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Multiple-Burst-Error Correction by Threshold Decoding*

L. R. BAHL† AND R. T. CHIEN

Coordinated Science Laboratory, University of Illinois, Urbana, Illinois 46990

A class of cyclic product codes capable of correcting multiple-burst errors is studied. A code of dimension p is constructed by forming the cyclic product of p one-dimensional single-parity-check codes of relatively prime block lengths. A consideration of the parity-check matrix shows that there are p orthogonal parity checks on each digit, and a burst of length b can corrupt at most one of the parity checks. The maximum allowable value of b can be easily calculated. The codes are completely orthogonal and $[p/2]$ bursts of length b or less can be corrected by one-step threshold decoding.

These codes have a very interesting geometric structure which is also discussed. Using the geometric structure, we show that the codes can also correct 2^{p-2} bursts of relatively short lengths. However, in this case the errors cannot be corrected by threshold decoding.

1. INTRODUCTION

Cyclic codes capable of correcting multiple bursts of errors have been constructed by Corr [*5*], Reed and Solomon [*10*], Stone [*12*], [*13*]), and Burton and Weldon [*3*]. The major problem in implementing any of these codes is that error correction requires complicated decoders. Of all the decoding methods available today, one-step threshold decoding (Massey, [*8*], Rudolph, [*11*]) is one of the easiest to implement. A class of multiple burst error correcting codes which is decodable by one-step threshold decoding is presented in this paper.

For an introduction to cyclic codes, the reader is referred to Peterson [*9*]. A codeword of block length n will be represented by the n tuple

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† L. R. Bahl is now at the T. J. Watson Research Center, IBM, Yorktown Heights, N.Y.

$\bar{v} = (v_0, v_1, \cdots, v_{n-1})$, where $v_i \in GF(q)$. A single-parity-check code consists of all n tuples which satisfy (1).

$$\sum_{i=0}^{n-1} v_i = 0. \tag{1}$$

A single-parity-check code can be considered as a cyclic code having the generator polynomial $g(x) = (x - 1)$.

The codes presented in this paper are cyclic product codes whose constituent subcodes are single-parity-check codes. Properties of cyclic product codes have been studied by Elias [6], Kautz [7], Burton and Weldon [3], Weng [14], Abramson [1], [2] and Calabi and Haefeli [4].

Let $n_1, n_2 \cdots n_p$ be integers which are pairwise relatively prime and $n_1 < n_2 \cdots < n_p$. Let $C_1, C_2, \cdots C_p$ be p single-parity-check codes having block lengths $n_1, n_2 \cdots n_p$ respectively. The code C of length $n = \prod_{i=1}^{p} n_i$ is constructed by forming the product of $C_1, C_2 \cdots C_p$. It follows from the results of Burton and Weldon [3] and Abramson [1] that C is a cyclic code with $k = \prod_{i=1}^{p} (n_i - 1)$ information symbols per block, whose generator polynomial $g_p(x)$ is given by (2).

$$g_p(x) = \text{l.c.m.}\ (x^{m_1} - 1, x^{m_2} - 1, \cdots x^{m_p} - 1) \tag{2}$$

where $m_i = n/n_i$, $i = 1, 2, \cdots p$.

A parity check matrix H_p for this code is given by (3)

$$H_p = \begin{bmatrix} I_{m_1} & I_{m_1} & I_{m_1} & \cdots \\ I_{m_2} & I_{m_2} & I_{m_2} & \cdots \\ \vdots & & & \\ I_{m_p} & I_{m_p} & I_{m_p} & \cdots \end{bmatrix}, \tag{3}$$

where I_{m_i} is the identity matrix of size $m_i \times m_i$. The matrix H_p has $\sum_{i=1}^{p} m_i$ rows and n columns. The rows of H_p are not independent and since the code has $k = \prod_{i=1}^{p} (n_i - 1)$ information symbols per block, H_p has rank $n - \prod_{i=1}^{p} (n_i - 1)$.

