Weak infinite-dimensionality in Cartesian products with the Menger Property

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Abstract

The Menger Property is a classical covering counterpart to \( \sigma \)-compactness. Assuming the Continuum Hypothesis we construct, for each natural \( n \), a separable metrizable space whose \( n \)th power is weakly infinite-dimensional and \((n + 1)\)th power has the Menger Property, but it is strongly infinite-dimensional.

Keywords: The Menger property, Weakly infinite-dimensional; C-spaces; Products

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1. Introduction and the main result

We shall deal in this paper with the following properties of separable metrizable spaces \( X \) (only such spaces will be considered in the sequel):

**Property 1.1 (The Menger Property).** For each sequence \( \mathcal{U}_1, \mathcal{U}_2, \ldots \) of open covers of \( X \) there exist finite open collections \( \mathcal{W}_i \subset \mathcal{U}_i \) with \( \bigcup_{i=1}^{n} \mathcal{W}_i \) covering \( X \).

**Property 1.2 (C-spaces).** For each sequence \( \mathcal{U}_1, \mathcal{U}_2, \ldots \) of open covers of \( X \) there exist disjoint open collections \( \mathcal{V}_i \) such that \( \mathcal{V}_i \) refines \( \mathcal{U}_i \) and \( \bigcup_{i=1}^{n} \mathcal{V}_i \) covers \( X \).

**Property 1.3 (Weakly infinite-dimensional spaces).** The property described in Property 1.2 holds for all sequences \( \mathcal{U}_1, \mathcal{U}_2, \ldots \) of two-element open covers of \( X \).

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The spaces without Property 1.3 are called strongly infinite-dimensional.
Let us briefly comment on these notions.

Remark 1.4. (A) The Menger Property [20,27] is a generalization of \( \sigma \)-compactness, and, as proved by Hurewicz [14], it coincides with \( \sigma \)-compactness in the class of analytic spaces. Similar properties were investigated from various points of view in [3,6,7,18,19]. Fremlin and Miller [7] gave, within the usual set theory, an example of a non \( \sigma \)-compact space with the Menger Property.

(B) The property described in Property 1.2 was introduced (in a slightly different form) by Haver [11,12], cf. Addis and Gresham [1]. All spaces which are countable unions of finite-dimensional sets are C-spaces, but there are compact C-spaces without this property.

(C) Property 1.3 is usually described in terms of partitions of the space, as in the original definition of Aleksandrov [2]. Our description, following [1,26], was chosen to make transparent that C-spaces are weakly infinite-dimensional (in fact, we do not know examples which would distinguish these two classes). A classical result of Hurewicz [1] can be viewed as the statement that the Hilbert cube is strongly infinite-dimensional [22].

The aim of this paper is to construct the following examples.

Theorem 1.5. Assuming the Continuum Hypothesis, for each natural \( n \geq 1 \) there exists a separable metrizable space \( X \) such that

(i) \( X^{n+1} \) has the Menger Property, and

(ii) \( X^n \) is a C-space, and hence weakly infinite-dimensional, but

(iii) \( X^{n+1} \) is strongly infinite dimensional.

This result was motivated by the following facts.

The product of two C-spaces, one of which is \( \sigma \)-compact, is a C-space, see [8–10,26] (it is an open problem if this is true for weakly infinite-dimensional spaces, cf. Remark 1.4(C)).

However, examples in [23,24] (cf. also [25]) show that in absence of \( \sigma \)-compactness the behaviour of C-spaces under the product operation is rather erratic.

Theorem 1.5 shows that (assuming the Continuum Hypothesis) the Menger Property, a classical covering analogue to \( \sigma \)-compactness (cf. Remark 1.4(A)), is not enough to eliminate such irregularities.

Our examples illustrate phenomena similar to those described by E. Pol [24]. The basic difference, however, is that her constructions are within the usual set theory, but do not provide compactness-like properties, while we relay heavily on the Continuum Hypothesis, but we get in effect spaces with the Menger Property.

