Note

Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, II

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Received 9 October 2005  
Available online 20 February 2006  
Submitted by L. Grafakos

Abstract

In [S.G. Samko, B.G. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, J. Math. Anal. Appl. 310 (2005) 229–246], Sobolev-type \( p(\cdot) \rightarrow q(\cdot) \)-theorems were proved for the Riesz potential operator \( I^\alpha \) in the weighted Lebesgue generalized spaces \( L^{p(\cdot)}(\mathbb{R}^n, \rho) \) with the variable exponent \( p(x) \) and a two-parameter power weight fixed to an arbitrary finite point \( x_0 \) and to infinity, under an additional condition relating the weight exponents at \( x_0 \) and at infinity. We show in this note that those theorems are valid without this additional condition. Similar theorems for a spherical analogue of the Riesz potential operator in the corresponding weighted spaces \( L^{p(\cdot)}(\mathbb{S}^n, \rho) \) on the unit sphere \( \mathbb{S}^n \) in \( \mathbb{R}^{n+1} \) are also improved in the same way.

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Keywords: Weighted Lebesgue spaces; Variable exponent; Riesz potentials; Spherical potentials; Stereographical projection

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doi:10.1016/j.jmaa.2006.01.069
1. Introduction

We consider the Riesz potential operator

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad 0 < \alpha < n,$$

(1.1)

in the weighted Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n, \rho)$ with a variable exponent $p(x)$ defined by the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n, \rho)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \rho(x) \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\},$$

(1.2)

where

$$\rho(x) = \rho_{\gamma_0, \gamma_\infty}(x) = |x|^\gamma_0 (1 + |x|)^{\gamma_\infty - \gamma_0}.$$  

(1.3)

We refer to [3–6] for the basics of the spaces $L^{p(\cdot)}$ with variable exponent.

We assume that the exponent $p(x)$ satisfies the standard conditions

$$1 < p_- \leq p(x) \leq p_+ < \infty, \quad x \in \mathbb{R}^n,$$

(1.4)

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^n,$$

(1.5)

and also the following condition at infinity

$$|p_+(x) - p_+(y)| \leq \frac{A_\infty}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^n,$$

(1.6)

where $p_+(x) = p\left(\frac{x}{|x|^p}\right)$. Conditions (1.5) and (1.6) taken together are equivalent to the following global condition:

$$|p(x) - p(y)| \leq C \frac{\sqrt{\ln(1 + |x|^2) + \ln(1 + |y|^2)}}{|x-y|}, \quad x, y \in \mathbb{R}^n.$$

(1.7)

Let

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

The following statement was proved in [8].

**Theorem 1.1.** Under assumptions (1.4)–(1.6) and the condition

$$p_+ < \frac{n}{\alpha},$$

(1.8)

the operator $I^\alpha$ is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n, \rho_{\gamma_0, \gamma_\infty})$ into the space $L^{q(\cdot)}(\mathbb{R}^n, \rho_{\mu_0, \mu_\infty})$, where

$$\mu_0 = \frac{q(0)}{p(0)} \gamma_0 \quad \text{and} \quad \mu_\infty = \frac{q(\infty)}{p(\infty)} \gamma_\infty,$$

(1.9)

if

$$\alpha p(0) - n < \gamma_0 < n\left[ p(0) - 1 \right], \quad \alpha p(\infty) - n < \gamma_\infty < n\left[ p(\infty) - 1 \right],$$

(1.10)
and the exponents $\gamma_0$ and $\gamma_\infty$ are related to each other by the equality
\[
\frac{q(0)}{p(0)}\gamma_0 + \frac{q(\infty)}{p(\infty)}\gamma_\infty = \frac{q(\infty)}{p(\infty)}[(n + \alpha)p(\infty) - 2n].
\] (1.11)

The goal of this note is to prove that Theorem 1.1 is valid without the additional condition (1.11). We consider also a similar statement for the spherical potential operators
\[
(K^\alpha f)(x) = \int_{S_n} \frac{f(\sigma)}{|x - \sigma|^{n-\alpha}} d\sigma, \quad x \in S_n, \ 0 < \alpha < n,
\] (1.12)
in the corresponding weighted spaces $L^p(\cdot)(S^n, \rho)$ on the unit sphere $S^n$ in $\mathbb{R}^{n+1}$.

