\mathbb{R}^2) \;\middle|\; f(x,t) = \varphi\left(\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right)\right) \text{ for some } \varphi \in \operatorname{Ind}_{Q^+}^G \chi \right\}.$$

By requiring that the map $\varphi \to f$ be an intertwining operator, $\mathcal{I}$ inherits an action of G so that $\mathcal{I} \cong \operatorname{Ind}_{Q^+}^G \chi$.

By writing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$
$$\begin{pmatrix} r & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$$

when $a \neq 0$, we see that φ can be reconstructed from f by

$$\begin{aligned} \varphi\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} r & v \\ 0 & 1 \end{pmatrix}\right)\right) &= \varphi\left(\left(\begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &= a^{-3} r^{-\frac{3}{4}} f\left(\frac{c}{a}, v\right). \end{aligned}$$

As a result, we see that $\mathcal{I} \subsetneq C^\infty(\mathbb{R}^2)$. For instance since $\left(\begin{smallmatrix} \frac{1}{x} & -1 \\ 1 & 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as $x \to \pm\infty$, any $f \in \mathcal{I}$ must satisfy

$$\lim_{x \to -\infty} x^3 f(x,t) = \lim_{x \to \infty} x^3 f(x,t).$$

In particular, $f(x,t)$ decays as least as fast as $\frac{1}{x^3}$ when $x \to \pm\infty$.

Theorem 4. *The* (*linear*) *action of* G *on* $\mathcal{I}$ *is given by*

$$\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} r & v \\ 0 & 1 \end{pmatrix}\right) \cdot f\right)(x,t) = (d-bx)^{-3} r^{\frac{3}{4}} f\left(\frac{ax-c}{d-bx}, \frac{-v+t}{r}\right)$$

whenever $d - bx \neq 0$.

Proof. By straightforward matrix multiplication,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{ax-c}{d-bx} & 1 \end{pmatrix} \begin{pmatrix} d-bx & -b \\ 0 & (d-bx)^{-1} \end{pmatrix}$$

when $d - bx \neq 0$ and

$$\begin{pmatrix} r & v \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-v+t}{r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} &\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} r & v \\ 0 & 1 \end{pmatrix}\right) \cdot f\right)(x,t) \\ &\quad = \varphi\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} r & v \\ 0 & 1 \end{pmatrix}\right)^{-1} \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right)\right) \\ &\quad = \varphi\left(\left(\begin{pmatrix} 1 & 0 \\ \frac{ax-c}{d-bx} & 1 \end{pmatrix}\begin{pmatrix} d-bx & -b \\ 0 & (d-bx)^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \frac{-v+t}{r} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &\quad = (d-bx)^{-3} r^{\frac{3}{4}} f\left(\frac{ax-c}{d-bx}, \frac{-v+t}{r}\right) \end{aligned}$$

where φ corresponds to f under the isomorphism $\mathcal{I} \cong \operatorname{Ind}_{Q^+}^G \chi$. $\square$

Corollary 4. *The linear action of G on $\mathcal{I}$ gives a globalization of the local one-parameter group action generated by the Lie point symmetries of the nonlinear heat equation $u_t = (u^{-\frac{4}{3}} u_x)_x$.*

Proof. We use Theorem 4 to calculate the various one parameter group actions. Beginning with the action of $SL(2, \mathbb{R})$, write $\{H, E, F\}$ for the standard $\mathfrak{sl}_2$-triple in $\mathfrak{sl}(2, \mathbb{R})$. We see that

$$\begin{aligned}
\big(e^{sH} \cdot f\big)(x, t) &= e^{3s} f\big(e^{2s}x, t\big), \\
\big(e^{sE} \cdot f\big)(x, t) &= (1 - sx)^{-3} f\left(\frac{x}{1 - sx}, t\right), \\
\big(e^{sF} \cdot f\big)(x, t) &= f(x - s, t).
\end{aligned}$$

Applying $\frac{d}{ds}|_{s=0}$, we see that $\{H, E, F\}$ act on $\mathcal{I}$ by the differential operators

$$\begin{aligned}
H &= 2x\frac{\partial}{\partial x} + 3, \\
E &= x^2\frac{\partial}{\partial x} + 3x, \\
F &= -\frac{\partial}{\partial x}.
\end{aligned}$$

