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Journal of Pure and Applied Algebra 183 (2003) 105–124
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Tilting complexes associated with a sequence of idempotents

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Received 28 May 2002; received in revised form 19 December 2002
Communicated by C.A. Weibel

Abstract

First, we show that a certain sequence of idempotents $e_0, e_1, \ldots, e_l$ in a ring A defines a tilting complex $P^\bullet$ for A of term length $l+1$ and that there exists a sequence of rings $B_0 = A, B_1, \ldots, B_l = \mathrm{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$ such that for any $0 \leqslant i < l$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of term length two defined by an idempotent. Next, in the case of A being a finite dimensional algebra over a field, we provide a construction of a two-sided tilting complex corresponding to $P^\bullet$. Simultaneously, we provide a sufficient condition for an algebra B containing A as a subalgebra to be derived equivalent to A.

MSC: Primary: 18E30; 16G30; secondary: 16E05

Rickard [13] showed that the Brauer tree algebras with the same numerical invariants are derived equivalent to each other. Let A be a Brauer tree algebra corresponding to a Brauer tree whose edges are labelled $1, 2, \ldots, n$. Note that there exists a partition of the edges $\{1, \ldots, n\} = E_0 \cup \cdots \cup E_l$, where E_s consists of the edges i for which there exists a sequence of edges $i_0, i_1, \ldots, i_s = i$ such that i_0 is adjacent to the exceptional vertex and for any $0 \leqslant r < l$, $i_r \neq i_{r+1}$ and i_r, i_{r+1} have a vertex in common. He constructed a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ such that $P^j = 0$ for $j > 0$ and $j < -l$, $P^{-j} \in \mathrm{add}(\bigoplus_{i \in E_{l-j}} e_i A)$, where $e_i \in A$ is a local idempotent corresponding to the edge i, for $0 \leqslant j \leqslant l$ and $\mathrm{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$ is a Brauer "star" algebra with the same numerical

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0022-4049/03/$ - see front matter
doi:10.1016/S0022-4049(03)00012-4

invariants as A. On the other hand, Okuyama [11] pointed out recently that for Brauer tree algebras A, B with the same numerical invariants there exists a sequence of Brauer tree algebras $B_0 = A, B_1, \ldots, B_l = B$ such that B_{r+1} is the endomorphism algebra of a tilting complex for B_r of term length two defined by an idempotent. See König and Zimmermann [8] for another example of derived equivalences which are iterations of derived equivalences induced by tilting complexes of term length two. We will formulate these results.

Let A be a noetherian ring and $e_0, e_1, \ldots, e_l \in A$ a sequence of idempotents such that $\mathrm{add}(e_0A_A) = \mathscr{P}_A$, $e_{i+1} \in e_iAe_i$ for $0 \leqslant i < l$ and $\mathrm{Ext}^j_A(A/Ae_iA, e_iA) = 0$ for $0 \leqslant j < i \leqslant l$. First, we will show that there exists a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$ (Proposition 2.4), and that such a tilting complex $P^\bullet$ is essentially unique (Remark 2.3). Next, we will show that there exists a sequence of rings $B_0 = A, B_1, \ldots, B_l = \mathrm{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$ such that for any $0 \leqslant i < l$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of term length two defined by an idempotent (Theorem 2.7). Furthermore, in case A is a selfinjective artin algebra over a commutative artin ring R and $\mathrm{add}(e_iA_A) = \mathrm{add}(D(_AAe_i))$ for $1 \leqslant i \leqslant l$, where $D = \mathrm{Hom}_R(-, E(R/\mathrm{rad}\, R))$, we will show that $\mathrm{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$ is a selfinjective artin R-algebra whose Nakayama permutation coincides with that of A (Proposition 3.3). Finally, we deal with the case where A is a finite dimensional algebra over a field k and $\mathrm{add}(e_iA_A) = \mathrm{add}(D(_AAe_i))$ for $1 \leqslant i \leqslant l$, where $D = \mathrm{Hom}_k(-, k)$. We will construct a two-sided tilting complex which corresponds to $P^\bullet$ (Section 4). Simultaneously, we will provide a sufficient condition for an algebra B containing A as a subalgebra to be derived equivalent to A (Theorem 5.3).

Throughout this note, rings are associative rings with identity and modules are unitary modules. Unless otherwise stated, modules are right modules. For a ring A, we denote by A^{op} the opposite ring of A and consider left A-modules as A^{op}-modules. In case A is a finite dimensional algebra over a field k, we denote by A^{e} the enveloping algebra $A^{\mathrm{op}} \otimes_k A$. Sometimes, we use the notation X_A (resp., $_AX$) to signify that the module X considered is a right (resp., left) A-module. We denote by Mod-A the category of A-modules and by $\mathscr{P}_A$ the full additive subcategory of Mod-A consisting of finitely generated projective modules. For an object X in an additive category $\mathscr{A}$, we denote by $\mathrm{add}(X)$ the full additive subcategory of $\mathscr{A}$ consisting of objects isomorphic to direct summands of finite direct sums of copies of X. For an additive category $\mathscr{A}$, we denote by $\mathsf{K}(\mathscr{A})$ the homotopy category of cochain complexes over $\mathscr{A}$ and by $\mathsf{K}^{\mathrm{b}}(\mathscr{A})$ the full subcategories of $\mathsf{K}(\mathscr{A})$ consisting of bounded complexes. In case $\mathscr{A}$ is an abelian category, we denote by $\mathsf{D}(\mathscr{A})$ the derived category of cochain complexes over $\mathscr{A}$. Also, we denote by $\mathrm{Z}^i(X^\bullet)$, $\mathrm{Z}'^i(X^\bullet)$ and $\mathrm{H}^i(X^\bullet)$ the ith cycle, the ith cocycle and the ith cohomology of a complex $X^\bullet$, respectively. Finally, we use the notation $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) to denote the single complex associated with the double hom (resp., tensor) complex (cf. Remark 1.12). We refer to [3,18,1] for basic results in the theory of derived categories. Also, we refer to [12,14] for definitions and basic properties of tilting complexes and two-sided tilting complexes, and to e.g. [2,14–17] and so on for recent progress.

1. Preliminaries

In this section, we collect several basic facts which we need in later sections. Let A be a ring and $e \in A$ an idempotent. We identify Mod-(A/AeA) with the full subcategory of Mod-A consisting of $X \in$ Mod-A with $Xe = 0$.

Lemma 1.1. *For any* $l \geqslant 1$ *the following are equivalent*:
(1) $\mathrm{Ext}^i_A(A/AeA, eA) = 0$ *for* $0 \leqslant i < l$.
(2) $\mathrm{Ext}^i_A(-, eA)$ *vanishes on* Mod-(A/AeA) *for* $0 \leqslant i < l$.

Remark 1.2. Let A be an artin algebra over a commutative artin ring R and $D = \mathrm{Hom}_R(-, E(R/\mathrm{rad}\, R))$. Assume $\mathrm{add}(eA_A) = \mathrm{add}(D({}_A Ae))$. Then $\mathrm{Ext}^i_A(A/AeA, eA) = 0$ for $i \geqslant 0$.

Remark 1.3. The functor $- \otimes_A Ae$: Mod-$A \to$ Mod-eAe is exact and has a fully faithful left adjoint $- \otimes_{eAe} eA$: Mod-$eAe \to$ Mod-A. Furthermore, these functors induce an equivalence $\mathrm{add}(eA_A) \xrightarrow{\sim} \mathscr{P}_{eAe}$.

Remark 1.4. If $X \in$ Mod-A is noetherian, so is $X \otimes_A Ae \in$ Mod-eAe.

Remark 1.5. Let $\mathscr{A}$ be an additive category, $X \in \mathscr{A}$ and $B = \mathrm{End}_{\mathscr{A}}(X)$. Then we have a fully faithful additive functor $\mathrm{Hom}_{\mathscr{A}}(X, -)$: $\mathrm{add}(X) \to \mathscr{P}_B$. Furthermore, if every idempotent $e \in B$ splits in $\mathscr{A}$, then $\mathrm{Hom}_{\mathscr{A}}(X, -)$ induces an equivalence $\mathrm{add}(X) \xrightarrow{\sim} \mathscr{P}_B$.

Remark 1.6. Let $\mathscr{A}$, $\mathscr{B}$ be additive categories and $F: \mathscr{A} \to \mathscr{B}$ an additive covariant functor. Let $X \in \mathscr{A}$ and put $B = \mathrm{End}_{\mathscr{A}}(X)$ and $C = \mathrm{End}_{\mathscr{B}}(FX)$. Then we have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{add}(X) & \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,-)} & \mathscr{P}_B \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle -\otimes_B C} \\ \mathrm{add}(FX) & \xrightarrow{\mathrm{Hom}_{\mathscr{B}}(FX,-)} & \mathscr{P}_C. \end{array}$$

In particular, if idempotents split in $\mathscr{A}$ and $\mathscr{B}$, and if F induces a ring isomorphism $\mathrm{End}_{\mathscr{A}}(X) \xrightarrow{\sim} \mathrm{End}_{\mathscr{B}}(FX)$, then F induces an equivalence $\mathrm{add}(X) \xrightarrow{\sim} \mathrm{add}(FX)$.

Remark 1.7. For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$, the following hold:
(1) We have equivalences

$$\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, -): \mathrm{add}(P^\bullet) \xrightarrow{\sim} \mathscr{P}_B,$$

$$\mathrm{Hom}_{\mathscr{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, -): \mathrm{add}(P^\bullet) \xrightarrow{\sim} \mathscr{P}_B,$$

where $B = \mathrm{End}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet) \cong \mathrm{End}_{\mathscr{D}(\mathrm{Mod}\text{-}A)}(P^\bullet)$.

