Compactness in Dual Spaces of Locally Compact Groups and Tensor Products of Irreducible Representations

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We deal with supports of tensor products of irreducible representations of locally compact groups. The main result describes the structure of almost connected groups for which tensor products of representations in the reduced dual have compact supports. An immediate consequence is that an infinite almost connected group cannot have a compact reduced dual. Furthermore, the continuity problem for these supports is studied.

Let \( G \) be a locally compact group and \( \hat{G} \) the set of all equivalence classes of irreducible unitary representations of \( G \). \( \hat{G} \), equipped with the usual topology, is called the dual space of \( G \). In this paper, we continue our studies [12, 13] of problems related to supports in \( \hat{G} \) of tensor products of irreducible representations of \( G \).

It is a consequence of the Pontrjagin duality theorem, that a locally compact abelian group with compact dual is discrete. However, easy examples show that a non-abelian locally compact group need not be discrete even though its dual space is compact. It was this fact which motivated us to investigate the connection between the compactness of supports of tensor products of irreducible representations and the group structure.

Section 1 contains some preliminary facts on compactness in dual spaces and a thorough look to groups with small invariant neighbourhoods of the identity. Our main result is proved in Section 2 and can roughly be stated as follows. Let \( G \) be an almost connected group, and suppose that for all \( \pi \) in the so-called reduced dual of \( G \), the tensor product \( \pi \otimes \bar{\pi} \) (where \( \bar{\pi} \) denotes the conjugate of \( \pi \)) has a compact support. Then \( G \) is a finite extension of a direct product of a vector group and a compact group.
Finally, in Section 3, we are concerned with the continuity of the mapping $(\pi, \rho) \mapsto \text{supp } \pi \otimes \rho$ from $\hat{G} \times \hat{G}$ into the space of all closed subsets of $\hat{G}$, endowed with Fell's topology [4]. This question has already been studied in [13], and the result we obtain here is largely based on [13]. It turns out that for an amenable Lie group $G$, the continuity of this mapping is equivalent to that every conjugacy class in $G$ is relatively compact.

1. PRELIMINARIES AND COMPACTNESS IN DUALS

Let $G$ be a locally compact group and $C^*(G)$ the group $C^*$-algebra of $G$. We use the same letter $\pi$ for a unitary representation of $G$ and the corresponding $*$-representation of $C^*(G)$, and $\ker \pi$ always denotes the $C^*$-kernel of $\pi$. The set $\text{Prim } C^*(G)$ of primitive ideals of $C^*(G)$ carries the hull-kernel topology, and the dual space $\hat{G}$ of $G$ is the set of equivalence classes of irreducible unitary representations of $G$ endowed with the weak topology with respect to the mapping $\pi \mapsto \ker \pi$ from $\hat{G}$ onto $\text{Prim } C^*(G)$. If $H$ is a closed subgroup of $G$ and $\tau$ a representation of $H$, then $\text{Ind}_H^G \tau$ denotes the representation of $G$ induced by $\tau$. Moreover, for $x \in G$, the representation $\tau^x$ of $xHx^{-1}$ is defined by $(\tau^x)(y) = \tau(x^{-1}yx)$. In case $H$ is normal, these $\tau^x$ form the $G$-orbit $G(\tau)$ of $\tau$. $1_G$ always denotes the trivial representation of $G$ and $\pi \otimes \rho$ the tensor product of $\pi$ and $\rho$.

If $S$ and $T$ are sets of unitary representations of $G$, then $S$ is weakly contained in $T$ ($S \subset T$) if

$$\bigcap_{\sigma \in S} \ker \sigma \supseteq \bigcap_{\tau \in T} \ker \tau.$$ 

$S$ and $T$ are called weakly equivalent ($S \sim T$) if $S \subset T$ and $T \subset S$. For any unitary representation $\pi$ of $G$ the support of $\pi$ is the closed subset $\text{supp } \pi = \{ \tau \in \hat{G}; \tau \prec \pi \}$ of $\hat{G}$. In particular, the support of the left regular representation of $G$ is the reduced dual $\hat{G}_r$ of $G$. Recall that $\hat{G}_r = \hat{G}$ iff $G$ is amenable [6]. For sets $\Pi$ and $\Omega$ of unitary representations, it is convenient to introduce the following notations:

$$\Pi \otimes \Omega = \{ \pi \otimes \omega; \pi \in \Pi, \omega \in \Omega \} \quad \text{and} \quad \text{supp } \Pi = \{ \rho \in \hat{G}; \rho \ll \Pi \}.$$

We list briefly some classes of locally compact groups which will occur in the sequel. Here $I(G)$ denotes the group of inner automorphisms of $G$.

[SIN]: $G$ has a fundamental system of $I(G)$-invariant neighbourhoods of the identity
[IN]: $G$ contains a compact $I(G)$-invariant neighbourhood of the identity

[FC]: every conjugacy class $I(G)(x), x \in G$, is relatively compact.

[FD]: the commutator subgroup of $G$ is relatively compact.

Then $[FD] \subseteq [FC] \subseteq [IN]$, and $G \in [IN]$ iff $G$ contains a compact normal subgroup $K$ such that $G/K \in [SIN]$ [7].

As is well known, for locally compact abelian groups $G$, compactness of $\hat{G}$ implies that $G$ is discrete. More generally, this conclusion holds for $[SIN]$ groups (see [11, Theorem 2] and Corollary 1.7), but not in general:

**Example 1.1.** An example of a nondiscrete group $G$ having a compact dual space was given in [1, p. 144]: the semi-direct product $G = \mathbb{Z} \rtimes \mathbb{R}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by $(n, x) \mapsto e^{nx}$. As a second example, consider the semi-direct product $G = \mathbb{Q}^\times \rtimes \hat{\mathbb{Q}}$, where the multiplicative group $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$ of the rational numbers $\mathbb{Q}$ acts on $\mathbb{Q} = \hat{\mathbb{Q}}$ by multiplication, hence on $\mathbb{Q}$ by the dual action, and where $\mathbb{Q}$ and $\mathbb{Q}^\times$ carry the discrete topology. In fact, there are only two orbits $\{0\}$ and $\mathbb{Q}^\times = G(1)$ in $\hat{\mathbb{Q}}$, so that by Mackey's theory

$$\hat{G} = \mathbb{Q}^\times \cup \{\text{Ind}_G 1\},$$

which is compact. Notice that $G \in [FD]^{-}$.