2. MULTIPLE BURST ERROR CORRECTION

Let $\bar{v}$ be a codeword of C. From H_p we see that there are p parity checks involving v_0. These can be represented by (4).

$$\sum_{i=0}^{n_j-1} v_{i \cdot m_j} = 0, \qquad j = 1, 2, \cdots p. \tag{4}$$

A set of parity check equations are called orthogonal on v_0 if the digit

v_0 is checked by each of these equations, but no other digit is checked by more than one of these equations. (Massey [8])

LEMMA 1. *The set of p parity check equations given by* (4) *are orthogonal on* v_0.

Proof. Assume that v_k, $0 < k < n$, is a digit which is checked by two or more of the parity check equations.

Let

$$v_0 + \sum_{i=1}^{n_r-1} v_{i \cdot m_r} = 0 \tag{5a}$$

and

$$v_0 + \sum_{j=1}^{n_s-1} v_{j \cdot m_s} = 0 \tag{5b}$$

be two parity check equations, both of which check v_k. Then for some i and j, $k = i \cdot m_r$ and $k = j \cdot m_s$. So k is divisible by m_r and m_s. Since $l \cdot c \cdot m(m_r, m_s) = n$, $n \mid k$. But $k < n$, so k must be 0. This is a contradiction to our assumption $k > 0$.

Hence, there is no digit v_k, $k > 0$, which is checked by more than one of the parity check equations given by (4). Q.E.D.

A burst error of length b is a set of errors confined to b consecutive digit position in the codeword. Since the codes discussed here are cyclic, digits in positions $n - 1$ and 0 are considered to be consecutive. So if a burst B of length b introduces errors in positions k and l, then these positions are at most b apart, i.e., $l - k \equiv s \bmod n$ where $0 \leqq |s| < b - 1$.

Let $t = [p/2]$, i.e., the integer part of $p/2$, and

$$b_t = \min_{i \neq j} [\text{g.c.d. } (m_i, m_j)] = n/n_p \cdot n_{p-1} = n_1 \cdot n_2 \cdots n_{p-2} .$$

THEOREM 1. *The code C corrects any t bursts of length b_t or less.*

Proof. Let $\bar{v} = (v_0, v_1, \cdots v_{n-1})$ be a codeword of C and

$$\bar{e} = (e_0, e_1, \cdots e_{n-1})$$

be an error pattern consisting of t or fewer bursts of length b_t. The received n tuple is $\bar{r} = (r_0, r_1, \cdots r_{n-1}) = \bar{v} + \bar{e}$, where $r_i = v_i + e_i$.

Using $\bar{r}$ and the p orthogonal parity checks on v_0 (Lemma 1), we can form $p + 1$ independent estimates of v_0, i.e.,

$$v_0^{(0)} = r_0 , \tag{6}$$

$$v_0^{(j)} = \sum_{i=1}^{n_j-1} r_{i \cdot m_j} , \qquad j = 1, 2, \cdots p. \tag{7}$$

We now show that at most t of these estimates can be wrong.

Let B_0 be a burst of length b_t or less which introduces an error in position 0. Since none of the digits r_k,

$$k = n - b_t + 1, \qquad n - b_t + 2, \cdots n - 1, \qquad 1, 2, \cdots b_t - 1$$

are involved in the equations represented by (7); any error introduced by B_0 cannot affect the p estimates given by (7).

Let B_1 be a burst of length b_t or less which does not include the digit in position 0 as one of its errors. Assume that B_1 introduces errors in positions k and $l = k + s$, $0 < s < b_t - 1$, and that r_k and r_l are involved in two different estimates of v_0 given by (7). Then $k = i_1 \cdot m_{j_1}$ for some i_1 and j_1 and $l = i_1 \cdot m_{j_2}$ for some i_2 and $j_2 \neq j_1$. Now

$$\text{g.c.d. } (m_{j_1}, m_{j_2}) = n/n_{j_1}n_{j_2} \geqq b_t ,$$

and

$$\text{g.c.d. } (m_{j_1}, m_{j_2}) | (l - k) \therefore (l - k) \geqq b_t , \quad \text{i.e.,} \quad s > b_t ,$$

which is a contradiction. Therefore, errors introduced by B_1 can affect at at most one of the $p + 1$ estimates of v_0.