The examples are based on a beautiful construction, due to Michael [21], of sets in the real line whose finite products are concentrated about "small" sets in the
corresponding Euclidean spaces. The construction of Michael is considered in Section 2.

An application of Michael's technique in dimension theory, given in Sections 5 and 6, uses some ideas from [23,25].

2. Michael's concentrated sets $B_n$

In this section we shall slightly modify Lemma 6.1 from the celebrated paper by Michael [21], following closely the original proof, and using also some ideas of Cox [4,5].

We shall adapt the following notation, cf. [21, p. 210].

- $R$ is the set of real numbers, $Q \subset R$ is the set of rationals,
- $\Delta(n) = \{(x_1, \ldots, x_n) \in R^n : x_i = x_j$ for some $i \neq j\}$,
- $D_n = \{(x_1, \ldots, x_n) \in R^n : x_1 + \cdots + x_n = \sqrt{2}\}$.

Lemma 2.1. Assume the Continuum Hypothesis. For each $n \geq 1$ there exists an uncountable collection $\mathcal{F}$ of pairwise disjoint $(n + 1)$-element sets in $R \setminus Q$, satisfying the following conditions, with

$$B_n = \bigcup \mathcal{F}, \quad E = \bigcup \{F^{n+1} : F \in \mathcal{F}\} \setminus \Delta(n + 1), \quad Y = Q \cup B_n,$$

(a) $Y^{n+1} \cap (D_{n+1} \setminus \Delta(n + 1)) = E$,
(b) each dense $G_\delta$-subset of $D_{n+1}$ contains all but countably many elements of $E$,
(c) for all $m \leq n + 1$, and each $G_\delta$-set $G$ dense in $R^m$, $Y^m \setminus (G \cup \Delta(m))$ is contained in the union of countably many hyperplanes $y_i = c$, and, in case $m = n + 1$, the hyperplane $D_{n+1}$.

Proof. (A) For each $T \subseteq \{1, \ldots, n + 1\}$, let

$$\Delta(T) = \{x \in R^T : \text{for some distinct } i, j \text{ in } T, x_i = x_j\}.$$

If the cardinality $|T| \leq n$, we denote by

$$\pi_T : D_{n+1} \rightarrow R^T$$

the projection. For any pair of disjoint nonempty sets $S, T$ in $\{1, \ldots, n + 1\}$ and $u \in R^S$, we define

$$\pi_T^u : D_{n+1} \rightarrow \{u\} \times R^T$$

by the formula

$$\pi_T^u(x) = (u, \pi_T(x)).$$

The mappings $\pi_T$ and $\pi_T^s$ are open, cf. [21, p. 211].

(B) Let $G$ be a dense $G_\delta$-set in $R^m$. For each $M \subseteq \{1, \ldots, m\}$ we set $N = \{1, \ldots, m\} \setminus M$, and let

$$G(M) = \{x \in R^M : G \cap ((x) \times R^N) \text{ is dense in } (x) \times R^N\};$$

in particular, $G(\{1, \ldots, m\}) = G$. 
Given disjoint subsets $S$, $T$ of $\{1, \ldots, m\}$, and $u \in R^S$, we define

$$G(u, T) = G(S \cup T) \cap \{\{u\} \times R^T\}.$$  

By the Kuratowski–Ulam theorem [17, §22.1.1], $G(M)$ is a dense $G_\delta$-set in $R^M$ and, if $u \in G(S)$, then $G(u, T)$ is a dense $G_\delta$-set in $\{u\} \times R^T$.

(C) Let $G_\alpha \subset R^{m(\alpha)}$, $\alpha < \omega_1$, $m(\alpha) \leq n + 1$, be all $G_\delta$-sets dense in corresponding spaces $R^{m(\alpha)}$, and let $H_\alpha \subset D_{n+1}$, $\alpha < \omega_1$, be all $G_\delta$-sets dense in the hyperplane $D_{n+1}$.