**Remark 1.2.** Let $A$ be a $C^*$-algebra and $B$ a dense $*$-subalgebra of $A$. Then $S \subseteq \hat{A}$ is contained in some compact subset of $\hat{A}$ iff there exist $f \in B$ and $\delta > 0$ such that $\|\pi(f)\| \geq \delta$ for all $\pi \in S$. The condition is sufficient by [3, (3.3.7)]. Conversely, let $C$ be a compact subset of $\hat{A}$ containing $S$, and for every $\rho \in C$ choose $f_\rho \in B$ such that $\rho(f_\rho) \neq 0$. Then $\pi(f_\rho) \neq 0$ for all $\pi$ in a neighbourhood $V(\rho)$ of $\rho$, and $C \subseteq \bigcup_{j=1}^r V(\rho_j)$ for certain $\rho_j$. Setting $f = \sum_{j=1}^r f_\rho f_\rho^*$, one easily checks that $\pi(f) \neq 0$ for all $\pi \in C$.

**Lemma 1.3.** Let $H$ be a closed subgroup of $G$ and $C$ a compact subset of $\hat{G}$. Then there exists a compact subset $K$ of $\hat{H}$ such that $K \cap \text{supp } \pi | H \neq \emptyset$ for all $\pi \in C$.

**Proof.** For any function $f$ on $G$ and $x, y \in G$, set $xf(y) = f(xy)$. There exists $f \in C_c(G)$ with the following properties:

(i) $\|\pi(f)\| \geq 4 \delta$ for some $\delta > 0$ and all $\pi \in C$;

(ii) $x \mapsto _x f | H, G \to L^1(H)$ is continuous.

Let $Q \subseteq G/H$ be compact such that $\text{supp } f \subseteq p^{-1}(Q)$, where $p$ denotes the
quotient map $G \to G/H$. If $\xi \in \mathcal{H}(\pi)$, the Hilbert space of $\pi$, such that $\|\xi\| \leq 1$ and $\|\pi(f)\xi\| \geq 3\delta$, then by Weil's formula

$$3\delta \leq \left\| \int_G f(x) \pi(x) \xi \, dx \right\| = \left\| \int_Q \pi(x) \int_H f(xh) \pi(h)\xi \, dh \, dx \right\| \leq \int_Q \left\| \pi \mid H(\cdot f \mid H) \right\| \, dx.$$

Hence, since $x \to \|\pi \mid H(\cdot f \mid H)\|$ is continuous,

$$\|\pi \mid H(\cdot f \mid H)\| \geq 3\delta \left| Q \right|^{-1}$$

for some $z \in p^{-1}(Q)$, where $|Q|$ denotes the Haar measure of $Q$. Applying the continuity of $x \to \cdot f \mid H, G \to L^1(H)$ once more, we get that

$$\|\rho(y,f \mid H) - \rho(z,f \mid H)\| \leq \delta \left| Q \right|^{-1}$$

for all $y$ in a neighbourhood $V(x)$ of $x$ and every representation $\rho$ of $H$. Now, if $Q \subseteq \bigcup_{j=1}^{\infty} D(V(x_j))$, then $h \in D(V(x_j))$ for some $j$ and $h \in H$, and hence

$$\left\| \pi \mid H(x_j f \mid H) \right\| \leq \left\| \pi \mid H(h \cdot f \mid H) \right\| - \left\| \pi \mid H(x_j \cdot f \mid H - x_j f \mid H) \right\| \geq \left\| \pi(h) \pi \mid H(z f \mid H) \right\| - \delta \left| Q \right|^{-1} \geq 2\delta \left| Q \right|^{-1}.$$

The assertion of Lemma 1.3 follows by setting

$$K = \bigcup_{j=1}^{\infty} \left\{ \tau \in \hat{G}; \left\| \tau(x_j f \mid H) \right\| \geq \delta \left| Q \right|^{-1} \right\}.$$

**Corollary 1.4.** Let $H$ be an open subgroup of $G$. Then $\hat{G}$ (resp. $\hat{G}_r$) is compact iff there exists a compact subset $C$ of $\hat{H}$ (resp. $\hat{H}_r$) such that $C \cap \text{supp } \pi \mid H \neq \emptyset$ for all $\pi \in \hat{G}$ (resp. $\hat{G}_r$). If moreover, $H$ is normal, then this condition is equivalent to $C \cap \overline{G(\lambda)} \neq \emptyset$ for all $\lambda \in \hat{H}$ (resp. $\hat{H}_r$).

**Proof.** The necessity of the condition follows from Lemma 1.3 applied to, say, the direct sum of all $\pi \in \hat{G}$ (resp. $\hat{G}_r$). Conversely, let $f \in C_c(H)$ and $\delta > 0$ such that $\|\tau(f)\| \geq \delta$ for all $\tau \in \hat{C}$. If $\pi \in \hat{G}$ and $\tau \in \hat{C} \cap \text{supp } \pi \mid H$, then

$$\|\pi(f)\| = \|\pi \mid H(f)\| \geq \|\tau(f)\| \geq \delta.$$

Suppose now that $H$ is normal. Note that if $\lambda \in \hat{H}$ and $\pi \in \text{supp } \text{Ind}_H^G \lambda$, then $\pi \mid H \sim \overline{G(\lambda)}$. Conversely, choose $\pi \in \text{supp } \pi \mid H$, then $C \cap \overline{G(\lambda)} \subseteq C \cap \text{supp } \pi \mid H$.

**Corollary 1.5.** Let $\rho$ be a representation of $G$ and $H$ a normal subgroup of finite index. If $\text{supp } \rho$ is compact, then so is $\text{supp } \rho \mid H$. 
Proof. Apply Lemma 1.3 to $C = \text{supp } \rho$. To conclude that $\text{supp } \rho \mid H$ is compact, it suffices to show that, given $\tau \in \text{supp } \rho \mid H$, there exists $\pi \in \text{supp } \rho$ such that $\pi \mid H \sim G(\tau)$. In fact, this can be seen as follows. Let $K \subseteq \hat{H}$ be as in Lemma 1.3, and let $f \in C_c(H)$ and $\delta > 0$ satisfy $\| \sigma(f) \| \geq \delta$ for all $\sigma \in K$. Moreover, choose a finite set $A$ in $G$ such that $G = AH$, and denote by $\Delta(x)$ the modulus of the automorphism $h \mapsto xhx^{-1}$ of $H$. Then

$$\max_{a \in A} \| \tau(\Delta(a)f^{a^{-1}}) \| = \max_{a \in A} \| \tau^a(f) \| = \| \pi \mid H(f) \| \geq \delta.$$ 

This proves $\text{supp } \rho \mid H \subseteq \bigcup_{a \in A} \{ \tau \in \hat{H}; \| \tau(\Delta(a)f^{a^{-1}}) \| \geq \delta \}$.