2. Preliminaries

We need the following theorem for bounded domains proved in [7].

**Theorem 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $x_0 \in \overline{\Omega}$ and let $p(x)$ satisfy conditions (1.4), (1.5) and (1.8) in $\Omega$. Then the following estimate
\[
\| I^\alpha f \|_{L^q(\Omega, |x-x_0|^\gamma)} \leq C \| f \|_{L^p(\Omega, |x-x_0|^{\mu})}
\] (2.1)
is valid, if
\[
\alpha p(x_0) - n < \gamma < n\left[p(x_0) - 1\right]
\] (2.2)
and
\[
\mu \geq \frac{q(x_0)}{p(x_0)}\gamma.
\] (2.3)

3. The case of the spatial potential operator

We prove the following theorem.

**Theorem A.** Under assumptions (1.4)–(1.6) and (1.8), the operator $I^\alpha$ is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n, \rho_{\gamma_0,\gamma_\infty})$ into the space $L^{q(\cdot)}(\mathbb{R}^n, \rho_{\mu_0,\mu_\infty})$, where
\[
\mu_0 = \frac{q(0)}{p(0)}\gamma_0 \quad \text{and} \quad \mu_\infty = \frac{q(\infty)}{p(\infty)}\gamma_\infty, \quad \mu_0 = \frac{q(0)}{p(0)}\gamma_0 \quad \text{and} \quad \mu_\infty = \frac{q(\infty)}{p(\infty)}\gamma_\infty,
\] (3.1)
if
\[
\alpha p(0) - n < \gamma_0 < n\left[p(0) - 1\right], \quad \alpha p(\infty) - n < \gamma_\infty < n\left[p(\infty) - 1\right].
\] (3.2)

**Proof.** Let $\| f \|_{L^{p(\cdot)}(\mathbb{R}^n, \rho)} \leq 1$. To estimate the integral $\int_{\mathbb{R}^n} \rho_{\mu_0,\mu_\infty}(x)|I^\alpha f(x)|^{q(x)} dx$, we split it, as in [8], in the following way:
\[
\int_{\mathbb{R}^n} \rho_{\mu_0,\mu_\infty}(x)|I^\alpha f(x)|^{q(x)} dx \leq c(A_{++} + A_{+-} + A_{-+} + A_{-}),
\]
\[ A_{++} = \int_{|x|<1} |x|^\mu_0 \left| \int_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx, \]
\[ A_{+-} = \int_{|x|<1} |x|^\mu_0 \left| \int_{|y|>1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx, \]
and
\[ A_{-+} = \int_{|x|>1} |x|^\mu_\infty \left| \int_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx, \]
\[ A_{--} = \int_{|x|>1} |x|^\mu_\infty \left| \int_{|y|>1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx. \]

The boundedness of the terms \( A_{++} \) and \( A_{--} \) was shown in [8] without condition (1.11). So we only have to treat the terms \( A_{+-} \) and \( A_{-+} \).

10. The term \( A_{-+} \). We split \( A_{-+} \) as
\[ A_{-+} = A_1 + A_2, \]
where
\[ A_1 = \int_{1<|x|<2} |x|^\mu_\infty \left| \int_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx \]
and
\[ A_2 = \int_{|x|>2} |x|^\mu_\infty \left| \int_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx. \]

The term
\[ A_1 \leq C \int_{1<|x|<2} |x|^\mu_0 \left| \int_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx \leq C \int_{|x|<2} |x|^\mu_0 \left| \int_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^q(x) \, dx \]
is covered by Theorem 2.1. For the term \( A_2 \) we have
\[ |x-y| \geq |x| - |y| \geq \frac{|x|}{2}, \]

Therefore,
\[ A_2 \leq C \int_{|x|>2} |x|^\mu_\infty + (\alpha-n)q(x) \left( \int_{|y|<1} |f(y)|\,dy \right)^q(x) \, dx. \]