We shall define the elements $F_1$, $F_2$, $\ldots$, $F_\alpha$, $\alpha < \omega_1$, of the family $\mathcal{F}$ by transfinite induction, securing at each stage $\alpha$ the following conditions (see (A) and (B)):

- for $\eta < \alpha$, $F^{n+1}_\eta \setminus \Delta(n + 1) \subset \bigcap \{H_\xi : \xi \leq \eta\}$, \hspace{1cm} (1)
- $(Q \cup \bigcup \{F_\eta : \eta < \alpha\})^{n+1} \cap (D_{n+1} \setminus \Delta(n + 1)) \subset \bigcup \{F^{n+1}_\eta : \eta < \alpha\}$, \hspace{1cm} (2)
- if $S \subset \{1, \ldots, m(\xi)\}$, with $\xi < \alpha$,
- then $\left( \bigcup \{F_\eta : \xi \leq \eta < \alpha\} \right)^S \setminus \Delta(S) \subset G_\xi(S) \cup D_{n+1}$. \hspace{1cm} (3)

We start with $F_0 = \emptyset$, $G_0 = R^{n+1}$, $H_0 = D_{n+1}$, and let us assume that the sets $F_\eta$ with $\eta < \alpha$ are defined. We shall consider all sets

$$\left(\pi^\eta \right)^{-1}(G_\xi(u, T)),$$

where $\xi < \alpha$, $u \in (\bigcup \{F_\eta : \xi \leq \eta < \alpha\})^S \setminus \Delta(S)$, and $S$, $T$ are disjoint nonempty subsets of $\{1, \ldots, m(\xi)\}$, and we shall consider also the sets

$$\left(\pi_T \right)^{-1}(G_\xi(T)),$$

where $\xi < \alpha$, $T \subset \{1, \ldots, m(\xi)\}$, $|T| \leq n$. Since the maps $\pi^\eta$ and $\pi_T$ are open (see (A)), and $u \in G_\xi(S)$, by (3), it follows from (B) that the sets (4) and (5) are dense $G_\delta$-sets in $D_{n+1}$. Finally, let

$$A = \left\{\sqrt{2} - (a_1 + \cdots + a_r) : a_i \in Q \cup \bigcup \{F_\xi : \eta < \alpha\}, r \leq n\right\},$$

$$U_T(a) = \left\{x \in R_T : \sum_{i \in T} x_i \neq a\right\},$$

with $T \subset \{1, \ldots, n + 1\}$, $|T| \leq n$, $a \in A$, and let us consider dense open sets in $D_{n+1}$,

$$\left(\pi_T \right)^{-1}(U_T(a)).$$

The intersection of all possible sets (4), (5), (6), and the set $\bigcap \{H_\xi : \xi \leq \alpha\}$ is a dense $G_\delta$-set in $D_{n+1}$, and so is the intersection $D^* \subset D$ of all sets obtained from $D$ by permutations of coordinates.
To choose $F_\alpha$, let $(x_1, \ldots, x_{n+1})$ be any point from $D^* \setminus \Delta(n+1)$, and let us set $F_\alpha = \{x_1, \ldots, x_{n+1}\}$. Since $D^*$ is invariant under coordinate permutations,

$$F_{\alpha + 1}^n \setminus \Delta(n+1) \subset D,$$

(D) the collection $\mathcal{F}$ defined in (C) has the required properties. Condition (a) follows from (1) and (2). Condition (b) follows from (1), as each $G_\delta$-set $H$ dense in $D_{n+1}$ was listed as some $H_\xi$. Finally, let $G \subset R^n$ be a dense $G_\delta$-set, and let $G = G_\xi$, $m = m(\xi)$ for some $\xi < \omega_1$. Let $u \in Y^m \setminus \Delta(m)$, and, in case $m = n + 1$, also $u \notin D_{n+1}$. By (3), where $S = \{1, \ldots, m\}$, either $u \in G_\xi(S) = G_\xi$, or else $u(S)$ intersects the countable set $C = Q \cup \bigcup\{F_\alpha : \eta < \xi\}$. It follows that $Y^m \setminus (G \cup \Delta(m))$ is contained in the union of countably many hyperplanes $y_i = c$, where $c \in C$, and if $m = n + 1$, the hyperplane $D_{n+1}$.