Set now $A = C^*(G) / \ker \rho$ and let $B$ denote the closed subalgebra $C^*(H) / \ker \rho = C^*(H) / \ker \rho \mid H$. For $\tau \in \text{supp } \rho \mid H = \hat{B}$ there exist $\pi \in \hat{A} = \text{supp } \rho$ and a closed linear subspace $L$ of $\mathcal{M}(\pi)$ such that $L$ is $\pi(B)$-invariant and $\tau$ is equivalent to $\pi \mid B$ restricted to $L$ [3, (2.10.2)]. Then $F = \sum_{x \in G} \pi(x) L$ is $\pi(G)$-invariant and hence dense in $\mathcal{M}(\pi)$. Now, if $f \in C^*(H)$ such that $0 = \tau^x(f) = \tau(\Delta(x)f^{x^{-1}})$ for all $x \in G$, then

$$0 = \pi(f^x)L = \pi(x^{-1})\pi(f)\pi(x)L$$

for all $x$, i.e., $\pi(f)F = 0$. This proves $f \in \ker \pi \mid N$. Conversely, $\ker \pi \mid N \subseteq \bigcap_{x \in G} \ker \tau^x$. Thus $\pi \mid N \sim G(\tau)$.

Interesting questions with respect to compactness are the following:

(i) When is $\text{supp } \pi \otimes \rho$ compact for all $\pi, \rho \in \hat{G}$?

(ii) For a closed subgroup $H$ of $G$ and a representation $\rho$ of $H$, when does $\text{Ind}_H^G \rho$ have a compact support?

In Section 2, we will solve (i) for almost connected groups. In the remaining part of this section, we are going to show that (i) and (ii) can be answered completely in the case of [SIN] groups. For that we recall an important fact for $G \in [\text{SIN}]$. Let $G_F$ denote the open normal subgroup of $G$, consisting of all elements with relatively compact conjugacy classes. If $N \subseteq G_F$ is a normal subgroup of $G$, then the group of inner automorphisms of $G$ restricted to $N$ has a compact closure in the full automorphism group of $N$ [7]. It follows, that every compact subset of $\hat{N}$ is contained in a $G$-invariant closed compact subset.

**Lemma 1.6.** Let $G$ be a [SIN] group and $N$ closed normal subgroup of $G$ contained in $G_F$. If $\Pi$ is a compact subset of $\hat{G}$, then $\text{supp } \Pi \mid N$ is also compact. The converse holds if, moreover, $N$ is open.

**Proof.** By Lemma 1.3 there exists a compact subset $K$ of $\hat{N}$ such that $K \cap \text{supp } \pi \mid N \not= \emptyset$ for all $\pi \in \Pi$. Since $N \subseteq G_F$, $K$ is contained in a compact closed $G$-invariant subset $S$ of $\hat{N}$. In addition, orbit closures in $\hat{N}$ are...
minimal closed invariant sets, and every \( \pi \mid N \) is weakly equivalent to some orbit. It follows that \( \text{supp} \pi \mid N \subseteq S \) for all \( \pi \in \Pi \). Hence \( \text{supp} \pi \mid N \) is compact.

Conversely, if \( N \) is open and \( \pi \mid N \) has a compact support, then \( \Pi \) is compact by the same argument as in the proof of Corollary 1.4.

**Corollary 1.7.** If \( G \in [\text{SIN}] \) has a compact reduced dual, then \( G \) is discrete.

**Proof.** By [7, Theorem 2.9] there exists an (open) normal subgroup \( N \) of \( G \) of the form \( N = V \times K \), where \( K \) is a compact group and \( V \) a vector group. Clearly, \( N \subseteq G_F \). Applying Lemma 1.6 to \( \Pi = \hat{G}_r \), we obtain that \( \hat{N} \) is compact and hence that \( N \) is finite.

**Corollary 1.8.** If \( G \in [\text{SIN}] \) and \( \Pi, \Omega \subseteq \hat{G} \) are compact, then \( \text{supp} \Pi \otimes \Omega \) is compact.

**Proof.** Let \( N \) be an open normal subgroup of \( G \) of the form \( N = V \times K \). By Lemma 1.6, \( \text{supp} \Pi \mid N \) and \( \text{supp} \Omega \mid N \) are compact. According to the structure of \( N \), this implies that

\[
\text{supp}(\Pi \mid N \otimes \Omega \mid N)
\]

is compact. Again by Lemma 1.6, it follows that \( \text{supp} \Pi \otimes \Omega \) is compact.

**Proposition 1.9.** Let \( G \) be a \([\text{SIN}]\) group, \( H \) a closed subgroup and \( \rho \) a representation of \( H \). Then \( \text{Ind}_{H}^{G} \rho \) has a compact support if and only if \( H \) is open and \( \text{supp} \rho \) is compact.

**Proof.** Let \( N \) be an open normal subgroup of the form \( N = V \times K \), where \( K \) is compact and \( V \) a vector group.

Suppose first that \( H \) is open and \( \text{supp} \rho \) is compact. Then \( H \cap N = V \times C \), where \( C \) is an open subgroup of \( K \). Since \( G \in [\text{SIN}] \), there exists an open subgroup \( D \) of \( C \) which is normal in \( G \). Lemma 1.6, applied to \( H \) and \( M = V \times D \), yields that \( \text{supp} \rho \mid M \) is compact. Hence \( G(\text{supp} \rho \mid M) \) is compact, and, using Lemma 1.6 once more, we get that

\[
\Pi = \{ \pi \in \hat{G}; \pi \mid M < G(\text{supp} \rho \mid M) \}
\]

is compact. Now, \( \text{Ind}_{H}^{G} \rho \prec \Pi \), since \( \text{Ind}_{H}^{G} \rho \mid M \) is weakly equivalent to \( G(\rho \mid M) \). Consequently, \( \text{Ind}_{H}^{G} \rho \) has a compact support.