(2) The inclusion $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A) \to \mathsf{K}(\text{Mod-}A)$ induces an equivalence between $\operatorname{add}(P^\bullet)$ in $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ and $\operatorname{add}(P^\bullet)$ in $\mathsf{K}(\text{Mod-}A)$.
(3) The canonical functor $\mathsf{K}(\text{Mod-}A) \to \mathsf{D}(\text{Mod-}A)$ induces an equivalence between $\operatorname{add}(P^\bullet)$ in $\mathsf{K}(\text{Mod-}A)$ and $\operatorname{add}(P^\bullet)$ in $\mathsf{D}(\text{Mod-}A)$.

Proof. According to [1, Proposition 3.2], idempotents split in $\mathsf{K}(\text{Mod-}A)$ and $\mathsf{D}(\text{Mod-}A)$. Also, by [1, Proposition 3.4], idempotents split in $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$. Thus the assertions follow by Remark 1.6. □

Lemma 1.8. *Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$. Assume $\operatorname{Hom}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$ and $\operatorname{add}(P^\bullet)$ generates $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ as a triangulated category. Then for any $Q^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ with $\operatorname{Hom}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet \oplus Q^\bullet, (P^\bullet \oplus Q^\bullet)[i]) = 0$ for $i > 0$ we have $Q^\bullet \in \operatorname{add}(P^\bullet)$.*

Proof. For a complex $X^\bullet \in \mathsf{D}(\text{Mod-}A)$ we denote by $\mathscr{C}(X^\bullet)$ the full subcategory of $\mathsf{D}(\text{Mod-}A)$ consisting of complexes $Z^\bullet$ with $\operatorname{Hom}_{\mathscr{D}(\text{Mod-}A)}(X^\bullet, Z^\bullet[i]) = 0$ for $i \neq 0$. We denote by f the composite of canonical homomorphisms $P^\bullet \oplus Q^\bullet \to P^\bullet \to P^\bullet \oplus Q^\bullet$ and set $B = \operatorname{End}_{\mathscr{D}(\text{Mod-}A)}(P^\bullet \oplus Q^\bullet)$. Note that by [7, Remark 1.3] $\{P^\bullet[i]\}_{i\in\mathbf{Z}}$ and $\{(P^\bullet \oplus Q^\bullet)[i]\}_{i\in\mathbf{Z}}$ are generating sets for $\mathsf{D}(\text{Mod-}A)$ in the sense of Neeman [10]. We have a commutative diagram

$$\begin{array}{ccc}
\mathscr{C}(P^\bullet) & \xrightarrow{\operatorname{Hom}_{\mathscr{D}(\text{Mod-}A)}(P^\bullet,-)} & \text{Mod-}fBf \\
\uparrow{\scriptstyle \text{inc.}} & & \uparrow{\scriptstyle -\otimes_B Bf} \\
\mathscr{C}(P^\bullet \oplus Q^\bullet) & \xrightarrow{\operatorname{Hom}_{\mathscr{D}(\text{Mod-}A)}(P^\bullet \oplus Q^\bullet,-)} & \text{Mod-}B.
\end{array}$$

By [6, Theorem 1.3] the horizontal functors are equivalences. Thus, since $-\otimes_B Bf : \text{Mod-}B \to \text{Mod-}fBf$ is dense, $\mathscr{C}(P^\bullet) = \mathscr{C}(P^\bullet \oplus Q^\bullet)$ and $-\otimes_B Bf : \text{Mod-}B \to \text{Mod-}fBf$ is an equivalence. On the other hand, we have a commutative diagram

$$\begin{array}{ccc}
\operatorname{add}(P^\bullet) & \xrightarrow{\operatorname{Hom}_{\mathscr{D}(\text{Mod-}A)}(P^\bullet,-)} & \mathscr{P}_{fBf} \\
\downarrow{\scriptstyle \text{inc.}} & & \uparrow{\scriptstyle -\otimes_B Bf} \\
\operatorname{add}(P^\bullet \oplus Q^\bullet) & \xrightarrow{\operatorname{Hom}_{\mathscr{D}(\text{Mod-}A)}(P^\bullet \oplus Q^\bullet,-)} & \mathscr{P}_B.
\end{array}$$

According to [1, Proposition 3.4], it follows by Remark 1.5 that the horizontal functors are equivalences. Now, since $-\otimes_B Bf : \mathscr{P}_B \to \mathscr{P}_{fBf}$ is an equivalence, $\operatorname{add}(P^\bullet) = \operatorname{add}(P^\bullet \oplus Q^\bullet)$ and $Q^\bullet \in \operatorname{add}(P^\bullet)$. □

Lemma 1.9. *Let $l \geqslant 0$ and $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ with $P^i = 0$ for $i > 0$ and $i < -l$. Assume $\mathrm{H}^{-i}(P^\bullet) \in \text{Mod-}(A/AeA)$ for $0 \leqslant i < l$. Then the following hold.*
(1) $\mathrm{H}^{-i}(\operatorname{Hom}^\bullet_A(eA, P^\bullet)) = 0$ for $0 \leqslant i < l$.
(2) If $\operatorname{Ext}^i_A(A/AeA, eA) = 0$ for $0 \leqslant i < l$, then $\mathrm{H}^i(\operatorname{Hom}^\bullet_A(P^\bullet, eA)) = 0$ for $0 \leqslant i < l$.

Proof. (1) We have $\mathrm{H}^{-i}(\operatorname{Hom}^\bullet_A(eA, P^\bullet)) \cong \operatorname{Hom}_A(eA, \mathrm{H}^{-i}(P^\bullet)) = 0$ for all $0 \leqslant i < l$.

(2) In case $l = 0$, we have nothing to prove. Assume $l \geqslant 1$. Since $\mathrm{H}^0(P^\bullet) \in \text{Mod-}(A/AeA)$, $\mathrm{H}^0(\mathrm{Hom}^\bullet_A(P^\bullet, eA)) \cong \mathrm{Hom}_A(\mathrm{H}^0(P^\bullet), eA) = 0$. For each $0 < i < l$, let $\pi^{-i} : P^{-i} \to \mathrm{Z}'^{-i}(P^\bullet)$ be the canonical epimorphism and $\varphi^{-i} : \mathrm{Z}'^{-i}(P^\bullet) \to P^{-i+1}$ the homomorphism with $d_P^{-i} = \varphi^{-i} \circ \pi^{-i}$. Then we have exact sequences in Mod-A

$$(c_i): \quad 0 \to \mathrm{H}^{-i}(P^\bullet) \to \mathrm{Z}'^{-i}(P^\bullet) \xrightarrow{\varphi^{-i}} P^{-i+1} \to \mathrm{Z}'^{-i+1}(P^\bullet) \to 0.$$

For any $0 < m < l$, by applying $\mathrm{Hom}_A(-, eA)$ to $(c_m), \ldots, (c_1)$ successively, we get

$$\begin{aligned} \mathrm{H}^m(\mathrm{Hom}^\bullet_A(P^\bullet, eA)) &\cong \mathrm{Cok}\,(\mathrm{Hom}_A(\varphi^{-m}, eA)) \\ &\cong \mathrm{Ext}^1_A(\mathrm{Z}'^{-m+1}(P^\bullet), eA) \\ &\quad\vdots \\ &\cong \mathrm{Ext}^m_A(\mathrm{Z}'^0(P^\bullet), eA) \\ &= 0. \qquad \square \end{aligned}$$

Lemma 1.10. *Let $l \geqslant 0$ and $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ a tilting complex with $P^i = 0$ for $i > 0$ and $i < -l$. Assume $\mathrm{H}^{-i}(P^\bullet) \in \text{Mod-}(A/AeA)$ for $0 \leqslant i < l$. Then $eA[l] \in \mathrm{add}(P^\bullet)$.*

Proof. Since $\mathrm{H}^{-i}(P^\bullet \oplus eA[l]) \in \text{Mod-}(A/AeA)$ for $0 \leqslant i < l$, by Lemma 1.9(1) $\mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet \oplus eA[l], (P^\bullet \oplus eA[l])[i]) = 0$ for $i > 0$. The assertion follows by Lemma 1.8. $\square$

Lemma 1.11 (cf. Hoshino and Kato [5, Lemma 2.1]). *Let $\mathscr{A}$ be an additive category. Let $l \geqslant 1$, $P_1, \ldots, P_l \in \mathscr{A}$ with $\mathrm{add}(P_{i+1}) \subset \mathrm{add}(P_i)$ for $1 \leqslant i < l$ and $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{A})$ with $P^i = 0$ for $i > 0$ and $i < -l$ and $P^{-i} \in \mathrm{add}(P_i)$ for $1 \leqslant i \leqslant l$. Then the following hold*:

(1) *If $\mathrm{H}^{-j}(\mathrm{Hom}^\bullet_{\mathscr{A}}(P_i, P^\bullet)) = 0$ for $0 \leqslant j < i \leqslant l$, then $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$.*
(2) *If $\mathrm{H}^j(\mathrm{Hom}^\bullet_{\mathscr{A}}(P^\bullet, P_i)) = 0$ for $0 \leqslant j < i \leqslant l$, then $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(P^\bullet, P^\bullet[i]) = 0$ for $i < 0$.*

Proof. Although this is essentially the same as [5, Lemma 2.1], we include a proof for the benefit of the reader.

(1) Let $0 < i \leqslant l$ and $f : P^\bullet \to P^\bullet[i]$ a cochain map. Put $h^0 = 0 : P^{-i+1} \to P^0$. Since $\mathrm{Hom}_{\mathscr{A}}(P^{-i}, d_P^{-1})$ is epic, there exists $h^{-1} : P^{-i} \to P^{-1}$ such that $f^{-i} - h^0 \circ d_P^{-i} = d_P^{-1} \circ h^{-1}$. Next, let $0 \leqslant j \leqslant l - i$ and assume there exist $h^{-k} : P^{-k-i+1} \to P^{-k}$ such that $f^{-k-i+1} - h^{-k+1} \circ d_P^{-k-i+1} = d_P^{-k} \circ h^{-k}$ for all $0 \leqslant k \leqslant j$. Then, since $d_P^{-j} \circ (f^{-j-i} - h^{-j} \circ d_P^{-j-i}) = 0$, and since

$$\mathrm{Hom}_{\mathscr{A}}(P^{-j-i}, P^{-j-1}) \to \mathrm{Hom}_{\mathscr{A}}(P^{-j-i}, P^{-j}) \to \mathrm{Hom}_{\mathscr{A}}(P^{-j-i}, P^{-j+1})$$

is exact, there exists $h^{-j-1} : P^{-j-i} \to P^{-j-1}$ such that $f^{-j-i} - h^{-j} \circ d_P^{-j-i} = d_P^{-j-1} \circ h^{-j-1}$. Thus by induction we get a homotopy $h : f \simeq 0$.