Remark 2.2. One can assume in Lemma 3.1, removing if necessary countably many elements of the collection $\mathcal{F}$, that for each set $U$ open in $D_{n+1}$, if $U \cap E \neq \emptyset$, then $U \cap E$ is uncountable (i.e., each point in $E$ is a condensation point of this set).

Let us also notice that condition (b) implies that the closure of each uncountable subset of $E$ has nonempty interior in $D_{n+1}$.

3. The spaces $Z(B_n)$

Let

$$p: Z \to R$$

be a perfect map of a separable metrizable space $Z$ onto the real line, and let

$$P_m = p \times \cdots \times p: Z^m \to R^m$$

be the product maps.

We shall adapt the notation of Section 2. Assume the Continuum Hypothesis, and let $B_n$ be the Michael's set defined in Lemma 2.1, with the additional property indicated in Remark 2.2. Let (see Lemma 2.1)

$$E^* = \{(x_1, \ldots, x_{n+1}) \in E : x_1 < \cdots < x_{n+1}\}.$$  

Let $\mathcal{H}$ be the collection of all $G_\delta$-sets $H$ in $Z^{n+1}$ with $E^* \subset P_{n+1}(H)$, and let us establish a correspondence $u \mapsto H(u)$ between the points in $E^*$ and the sets in $\mathcal{H}$, ensuring the property (cf. Remark 2.2) that for each $H \in \mathcal{H}$,

$$\{u : H(u) = H\} \text{ is dense in } E^*.$$  

We shall consider a Hilgers function [13] related to this correspondence, $h: E^* \to Z^{n+1}$, defined by

$$h(u) \in P_{n+1}^{-1}(u) \setminus H(u),$$

whenever such a choice is possible.
The Hilgers function $h$ has the following property:

For each $G_δ$-set $H$ in $Z^{n+1}$, if $h(E^*) \subseteq H$, then there exists a set $A$ dense in $E^*$ with $P_{n+1}^{-1}(A) \subseteq H$.  \hfill (12)

Indeed (cf. [13]), $H = H(u)$ for $u$ in a dense subset $A$ of $E^*$, by (10), and since $h(A) \subseteq H$, (12) follows from (11).

We shall associate with the function $h$ a selection function

$$s : B_n \to Z, \quad \text{where } s(x) \in p^{-1}(x)$$

in the following way. Let (see Lemma 2.1)

$$F = \{x_1, \ldots, x_{n+1}\} \in \mathcal{F} \quad \text{with } x_1 < \cdots < x_{n+1},$$

i.e., $(x_1, \ldots, x_{n+1}) \in E^*$; then we select $s(x_i) \in p^{-1}(x_i)$ so that

$$\left(s(x_1), \ldots, s(x_{n+1})\right) = h(x_1, \ldots, x_{n+1}) \in P_{n+1}^{-1}(x_1, \ldots, x_{n+1}).$$  \hfill (13)

Finally, the space $Z(B_n)$ is defined by

$$Z(B_n) = p^{-1}(Q) \cup \{s(x) : x \in B_n\}. \hfill (14)$$

We shall check (in Section 4) that $Z(B_n)^{n+1}$ has the Menger Property. To ensure that $Z(B_n)^n$ is a C-space (in Section 5) we shall assume in addition that

$$p^{-1}(q) \text{ is finite-dimensional, for } q \in Q. \hfill (15)$$

And, to get $Z(B_n)^{n+1}$ strongly infinite-dimensional (in Section 6) we shall deal with a specific map (7) defined as follows.