We have shown that an error burst of length b_t or less can affect at most one of the estimates of v_0. So if the error pattern $\bar{e}$ consists of t or fewer bursts of length b_t, at most t of the estimates of v_0 can be incorrect, hence there will be at least $t + 1$ estimates which are correct and v_0 can be recovered correctly by a majority vote.

Since the code is cyclic, this is true for all the digits v_i, $i = 0, 1, \cdots n - 1$.

Q.E.D.

It follows directly from Theorem 1 that the code C can be one-step threshold decoded. (Massey [8])

EXAMPLE 1. Consider the cove with parameters $n_1 = 5$, $n_2 = 6$, $n_3 = 7$ and $n_4 = 11$. Then $n = 2310$, $k = 1200$, $t = 2$, and $b_2 = 30$, i.e., the code corrects any 2 bursts of length 30 or less.

In the discussion above, b_2 is the maximum allowable length of a burst that affects at most one of the estimates of v_0 given by Eqs. (6) and (7). It is evident that if t' bursts occurs, where $t' < t$, then the code is capable of correcting longer bursts, i.e., bursts of length $b_{t'} > b_t$. A generalization of some of the ideas of Theorem 1 is presented here.

There are p orthogonal parity checks on each digit position. If a burst of length $b_{t'}$ affects at most s' estimates then the code can correct $t' = [p/2s']$ bursts of length $b_{t'}$ by one-step threshold decoding. If we take $s' = 1$, then the results of Theorem 1 follow. It is not possible in general to write down an explicit formula for $b_{t'}$ when $s' > 1$. However, a simple algorithm is developev here for calculating $b_{t'}$ for any $t' < t$.

In equation (7), let T_j be the digit positions involved in the estimate $v_0^{(j)}$. Then

$$T_j = \{a \cdot m_j \mid a = 1, 2, \cdots n_j - 1\} \qquad j = 1, 2, \cdots p.$$

Let B be a burst of length b starting in position a_1 and ending in position $a_2 = a_1 + b - 1$. If B affects more than s' estimate of v_0, then there must exist at least $s' + 1$ integers t_j such that $t_j \in T_j$ and $a_1 \leqq t_j \leqq a_2$. Define $\bar{T} = [\bar{a}_1, \bar{a}_2]$ to be the smallest interval such that $\bar{a}_1 \leqq t_j \leqq \bar{a}_2$ for $s' + 1$ of the j's. Then no burst of length $b_{t'} = \bar{a}_2 - \bar{a}_1$ or less can affect more than s' estimates since this would imply the existence of an interval shorter than $\bar{T}$ satisfying the same conditions as $\bar{T}$, which is a contradiction.

The problem is now to find the shortest interval $\bar{T} = [\bar{a}_1, \bar{a}_2]$ for different value of s'. If $s' = 1$, then this interval is of length $b_t = n/n_p n_{p-1}$ as proved in Theorem 1. For $s' > 1$, the value of $b_{t'}$ can be obtained by a simple computer program. The T_j's can be generated quite easily and a simple search procedure will find $\bar{T}$ for all values of s'.

Example 2. Using the same n_1, n_2, n_3 and n_4 as in Example 1, one obtains $b_1 = 77$, i.e., the code can correct a single burst of length 77 by threshold decoding. This of course exceeds $2 \cdot b_2 = 60$ which is guaranteed from the results of Example 1.

3. GEOMETRIC STRUCTURE OF THE CODES

The code C is the cyclic product of single-parity-check codes. Using this fact, one can construct a geometric model of this code. Kautz [7] and Calabi and Haefeli [4] have investigated the geometric structure of such codes extensively and determined their random error and single burst error correction capabilities. In this section, we study the multiple burst correction properties of these codes using their geometric structure.