(2) Similar to (1). □

Remark 1.12 (Rickard [14, Section 4]). Let A, B and C be algebras over a field k. Then the following hold.

(1) Let $X^\bullet \in \mathsf{K}^-(\text{Mod-}(B^{\text{op}} \otimes_k A))$ and $Y^\bullet \in \mathsf{K}^+(\text{Mod-}(C^{\text{op}} \otimes_k A))$. If either each term of $X^\bullet$ is projective as an A-module or each term of $Y^\bullet$ is injective as an A-module, the canonical homomorphism in $\mathsf{D}(\text{Mod-}(C^{\text{op}} \otimes_k B))$

$$\text{Hom}_A^\bullet(X^\bullet, Y^\bullet) \to \boldsymbol{R}\,\text{Hom}_A^\bullet(X^\bullet, Y^\bullet)$$

is an isomorphism.

(2) Let $X^\bullet \in \mathsf{K}^-(\text{Mod-}(B^{\text{op}} \otimes_k A))$ and $Y^\bullet \in \mathsf{K}^-(\text{Mod-}(A^{\text{op}} \otimes_k C))$. If either each term of $X^\bullet$ is flat as an A-module or each term of $Y^\bullet$ is flat as an A^{op}-module, the canonical homomorphism in $\mathsf{D}(\text{Mod-}(B^{\text{op}} \otimes_k C))$

$$X^\bullet \otimes_A^{\boldsymbol{L}} Y^\bullet \to X^\bullet \otimes_A^\bullet Y^\bullet$$

is an isomorphism.

2. General case

In this section, we will show that a certain sequence of idempotents $e_0, e_1, \ldots, e_l$ in a ring A defines a tilting complex $P^\bullet \in \mathsf{K}^{\text{b}}(\mathscr{P}_A)$ of term length $l+1$ and that there exists a sequence of rings $B_0 = A, B_1, \ldots, B_l = \text{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$ such that for any $0 \leqslant i < l$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of term length two defined by an idempotent.

Definition 2.1 (Hartshorne [3]). For a complex $X^\bullet$ and $n \in \mathbf{Z}$, we define the following truncation:

$$\tau_{\geqslant n}(X^\bullet): \quad \cdots \to 0 \to X^n \to X^{n+1} \to X^{n+2} \to \cdots .$$

Lemma 2.2. *Let $l \geqslant 0$ and $P^\bullet \in \mathsf{K}^{\text{b}}(\mathscr{P}_A)$ a tilting complex with $P^i = 0$ for $i > 0$ and $i < -l$. Assume there exists an idempotent $e \in A$ such that $\text{H}^{-i}(P^\bullet) \in \text{Mod-}(A/AeA)$ for $0 \leqslant i < l$, $\text{Z}^{-l}(P^\bullet) \otimes_A Ae_{eAe}$ is finitely generated and $\text{Ext}_A^i(A/AeA, eA) = 0$ for $0 \leqslant i \leqslant l$. Then there exists a tilting complex $\hat{P}^\bullet \in \mathsf{K}^{\text{b}}(\mathscr{P}_A)$ such that $\hat{P}^i = 0$ for $i > 0$ and $i < -l-1$, $\hat{P}^{-l-1} \in \text{add}(eA)$, $\text{H}^{-l}(\hat{P}^\bullet) \in \text{Mod-}(A/AeA)$ and $\tau_{\geqslant -l}(\hat{P}^\bullet) = P^\bullet$.*

Proof. Let $f: Q^{-l-1} \to \text{Z}^{-l}(P^\bullet) \otimes_A Ae$ be an epimorphism in Mod-eAe with $Q^{-l-1} \in \mathscr{P}_{eAe}$ and set

$$g = \varphi \circ \mu \circ (f \otimes_{eAe} eA): Q^{-l-1} \otimes_{eAe} eA \to P^{-l},$$

where $\varphi\colon \mathrm{Z}^{-l}(P^\bullet) \to P^{-l}$ is the inclusion and $\mu\colon \mathrm{Z}^{-l}(P^\bullet)\otimes_A Ae \otimes_{eAe} eA \to \mathrm{Z}^{-l}(P^\bullet)$ is the multiplication map. Denote by $P'^\bullet$ the mapping cone of $g[l]\colon Q^{-l-1} \otimes_{eAe} eA[l] \to P^\bullet$ and set $\hat{P}^\bullet = P'^\bullet \oplus eA[l+1]$. We claim that $\hat{P}^\bullet$ is a complex as desired. It is obvious that $\hat{P}^{-l-1} \in \mathrm{add}(eA)$ and $\tau_{\geqslant -l}(\hat{P}^\bullet) = P^\bullet$. Since $P'^\bullet \otimes^\bullet_A Ae$ is isomorphic to the mapping cone of $((\varphi\otimes_A Ae)\circ f)[l]\colon Q^{-l-1}[l] \to P^\bullet \otimes^\bullet_A Ae$, we have $\mathrm{H}^i(P'^\bullet)\otimes_A Ae \cong \mathrm{H}^i(P'^\bullet \otimes^\bullet_A Ae) = 0$ for $i \neq -l-1$ and $\mathrm{H}^i(\hat{P}^\bullet) \in \text{Mod-}(A/AeA)$ for $i \neq -l-1$. Note that we have a distinguished triangle in $\mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ of the form

$$\hat{P}^{-l-1}[l] \to P^\bullet \to \hat{P}^\bullet \to .$$

Since $\hat{P}^{-l-1}[l+1] \in \mathrm{add}(\hat{P}^\bullet)$, it follows that $\mathrm{add}(\hat{P}^\bullet)$ generates $\mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ as a triangulated category. It only remains to prove the following.

Claim. $\mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(\hat{P}^\bullet, \hat{P}^\bullet[i]) = 0$ *for* $i \neq 0$.

Proof. We set $\mathrm{Ext}^i(-,-) = \mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(-,-[i])$ for $i \in \mathbf{Z}$. Then, by applying $\mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet, -)$ to the above distinguished triangle, we get exact sequences

$$\mathrm{Ext}^i(P^\bullet, P^\bullet) \to \mathrm{Ext}^i(P^\bullet, \hat{P}^\bullet) \to \mathrm{Ext}^{i+l+1}(P^\bullet, \hat{P}^{-l-1}).$$

Since $\hat{P}^{-l-1} \in \mathrm{add}(eA)$, by Lemma 1.9 $\mathrm{Ext}^{i+l+1}(P^\bullet, \hat{P}^{-l-1}) = 0$ for $i \neq -1$ and $\mathrm{Ext}^i(P^\bullet, \hat{P}^\bullet) = 0$ for $i \neq 0, -1$. Next, apply $\mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(-, \hat{P}^\bullet)$ to the above distinguished triangle. Then we get exact sequences

$$\mathrm{Ext}^{i-l-1}(\hat{P}^{-l-1}, \hat{P}^\bullet) \to \mathrm{Ext}^i(\hat{P}^\bullet, \hat{P}^\bullet) \to \mathrm{Ext}^i(P^\bullet, \hat{P}^\bullet).$$

Since $\mathrm{H}^i(\hat{P}^\bullet) \in \text{Mod-}(A/AeA)$ for $i \neq -l-1$, by Lemma 1.9 $\mathrm{Ext}^{i-l-1}(\hat{P}^{-l-1}, \hat{P}^\bullet) = 0$ for $i \neq 0$. Thus $\mathrm{Ext}^i(\hat{P}^\bullet, \hat{P}^\bullet) = 0$ for $i \neq 0, -1$. By the dual argument, we also have $\mathrm{Ext}^i(\hat{P}^\bullet, \hat{P}^\bullet) = 0$ for $i \neq 0, 1$. □

Throughout the rest of this section, $e_0, e_1, \ldots$ are idempotents in A such that $\mathrm{add}(e_0A) = \mathscr{P}_A$ and $e_{i+1} \in e_iAe_i$ for all $i \geqslant 0$.

Remark 2.3. Let $l \geqslant 0$ and $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ a complex such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$. Assume $\mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$ and $\mathrm{add}(P^\bullet)$ generates $\mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ as a triangulated category. Then $\mathrm{add}(P^\bullet)$ is uniquely determined.

Proof. Let $P'^\bullet \in \mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ be another complex satisfying the same conditions as $P^\bullet$. Then, since $\mathrm{H}^{-j}(P^\bullet \oplus P'^\bullet) \cong \mathrm{H}^{-j}(P^\bullet) \oplus \mathrm{H}^{-j}(P'^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$, by Lemmas 1.9(1) and 1.11(1) $\mathrm{Hom}_{\mathscr{D}(\text{Mod-}A)}(P^\bullet \oplus P'^\bullet, (P^\bullet \oplus P'^\bullet)[i]) = 0$ for $i > 0$. It follows by Lemma 1.8 that $\mathrm{add}(P^\bullet) = \mathrm{add}(P'^\bullet)$. □

Proposition 2.4. *Assume A is right noetherian. Let $l \geqslant 0$ and assume $\mathrm{Ext}^j_A(A/Ae_iA, e_iA) = 0$ for $0 \leqslant j < i \leqslant l$. Then there exists a tilting complex $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$.*

Proof. We make use of induction on $l \geqslant 0$. Note first that $\mathrm{add}(e_{i+1}A) \subset \mathrm{add}(e_iA)$ for $i \geqslant 0$. In case $l = 0$, e_0A_A is a required tilting complex. Let $l \geqslant 1$ and assume there exists a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l+1$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l-1$ and $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l-1$. Assume $\mathrm{Ext}^j_A(A/Ae_iA, e_iA) = 0$ for $0 \leqslant j < i \leqslant l$. For any $0 \leqslant j \leqslant l-1$, since $e_l \in e_jAe_j$, we have $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae_lA)$. Thus it follows by Lemma 2.2 that there exists a tilting complex $\hat{P}^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ such that $\hat{P}^i = 0$ for $i > 0$ and $i < -l$, $\hat{P}^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(\hat{P}^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$. □

Remark 2.5. In case $e_1 = \cdots = e_l$, the construction above is the same as in [5, Theorem 2.3].