Let $\sigma : J^\omega \to 2^\omega$ map the countable product of $J = \{0\} \cup [\frac{1}{2}, 1]$ onto the countable product of $\{0, 1\}$ according to the formula

$$\sigma(x_1, x_2, \ldots) = (y_1, y_2, \ldots), \quad \text{where } y_i = \min\{2x_i, 1\}. $$

We let

$$Z = \{(x, y) \in J^\omega \times 2^\omega : y = \sigma(x) \text{ and } y \text{ has at least two distinct coordinates}\}. \quad (16)$$

Before we define $p : Z \to R$, let us make an observation which will be useful in Section 6.

Let $\mathcal{Z}_i = \{U_{i0}, U_{i1}\}$, for $i = 1, 2, \ldots$, be a sequence of two-element open families in $Z$ defined by

$$U_{i0} = \{(x, y) \in Z : 0 < x_j < \frac{5}{6}, y_1 + \cdots + y_j = i\},$$

$$U_{i1} = \{(x, y) \in Z : x_j > \frac{4}{6}, y_1 + \cdots + y_j = i\}.$$

Let

$$\Sigma = \{y \in 2^\omega : \text{all but finitely many coordinates of } y \text{ are zero}\}.$$
Let us notice that

if \( y \in \Sigma \) then \( \sigma^{-1}(y) \) is finite-dimensional, \( (17) \)

if \( y \notin \Sigma \) then the sequence \( \mathcal{Z}_1, \mathcal{Z}_2, \ldots \) restricted to \( \sigma^{-1}(y) \) has no refinement required by Property 1.2 and \( \sigma^{-1}(y) \subseteq \cap_{i=1}^{\infty} (\cup \mathcal{Z}_i) \). \( (18) \)

Indeed, let us fix \( y \in 2^{\infty} \) and let \( S = \{ i : y_i \neq 0 \} \). Then one can identify \( \sigma^{-1}(y) \) with the cube \( [\frac{1}{2}, 1]^3 \).

If \( y \in \Sigma \), the cube is finite-dimensional.

If \( y \notin \Sigma \), \( \sigma^{-1}(y) \) is a Hilbert cube with pairs of opposite faces

\[ A_{i0} = \{ x \in \sigma^{-1}(y) : x_j = \frac{1}{2} \}, \quad A_{i1} = \{ x \in \sigma^{-1}(y) : x_j = 1 \}, \]

where \( y_1 + \cdots + y_j = i \). Since \( A_{ik} \subseteq U_{ik} \), \( k = 0, 1 \), and \( A_{i0} \cap \overline{U_{i1}} = \emptyset = A_{i1} \cap \overline{U_{i0}} \), a reasoning of Addis and Gresham [1], proof of Theorem 3.2, can be repeated to justify (18), cf. also [16, p.49].

To define the map \( p \), let us consider first a map \( \nu : 2^{\infty} \rightarrow \overline{R} \), where \( \overline{R} \) is the real line extended by two points at infinity compactifying \((-\infty, 0] \) and \([0, +\infty) \), respectively, such that: \( \nu^{-1}(t) \) is a singleton, if \( t \in \mathbb{Q} \) or \( t \) is a point at infinity, and at most a two-element set otherwise, \( \nu^{-1}(Q) = \Sigma \) and the two constant sequences in \( 2^{\infty} \) are mapped onto the points at infinity, cf. [16, p.44]. Now (cf. (16)) \( p : Z \rightarrow R \) is defined by

\[ p(x, y) = \nu \circ \sigma(x). \]