It follows from condition (1.6) (see also (1.7)) that
\[ |p(x) - p(\infty)| \leq \frac{C}{\ln |x|}, \quad |x| \geq 2, \]
and then the same is valid for $q(x)$, so that

$$A_2 \leq C \int_{|x|>2} |x|^\mu + (\alpha - n)q(\infty) \left( \int_{|y|<1} |f(y)| \, dy \right)^{q(x)} \, dx.$$  

Observe that

$$\int_{|y|<1} |f(y)| \, dy \leq C \|f\|_{L^p(\mathbb{R}^n, \rho)}.$$  

(3.3)

Indeed, denote $g(y) = [\rho(y)]^{-1/p(y)}$; by the Hölder inequality for variable $L^p(\cdot)$-spaces we get

$$\int_{|y|<1} |f(y)| \, dy = \int_{|y|<1} g(y) [\rho(y)]^{1/p(y)} |f(y)| \, dy \leq k \|g\|_{L^{p'(\cdot)}} \|\rho^{1/p} f\|_{L^{p(\cdot)}} = k \|g\|_{L^{p'(\cdot)}} \|f\|_{L^{p(\cdot)(\mathbb{R}^n, \rho)}}.$$  

(3.4)

To arrive at (3.3), we have to show that

$$\|g\|_{L^{p'(\cdot)}} < \infty \iff \int_{|y|<1} |g(y)|^{p'(y)} \, dy < \infty.$$  

(3.5)

As is easily seen, the last integral is finite since $\gamma_0 < n[p(0) - 1]$. Therefore, from (3.4) there follows (3.3).

Then $A_2 \leq C < \infty$ if we take into account that $\mu + (\alpha - n)q(\infty) < -n$ under the condition $\gamma_\infty < n[p(\infty) - 1]$.

2. The term $A_{+\cdot}$ is estimated similarly to $A_{-\cdot}$: we split $A_{+\cdot}$ as

$$A_{+\cdot} = A_3 + A_4,$$

where

$$A_3 = \int_{|x|<1} |x|^{\mu_0} \int_{1<|y|<2} \frac{f(y) \, dy}{|x-y|^{n-\alpha}} \, dq(x) \, dx$$

and

$$A_4 = \int_{|x|<1} |x|^{\mu_0} \int_{|y|>2} \frac{f(y) \, dy}{|x-y|^{n-\alpha}} \, dq(x) \, dx.$$  

The term $A_3$ is covered by Theorem 2.1 similarly to the term $A_1$ in 1. For the term $A_4$, we have $|x-y| \geq |y| - |x| \geq |y|/2$. Then

$$\int_{|y|>2} \frac{f(y) \, dy}{|x-y|^{n-\alpha}} \leq C \int_{|y|>2} \frac{|f(y)| \, dy}{|y|^{n-\alpha}} = C \int_{|y|>2} \frac{|f_0(y)| \, dy}{|y|^{n-\alpha + \frac{n}{p(\infty)}}},$$

where $f_0(y) = |y|^{\frac{n}{p(\infty)}} f(y)$. It is easily seen that $f_0(y) \in L^{p(\cdot)}(\mathbb{R}^n \setminus B(0, 2))$, since $[\rho(y)]^{1/p(\cdot)} \times f(y) \in L^{p(\cdot)}(\mathbb{R}^n)$ and $[\rho(y)]^{1/p(\cdot)} \sim |y|^{\frac{n}{p(\infty)}}$ for $|y| \geq 2$ under the log-condition at infinity. Hence by the Hölder inequality and the same log-condition at infinity,
\[
\left| \int_{|y| > 2} \frac{f(y) \, dy}{|x - y|^{n - \alpha}} \right| \leq C_1 \|f_0\|_{L^p(\mathbb{R}^n \setminus B(0,2))} \|y|^{\alpha - n - \frac{\gamma}{p(\sigma)}}\|_{L^p(\mathbb{R}^n \setminus B(0,2))}
\]
\[
\leq C \|f\|_{L^p(\mathbb{R}^n, \rho_{\beta_a, \beta_b}(\sigma))} \|y|^{\alpha - n - \frac{\gamma}{p(\sigma)}}\|_{L^p(\mathbb{R}^n \setminus B(0,2))}
\]
\[
\leq C \|y|^{\alpha - n - \frac{\gamma}{p(\sigma)}}\|_{L^p(\mathbb{R}^n \setminus B(0,2))},
\]
where the last norm is finite under the condition \(\alpha p(\infty) - n < \gamma_{\infty}\) (use the argument given in (3.5)).