Remark 2.6. Assume in Proposition 2.3 that $\mathrm{Ext}^j_A(A/Ae_iA, A) = 0$ for $0 \leqslant j < i \leqslant l$. Then $\mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet, A[l]) \in \text{Mod-}A^{\mathrm{op}}$ is a tilting module of projective dimension at most l. This construction would be the same as in [9, Theorem 2.2].

Theorem 2.7. *Let $l \geqslant 0$ and assume $\mathrm{Ext}^j_A(A/Ae_iA, e_iA) = 0$ for $0 \leqslant j < i \leqslant l$. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ be a tilting complex such that $P^i = 0$ for $i > 0$ and $i < -l-1$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l+1$ and $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l+1$. Then the following hold.*

(1) *There exists a tilting complex $\bar{P}^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ such that $\bar{P}^i = 0$ for $i > 0$ and $i < -l$, $\bar{P}^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$, $\mathrm{H}^{-j}(\bar{P}^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$ and $e_{l+1}A[l]$ is a direct summand of $\bar{P}^\bullet$. Furthermore, we have a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ of the form*

$$P[l] \to \bar{P}^\bullet \to P^\bullet \to$$

with $P = P^{-l-1} \oplus e_{l+1}A$.

(2) *Let $B = \mathrm{End}_{\mathscr{K}(\text{Mod-}A)}(\bar{P}^\bullet)$ and $f \in B$ the composite of canonical homomorphisms $\bar{P}^\bullet \to e_{l+1}A[l] \to \bar{P}^\bullet$. Then $\mathrm{Hom}_B(B/BfB, fB) = 0$ and there exists a tilting complex $Q^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_B)$ such that $Q^i = 0$ for $i \neq 0, -1$, $Q^{-1} \in \mathrm{add}(fB)$, $\mathrm{H}^0(Q^\bullet) \in \text{Mod-}(B/BfB)$ and $\mathrm{End}_{\mathscr{K}(\text{Mod-}B)}(Q^\bullet) \cong \mathrm{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$.*

Proof. (1) Denote by $X^\bullet$ the mapping cone of $\mathrm{id}_{e_{l+1}A} : e_{l+1}A \to e_{l+1}A$. Set $\bar{P}^\bullet = \tau_{\geqslant -l}(P^\bullet \oplus X^\bullet[l]) = \tau_{\geqslant -l}(P^\bullet) \oplus e_{l+1}A[l]$ and $P = P^{-l-1} \oplus e_{l+1}A \in \mathrm{add}(e_{l+1}A)$. Then, since $P^\bullet \oplus X^\bullet[l] \cong P^\bullet$ in $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$, we have a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ of the form

$$P[l] \xrightarrow{\varphi} \bar{P}^\bullet \to P^\bullet \to,$$

where $\varphi = \mathrm{diag}(d_P^{-l-1}, \mathrm{id}_{e_{l+1}A})[l]$. Note that $e_{l+1}A[l]$ is a direct summand of $\bar{P}^\bullet$. Thus $P[l] \in \mathrm{add}(\bar{P}^\bullet)$ and $\mathrm{add}(\bar{P}^\bullet)$ generates $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ as a triangulated category. Note also that $\bar{P}^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(\bar{P}^\bullet) \in \text{Mod-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$. Thus by Lemmas 1.9 and 1.11 $\mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(\bar{P}^\bullet, \bar{P}^\bullet[i]) = 0$ for $i \neq 0$.

(2) Note first that we have an equivalence $\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(\bar{P}^\bullet, -) : \mathrm{add}(\bar{P}^\bullet) \xrightarrow{\sim} \mathscr{P}_B$. Denote by $Y^\bullet \in \mathsf{K}(\mathsf{K}(\mathrm{Mod}\text{-}A))$ the complex

$$\cdots \to 0 \to P[l] \xrightarrow{\varphi} \bar{P}^\bullet \to 0 \to \cdots,$$

where $\bar{P}^\bullet$ is in degree 0, and set $Q^\bullet = \mathrm{Hom}^\bullet_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(\bar{P}^\bullet, Y^\bullet) \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_B)$. Namely, $Q^i = 0$ for $i \neq 0, -1$, $Q^0 = \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(\bar{P}^\bullet, \bar{P}^\bullet) = B$, $Q^{-1} = \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(\bar{P}^\bullet, P[l])$ and $d_Q^{-1} = \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(\bar{P}^\bullet, \varphi)$. Note that $fB[1]$ is a direct summand of $Q^\bullet$ and that, since $\mathrm{add}(P) = \mathrm{add}(e_{l+1}A)$, we have $\mathrm{add}(Q^{-1}) = \mathrm{add}(fB)$. In particular, $Q^{-1}[1] \in \mathrm{add}(Q^\bullet)$ and $\mathrm{add}(Q^\bullet)$ generates $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_B)$ as a triangulated category. We claim first that $\mathrm{H}^0(Q^\bullet) \in \mathrm{Mod}\text{-}(B/BfB)$. Note that $\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P[l], P^\bullet) = 0$. Thus, by applying $\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P[l], -)$ to the distinguished triangle in (1), we conclude that $\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P[l], \varphi)$ is epic and so is $\mathrm{Hom}_B(Q^{-1}, d_Q^{-1})$. It follows that $\mathrm{H}^0(Q^\bullet) \in \mathrm{Mod}\text{-}(B/BfB)$ and hence, since $\mathrm{B}^0(Q^\bullet) \subset BfB$, $\mathrm{H}^0(Q^\bullet) = B/BfB$. Similarly, since $\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P[l]) = 0$, $\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(\varphi, P[l])$ is monic and so is $\mathrm{Hom}_B(d_Q^{-1}, Q^{-1})$. It follows that $\mathrm{Hom}_B(B/BfB, fB) = 0$. Consequently, by Lemmas 1.9 and 1.11 $Q^\bullet$ is a tilting complex for B. Finally, let $C = \mathrm{End}_{\mathscr{K}(\mathrm{Mod}\text{-}B)}(Q^\bullet)$. Then we have an equivalence of triangulated categories $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_C) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ which sends C to $P^\bullet$. Thus $\mathrm{End}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet) \cong C$. □

3. The case of artin algebras

In this section, we will apply the results of the preceding section to the case where A is an artin algebra over a commutative artin ring R. We set $D = \mathrm{Hom}_R(-, E(R/\mathrm{rad}\, R))$.

According to Proposition 2.4 together with Remark 1.2, we have the following.

Proposition 3.1. *Let $e_0, e_1, e_2, \ldots$ be idempotents in A such that $\mathrm{add}(e_0A) = \mathscr{P}_A$ and $e_{i+1} \in e_iAe_i$ for $i \geqslant 0$. Let $l \geqslant 0$ and assume $\mathrm{add}(e_iA_A) = \mathrm{add}(D({}_AAe_i))$ for $1 \leqslant i \leqslant l$. Then there exists a tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$.*

Proposition 3.2. *Let $e_0, e_1, e_2, \ldots$ be idempotents in A such that $\mathrm{add}(e_0A) = \mathscr{P}_A$ and $e_{i+1} \in e_iAe_i$ for $i \geqslant 0$. Let $l \geqslant 0$ and assume $\mathrm{add}(e_iA_A) = \mathrm{add}(D({}_AAe_i))$ for $1 \leqslant i \leqslant l+1$. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ be a tilting complex such that $P^i = 0$ for $i > 0$ and $i < -l-1$, $P^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l+1$ and $\mathrm{H}^{-j}(P^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l+1$. Then the following hold.*

(1) *There exists a tilting complex $\bar{P}^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ such that $\bar{P}^i = 0$ for $i > 0$ and $i < -l$, $\bar{P}^{-i} \in \mathrm{add}(e_iA)$ for $0 \leqslant i \leqslant l$, $\mathrm{H}^{-j}(\bar{P}^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_iA)$ for $0 \leqslant j < i \leqslant l$ and $e_{l+1}A[l]$ is a direct summand of $\bar{P}^\bullet$. Furthermore, we have a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ of the form*

$$P[l] \to \bar{P}^\bullet \to P^\bullet \to$$

with $P = P^{-l-1} \oplus e_{l+1}A$.

(2) *Let* $B = \operatorname{End}_{\mathscr{K}(\text{Mod-}A)}(\bar{P}^\bullet)$ *and* $f \in B$ *the composite of canonical homomorphisms* $\bar{P}^\bullet \to e_{l+1}A[l] \to \bar{P}^\bullet$. *Then* $\operatorname{add}(fB_B) = \operatorname{add}(D({}_BBf))$ *and there exists a tilting complex* $Q^\bullet \in \mathsf{K}^\mathrm{b}(\mathscr{P}_B)$ *such that* $Q^i = 0$ *for* $i \neq 0, -1$, $Q^{-1} \in \operatorname{add}(fB)$, $\mathrm{H}^0(Q^\bullet) \in \text{Mod-}(B/BfB)$ *and* $\operatorname{End}_{\mathscr{K}(\text{Mod-}B)}(Q^\bullet) \cong \operatorname{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$.