Conversely, suppose that \( \text{supp}(\text{Ind}_{H}^{G} \rho) \) is compact. By Lemma 1.6 \( \text{Ind}_{H}^{G} \rho \mid N \) has a compact support. Since \( N \) is open,

\[
\text{Ind}_{H}^{G} \rho \mid N \succ \text{Ind}_{H \cap N}^{N} \rho \mid H \cap N.
\]
Set $L = H \cap N$, choose any $\gamma \in \text{supp} \rho | L$ and $\alpha \in \hat{N}$ such that $\alpha | L > \gamma$. Then $\alpha \otimes \text{Ind}_L^N \rho | L$ has a compact support and
\[
\text{Ind}_L^N 1_L < \text{Ind}_L^N (\gamma \otimes \tilde{\gamma}) < \text{Ind}_L^N (\rho | L \otimes \alpha | L) = \alpha \otimes \text{Ind}_L^N \rho | L.
\]
Now, $\text{supp}(\text{Ind}_L^N 1_L)$ being compact, we can assume that $K$ is a Lie group. It follows that
\[
V/V \cap L \sim \text{Ind}_{V \cap L}^V 1_{V \cap L} < \text{Ind}_L^N 1_L | V
\]
[5, Theorem 5.2]. $\text{Ind}_L^N 1_L | V$ has a compact support (Lemma 1.6), and therefore $V \cap L$ is open in $V$. This shows that $H \supseteq V$, hence $L = V \times C$ for some subgroup $C$ of $K$. Finally, compactness of
\[
\text{supp}(\text{Ind}_L^N 1_L) = \{1, \nu\} \times \text{supp}(\text{Ind}_C^K 1_C)
\]
implies that $C$ has finite index in $K$. Thus $H$ is open in $G$. Since $H$ is open and $G \in [\text{[SIN]}]$, $H$ contains an open normal subgroup $N$ of $G$ such that $N \subseteq G_F$. As $\rho | N < \text{Ind}_H^G \rho | N$, a further application of Lemma 1.6 gives that $\rho | N$, and then also $\rho$, has a compact support.

2. Almost Connected Groups and Compactness of Supports of Tensor Products of Group Representations

In this section we are studying the effect that the compactness of supports of tensor products of irreducible representations may have on the structure of groups. Our final result will be that mentioned in the introduction.

**Lemma 2.1.** Let $G \in [\text{FD}]^-$ be a connected Lie group such that $\text{supp} \pi \otimes \tilde{\pi}$ is compact for all $\pi \in \hat{G}$. Then $G$ is the direct product of a compact group and a vector group.

**Proof.** Choose a compact normal subgroup $C$ of $G$ such that $G/C$ is abelian. Given $\pi \in \hat{G}$, by [9, Proposition 2.1] there exists a subgroup $H$ of $G$ containing $C$ and a finite-dimensional (irreducible) representation $\tau$ of $H$ such that $\pi \sim \text{Ind}_H^G \tau$. Then, since $H$ is normal and $\dim \tau < \infty,$
\[
\pi \otimes \tilde{\pi} = \text{Ind}_H^G (\text{Ind}_H^G \tau | H \otimes \tilde{\tau}) > \text{Ind}_H^G (\tau \otimes \tilde{\tau}) > \text{Ind}_H^G 1_H \sim \hat{G}/H.
\]
Therefore, $\hat{G}/H$ is compact, and $G$ being connected, this implies $H = G$. Hence $G$ is a connected group, all of whose irreducible representations are
finite-dimensional. By the Freudenthal–Weil theorem [3, (16.4.6)],
\( G = \mathbb{R}^d \times K, \) where \( K \) is compact.

**Lemma 2.2.** Let \( G \) be a connected Lie group, and suppose that \( G \) contains
a normal vector subgroup \( N \) such that \( G/N = V \times K \), where \( V \) is a vector
group and \( K \) is compact. If \( \text{supp} \pi \otimes \bar{\pi} \) is compact for all \( \pi \in \hat{G} \), then \( G \) is the
direct product of a vector group and a compact group.

**Proof.** The main step is to prove that \( N \) is contained in the center of \( G \).
For that it suffices to show that for every \( \lambda \in \hat{N} \), the stability subgroup \( G_{\lambda} \)
of \( \lambda \) equals \( G \). We define subgroups \( L \) and \( M \) of \( G \) by

\[
M \supseteq N, \quad M/N = V, \quad \text{and} \quad L = \overline{G_{\lambda}M}.
\]

Since \( L/M \) is a subgroup of the compact group \( K \), \( L \) is clearly regularly
related to itself in the sense of Mackey. Recall that Fell's weak Frobenius
property (WF1) [5, p. 442] is equivalent to amenability. Therefore, it
follows from [5, Theorem 5.2] that \( \sigma < \text{Ind}_{L}^{G} \sigma \mid L \) for every representation
\( \sigma \) of \( L \). Moreover, \( \tau < \text{Ind}_{G_{\lambda}}^{L} \tau \mid G_{\lambda} \) for every representation \( \tau \) of \( G_{\lambda} \). In fact,
\( G_{\lambda} \) is normal in \( L \) as \( M/N \) is central in \( G/N \).

Now, choose \( \tau \in \hat{G}_{\lambda} \) such that \( \tau \mid N \sim \lambda \), and set \( \sigma = \text{Ind}_{G_{\lambda}}^{L} \tau \) and
\( \pi = \text{Ind}_{G_{\lambda}}^{G} \tau \). Then \( \pi \in \hat{G} \), and

\[
\pi \otimes \pi = \text{Ind}_{L}^{G} \sigma \otimes \text{Ind}_{L}^{G} \sigma = \text{Ind}_{L}^{G}(\text{Ind}_{L}^{G} \sigma \mid L \otimes \sigma)
\]

\[
> \text{Ind}_{L}^{G}(\sigma \otimes \sigma) = \text{Ind}_{L}^{G}(\text{Ind}_{G_{\lambda}}^{L} \tau \otimes \text{Ind}_{G_{\lambda}}^{L} \bar{\tau})
\]

\[
= \text{Ind}_{L}^{G}(\text{Ind}_{G_{\lambda}}^{L}(\text{Ind}_{G_{\lambda}}^{G} \tau \mid G_{\lambda} \otimes \bar{\tau}))
\]

\[
> \text{Ind}_{L}^{G}(\text{Ind}_{G_{\lambda}}^{G}(\tau \otimes \bar{\tau})) = \text{Ind}_{G_{\lambda}}^{G}(\tau \otimes \bar{\tau}).
\]

Note that, if \( N_{\lambda} = \{ x \in N; \lambda(x) = 1 \} \), then \( \alpha \otimes \bar{\alpha} > 1 \) for every representation
\( \alpha \) of \( G_{\lambda} \) satisfying \( \alpha(N_{\lambda}) = I \). Indeed, \( N_{\lambda} \) is normal in \( G_{\lambda} \), \( N/N_{\lambda} \) is contained
in the center of \( G_{\lambda}/N_{\lambda} \), and \( G_{\lambda}/N \in [FC]^{\sim} \), so that [12, Theorem 3]
implies the stated weak containment property. We conclude that \( \text{Ind}_{G_{\lambda}}^{G} 1 \)
has a compact support. Finally, since \( G/N \) is the direct product of a vector
group and a compact connected group, the last paragraph of the proof of
Proposition 1.9 shows that \( G_{\lambda} = G \).