**Corollary 3.1.** Let \(0 < \alpha < n\), \(p(x)\) satisfy conditions (1.4)–(1.6) and (1.8). Then the operator \(I^\alpha\) is bounded from the space \(L^{p(\cdot)}(\mathbb{R}^n)\) into the space \(L^{q(\cdot)}(\mathbb{R}^n)\), \(\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}\).

The statement of the corollary was proved in [1,2] under a weaker than (1.6) version of the log-condition at infinity.

### 4. The case of the spherical potential operator

#### 4.1. The space \(L^{p(\cdot)}(\mathbb{S}^n, \rho)\)

We consider the weighted space \(L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a, \beta_b})\) with a variable exponent on the unit sphere \(\mathbb{S}^n = \{\sigma \in \mathbb{R}^{n+1}: |\sigma| = 1\}\), defined by the norm

\[
\|f\|_{L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a, \beta_b})} = \left\{ \lambda > 0: \int_{\mathbb{S}^n} |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b} \left| \frac{f(\sigma)}{\lambda} \right|^{p(\sigma)} d\sigma \leq 1 \right\},
\]

where \(\rho_{\beta_a, \beta_b}(\sigma) = |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b}\) and \(a \in \mathbb{S}^n\) and \(b \in \mathbb{S}^n\) are arbitrary points on \(\mathbb{S}^n\), \(a \neq b\).

We assume that \(0 < \alpha < n\) and

\[
1 < p_- \leq p(\sigma) \leq p_+ < \frac{n}{\alpha}, \quad \sigma \in \mathbb{S}^n,
\]

\[
|p(\sigma_1) - p(\sigma_2)| \leq \frac{A}{\ln \frac{3}{|\sigma_1 - \sigma_2|}}, \quad \sigma_1 \in \mathbb{S}^n, \sigma_2 \in \mathbb{S}^n.
\]

The following theorem is valid.

**Theorem B.** Let the function \(p: \mathbb{S}^n \to [1, \infty)\) satisfy conditions (4.1) and (4.2). The spherical potential operator \(K^\alpha\) is bounded from the space \(L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a, \beta_b})\) with \(\rho_{\beta_a, \beta_b}(\sigma) = |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b}\), where \(a \in \mathbb{S}^n\) and \(b \in \mathbb{S}^n\) are arbitrary points on the unit sphere \(\mathbb{S}^n\), \(a \neq b\), into the space \(L^{q(\cdot)}(\mathbb{S}^n, \rho_{\beta_a, \beta_b})\) with \(\rho_{\nu_a, \nu_b}(\sigma) = |\sigma - a|^{\nu_a} \cdot |\sigma - b|^{\nu_b}\), where \(\frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n}\), and

\[
\alpha p(a) - n < \beta_a < np(a) - n, \quad \alpha p(b) - n < \beta_b < np(b) - n,
\]

\[
v_a = \frac{q(a)}{p(a)} \beta_a, \quad v_b = \frac{q(b)}{p(b)} \beta_b.
\]

This theorem was proved in [8] under the additional assumption that the weight exponents \(\beta_a\) and \(\beta_b\) are related to each other by the connection

\[
\frac{q(a)}{p(a)} \beta_a = \frac{q(b)}{p(b)} \beta_b.
\]
Now Theorem B without this condition follows from Theorem A by means of the stereographic projection exactly in the same way as in [8, Section 5].

**Corollary 4.1.** Under assumptions (4.1) and (4.2), the spherical potential operator $K^\alpha$ is bounded from $L^{p(\cdot)}(\mathbb{S}^n)$ into $L^{q(\cdot)}(\mathbb{S}^n)$, \( \frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n} \).

**References**