Then (17) and (18) yield (15) and

if \( t \in R \setminus Q \) then the sequence \( \mathcal{Z}_1, \mathcal{Z}_2, \ldots \) restricted to \( p^{-1}(t) \) has no refinement required by Property 1.2 and \( p^{-1}(t) \subseteq \cap_{i=1}^{\infty} (\cap \mathcal{Z}_i) \). \( (19) \)

4. The product \( Z(B_n)^{n+1} \) has the Menger Property

We adapt the notation from Sections 2 and 3. Let us begin with the following observations:

(a) the Menger Property is inherited by closed subspaces, and countable unions of sets with the Menger Property have this property;

(b) the product of a space with the Menger Property and a \( \sigma \)-compact space has the Menger Property;

(c) if \( N \subseteq M \) are such that \( N \), as well as each closed subset of \( M \) disjoint from \( N \), have the Menger Property, then so does \( M \).
We shall use (c) setting

\[ M = Z(B_n)^m, \quad N = P_m^{-1}(Q^m), \quad m \leq n + 1. \]

Then \( N \) is \( \sigma \)-compact and for each set \( F \) closed in \( M \) and disjoint from \( N \), \( P_m(F) \) is contained in the union of \( \Delta(m) \), countably many hyperplanes \( y_i = c \), and, if \( m = n + 1 \), the hyperplane \( D_{n+1} \).

Indeed, \( P_m \) being perfect, the closure of \( P_m(F) \) in \( R^m \) is disjoint from \( Q^m \), and it is disjoint from an open set dense in \( R^m \).

Therefore, by (a), it is enough to check that, for \( m \leq n + 1 \),

\[ P_m^{-1}(\Delta(m)) \cap Z(B_n)^m \text{ has the Menger Property,} \]  
\[ P_m^{-1}(L) \cap Z(B_n)^m \text{ has the Menger Property,} \]

where \( L \) is any hyperplane \( y_i = c \) in \( R^m \), and, finally,

\[ P_{n+1}^{-1}(D_{n+1} \setminus \Delta(n + 1)) \cap Z(B_n)^{n+1} \text{ has the Menger Property.} \]

For \( m = 1 \), \( \Delta(m) = \emptyset \) and each \( L \) in (21) is a singleton \( \{c\} \), while \( P_1^{-1}(c) \cap Z(B_1)^m \) is either the compactum \( p^{-1}(c) \), if \( c \in Q \), or the singleton \( \{s(c)\} \), if \( c \in B_1 \), see (14).

This establishes the Menger Property of \( Z(B_1) \), and let us assume that \( Z(B_k)^k \) has this property for \( k < m \).

To check (20) consider, for \( 1 \leq i < j \leq m \), the sets

\[ K_{ij} = \{(z_1, \ldots, z_m) \in Z^m: p(z_i) \in Q, p(z_j) \in Q\}, \]

and

\[ Z_{ij} = \{(z_1, \ldots, z_m) \in Z^m: z_i = z_j\}. \]

Since \( p^{-1}(Q) \times p^{-1}(Q) \) is \( \sigma \)-compact, and \( K_{ij} \cap Z(B_n)^m \) is homeomorphic to \( p^{-1}(Q) \times p^{-1}(Q) \times Z(B_n)^{m-2} \), these sets have the Menger Property, by (b) and the inductive hypothesis. The sets \( Z_{ij} \cap Z(B_n)^m \), being homeomorphic to \( Z(B_n)^m \), have also the Menger Property.

It remains to verify (22). By Lemma 2.1(a), \( P_{n+1}^{-1}(D_{n+1} \setminus \Delta(n + 1)) \cap Z(B_n)^{n+1} \subset P_{n+1}^{-1}(E) \), and since \( E \) is obtained from the set \( E^* \) (defined in (9)) by permutations of the coordinates, one can concentrate on \( P_{n+1}^{-1}(E^*) \cap Z(B_n)^{n+1} \).