We claim that even \( M \) is central in \( G \). To this end choose a sequence of
closed normal subgroups

\[
N = N_{0} \subseteq N_{1} \subseteq \cdots \subseteq N_{d} = M, \quad d = \text{dim } V,
\]

such that \( N_{j}/N_{j-1} = \mathbb{R} \). Then \( N_{j} \) is a semi-direct product of \( \mathbb{R} \) and \( N_{j-1} \).
An easy induction argument now gives that \( N_{j} \) is contained in the center.
In fact, \( N_{0} \) is central by what we have shown above, and assuming that
is central, we obtain that \( N_j = \mathbb{R} \times N_{j-1} \) is a normal vector subgroup of \( G \) and hence contained in the center of \( G \).

Thus \( G \) has a cocompact center. An application of the Freudenthal-Weil theorem completes the proof.

**Proposition 2.3.** Let \( G \) be an amenable connected Lie group, and suppose that \( \text{supp} \pi \otimes \bar{\pi} \) is compact for all \( \pi \in \hat{G} \). Then \( G \) is the direct product of a compact group and a vector group.

**Proof.** Note first that we can assume that \( G \) contains no nontrivial compact normal subgroup. In fact, if \( C \) denotes the maximal compact normal subgroup of \( G \), and if \( G/C \) has been shown to be the direct product of a compact group and a vector group, then the same is true for \( G \) by Lemma 2.1 since \( G \in [FD]^- \).

The proposition follows by induction on the dimension of the radical \( R \) of \( G \). Indeed, supposing that \( \dim R \geq 1 \), there exists a normal vector subgroup \( N \) of \( G \) of positive dimension. By induction hypothesis, \( G/N \) is the direct product of a vector group and a compact group, and Lemma 2.2 now implies that \( G \) is the direct product of a compact group and a vector group.

**Lemma 2.4.** Let \( G \) be a noncompact connected semisimple Lie group. Then, for some \( \pi \in \hat{G} \), \( \text{supp} \pi \otimes \bar{\pi} \) is noncompact.

**Proof.** Suppose first that \( G \) has a finite center and is acceptable in the sense of Harish-Chandra (see [23, Sect. 8.1.1]). Let \( G = KAN \) be an Iwasawa decomposition of \( G \), \( M \) the centralizer of \( A \) in \( K \) and \( P = MAN \) the corresponding minimal parabolic subgroup.

For \( \lambda \in \hat{A} \) and \( \delta \in \hat{M} \) let \( \sigma_{\lambda,\delta} \in \hat{P} \) denote the representation \( \rho(\lambda) \delta(\mu) \) and \( \pi_{\lambda,\delta} \) the representation of \( G \) induced by \( \sigma_{\lambda,\delta} \). The equivalence classes of the principal \( P \)-series representations \( \pi_{\lambda,\delta} \) are in one-to-one correspondence with the points of the quotient space \( \hat{A} \times \hat{M}/W \), where \( W \) is a finite (Weyl) group acting on \( \hat{A} \) and \( \hat{M} \) [14, Sect. 4, 22, Sect. 5.5.2]. Let \( \hat{G}_p \subseteq \hat{G} \) denote the set of equivalence classes of irreducible principal \( P \)-series representations. By a result of Kostant, the \( \pi_{\lambda,1} \) are irreducible [22, Theorem 5.5.2.3]. \( S = \{ \pi_{\lambda,1} : \lambda \in \hat{A} \} \subseteq \hat{G}_p \) corresponds to

\[
\hat{A} \times \{ 1_M \}/W = \hat{A}/W \times \{ 1_M \},
\]

and it follows from [14, Theorem 6.2] that \( S \) is closed in \( \hat{G} \). Moreover, since \( A \) is a nontrivial vector group and \( W \) is finite, \( S \) is noncompact. But \( S \) is weakly equivalent to \( \pi \otimes \bar{\pi} \), where \( \pi = \pi_{1,1} = \pi \). Indeed, if for \( \lambda \in \hat{A} \), \( \sigma_{\lambda} \)
denotes the character of $MA$ defined by $\sigma_\lambda(ma) = \lambda(a)$, then by Theorems 1 and 2 of [17]

$$\pi \otimes \bar{\pi} = \text{Ind}_{MA}^G \sigma_1 \text{ and } \text{Ind}_{MA}^G \sigma_\lambda = \text{Ind}_{MA}^G \sigma_1 \quad \text{for all } \lambda,$$

so that

$$\pi \otimes \bar{\pi} \sim \{\text{Ind}_{MA}^G \sigma_\lambda; \lambda \in \hat{A}\} \sim \{\pi_\lambda; \lambda \in \hat{A}\}.$$ 

Let now $H$ be a connected semisimple Lie group with finite center. Then $H$ has a finite covering group $G$ which is acceptable [23, Sect. 8.1.1]. Denoting by $C$ the kernel of the covering homomorphism, $C$ is a finite central subgroup of $G$. Hence $G \subseteq M$ and $\pi \in \hat{G} \cap \hat{G}_r = \hat{H}$, and $\text{supp } \pi \otimes \bar{\pi} \subseteq \hat{H}$ is noncompact.

Finally, consider an arbitrary noncompact connected semisimple Lie group $F$, and let $Z(F)$ denote its center. Then $H = F/Z(F)$ has a trivial center and is noncompact, for otherwise $F$ would be the direct product of a vector group and a compact group, hence compact by semisimplicity of $F$. By the preceding paragraph, there exists $\pi \in \hat{H}, \subseteq \hat{F}$ such that $\text{supp } \pi \otimes \bar{\pi}$ is noncompact.