Consider a set of $n = \prod_{j=1}^{p} n_j$ digits arranged in a p-dimensional array of size $n_1 \times n_2 \cdots \times n_p$. The array corresponds to a lattice in the shape of a rectangular parallelopiped in p-dimensional Euclidean space with the lattice points having integral coordinates $(i_1, i_2, \cdots i_p)$ such that $0 \leqq i_j < n_j$. The $k = \prod_{j=1}^{p} (n_j - 1)$ data digits are placed at the lattice points with coordinates $(i_1, i_2, \cdots i_p)$ with $0 \leqq i_j < j_n - 1$. The digit in position $(i_1, i_2, \cdots i_p)$ is denoted by $u_{(i_1, i_2, \cdots, i_p)}$. We complete the lattice by inserting parity check digits at the remaining $r = n - k$ positions in such a manner that the parity along any line parallel to a coordinate axis is even. The resultant lattice L is then a codeword of the p-dimensional product code C' whose constituent subcodes are single-parity-check codes of block lengths $n_1, n_2, \cdots n_p$.

Using the approach of Burton and Weldon [3], one can define a mapping F from the lattice L to the n tuple $\bar{v}$. The mapping F is derived in the following manner.

Since g.c.d. $(m_j, n_j) = 1, j = 1, 2, \cdots p$, there exist integers $t_1, t_2, \cdots t_p$ such that

$$t_j \cdot m_j \equiv 1 \bmod n_j, \qquad j = 1, 2, \cdots p. \tag{8}$$

Then

$$i = F(i_1, i_2, \cdots i_p) = \sum_{j=1}^{p} t_j \cdot m_j \cdot i_j \bmod n. \tag{9}$$

The digit $u_{(i_1, i_2, \cdots i_p)}$ in L is mapped to v_i in $\bar{v}$ where i is given by (9). This mapping is 1–1 and the inverse mapping G is given by

$$G(i) = (i_1, i_2, \cdots i_p) \quad \text{where} \quad i_j = i \bmod n_j. \tag{10}$$

(Residue classes are represented by the smallest nonnegative integer in that class).

Under the mapping F the p-dimensional code C' is equivalent to the cyclic code C whose generator polynomial is given by (2). (Burton and Weldon, [3])

It is known that the minimum distance of the code C' is 2^p and that the codewords of weight exactly 2^p are those patterns having nonzero digits at the corners of a p-dimensional rectangular parallelopiped sublattice of L, [7].

Each p'-dimensional hyperplane of L parallel to the coordinate axes is a p'-dimensional subcode having a structure similar to C'. Each such hyperplane can be considered as a p'-dimensional product code having minimum distance $2p'$.

LEMMA 2. *If e_k and e_l, $k \neq l$ are errors belonging to the same burst of length n_1, then the points corresponding to k and l in L cannot both lie on any p'-dimensional hyperplane of L for $p' < p$.*

Proof. Since e_k and e_l belong to a burst of length n_1, either (i) $l - k \equiv a \bmod n$ or (ii) $l - k \equiv -a \bmod n$ for some $0 < a < n_1$.

Let $G(k) = (k_1, k_2, \cdots k_p)$ and $G(l) = (l_1, l_2, \cdots l_p)$.

In case (i),
$$l_j - k_j \equiv a \bmod n_j. \tag{11a}$$
In case (ii),
$$l_j - k_j \equiv a \bmod n_j \equiv (n_j - a) \bmod n_j. \tag{11b}$$
for $j = 1, 2, \cdots p$, since n_1 is the smallest of the n_j's.

For $u_{(k_1,k_2,\cdots k_p)}$ and $u_{(l_1,l_2,\cdots l_p)}$ to be on the same $(p-1)$-dimensional hyperplane we must have

$$k_{j_1} = l_{j_1} \quad \text{for some} \quad j_1, \quad \text{i.e.,} \quad l_{j_1} - k_{j_1} \equiv 0 \bmod n_{j_1}.$$

But from (11a) and (11b) we see that $l_j - k_j \not\equiv 0 \bmod n_j$ for any j. Therefore, the points corresponding to k and l in L cannot both lie on the same $(p-1)$-dimensional hyperplane, and consequently on any p'-dimensional hyperplane of L for $p' < p$. Q.E.D.