Proof. According to Theorem 2.7 together with Remark 1.2, we have only to show that $\operatorname{add}(fB) = \operatorname{add}(D(Bf))$. We have $fB_B \cong \operatorname{Hom}_{\mathscr{K}(\text{Mod-}A)}(\bar{P}^\bullet, e_{l+1}A[l])$: On the other hand, by [4, Lemma 3.1]

$$\begin{aligned} D({}_BBf) &\cong D\operatorname{Hom}_{\mathscr{K}(\text{Mod-}A)}(e_{l+1}A[l], \bar{P}^\bullet) \\ &\cong \operatorname{Hom}_{\mathscr{K}(\text{Mod-}A)}(\bar{P}^\bullet, D(Ae_{l+1})[l]). \end{aligned}$$

Thus, since $\operatorname{add}(e_{l+1}A) = \operatorname{add}(D(Ae_{l+1}))$, $\operatorname{add}(fB) = \operatorname{add}(D(Bf))$. □

Consider next the case of A being selfinjective. Let $\{e_1, \ldots, e_n\}$ be a basic set of orthogonal local idempotents in A and $I_0 = \{1, \ldots, n\}$. Set $\nu = D \circ \operatorname{Hom}_A(-, A)$. Then there exists a permutation σ of I_0, called the Nakayama permutation, such that $\nu(e_iA) \cong e_{\sigma(i)}A$ for all $i \in I_0$.

Proposition 3.3. *Let* $I_0 \supset I_1 \supset I_2 \supset \cdots$ *be a descending sequence of nonempty* σ*-stable subsets of* I_0 *and* $e^{(i)} = \sum_{j \in I_i} e_j$ *for* $i \geqslant 0$. *Then for any* $l \geqslant 0$ *there exists a tilting complex* $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ *such that* $P^i = 0$ *for* $i > 0$ *and* $i < -l$, $P^{-i} \in \operatorname{add}(e^{(i)}A)$ *for* $0 \leqslant i \leqslant l$ *and* $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae^{(i)}A)$ *for* $0 \leqslant j < i \leqslant l$. *Furthermore,* $\operatorname{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$ *is a selfinjective artin algebra whose Nakayama permutation coincides with* σ.

Proof. By Proposition 3.1 there exists a tilting complex $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \operatorname{add}(e^{(i)}A)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae^{(i)}A)$ for $0 \leqslant j < i \leqslant l$. Note also that $\nu(P^\bullet) \in \mathsf{K}^\mathrm{b}(\mathscr{P}_A)$ is a tilting complex and $\nu(P^{-i}) \in \operatorname{add}(e^{(i)}A)$ for $0 \leqslant i \leqslant l$. For any $P \in \mathscr{P}_A$ and $X \in \text{Mod-}A$ we have an isomorphism

$$P \otimes_A \operatorname{Hom}_A(X, A) \xrightarrow{\sim} \operatorname{Hom}_A(X, P), \qquad p \otimes h \mapsto (x \mapsto ph(x)).$$

Thus, since $\nu(e^{(i)}A) \cong e^{(i)}A$ for $0 \leqslant i \leqslant l$, we have

$$\begin{aligned} \operatorname{Hom}_A(e^{(i)}A, \mathrm{H}^{-j}(\nu(P^\bullet))) &\cong \operatorname{Hom}_A(e^{(i)}A, \nu(\mathrm{H}^{-j}(P^\bullet))) \\ &\cong D(e^{(i)}A \otimes_A \operatorname{Hom}_A(\mathrm{H}^{-j}(P^\bullet), A)) \\ &\cong D\operatorname{Hom}_A(\mathrm{H}^{-j}(P^\bullet), e^{(i)}A) \\ &\cong D\operatorname{Hom}_A(\mathrm{H}^{-j}(P^\bullet), \nu(e^{(i)}A)) \\ &\cong \mathrm{H}^{-j}(P^\bullet) \otimes_A \operatorname{Hom}_A(e^{(i)}A, A) \\ &\cong \mathrm{H}^{-j}(P^\bullet) \otimes_A Ae^{(i)} \\ &= 0 \end{aligned}$$

for $0 \leqslant j < i \leqslant l$. Thus by Remark 2.3 $\mathrm{add}(P^\bullet) = \mathrm{add}(v(P^\bullet))$. Since by [4, Lemma 3.1] we have an isomorphism in Mod-B

$$D\,\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet) \cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, v(P^\bullet)),$$

it follows that $\mathrm{End}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is selfinjective. It remains to show that the Nakayama permutation of $\mathrm{End}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ coincides with σ. In case $l=0$, this is obvious because $\mathrm{End}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ is Morita equivalent to A. Assume $l \geqslant 1$. Then, according to [4, Theorem 3.4], Proposition 3.2 enables us to make use of induction. □

4. Two-sided tilting complexes

Let A be a finite dimensional algebra over a field k and $D = \mathrm{Hom}_k(-, k)$. Our aim is to construct two-sided tilting complexes which correspond to tilting complexes constructed in Proposition 3.1 (see Remark 4.7). According to Proposition 3.2, we have only to deal with tilting complexes of term length two.

Taking Remark 2.3 into account, we will first construct a two-sided tilting complex $T^\bullet$ corresponding to the following tilting complex $S^\bullet$ (see Theorem 4.6). Recall that an idempotent $e \in A$ is called local if eAe is a local ring. Let $\{e_1, \ldots, e_n\}$ be a basic set of orthogonal local idempotents in A and J the Jacobson radical of A. We fix a nonempty subset I_0 of $I = \{1, \ldots, n\}$ and define $S^\bullet$ as the mapping cone of the multiplication map

$$\rho : \bigoplus_{i \in I_0} Ae_i \otimes_k e_iA \to A.$$

We set $e = \sum_{i \in I_0} e_i$, $B = \mathrm{End}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^\bullet)$ and $d_{ij} = \dim_k e_iAe_j$ for $i, j \in I_0$. We assume the following conditions are satisfied:

(a_1) there exists a permutation σ of I_0 such that $e_iA_A \cong D({}_AAe_{\sigma(i)})$ for all $i \in I_0$;
(a_2) $e_iJe_{\sigma(i)} \neq 0$ for all $i \in I_0$; and
(a_3) $e_iAe_i/e_iJe_i \cong k$ for all $i \in I_0$.

Remark 4.1. In case k is an algebraically closed field and A is a symmetric k-algebra without semisimple algebra summands, the conditions (a_1) $\sim$ (a_3) are satisfied for any nonempty subset I_0 of I.

Remark 4.2. For any $i, j \in I_0$ the following hold.

(1) $e_iAe_j \cong D(e_jAe_{\sigma(i)}) \cong e_{\sigma(i)}Ae_{\sigma(j)}$.
(2) ${}_A\mathrm{Hom}_A(Ae_{\sigma(j)} \otimes_k e_{\sigma(i)}A_A, A_A)_A \cong {}_AAe_{\sigma(i)} \otimes_k e_jA_A \cong D({}_AAe_{\sigma(j)} \otimes_k e_iA_A)$.
(3) $e_i \otimes e_j \in A^{\mathrm{e}}$ is a local idempotent.

Proof. (1) and (2) follow by condition (a_1) and (3) follows by the condition (a_3). □

Remark 4.3. For any $i, j \in I_0$ the following hold.

(1) $d_{ij} = d_{j,\sigma(i)} = d_{\sigma(i),\sigma(j)}$.

(2) $d_{ij} \geqslant 1$ if either $j = i$ or $j = \sigma(i)$.
(3) $d_{ij} \geqslant 2$ if $j = i = \sigma(i)$.

Proof. (1) follows by Remark 4.2(1) and (2), (3) follow by the condition (a_2). □

Proposition 4.4. *The following hold.*

(1) $S^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ *is a tilting complex with* $\mathrm{H}^0(S^\bullet) \in \text{Mod-}(A/AeA)$.
(2) *The left multiplication of* A *on each homogeneous component of* $S^\bullet$ *gives rise to an injective* k*-algebra homomorphism* $\varphi : A \to B$.
(3) ${}_A(B/A)_A \cong \bigoplus_{i,j \in I_0} ({}_A Ae_i \otimes_k e_j A_A)^{(\alpha_{ij})}$, *where*

$$\alpha_{ij} = \begin{cases} d_{ji} - 2 & \text{if } i = j = \sigma(j), \\ d_{ji} - 1 & \text{if } j \neq \sigma(j) \text{ and } i \in \{j, \sigma(j)\}, \\ d_{ji} & \text{otherwise.} \end{cases}$$

(4) *For any* $i \in I_0$, $e_i B_B \cong \bigoplus_{j \in I_0} \mathrm{Hom}_{\mathscr{K}(\text{Mod-}A)}(S^\bullet, e_{\sigma(j)} A[1])^{(\mu_{ij})}$, *where*

$$\mu_{ij} = \begin{cases} d_{ji} - 1 & \text{if } i = \sigma(j), \\ d_{ji} & \text{otherwise.} \end{cases}$$

Proof. (1) See [4, Propositions 2.4(1) and 5..1(1)].

(2) We have to show the injectivity of φ. Set $V = \bigoplus_{i \in I_0} Ae_i \otimes_k e_i A$ and $\mathfrak{A} = \{g \in \mathrm{End}_A(V_A) \mid \rho \circ g = 0\}$. Then we have homomorphisms in Mod-A^{e}

$$\xi : V \to \mathfrak{A}, \quad v \mapsto (v' \mapsto v\rho(v') - \rho(v)v'),$$

$$\zeta : \mathfrak{A} \to B, \quad g \mapsto g[1].$$

We need the following.