Under the prolongation formalism, an easy application of the chain rule shows that the Lie symmetry vector field

$$h_1(x, t)\frac{\partial}{\partial x} + h_2(x, t)\frac{\partial}{\partial t} + h_3(x, t)u\frac{\partial}{\partial u}$$

on $\mathbb{R}^2 \times \mathbb{R}$ corresponds to the differential operator

$$-h_1(x, t)\frac{\partial}{\partial x} - h_2(x, t)\frac{\partial}{\partial t} + h_3(x, t)$$

on $C^1(\mathbb{R}^2)$. Therefore the one parameter groups corresponding to $\{H, E, F\}$ give rise to the symmetry vector fields

$$\begin{aligned}
\frac{1}{3}X_4 &= -2x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}, \\
X_5 &= -x^2\frac{\partial}{\partial x} + 3xu\frac{\partial}{\partial u}, \\
X_2 &= \frac{\partial}{\partial x}.
\end{aligned}$$

Turning to the action of A, let $R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ be a basis for the Lie algebra of A. It follows that

$$\begin{aligned}
\big(e^{sR} \cdot f\big)(x, t) &= e^{\frac{3}{4}s} f\big(x, e^{-s}t\big), \\
\big(e^{sV} \cdot f\big)(x, t) &= f(x, t - s).
\end{aligned}$$

Therefore $\{R, V\}$ act on $\mathcal{I}$ by the differential operators

$$\begin{aligned}
R &= -t\frac{\partial}{\partial t} + \frac{3}{4}, \\
V &= -\frac{\partial}{\partial t}.
\end{aligned}$$

Under the prolongation formalism, this gives rise to the symmetry vector fields

$$\begin{aligned}
\frac{1}{2}X_3 + \frac{3}{4}X_4 &= t\frac{\partial}{\partial t} + \frac{3}{4}\frac{\partial}{\partial u}, \\
X_1 &= \frac{\partial}{\partial t}.
\end{aligned}$$

Since the Lie point symmetries of $u_t = (u^{-\frac{4}{3}} u_x)_x$ are spanned by $\{X_1, X_2, X_3, X_4, X_5\}$, we are done. □

References

[1] G.W. Bluman, J.D. Cole, Similarity Methods for Differential Equations, Springer, Berlin, 1974.
[2] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Springer, Berlin, 1989.
[3] P. Broadbridge, D. Arrigo, All solutions of standard symmetric linear partial differential equations have classical Lie symmetry, J. Math. Anal. Appl. 234 (1) (1999) 109–122.
[4] R. Cherniha, M. Serov, I. Rassokha, Lie symmetries and form-preserving transformations of reaction–diffusion–convection equations, J. Math. Anal. Appl. 342 (2) (2008) 1363–1379.
[5] M. Craddock, The symmetry groups of linear partial differential equations and representation theory, I, J. Differential Equations 116 (1) (1995) 202–247.
[6] M. Craddock, Symmetry groups of linear partial differential equations and representation theory: The Laplace and axially symmetric wave equations, J. Differential Equations 166 (1) (2000) 107–131.
[7] L. Dresner, Similarity Solutions of Nonlinear Partial Differential Equations, Pitman, London, 1983.
[8] J.M. Hill, Differential Equations and Group Methods for Scientists and Engineers, CRC Press, Boca Raton, FL, 1992.
[9] M. Hunziker, M.R. Sepanski, R. Stanke, The minimal representation of the conformal group and classic solutions to the wave equation, submitted for publication, arXiv:0901.2280v1.
[10] N.H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, vol. 1, CRC Press, Boca Raton, FL, 1994.
[11] S. Lie, Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen, Arch. Math. 6 (1881) 328–368.
[12] A. Knapp, Representation Theory of Semisimple Groups, An Overview Based on Examples, Princeton University Press, Princeton, 1986.
[13] S. Lie, Vorlesungen über Differentialgleichungen mit bekkanten infinitesimalen Transformationen, Teubner, Leipzig, 1891.
[14] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer, New York, 1993.
[15] L.V. Osviannikov, Group Analysis of Differential Equations, Nauka, Moscow, 1978; English translation by W.F. Ames, Academic Press, New York, 1982.
[16] L.V. Osviannikov, Group properties of nonlinear heat equation, Dokl. Akad. Nauk SSSR 125 (3) (1959) 492.
[17] L.V. Ovsiannikov, Gruppovye svoystva differentsialnyh uravneniy (Group properties of differential equations), Soviet Academy of Sciences, Novosibirsk, 1962; English translation by G.W. Bluman, 1967.
[18] C. Rogers, W.F. Ames, Nonlinear Boundary Value Problems in Science and Engineering, Academic Press, New York, 1989.
[19] M.R. Sepanski, R. Stanke, On global $SL(2,\mathbb{R})$ symmetries of differential operators, J. Funct. Anal. 224 (1) (2005) 1–21.
[20] M.R. Sepanski, R. Stanke, Symmetry analysis of the heat and Schrödinger equations, in preparation.
[21] H. Stephani, Differential Equations, Their Solution Using Symmetries, Cambridge University Press, Cambridge, UK, 1989.