**Remark 2.5.** We need that a projective limit $G$ of [SIN] groups is also [SIN]. To see this, let $U$ be a neighbourhood of $e$ in $G$. Then there exists a compact normal subgroup $K$ of $G$ such that $K \subseteq U$ and $G/K \in [SIN]$. Choose a neighbourhood $V$ of $e$ in $G$ with $KV \subseteq U$, and denote by $p$ the homomorphism $G \to G/K$. There exists an open $G/K$-invariant neighbourhood $W$ of $\{K\}$ in $G/K$ contained in $p(V)$. Then $p^{-1}(W)$ is an open $G$-invariant neighbourhood of $e$ in $G$, and $p^{-1}(W) \subseteq p^{-1}(p(V)) = VK \subseteq U$.

Now it is easy to prove the following

**Theorem 2.6.** For an almost connected group $G$ the following conditions are equivalent:

(i) $\text{supp } \Pi \otimes \Omega$ is compact for all compact subsets $\Omega$ and $\Pi$ of $\hat{G}$.

(ii) $\text{supp } \pi \otimes \bar{\pi}$ is compact for all $\pi \in \hat{G}_r$.

(iii) $G$ contains a subgroup of finite index of the form $V \times K$, where $K$ is a compact group, $V$ a vector group and both are normal in $G$.

**Proof.** (iii) $\Rightarrow$ (i) follows immediately from Corollary 1.8. To show (ii) $\Rightarrow$ (iii), we note first that if $G$ is a connected Lie group satisfying (ii), then $G$ is a direct product of a vector group and a compact group. In fact, if $R$ denotes the radical of $G$, then (ii) holds for the semisimple group $G/R$ since $(\hat{G}/R) \subseteq \hat{G}_r$. By Lemma 2.4, $G/R$ is compact, and hence $G$ is amenable. The assertion then follows from Proposition 2.3.
Now, let $G$ be an arbitrary almost connected group. $G$ is a projective limit of Lie groups $G_i = G/K_i$, $i \in I$ [19, p. 175]. Then $(\hat{G}_i), \subseteq \hat{G}$, as $K_i$ is compact, and $G_i$ is a finite extension of its connected component.

If $H$ is a locally compact group with (ii), then (ii) is inherited by every normal subgroup $N$ of finite index. Indeed, given $\tau \in \hat{N}$, there exists $\pi \in \hat{G}$ such that $\tau < \pi | N$, and then the compactness of $\text{supp} \, \tau \otimes \bar{\tau}$ follows from $\tau \otimes \bar{\tau} < \pi \otimes \bar{\pi} | N$ and Corollary 1.5.

We conclude from this and from the first part of the proof, that every $G_i$ is a finite extension of a direct product of a vector group and a compact group. In particular, $G_i \in \text{[SIN]}$ for all $i$, and therefore $G \in \text{[SIN]}$ by Remark 2.5. By the structure theory of almost connected [SIN] groups, $G$ contains an open subgroup $H$ of finite index of the form $H = V \times K$, where $V$ and $K$ are as in (iii) [7, Theorem 2.9].

As an immediate consequence of Theorem 2.6 and Corollary 1.7 we obtain

**Corollary 2.7.** An infinite almost connected group cannot have a compact reduced dual.

3. Continuity of $(\pi, \rho) \to \text{supp} \, \pi \otimes \rho$

In [13, Sect. 2] we studied the continuity problem for the mapping

$$(\pi, \rho) \to \text{supp} \, \pi \otimes \rho$$

from $\hat{G} \times \hat{G}$ into the space $\mathcal{H}(\hat{G})$ of all closed subsets of $\hat{G}$, where $\mathcal{H}(\hat{G})$ carries Fell's topology [4]. Recall that a subbasis of this topology is given by the sets

$$U(C, V) = \{A \subset \mathcal{H}(\hat{G}); A \cap C = \emptyset, A \cap V \neq \emptyset\},$$

where $C$ is compact and $V$ open in $\hat{G}$. We are going to prove the following theorem, which has already been announced in [13]:

**Theorem 3.1.** Let $G$ be an amenable group containing a compact normal subgroup $K$ such that $G/K$ is a Lie group. Then the following are equivalent:

(i) $G \in \text{[FC]}$

(ii) The mapping $(\pi, \rho) \to \text{supp} \, \pi \otimes \rho$ from $\hat{G} \times \hat{G}$ into $\mathcal{H}(\hat{G})$ is continuous.

(iii) The mapping $\pi \to \text{supp} \, \pi \otimes \bar{\pi}$ from $\hat{G}$ into $\mathcal{H}(\hat{G})$ is continuous at $1_G$. 

(i) ⇒ (ii) has been shown in [13, Proposition 2]. For the readers convenience, we repeat [13, Lemma 4]:

**Lemma 3.2.** Let \( G \in [\text{SIN}] \) be amenable and first countable. If \( \pi \to \text{supp } \pi \otimes \bar{\pi} \) is continuous at \( 1_G \), then \( G \in [\text{FC}]^- \).

Given a locally compact group \( G \), a closed subgroup \( H \) and a representation \( \pi \) of \( H \), at some points it will be crucial to have \( \pi < \text{Ind}_H^G \pi \mid H \), as it was in Sections 1 and 2. This is Fell's weak Frobenius property (WF2), which does not hold in general [5, Sect. 6], but is known to be true when \( H \) is open or normal, when \( G \in [\text{SIN}] \) [10, Theorem 3.1], or when \( G \) is second countable and amenable and \( H \) is regularly related to itself [5, Theorem 5.2]. Unfortunately, in proving Theorem 3.1, we did not succeed to dispense with some other sufficient conditions which are given in the lemma below.

Recall that a locally compact group \( H \) is called \( * \)-regular if the canonical mapping

\[ \text{Prim } C^*(H) \to \text{Prim } L^1(H), \quad \ker \pi \to \ker \pi \cap L^1(H) \]

between the primitive ideal spaces of \( C^*(H) \) and \( L^1(H) \) is a homeomorphism.

**Lemma 3.3.** Let \( H \) be a \( * \)-regular closed subgroup of \( G \), and suppose that the modular functions of \( G \) and \( H \) coincide on \( H \). Then \( \pi < \text{Ind}_H^G \pi \mid H \) for every representation \( \pi \) of \( H \).

**Proof.** Since \( H \) is \( * \)-regular, for \( \pi < \text{Ind}_H^G \pi \mid H \) it suffices to show that \( \text{Ind}_H^G \pi \mid H(f) = 0 \) implies \( \pi(f) = 0 \) for \( f \in L^1(H) \). Let \( \varphi \in C_c(G) \) and consider \( g = \varphi * \mu_f \), where \( \mu_f \) denotes the Radon measure on \( G \) defined by \( f \).