Claim. We have a pull-back diagram in Mod-A^{e} of the form

$$(*) \quad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{-\zeta} & B \\ {\scriptstyle \xi}\uparrow & & \uparrow{\scriptstyle \varphi} \\ V & \xrightarrow{\rho} & A. \end{array}$$

Proof. For any $u \in B$, since $\rho(u^{-1}(v) - u^0(1)v) = u^0(\rho(v)) - u^0(1)\rho(v) = 0$ for all $v \in V$, there exists $g \in \mathfrak{A}$ such that $u - \varphi(u^0(1)) = \zeta(g)$. Also, for any $v \in V$, the cochain map $\zeta(\xi(v)) + \varphi(\rho(v)) : S^\bullet \to S^\bullet$ is homotopic to zero by the homotopy $h : A_A \to V_A$ which sends 1 to v. Therefore, the square (*) is commutative. Next, let $(g, a) \in \mathfrak{A} \times A$ and assume $\varphi(a) = -\zeta(g)$ in B, i.e., there exists a homotopy $h : \zeta(g) + \varphi(a) \simeq 0$. Then $a = \rho(h(1))$ and, since the fact that h is a homotopy, $g(v') + av' = h(1)\rho(v')$ for any $v' \in V$ which implies that we have $g = \xi(h(1))$. Thus we have $\mathrm{Im}\,({}^t[\xi \ \ \rho]) = \mathrm{Ker}\,([-\zeta \ \ \varphi])$. It remains to show that $\mathrm{Ker}\,({}^t[\xi \ \ \rho]) = 0$. Let $K = \mathrm{Ker}\,\rho$ and $\eta : K \to V$ the inclusion.

Denote by

$$\phi : K \to \mathrm{Hom}_A(A_A, K_A), \quad v \mapsto (a \mapsto va),$$

the canonical isomorphism. Then $\mathrm{Hom}_A(\rho, K_A) \circ \phi = \xi \circ \eta$ and by (1) we have

$$\begin{aligned}\mathrm{Ker}\,({}^t[\xi \;\; \rho]) &\cong \mathrm{Ker}\,(\xi \circ \eta)\\ &\cong \mathrm{Ker}\,(\mathrm{Hom}_A(\rho, K_A))\\ &\cong \mathrm{Hom}_A(\mathrm{Cok}\,\rho, K_A)\\ &\cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^\bullet, S^\bullet[-1])\\ &= 0.\end{aligned}$$

Now, according to Claim, it suffices to prove that $\mathrm{Ker}\,\xi = 0$. Note that $\mathrm{Ker}\,\xi \cap \mathrm{Ker}\,\rho = \mathrm{Ker}\,({}^t[\xi \;\; \rho]) = 0$. Thus it suffices to show that $\mathrm{soc}({}_AV_A) \subset \mathrm{Ker}\,\rho$. Note that $\mathrm{soc}({}_AV_A) = \bigoplus_{i \in I_0} \mathrm{soc}({}_AAe_i \otimes_k e_iA_A)$. We claim $\rho(\mathrm{soc}({}_AAe_i \otimes_k e_iA_A)) = 0$ for all $i \in I_0$. Let $i \in I_0$. It is obvious that $\mathrm{soc}({}_AAe_i) \otimes_k \mathrm{soc}(e_iA_A) \subset \mathrm{soc}({}_AAe_i \otimes_k e_iA_A)$. On the other hand, since by Remark 4.2 ${}_AAe_i \otimes_k e_iA_A$ is indecomposable injective, $\mathrm{soc}({}_AAe_i \otimes_k e_iA_A)$ is simple and hence $\mathrm{soc}({}_AAe_i \otimes_k e_iA_A) = \mathrm{soc}({}_AAe_i) \otimes_k \mathrm{soc}(e_iA_A)$. Finally, for any $a \in \mathrm{soc}({}_AAe_i) \subset Je_i$, since e_iA_A is indecomposable injective, the homomorphism $e_iA_A \to A_A$, $x \mapsto ax$, cannot be monic and $a(\mathrm{soc}(e_iA_A)) = 0$.

(3) Note that $\mathfrak{A} \cong \mathrm{Hom}_A(V_A, K_A)$ and that by Remark 4.2(2) V and $\mathrm{Hom}_A(V_A, A)$ are projective-injective in Mod-A^{e}. Thus ξ is a split monomorphism. Also, by (1) and Remark 4.2(2) $\mathrm{Hom}_A(V_A, \rho)$ is a split epimorphism. Thus

$$\begin{aligned}B/A \oplus V \oplus \mathrm{Hom}_A(V_A, A_A) &\cong \mathrm{Cok}\,\xi \oplus V \oplus \mathrm{Hom}_A(V_A, A_A)\\ &\cong \mathrm{Hom}_A(V_A, K_A) \oplus \mathrm{Hom}_A(V_A, A_A)\\ &\cong \mathrm{Hom}_A(V_A, V_A)\\ &\cong V \otimes_A \mathrm{Hom}_A(V_A, A_A)\end{aligned}$$

in Mod-A^{e}. Also, by Remark 4.2(2) we have

$$\mathrm{Hom}_A(V_A, A_A) \cong \bigoplus_{j \in I_0} Ae_{\sigma(j)} \otimes_k e_jA,$$

$$V \otimes_A \mathrm{Hom}_A(V_A, A_A) \cong \bigoplus_{i,j \in I_0} Ae_i \otimes_k e_iAe_{\sigma(j)} \otimes_k e_jA$$

in Mod-A^{e}. Thus

$$B/A \oplus \left(\bigoplus_{j \in I_0} Ae_j \otimes_k e_jA \right) \oplus \left(\bigoplus_{j \in I_0} Ae_{\sigma(j)} \otimes_k e_jA \right) \cong \bigoplus_{i,j \in I_0} \left(Ae_i \otimes_k e_jA \right)^{(d_{ji})}$$

in Mod-A^{e} and the assertion follows by the Krull–Schmidt theorem.

(4) Since $e_iA \otimes_A \rho$ is a split epimorphism, and since $d_{i,\sigma(j)} = d_{ji}$ for all $j \in I_0$, $e_iA \otimes_A^{\bullet} S^{\bullet} \cong \bigoplus_{j \in I_0} e_{\sigma(j)}A[1]^{(\mu_{ij})}$ in $\mathsf{K}(\text{Mod-}A)$ and

$$\begin{aligned} e_iB_B &\cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^{\bullet}, e_iA \otimes_A^{\bullet} S^{\bullet}) \\ &\cong \bigoplus_{j \in I_0} \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^{\bullet}, e_{\sigma(j)}A[1])^{(\mu_{ij})} \end{aligned}$$

in Mod-B. □

Proposition 4.5. *For any $i \in I_0$ there exists a local idempotent $f_i \in e_iBe_i$ such that $f_iB_B \cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^{\bullet}, e_{\sigma(i)}A[1])$. Furthermore, the following hold.*

(1) *$f_iB_B \ncong f_jB_B$ unless $i = j$.*
(2) *$f_iB_B \cong D({}_BBf_{\sigma(i)})$ for all $i \in I_0$.*
(3) *$f_iBf_j \cong e_iAe_j$ for all $i, j \in I_0$.*
(4) *$e_iB_B \cong \bigoplus_{j \in I_0} f_jB_B^{(\mu_{ij})}$ for all $i \in I_0$.*
(5) *$f_iB_A \cong \bigoplus_{j \in I_0} e_jA_A^{(\mu_{ji})}$ for all $i \in I_0$.*

Proof. The existence of a desired f_i follows by the fact that $\mu_{ii} \geqslant 1$.

(1) By the fact that $\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^{\bullet}, -)$ induces an equivalence $\mathrm{add}(S^{\bullet}) \xrightarrow{\sim} \mathscr{P}_B$.

(2) By [4, Lemma 3.1] we have

$$\begin{aligned} f_{\sigma^{-1}(i)}B_B &\cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^{\bullet}, e_iA[1]) \\ &\cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^{\bullet}, D\,\mathrm{Hom}_A(e_{\sigma(i)}A, A)[1]) \\ &\cong D\,\mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(e_{\sigma(i)}A[1], S^{\bullet}) \\ &\cong D({}_BBf_i). \end{aligned}$$

(3) By Remark 4.2(1) we have

$$\begin{aligned} f_iBf_j &\cong \mathrm{Hom}_B(f_jB, f_iB) \\ &\cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(e_{\sigma(j)}A[1], e_{\sigma(i)}A[1]) \\ &\cong e_{\sigma(i)}Ae_{\sigma(j)} \\ &\cong e_iAe_j. \end{aligned}$$

(4) Immediate by Proposition 4.4(4).

(5) We have

$$\begin{aligned} f_iB_A &\cong \mathrm{Hom}_{\mathscr{K}(\mathrm{Mod}\text{-}A)}(S^{\bullet}, e_{\sigma(i)}A[1]) \\ &\cong \mathrm{H}^1(\mathrm{Hom}_A^{\bullet}(S^{\bullet}, e_{\sigma(i)}A)) \\ &\cong \mathrm{Cok}\,(\mathrm{Hom}_A(\rho, e_{\sigma(i)}A)). \end{aligned}$$

Also, by Remark 4.3(1) we have

$$\begin{aligned}\mathrm{Hom}_A(Ae_{\sigma(j)} \otimes_k e_{\sigma(j)}A, e_{\sigma(i)}A_A) &\cong \mathrm{Hom}_k(Ae_{\sigma(j)}, e_{\sigma(i)}Ae_{\sigma(j)})\\ &\cong e_{\sigma(i)}Ae_{\sigma(j)} \otimes_k D({}_AAe_{\sigma(j)})\\ &\cong e_jA_A^{(d_{ij})}\end{aligned}$$

for all $j \in I_0$. Thus we have a split monomorphism

$$\mathrm{Hom}_A(\rho, e_{\sigma(i)}A) : e_{\sigma(i)}A_A \to \bigoplus_{j\in I_0} e_jA_A^{(d_{ij})}$$

and the desired isomorphism follows by the Krull–Schmidt theorem. □

Theorem 4.6. *The mapping cone $T^\bullet$ of the multiplication map*

$$\bigoplus_{i\in I_0} {}_BBf_i \otimes_k e_iA_A \to {}_BB_A$$

is a two-sided tilting complex with $T^\bullet \cong S^\bullet$ in $\mathsf{K}(\text{Mod-}A)$.