Let \( \mathcal{Z}_1, \mathcal{Z}_2, \ldots \) be a sequence of open collections in \( Z^{n+1} \) covering \( h(E^*) \), and let \( H = \bigcap_{i=1}^{\infty} (\bigcup \mathcal{Z}_i) \). Then \( H \) is a \( G_\delta \)-set in \( Z^{n+1} \) containing \( h(E^*) \) and, by (12),
there is a set $A = \{u_1, u_2, \ldots\}$ dense in $E^*$ with $P_{n+1}^{-1}(A) \subseteq H$. For each $j$, $P_{n+1}^{-1}(u_j) \subseteq H \cup \mathcal{W}_2$, and therefore there are finite collections $\mathcal{W}_{2j} \subseteq \mathcal{W}_2$ with $P_{n+1}^{-1}(u_j) \subseteq \bigcup \mathcal{W}_{2j}$. Furthermore, $P_{n+1}$ being perfect, there is an open set $U$ in $E^*$ with $A \subseteq U$ and $P_{n+1}^{-1}(U) \subseteq \bigcup_{j=1}^{\infty} (\bigcup \mathcal{W}_{2j}) = W$. The set $C = E^* \setminus U$ is closed in $E^*$ and disjoint from the set $A$ dense in $E^*$, and by Remark 2.2, $C$ is countable. It follows that the set $h(E^*) \setminus W \subseteq h(C)$ is countable, and choosing one element from each collection $\mathcal{W}_{2j+1}$ one can complete the family $\bigcup_{j=1}^{\infty} \mathcal{W}_{2j}$ to a covering of $h(E^*)$.

5. The product $Z(B_n)^n$ is a C-space

In (a) and (c) of Section 4, the Menger Property can be replaced by Property 1.2. Also, by results quoted in Section 1, the product of two C-spaces one of which is $\sigma$-compact, is a C-space.

Since we assume (see (15)) that $N = P_{m}^{-1}(Q^{m})$ is a countable union of finite-dimensional compacta, to check that $Z(B_n)^{m}$ is a C-space for $m \leq n$, we can repeat the inductive arguments from Section 4, replacing the Menger Property by Property 1.2.

However, the analogue of (22) for C-spaces is no longer valid, as we shall demonstrate in the next section.

6. Strong infinite-dimensionality of $Z(B_n)^{n+1}$

By arguments preceding (23) in Section 4, $h(E^*)$ is an $F_\sigma$-set in $Z(B_n)^{n+1}$. Since weak infinite-dimensionality is inherited by $F_\sigma$-subspaces, it is enough to show that $h(E^*)$ is strongly infinite-dimensional. A sequence of covers witnessing this fact can be obtained from the open families $\mathcal{U}_i = \{U_{i0}, U_{i1}\}$ in $Z$ defined in Section 3 after formula (16), setting

$$\mathcal{W}_i = \{U_{i0} \times Z^n, U_{i1} \times Z^n\}, \quad i = 1, 2, \ldots.$$

Aiming at a contradiction, suppose that for each $i$, $\mathcal{W}_i$ is an open in $Z^{n+1}$ disjoint collection which refines $\mathcal{W}_i$ such that $H = \bigcap_{i=1}^{\infty} (\bigcup \mathcal{W}_i)$ contains $h(E^*)$. Then property (12) provides a point $u = (x_1, x_2, \ldots, x_{n+1}) \in E^*$ with $P_{n+1}^{-1}(u) = P_{n+1}^{-1}(x_1) \times P_{n+1}^{-1}(x_2) \times \cdots \times P_{n+1}^{-1}(x_{n+1}) \subseteq H$. Let $p(a_i) = x_i$. The fiber $P_{n+1}^{-1}(u)$ contains the set $p^{-1}(x_1) \times \{a_2\} \times \cdots \times \{a_{n+1}\}$, and since $x_1 \notin Q$, the families $\mathcal{W}_i$ restricted to this set would violate property (19) established in Section 3.

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