Then

\[ \text{Ind}_H^G \pi(g) = \text{Ind}_H^G \pi(\varphi) \text{Ind}_H^G \pi \mid H(f) = 0 \]

and Ludwig [15, Proof of Lemma 2] has shown that there exists a local null set \( N_g \) in \( G \) such that \( x, g \mid H \in L^1(H) \) and \( \pi(x, g \mid H) = 0 \) for all \( x \notin N_g \). Now

\[ xg \mid H = (x\varphi * \mu_f) \mid H = x\varphi \mid H \ast f. \]

It follows that \( 0 = \pi(\varphi \mid H) \pi(f) \) provided that the mapping \( x \to x\varphi \mid H \) form \( G \) into \( L^1(H) \) is continuous. Denote by \( \phi \) the set of all those \( \varphi \in C_c(G) \) having this property. Then \( \phi \mid H \) contains an approximate identity for \( L^1(H) \). In fact, for every neighbourhood \( V \) of \( e \) in \( G \) choose \( \varphi_v \in C_c^+(G) \) such that \( \varphi_v(e) > 0 \) and \( \varphi_v \mid G \setminus V = 0 \) and set

\[ \varphi'_v = \| \varphi_v * \varphi_v \mid H \|_1^{-1} (\varphi_v * \varphi_v), \]
then \( \phi'_v \in \phi \) and \( \phi'_v | H * f \to f \) for every \( f \in L^1(H) \). This shows that \( \pi(f) = 0 \).

**Lemma 3.4.** Let \( G \) be a Lie group and \( A \) an abelian connected normal subgroup of \( G \) such that \( G/A \in [FC]^- \). Let \( \lambda \in \hat{A} \), and suppose that \( \tau \) is a representation of the stability group \( G_\lambda \) of \( \lambda \) with \( \tau | A \sim \lambda \). Then \( \tau < \text{Ind}^G_A \phi \).

**Proof.** The connected component \( (G/A)_0 = G_0/A \) of \( G/A \) is a compactly generated \([FC]^-\) group, hence the subgroup \((G_0/A)^e\) consisting of all compact elements in \( G_0/A \) is compact [7, Theorem 3.20]. Define a normal subgroup \( B \) of \( G \) by \( B \supseteq A \) and \( B/A = (G_0/A)^e \), and set

\[
A_\lambda = \{ a \in A : \lambda(a) = 1 \}, \quad C_\lambda = G_\lambda B \quad \text{and} \quad D = G_\lambda G_0.
\]

\( C \) and \( D \) are subgroups of \( G \), \( D \) is open, and \( C \) is closed, since \( B/A \) is compact and \( G_\lambda \supset A \). \( D \) being open in \( G \), we have \( \pi < \text{Ind}^G_A \pi | H \) for every representation \( \pi \) of \( G \). By definition of \( B \), \( G_0/B \) is a normal vector subgroup of \( G/B \). But a normal vector subgroup \( V \) in an \([FC]^-\) group \( H \) is central. In fact, \( H/H^e \) is abelian [7, Theorem (3.18)], hence \( vxv^{-1}x^{-1} \in V \cap H^e = \{0\} \) for \( v \in V \) and \( x \in H \). Using that \( G_0/B \) is contained in the center of \( G/B \), it is easily verified that \( C \) is normal in \( D \). Thus, for every representation \( \rho \) of \( C \), \( \rho < \text{Ind}^G_C \rho | C \), and therefore

\[
\rho < \text{Ind}^C_G \rho | C \sim \text{Ind}^C_G \rho | C.
\]

It remains to show that \( \tau < \text{Ind}^C_G \tau | G_\lambda \). Set

\[
K = \bigcap_{x \in C} xA_\lambda x^{-1} = \bigcap_{x \in C} A_\lambda x.
\]

Then \( K \) is normal in \( C \), \( \lambda \in \hat{C/K} \), and \( \tau \) is a representation of \( G_\lambda/K \). We are going to show that \( C/K \) has polynomial growth. It will follow then from Lemma 3.3 that \( \tau < \text{Ind}^C_G \tau | G_\lambda \). In fact, if \( C/K \) has polynomial growth, then the same is true for \( G_\lambda/K \), and hence \( G_\lambda/K \) is \(*\)-regular [2], and both groups, \( C/K \) and \( G_\lambda/K \), are unimodular [8]. Now, it follows from [8, Corollaire III.2] that \( C/K \) has polynomial growth as soon as we have shown that every compact subset of \( A/K \) is contained in a compact \( C/K \)-invariant set. For that it suffices to prove that every element of \( A/K \) has a relatively compact orbit under the action of \( C/K \) by inner automorphisms.

Consider the set \( \Delta \) of all \( \delta \in \hat{A/K} \) such that \( \Delta \) is compact. Since for \( x \in C \),

\[
G_\lambda B = xG_\lambda x^{-1}B = xG_\lambda Bx^{-1} = C,
\]
$C(\lambda^x) = B(\lambda^x)$ is compact. Hence $\lambda^x \in A$ for all $x \in C$. Thus, by definition of $K$, $A$ separates the points of $A/K$ and therefore is dense in $A/K$. Now, $A/K = \mathbb{R}^d \times F$, where $F$ is compact. Therefore, $A \cap \mathbb{R}^d$ is dense in $\mathbb{R}^d$, and we can assume that $A/K = \mathbb{R}^d$. But every continuous automorphism of a vector group is linear, so that $A$ is a linear subspace of $\mathbb{R}^d$. It follows that $A = \mathbb{R}^d$. Looking at the dual action on $\mathbb{R}^d$, i.e., that by inner automorphisms on $\mathbb{R}^d$, we conclude that every $v \in \mathbb{R}^d$ has a relatively compact $C/K$-orbit.

Lemma 3.5. Let $G$ be a Lie group and $V$ a normal vector subgroup of $G$, and suppose that $G/V \in [FC]^-$. If $\pi \to \text{supp } \pi \otimes \bar{\pi}$ is continuous at $1_G$, then $V$ is contained in the center of $G$.