We will prove this in the next section (see Theorem 5.3).

Remark 4.7. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ be a tilting complex such that $P^i = 0$ for $i \neq 0, -1$, $\mathrm{add}(P^0) = \mathscr{P}_A$, $P^{-1} \in \mathrm{add}(eA)$ and $\mathrm{H}^0(P^\bullet) \in \text{Mod-}(A/AeA)$. Then by Remark 2.3 $C = \mathrm{End}_{\mathscr{K}(\text{Mod-}A)}(P^\bullet)$ is Morita equivalent to B, so that there exists $V \in \text{Mod-}(C^{\mathrm{op}} \otimes_k B)$ such that $V \otimes^{\boldsymbol{L}}_B T^\bullet$ is a two-sided tilting complex corresponding to $P^\bullet$.

Remark 4.8. Consider the case where k is an algebraically closed field and A is a k-algebra without semisimple algebra summands. Let $e^{(0)} = 1, e^{(1)}, e^{(2)}, \ldots, e^{(l)}$ be a sequence of idempotents in A such that $\mathrm{add}(e^{(i)}A_A) = \mathrm{add}(D({}_AAe^{(i)}))$ for $1 \leqslant i \leqslant l$ and $e^{(i+1)} \in e^{(i)}Ae^{(i)}$ for $0 \leqslant i < l$. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathscr{P}_A)$ be a tilting complex such that $P^i = 0$ for $i > 0$ and $i < -l$, $P^{-i} \in \mathrm{add}(e^{(i)}A)$ for $0 \leqslant i \leqslant l$ and $\mathrm{H}^{-j}(P^\bullet) \in \text{Mod-}(A/Ae^{(i)}A)$ for $0 \leqslant j < i \leqslant l$. It then follows by Proposition 3.2 and Remark 4.7 that there exists a two-sided tilting complex, corresponding to $P^\bullet$, of the form

$$T_l^\bullet \otimes^{\boldsymbol{L}}_{B_{l-1}} \cdots \otimes^{\boldsymbol{L}}_{B_1} T_1^\bullet$$

with the $T_i^\bullet$ two-sided tilting complexes of term length two.

5. Derived equivalent extension algebras

Let A be the same as in Section 4. We will show that an algebra B containing A as a subalgebra satisfying (3) of Proposition 4.4 and (1)–(5) of Proposition 4.5 is derived equivalent to A.

More precisely, let B be a finite dimensional k-algebra containing A as a subalgebra and for each $i \in I_0$ take a local idempotent $f_i \in e_iBe_i$. We assume the following conditions are satisfied:

(b$_1$) ${}_A(B/A)_A \cong \bigoplus_{i,j\in I_0}({}_AAe_i \otimes_k e_jA_A)^{(\alpha_{ij})}$;
(b$_2$) $f_iB_B \not\cong f_jB_B$ unless $i=j$ and $f_iB_B \cong D({}_BBf_{\sigma(i)})$ for all $i\in I_0$;
(b$_3$) $f_iBf_j \cong e_iAe_j$ for all $i,j\in I_0$;
(b$_4$) $e_iB_B \cong \bigoplus_{j\in I_0} f_jB_B^{(\mu_{ij})}$ for all $i\in I_0$; and
(b$_5$) $f_iB_A \cong \bigoplus_{j\in I_0} e_jA_A^{(\nu_{ij})}$ for all $i\in I_0$.

Remark 5.1. The following hold:

(1) ${}_BBf_{\sigma(i)} \otimes_k f_jB_B \cong D({}_BBf_{\sigma(j)} \otimes_k f_iB_B)$ for all $i,j\in I_0$.
(2) ${}_BBe_i \cong \bigoplus_{j\in I_0} {}_BBf_j^{(\mu_{ij})}$ for all $i\in I_0$.
(3) ${}_ABf_i \cong \bigoplus_{j\in I_0} {}_AAe_j^{(\nu_{\sigma^{-1}(i),\sigma^{-1}(j)})}$ for all $i\in I_0$.

Proof. (1) follows by the condition (b$_2$), (2) follows by condition (b$_4$) and (3) follows by conditions (b$_2$), (b$_5$). □

Remark 5.2. For any $i,j\in I_0$ the following hold.

(1) ${}_BBf_{\sigma(i)} \otimes_k e_jA_A \cong D({}_AAe_{\sigma(j)} \otimes_k f_iB_B)$.
(2) $f_i\otimes e_j \in B^{\mathrm{op}}\otimes_k A$ and $e_i\otimes f_j\in A^{\mathrm{op}}\otimes_k B$ are local idempotents.

Proof. (1) follows by conditions (a$_1$), (b$_2$) and (2) follows by conditions (a$_3$), (b$_3$). □

Theorem 5.3. *Denote by $T^\bullet$ the mapping cone of the multiplication map*

$$\delta : \bigoplus_{i\in I_0} {}_BBf_i \otimes_k e_iA_A \to {}_BB_A.$$

Then $T^\bullet$ is a two-sided tilting complex with $T^\bullet \cong S^\bullet$ in $\mathsf{K}(\text{Mod-}A)$ *if*

$$\alpha_{ij} = \begin{cases} d_{ji}-2 & \text{if } i=j=\sigma(j),\\ d_{ji}-1 & \text{if } j\neq\sigma(j) \text{ and } i\in\{j,\sigma(j)\},\\ d_{ji} & \text{otherwise,}\end{cases}$$

$$\mu_{ij}=\nu_{ji}=\begin{cases} d_{ji}-1 & \text{if } i=\sigma(j),\\ d_{ji} & \text{otherwise.}\end{cases}$$

Proof. Set $\Delta = \operatorname{Cok}\delta$ and $\tilde{T}^\bullet = \operatorname{Hom}^\bullet_B(T^\bullet, {}_BB) \in \mathsf{K}(\text{Mod-}(A^{\mathrm{op}}\otimes_k B))$. We divide the proof into several steps.

Claim 1. *There exists a homomorphism in* Mod-$(A^{\mathrm{op}}\otimes_k B)$

$$\tilde{\delta} : {}_AB_B \to \bigoplus_{i\in I_0} {}_AAe_{\sigma(i)} \otimes_k f_iB_B$$

the mapping cone of which is isomorphic to $\tilde{T}^\bullet[1]$. In particular, $\operatorname{Ker}\tilde{\delta} \cong \operatorname{Hom}_B(\Delta, {}_BB)$.

Proof. By condition (a_1), $\mathrm{Hom}_B({}_BBf_i \otimes_k e_jA, {}_BB) \cong Ae_{\sigma(j)} \otimes_k f_iB$ in Mod-$(A^{\mathrm{op}} \otimes_k B)$ for all $i, j \in I_0$.

Claim 2. *$\mathrm{Im}\,\delta = \sum_{i\in I_0} Be_iA = \sum_{i\in I_0} Bf_iB$, so that $\Delta e_i = 0 = f_i\Delta$ for all $i \in I_0$.*

Proof. We claim first that $e_i \in \mathrm{Im}\,\delta$ for all $i \in I_0$. Let $i \in I_0$ and $e_i = \sum_{s=1}^{m} f_{is}$ in B with the f_{is} orthogonal local idempotents. For any $1 \leqslant s \leqslant m$, by the condition (b_4) there exists $l \in I_0$ such that $f_lB_B \cong f_{is}B_B$ and then there exists $b \in f_{is}Bf_l$ such that $f_{is} = bf_l \in \sum_{j\in I_0} Bf_jA = \mathrm{Im}\,\delta$. Next, we claim that $f_iB \subset \mathrm{Im}\,\delta$ for all $i \in I_0$. Note that we have proved $\mathrm{Im}\,\delta = \sum_{j\in I_0} Be_jA$. For any $i \in I_0$, by the condition (b_5) there exists $\phi : \bigoplus_{j\in I_0} e_jA_A{}^{(v_{ij})} \to B_A$ such that $f_iB = \mathrm{Im}\,\phi \subset \sum_{j\in I_0} Be_jA = \mathrm{Im}\,\delta$.

Claim 3. *For any $i \in I_0$ the following hold*:

(1) *$\delta \otimes_A Ae_i$ and $f_iB \otimes_B \delta$ are epic.*
(2) *$e_iA \otimes_A \tilde{\delta}$ and $\tilde{\delta} \otimes_B Bf_i$ are monic.*

Proof. (1) Immediate by Claim 2.
(2) By Claim 2 we have $e_i\mathrm{Hom}_B(\Delta, {}_BB) \cong \mathrm{Hom}_B(\Delta e_i, {}_BB) = 0$ and

$$\begin{aligned}\mathrm{Hom}_B(\Delta, {}_BB)f_i &\cong \mathrm{Hom}_B(\Delta, {}_BBf_i)\\ &\cong \mathrm{Hom}_B(\Delta, D(f_{\sigma^{-1}(i)}B_B))\\ &\cong D(f_{\sigma^{-1}(i)}\Delta)\\ &= 0.\end{aligned}$$

Claim 4. *The following hold.*

(1) *$\tilde{T}^\bullet \otimes_B^\bullet T^\bullet \cong \mathrm{H}^0(\tilde{T}^\bullet \otimes_B^\bullet T^\bullet)$ in $\mathsf{K}(\text{Mod-}A^{\mathrm{e}})$.*
(2) *$\mathrm{H}^0(\tilde{T}^\bullet \otimes_B^\bullet T^\bullet) \cong A$ in Mod-A^{e} if and only if $v_{ij} + v_{\sigma^{-1}(j),i} = \alpha_{\sigma(i),j} + d_{ij}$ for all $i, j \in I_0$.*