Lemma 3. *Let e_k and e_l be errors belonging to the same burst of length n_1 with $G(k) = (k_1, k_2, \cdots k_p)$ and $G(l) = (l_1, l_2, \cdots l_p)$.*

If $(l_{j_1} - k_{j_1}) \equiv a' \bmod n_{j_1}$ and $(l_{j_2} - k_{j_2}) \equiv a' \bmod n_{j_2}$ for some $0 < a' < n_1$, $j_1 \neq j_2$, then $(l_j - k_j) \equiv a' \bmod n_j$ for all $j = 1, 2, \cdots p$.

Proof. As in Lemma 2, either (i) $l - k \equiv a \bmod n$ or (ii) $l - k \equiv -a \bmod n$. In case (ii) $(l_j - k_j) \equiv n_j - a \bmod n_j$, $j = 1, 2, \cdots p$ and since $n_{j_1} \neq n_{j_2}$ for all $j_1 \neq j_2$, we need not consider this case.

In case (i), $l_j - k_j \equiv a \bmod n_j$, $j = 1, 2, \cdots p$. Therefore, if $(l_{j_1} - k_{j_1}) \equiv a' \bmod n_{j_1}$, and $(l_{j_2} - k_{j_2}) \equiv a' \bmod n_{j_2}$ then $a' = a$ and $(l_j - k_j) \equiv a' \equiv a \bmod n_j$, $j = 1, 2, \cdots p$. Q.E.D.

Theorem 2. *For $p \geqq 3$, the code C can correct all burst patterns consisting of 2^{p-2} or less bursts of length n_1.*

Proof. To prove this theorem, we need to show that there is no nonzero codeword in C which is a pattern of $2 \cdot 2^{p-2}$ or fewer bursss of length n_1. This is proved by contradiction.

Let $\bar{v}$ be a nonzero codeword of C which is a pattern of 2^{p-1} or fewer bursts of length n_1. Since the code is cyclic, we may assume, without loss of generality, that one of the bursts, call it B_1, starts in position 0. Using the mapping G we construct the lattice L which is the equivalent of $\bar{v}$. Then L must be a codeword of C'.

Now $\bar{v}_0 \neq 0$, therefore, $u_{(0,0,\cdots 0)} \neq 0$. Consider the $p-1$ dimensional hyperplane of L which contains all points $(i_1, i_2, \cdots i_p)$ with $i_1 = 0$. This hyperplane contains a nonzero digit $u_{(0,0,\cdots 0)}$. and corresponds to a $(p-1)$-dimensional code having minimum distance 2^{p-1}, therefore it must contain at least 2^{p-1} nonzero digits. It cannot contain more than 2^{p-1} nonzero digits, since a burst of length n_1 can contribute at most 1 nonzero digit to a $(p - 1)$-dimensional hyperplane (Lemma 2). This hyperplane, therefore, contains exactly 2^{p-1} nonzero digits and $\bar{v}$ must be a pattern of exactly 2^{p-1} bursts. The 2^{p-1} nonzero points in the hyperplane must lie on the corners of a $(p - 1)$-dimensional rectangular parallelopiped, which can be represented by the set

$$T_{i_1=0} = \{(i_1, i_2, \cdots i_p) | i_1 = 0, i_j = 0 \quad \text{or} \quad a_j \quad \text{for} \quad j = 2, 3, \cdots p\}$$

where $a_2, a_3, \cdots a_p$ are $p - 1$ integers with $0 < a_j < n_j$. Then

$$u_{(0,0,\cdots 0)} \in B_1, \; u_{(0,a_2,0,\cdots 0)} \in B_2, \; u_{(0,0,a_3,0,\cdots 0)} \in B_3, \qquad \text{etc.},$$

where $B_1, B_2, \cdots B_2(p - 1)$ are different bursts of length n_1.