Proof. Assume that $G_\lambda \neq G$ for some $\lambda \in V$, and set $H = G_\lambda$ and $\lambda_n = (1/n)\lambda$. Then $H = G_{\lambda_n}$, as automorphisms of $V$ are linear. Now,

$$1_H < \bigcup_{n=1}^{\infty} \{ \tau \in \hat{H}; \tau \mid V \sim \lambda_n \},$$

and this implies

$$\text{Ind}_{H}^{G} 1 < \bigcup_{n=1}^{\infty} \{ \text{Ind}_{H}^{G} \tau; \tau \in \hat{H}, \tau \mid V \sim \lambda_n \} \subseteq \hat{G}.$$ 

Choose $\rho \in \text{supp}(\text{Ind}_{H}^{G} 1), \rho \neq 1_G$. Since $G/V$ has a $T_1$ primitive ideal space $U = \{ A \in \mathcal{K}(\hat{G}); \rho \notin A \}$ is a neighbourhood of $\{1_G\}$ in $\mathcal{K}(\hat{G})$. Therefore, by assumption, $\rho \notin \text{supp } \pi \otimes \bar{\pi}$ for all $\pi$ in some neighbourhood $W$ of $1_G$ in $\hat{G}$. By the above, there exists $\tau \in \hat{H}$ such $\tau \mid V \sim \lambda_n$ for some $n$ and $\pi = \text{Ind}_{H}^{G} \tau \in W$. Setting $V_n = \{ v \in V; \lambda_n(v) = 1 \}$, we have $\tau \in \hat{H}/\hat{V}_n$, and $V/V_n$ is central in $H/\hat{V}_n$. Since $H/V \in [FC]^{-}$, [12, Theorem 3] yields $1_H < \tau \otimes \bar{\tau}$. By Lemma 3.4, we conclude

$$\pi \otimes \bar{\pi} = \text{Ind}_{H}^{G} \tau \otimes \text{Ind}_{H}^{G} \tau$$

$$= \text{Ind}_{H}^{G}(\tau \otimes \bar{\tau}) \supset \text{Ind}_{H}^{G}(\tau \otimes \bar{\tau}) \supset \text{Ind}_{H}^{G} 1 \geq \rho,$$

a contradiction.

Proof of Theorem 3.1. To prove (iii) $\Rightarrow$ (i), we can of course assume that $G$ is a Lie group. Let $K$ denote the unique maximal compact normal subgroup of $G_0$. Then $K$ is normal in $G$, and replacing $G$ by $G/K$ we can assume that $G_0$ has no nontrivial compact normal subgroups. The proof is now done by induction on $\text{dim } G_0$, the case $\text{dim } G_0 = 0$ being covered by Lemma 3.2. Let $\text{dim } G_0 \geq 1$ and denote by $R(G_0)$ the radical of $G_0$. Then $\text{dim } R(G_0) \geq 1$, since $G$ is amenable and $G_0$ contains no compact normal subgroup. Thus there exists a nontrivial normal vector subgroup $V$ of $G$. 
By induction hypothesis, $G/V \in [\text{FC}]^-$, and Lemma 3.5 yields that $V$ is central in $G$. Now, by the structure theory of $[\text{FC}]^-$ groups, there exists a closed connected normal subgroup $N$ of $G$ such that $V \subseteq N \subseteq G_0$, $N/V$ is compact and $G_0/N$ is a vector group [7, Theorem 3.20]. $N$ has a compact center, hence a relatively compact commutator subgroup [7, Proposition 4.4]. Again, since $G_0$ has no nontrivial compact normal subgroup, $N$ turns out to be a central vector subgroup of $G$. It follows now by the same argument as in the proof of Lemma 2.2, that $G_n$ is a vector group and hence contained in the center of $G$ (Lemma 3.5). In particular, $G \in [\text{SIN}]$, and Lemma 3.2 shows that $G \in [\text{FC}]^-$. 

It is worth mentioning that Theorem 3.1 does not hold for arbitrary locally compact groups. In fact, every group with the so-called Kazhdan property $(T)$ (i.e., $\{1_G\}$ is open in $\hat{G}$) satisfies (iii). Note that $SL(n, \mathbb{C})$ has property $(T)$ for $n \geqslant 3$ (see [21]). We conclude the paper by looking at $SL(2, \mathbb{C})$.

**Example 3.6.** The irreducible unitary representations of $G = SL(2, \mathbb{C})$ are well known: $\hat{G} = \hat{G}_p \cup \hat{G}_c \cup \{1_G\}$, where $\hat{G}_p$ and $\hat{G}_c$ denotes the principal and complementary series, respectively. $\hat{G}_p$ consists of representations $\pi_{m,t}$, where $m$ is an integer $\geqslant 0$ and $t \in \mathbb{R}$ such that $t \geqslant 0$ if $m = 0$, while representations in the complementary series are indexed by a real number $s$ with $-1 < s < 0$. The topology on $\hat{G}$ has been worked out by Fell (see [23, Sect. 7.1]): $\hat{G}_p$ and $\hat{G}_c$ carry the topology of their parameter spaces. $\hat{G}_p$ is closed in $\hat{G}$, and 

$$S \subseteq \hat{G}_c \cup \{1_G, \pi_{0,0}, \pi_{2,0}\} \quad \text{for} \quad S \subseteq \hat{G}_c.$$ 

More precisely, if $s \rightarrow 0$, then $\pi_s \rightarrow \pi_{0,0}$, and if $s \rightarrow -1$, then $\pi_s \rightarrow \pi_{2,0}$. Tensor products of irreducible representations of $G$ have been computed in [16] and [20] (see also [18]): Denoting by $\hat{G}_p^+$ (resp. $\hat{G}_p^-$) the set of all $\pi_{k,r} \in \hat{G}_p$ with $k$ even (resp. odd), the results show that

$$\text{supp}(\pi_{m,s} \otimes \pi_{n,t}) = \begin{cases} \hat{G}_p^+ & \text{if } m, n \text{ are both even or odd} \\ \hat{G}_p^- & \text{otherwise} \end{cases}$$

$$\text{supp}(\pi_{m,s} \otimes \pi_s) = \begin{cases} \hat{G}_p^+ & \text{if } m \text{ is even} \\ \hat{G}_p^- & \text{if } m \text{ is odd} \end{cases}$$

and

$$\text{supp}(\pi_s \otimes \pi_t) = \begin{cases} \hat{G}_p^+ & \text{if } s + t \geqslant -1 \\ \hat{G}_p^+ \cup \{\pi_{s+t+1}\} & \text{if } s + t < -1 \end{cases}.$$
By easy calculations, it follows from these equations and the description of the topology on $\mathcal{G}$, that mapping $(\pi, \rho) \rightarrow \text{supp } \pi \otimes \rho$ is discontinuous in exactly those points $(\pi, \rho)$ with $\pi = 1$ or $\rho = 1$. In particular, $\pi \rightarrow \text{supp } \pi \otimes \bar{\pi}$ fails to be continuous only in $1_G$.

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