Proof. (1) Note first that $\tilde{T}^\bullet \otimes_B^\bullet T^\bullet$ is isomorphic to the total complex of the following commutative square in Mod-A^{e}:

$$\begin{array}{ccc}\bigoplus_{i,j\in I_0} Ae_{\sigma(i)} \otimes_k f_iBf_j \otimes_k e_jA & \xrightarrow{\mathrm{id}\otimes\delta} & \bigoplus_{i\in I_0} Ae_{\sigma(i)} \otimes_k f_iB\\ \uparrow{\scriptstyle \tilde{\delta}\otimes\mathrm{id}} & & \uparrow{\scriptstyle \tilde{\delta}}\\ \bigoplus_{j\in I_0} Bf_j \otimes_k e_jA & \xrightarrow{\delta} & B.\end{array}$$

Also, by Claim 3, $\mathrm{id}\otimes_B\delta$ is epic and $\tilde{\tilde{\delta}}\otimes_B$ id is monic. Furthermore, by condition (b_5) and Remark 5.1(3) we have

$$\bigoplus_{i\in I_0} Ae_{\sigma(i)}\otimes_k f_iB \cong \bigoplus_{i,j\in I_0}(Ae_{\sigma(i)}\otimes_k e_jA)^{(v_{ij})},$$
$$\bigoplus_{j\in I_0} Bf_j\otimes_k e_jA \cong \bigoplus_{i,j\in I_0}(Ae_{\sigma(i)}\otimes_k e_jA)^{(v_{\sigma^{-1}(j),i})}$$

in Mod-A^{e}. Thus, since by Remark 4.2 the ${}_AAe_i\otimes_k e_jA_A$ are projective-injective, it follows that $\tilde{T}^\bullet\otimes^\bullet_B T^\bullet\cong \mathrm{H}^0(\tilde{T}^\bullet\otimes^\bullet_B T^\bullet)$ in K(Mod-A^{e}).

(2) According to condition (b_1), by (1) we have

$$\mathrm{H}^0\left(\tilde{T}^\bullet\otimes^\bullet_B T^\bullet\right)\oplus\left(\bigoplus_{i,j\in I_0}(Ae_{\sigma(i)}\otimes_k e_jA)^{(v_{ij})}\right)\oplus\left(\bigoplus_{i,j\in I_0}(Ae_{\sigma(i)}\otimes_k e_jA)^{(v_{\sigma^{-1}(j),i})}\right)$$
$$\cong A\oplus\left(\bigoplus_{i,j\in I_0}(Ae_{\sigma(i)}\otimes_k e_jA)^{(\alpha_{\sigma(i),j})}\right)\oplus\left(\bigoplus_{i,j\in I_0}(Ae_{\sigma(i)}\otimes_k e_jA)^{(d_{ij})}\right)$$

in Mod-A^{e} and the assertion follows by the Krull–Schmidt theorem.

Claim 5. *The following hold*:

(1) $T^\bullet\otimes^\bullet_A\tilde{T}^\bullet\cong \mathrm{H}^0(T^\bullet\otimes^\bullet_A\tilde{T}^\bullet)$ *in* K(Mod-B^{e}).

(2) $\mathrm{H}^0(T^\bullet\otimes^\bullet_A\tilde{T}^\bullet)\cong B$ *in* Mod-($B^{\mathrm{op}}\otimes_k A$) *if and only if*

$$\sum_{s\in I_0} d_{i,\sigma(s)}v_{sj}+\sum_{s\in I_0}\mu_{si}\alpha_{sj}=\sum_{s\in I_0}\mu_{is}v_{sj}+\sum_{s\in I_0}\mu_{\sigma(s),i}v_{sj}$$

for all $i,j\in I_0$.

(3) $\mathrm{H}^0(T^\bullet\otimes^\bullet_A\tilde{T}^\bullet)\cong B$ *in* Mod-($A^{\mathrm{op}}\otimes_k B$) *if and only if*

$$\sum_{s\in I_0} v_{\sigma^{-1}(s),\sigma^{-1}(i)}d_{s,\sigma(j)}+\sum_{s\in I_0}\alpha_{is}\mu_{sj}$$
$$=\sum_{s\in I_0} v_{\sigma^{-1}(s),\sigma^{-1}(i)}\mu_{sj}+\sum_{s\in I_0} v_{\sigma^{-1}(s),\sigma^{-1}(i)}\mu_{\sigma(j),s}$$

for all $i,j\in I_0$.

Proof. (1) Note first that $T^\bullet\otimes^\bullet_A\tilde{T}^\bullet$ is isomorphic to the total complex of the following commutative square in Mod-B^{e}

$$\begin{array}{ccc}
\bigoplus_{i,j\in I_0} Bf_i\otimes_k e_iAe_{\sigma(j)}\otimes_k f_jB & \xrightarrow{\delta\otimes\mathrm{id}} & \bigoplus_{j\in I_0} Be_{\sigma(j)}\otimes_k f_jB \\
\uparrow{\scriptstyle \mathrm{id}\otimes\tilde{\delta}} & & \uparrow{\scriptstyle B\otimes\tilde{\delta}} \\
\bigoplus_{i\in I_0} Bf_i\otimes_k e_iB & \xrightarrow{\delta\otimes B} & B\otimes_A B.
\end{array}$$

Also, by Claim 3, $\delta \otimes_A \mathrm{id}$ is epic and $\mathrm{id} \otimes_A \tilde{\delta}$ is monic. Furthermore, by condition (b_4) and (1), (2) of Remark 5.1, both $\bigoplus_{i\in I_0} Bf_i \otimes_k e_iB$ and $\bigoplus_{j\in I_0} Be_{\sigma(j)} \otimes_k f_jB$ are projective-injective. Thus $T^\bullet \otimes^\bullet_A \tilde{T}^\bullet \cong \mathrm{H}^0(T^\bullet \otimes^\bullet_A \tilde{T}^\bullet)$ in $\mathsf{K}(\text{Mod-}B^{\mathrm{e}})$.

(2) By conditions (b_1), (b_4), (b_5) and Remark 5.1(2) we have

$$\bigoplus_{i,s\in I_0} Bf_i \otimes_k e_iAe_{\sigma(s)} \otimes_k f_sB \cong \bigoplus_{i,j\in I_0} (Bf_i \otimes_k e_jA)^{(\sum_s d_{i,\sigma(s)}\nu_{sj})},$$

$$\bigoplus_{i\in I_0} Bf_i \otimes_k e_iB \cong \bigoplus_{i,j\in I_0} (Bf_i \otimes_k e_jA)^{(\sum_s \mu_{is}\nu_{sj})},$$

$$\bigoplus_{i\in I_0} Be_{\sigma(i)} \otimes_k f_iB \cong \bigoplus_{i,j\in I_0} (Bf_i \otimes_k e_jA)^{(\sum_s \mu_{\sigma(s),i}\nu_{sj})},$$

$$B \otimes_A B \cong B \oplus \left(\bigoplus_{i,j\in I_0} (Bf_i \otimes_k e_jA)^{(\sum_s \mu_{si}\alpha_{sj})}\right),$$

in $\text{Mod-}(B^{\mathrm{op}} \otimes_k A)$. Thus by (1) we have

$$\mathrm{H}^0(\tilde{T}^\bullet \otimes^\bullet_B T^\bullet) \oplus \left(\bigoplus_{i,j\in I_0} \left(Bf_i \otimes_k e_jA\right)^{(\sum_s \mu_{is}\nu_{sj})}\right) \oplus \left(\bigoplus_{i,j\in I_0} \left(Bf_i \otimes_k e_jA\right)^{(\sum_s \mu_{\sigma(s),i}\nu_{sj})}\right)$$
$$\cong B \oplus \left(\bigoplus_{i,j\in I_0} \left(Bf_i \otimes_k e_jA\right)^{(\sum_s \mu_{si}\alpha_{sj})}\right) \oplus \left(\bigoplus_{i,j\in I_0} \left(Bf_i \otimes_k e_jA\right)^{(\sum_s d_{i,\sigma(s)}\nu_{sj})}\right)$$

in $\text{Mod-}(B^{\mathrm{op}} \otimes_k A)$ and the assertion follows by the Krull–Schmidt theorem.

(3) Similar to (2).

Now, assume

$$\alpha_{ij} = \begin{cases} d_{ji} - 2 & \text{if } i = j = \sigma(j), \\ d_{ji} - 1 & \text{if } j \neq \sigma(j) \text{ and } i \in \{j, \sigma(j)\}, \\ d_{ji} & \text{otherwise,} \end{cases}$$

$$\mu_{ij} = \nu_{ji} = \begin{cases} d_{ji} - 1 & \text{if } i = \sigma(j), \\ d_{ji} & \text{otherwise.} \end{cases}$$

Then, according to Remark 4.3(1), it follows by Claim 4 that $\tilde{T}^\bullet \otimes^\bullet_B T^\bullet \cong A$ in $\mathsf{K}(\text{Mod-}A^{\mathrm{e}})$. Also, it follows by Claim 5 that $T^\bullet \otimes^\bullet_A \tilde{T}^\bullet$ is a two-sided tilting complex for B^{e}. Thus $T^\bullet$ has a right inverse as well as a left inverse, so that $T^\bullet$ is a two-sided tilting complex with $\tilde{T}^\bullet$ the inverse. Namely, $T^\bullet \otimes^\bullet_A \tilde{T}^\bullet \cong B$ in $\mathsf{K}(\text{Mod-}B^{\mathrm{e}})$ and $\tilde{T}^\bullet \cong \mathrm{Hom}^\bullet_A(T^\bullet, A_A)$ in $\mathsf{K}(\text{Mod-}(A^{\mathrm{op}} \otimes_k B))$. Set $\tilde{S}^\bullet \cong \mathrm{Hom}^\bullet_A(S^\bullet, A_A)$. Then it is easy to see that $T^\bullet \otimes_A^\bullet \tilde{S}^\bullet \cong B$ in $\mathsf{K}(\text{Mod-}(B^{\text{op}} \otimes_k A))$. Thus $\tilde{T}^\bullet \cong \tilde{S}^\bullet$ in $\mathsf{K}(\text{Mod-}A^{\text{op}})$ and $T^\bullet \cong S^\bullet$ in $\mathsf{K}(\text{Mod-}A)$. □

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