Considering other hyperplanes of dimension $p - 1$ and using the condition that each hyperplane is a codeword of minimum distance 2^{p-1}, it follows that there are nonzero digits at the 2^p points which are represented by the set

$$T = \{(i_1, i_2, \cdots i_p) | i_j = 0 \quad \text{or} \quad a_j, j = 1, 2, \cdots p\}.$$

(i). Now consider the nonzero digit in position $(i_1, i_2, \cdots i_p)$, $i_j = a_j$ for all $j = 1, 2, \cdots p$. This point is coplanar (i.e., lies on the same $(p - 1)$-dimensional hyperplane) with all points belonging to $T_{i_1=0}$, except the point $(0, 0, \cdots 0)$. By Lemma 2, $u_{(0,0,\cdots 0)}$ and $u_{(a_1,a_2,\cdots a_p)}$ must both belong to B_1. Since B_1 starts in position v_0, the only digits that can belong to B_1 are $v_1, v_2, \cdots v_{n_1-1}$. Therefore $(a_1, a_2, \cdots a_p) = G(a) = (a, a, \cdots a)$, where $0 < a < n_1 - 1$.

$$a_j = a, \qquad j = 1, 2, \cdots p. \tag{12}$$

(ii). Now consider the point $(i_1, i_2, \cdots i_p)$ with $i_2 = 0$ and $i_j = a_j$ for $j \neq 2$. This point is coplanar with all points belonging to $T_{i_1=0}$ except the point $(i_1', i_2', \cdots i_p')$ with $i_2' = a_2$ and $i_j' = 0$ for $j \neq 2$.

Now

$$i_j - i_{j'} \equiv a_j \equiv a \bmod n_j, \qquad j \neq 2, \tag{13a}$$

and

$$i_2 - i_{2'} \equiv -a_2 \equiv n_2 - a \bmod n_2. \tag{13b}$$

By Lemma 3

$$n_2 - a \equiv a \bmod n_2, \tag{14}$$

since $0 < a < n_2 \qquad n_2 = 2a.$

(iii). Now consider the point $(i_1, i_2, \cdots i_p)$ with $i_3 = 0$ and $i_j = a_j$ for $j \neq 3$. This point is coplanar with all points of $T_{i_1=0}$ except the point $(i_1'', i_2'', \cdots i_p'')$ with $i_3'' = a_3$ and $i_j'' = 0$ for $j \neq 3$. Then as in (ii), we have

$$n_3 = 2a. \tag{15}$$

From Eq. (15) and (16) $n_2 = n_3$ which is a contradiction.

Therefore, our assumption that $\bar{v}$ is a codeword of C, consisting of 2^{p-2} or fewer bursts of length n_1 must be false. Q.E.D.

EXAMPLE 3. Consider the code of example 1 with $n_1 = 5, n_2 = 6, n_3 = 7$ and $n_4 = 11$. Then by Theorem 2, this code corrects 4 bursts of length 5.

We can consider the code C' as the interlace of n_1 codewords of a code C'' of block length $n'' = \prod_{i=2}^{p} n_i$ which has dimension $p - 1$ and minimum distance 2^{p-1}. Then from the burst error correcting capability of interlace codes (Corr, [5], Burton and Weldon, [3]) we know that C' can correct $2^{p-2} - 1$ bursts of length n_1. Theorem 2 proves that the code is capable of correcting at least one more burst than is indicated by the interlace code bound. However, there does not seem to be any simple decoding algorithm for correcting 2^{p-2} bursts of length n_1.

4. CONCLUSIONS

We have studied the multiple burst correcting properties of a class of cyclic product codes. Kautz [7] investigated the random error and single burst correction capabilities of these codes. Theorem 1 shows that a p-dimensional code of block length $n = n_1 \times n_2 \times \cdots n_p$ can correct $[p/2]$ bursts of length $n_1 \times n_2 \times \cdots n_{p-2}$, and that such error correction can be achieved in a very simple way. Theorem 2, based on the geometric structure of the code, shows that the code is also capable of correcting 2^{p-2} bursts of length n_1, which is one burst more than is indicated by the interlace code bound.

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